A NEW AND EFFICIENT METHOD FOR THE COMPUTATION OF LEGENDRE COEFFICIENTS

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ABSTRACT. An efficient procedure for the computation of the coefficients of Legendre expansions is here presented. We prove that the Legendre coefficients associated with a function $f(x)$ can be represented as the Fourier coefficients of an Abel–type transform of $f(x)$. The computation of N Legendre coefficients can then be performed in $\mathcal{O}(N \log N)$ operations with a single Fast Fourier Transform of the Abel–type transform of $f(x)$.

1. INTRODUCTION

The efficient computation of the coefficients of Legendre expansions is a very important problem in numerical analysis and applied mathematics with a wide range of applications including, just to mention a few, approximation theory, solution of partial differential equations and quadratures. Recently its relevance emerged also in connection with the computation of spectra of highly oscillatory Fredholm integral operators, which play an important role in laser engineering [2].

The difficulty of the problem lies essentially in the fact that these coefficients are represented by integrals whose integrands oscillate rapidly for large values of the index of the polynomials. Using standard quadrature procedures for the calculation of N Legendre coefficients leads only to *slow* $\mathcal{O}(N^2)$ algorithms (see, e.g., Ref. [5]). More efficiently, in Ref. [1] (see also [13, 14]) the Legendre coefficients are obtained by a suitable transformation of the corresponding Chebyshev coefficients, which yields *faster* $\mathcal{O}(N(\log N)^2)$ algorithms. Recently this problem has been also clearly

discussed in a paper by A. Iserles [10], in which an algorithm for the computation of the Legendre coefficients, which is certainly fast and brilliant, is presented.

In this paper we present an alternative procedure. The basic idea of our method consists in exploiting the Dirichlet–Murphy integral representation of the Legendre polynomials. Next, we prove that the coefficients of the Legendre expansion of a function $f(x)$ are connected with a subset of the Fourier coefficients (the ones with nonnegative index) of an Abel–type transform of $f(x)$.

The numerical implementation of the algorithm follows straightforwardly and is very efficient. The aforementioned Fourier coefficients (which represent the searched Legendre coefficients) can be computed in $\mathcal{O}(N \log N)$ operations by a single Fast Fourier Transform after the evaluation of the Abel–type integral by means of standard quadrature techniques.

2. Connection of Legendre expansions to Fourier series

The standard form of the Legendre expansion reads:

$$(2.1) \qquad f(x) = \sum_{n=0}^{\infty} c_n\, P_n(x) \qquad x \in [-1,1],$$

where $P_n(x)$ are the Legendre polynomials, which can be defined by the generating function [6]:

$$(2.2) \qquad \sum_{n=0}^{\infty} P_n(x)\, t^n = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}},$$

and the coefficients $\{c_n\}_{n=0}^{\infty}$ are given by:

$$(2.3) \qquad c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_n(x)\, \mathrm{d}x \qquad (n \geqslant 0).$$

The conditions to be satisfied by $f(x)$ to guarantee the uniform convergence of the series in (2.1) are discussed in [8]. However, for our purpose of computing the Legendre coefficients c_n it is sufficient to assume that $f(x)$ be summable in the interval $[-1,1]$.

We can now state the following theorem.

Theorem 1. *The coefficients $\{a_n\}_{n=0}^{\infty}$, defined as $a_n \doteq \frac{c_n}{(2n+1)}$, coincide with the Fourier coefficients (with $n \geqslant 0$) of an Abel–type transform of $f(x)$, that is:*

$$(2.4) \qquad a_n \doteq \frac{c_n}{(2n+1)} = \int_{-\pi}^{\pi} \widehat{f}(y)\, e^{iny}\, \mathrm{d}y \qquad (n \geqslant 0),$$

where the 2π-periodic function $\widehat{f}(y)$ is defined by

$$(2.5) \qquad \widehat{f}(y) = \frac{1}{2\pi i}\, \varepsilon(y)\, e^{i\frac{y}{2}} \int_{\cos y}^{1} \frac{f(x)}{[2(x - \cos y)]^{\frac{1}{2}}}\, \mathrm{d}x \qquad (y \in \mathbb{R}),$$

$\varepsilon(y)$ being the sign function.

Proof. Plugging the Dirichlet–Murphy integral representation of the Legendre polynomials [15, Ch. III, §5.4]:

$$(2.6) \qquad P_n(\cos x) = -\frac{i}{\pi} \int_{x}^{(2\pi - x)} \frac{e^{i(n+\frac{1}{2})y}}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, \mathrm{d}y,$$

into equality (2.3) (after the change of variable $x \to \cos x$), we have:

$$(2.7) \qquad 2\pi i\, a_n = \int_0^\pi dx\, f(\cos x) \sin x \int_x^{(2\pi - x)} \frac{e^{i(n+\frac{1}{2})y}}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, dy.$$

Interchanging the order of integration in (2.7) we have:

$$(2.8) \qquad \begin{aligned} 2\pi i\, a_n &= \int_0^\pi dy\, e^{i(n+\frac{1}{2})y} \int_0^y f(\cos x)\, \frac{\sin x}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, dx \\ &\quad + \int_\pi^{2\pi} dy\, e^{i(n+\frac{1}{2})y} \int_0^{(2\pi - y)} f(\cos x)\, \frac{\sin x}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, dx. \end{aligned}$$

Next, if we make the change of variables: $y \to y - 2\pi$ and $x \to -x$, the second integral on the r.h.s. of (2.8) becomes:

$$(2.9) \qquad e^{i\pi} \int_{-\pi}^0 dy\, e^{i(n+\frac{1}{2})y} \int_0^y f(\cos x)\, \frac{\sin x}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, dx.$$

Finally, we obtain:

$$(2.10) \qquad \begin{aligned} 2\pi i\, a_n &= \int_0^\pi dy\, e^{i(n+\frac{1}{2})y} \int_0^y f(\cos x)\, \frac{\sin x}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, dx \\ &\quad + e^{i\pi} \int_{-\pi}^0 dy\, e^{i(n+\frac{1}{2})y} \int_0^y f(\cos x)\, \frac{\sin x}{[2(\cos x - \cos y)]^{\frac{1}{2}}}\, dx, \end{aligned}$$

which, after the change of variable $\cos x \to x$ into the integrals on the r.h.s., yields:

$$(2.11) \qquad a_n = \int_{-\pi}^\pi \widehat{f}(y)\, e^{iny}\, dy \qquad (n \geqslant 0),$$

with $\widehat{f}(y)$ given by (2.5). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is easy to check from (2.5) that $\widehat{f}(y)$ satisfies the following symmetry relation:

$$(2.12) \qquad \widehat{f}(y) = -e^{iy}\, \widehat{f}(-y).$$

This latter, along with formulae (2.4) and (2.5), allows us to write the Legendre coefficients c_n in the following form:

$$(2.13) \qquad c_n = \frac{2}{\pi}\left(n + \frac{1}{2}\right) \int_0^\pi \phi(y) \sin\left[\left(n + \frac{1}{2}\right)y\right]\, dy,$$

where

$$(2.14) \qquad \phi(y) = \int_{\cos y}^1 \frac{f(x)}{[2(x - \cos y)]^{\frac{1}{2}}}\, dx.$$

The numerical implementation of the algorithm first requires the computation of the Abel–type integral $\widehat{f}(y)$ defined in (2.5) (or, equivalently, of the function $\phi(y)$ in (2.14)). The integrand presents a weak algebraic singularity at the end point of the domain of integration, which can be effectively handled by means of a proper nonlinear change of variable. This technique, along with the use of a standard quadrature formula (e.g., the Gauss-Legendre one), allows obtaining high accuracy with a small number of nodes [12].

TABLE 1. *True and computed Legendre coefficients c_n for the function $f(x) = |x|^{3/2}$.*

n	True c_n	Computed c_n	Error
0	0.4000000000000000000	0.4000000000268187694	2.68_{-12}
2	0.6666666666666666666	0.6666666671084839901	4.42_{-11}
4	-0.09230769230769230769	-0.09230769246022409169	1.53_{-10}
6	0.03921568627450980392	0.03921568630769028951	3.32_{-11}
8	-0.02197802197802197802	-0.02197802226184656509	2.84_{-10}
10	0.01411764705882352941	0.01411764762633771833	5.68_{-10}
12	-0.00985221674876847290	-0.00985221698441776129	2.36_{-10}
14	0.00727272727272727272	0.00727272772068757959	4.48_{-10}
16	-0.00559179869524697110	-0.00559179938291245667	6.88_{-10}
18	0.00443458980044345898	0.00443458980116155730	7.18_{-13}
20	-0.00360360360360360360	-0.00360360382001508440	2.16_{-10}
22	0.00298656047784967645	0.00298656131882347585	8.41_{-10}
24	-0.00251572327044025157	-0.00251572347582495201	2.05_{-10}
26	0.00214822771213748657	0.00214822836757538881	6.55_{-10}
28	-0.00185586142901330034	-0.00185586305079012375	1.62_{-09}
30	0.00161943319838056680	0.00161943394840411781	7.50_{-10}

Finally, formula (2.4) makes it possible to take full advantage of the computational efficiency of the Fast Fourier Transform both in terms of speed of computation and of accuracy [3, 9]. The calculation of the first N coefficients of the expansion can consequently be accomplished in $\mathcal{O}(N \log N)$ operations.

The algorithm described has been implemented in double precision arithmetics using the open source GNU Scientific Library (GSL) [7], and its performance has been tested on a variety of functions.

First, feasibility and accuracy of the algorithm have been verified by direct comparison of the obtained numerical results with the true Legendre coefficients for the function $f(x) = |x|^{3/2}$, whose Legendre coefficients are known to be [4, p. 78]:

$$(2.15) \qquad c_n = \begin{cases} 0 & \text{if } n \text{ odd,} \\ (\alpha+1)^{-1} & \text{if } n = 0, \\ \dfrac{(2n+1)\,\alpha\,(\alpha-2)\cdots(\alpha-n+2)}{(\alpha+1)(\alpha+3)\cdots(\alpha+n+1)} & \text{otherwise,} \end{cases}$$

where $\alpha = 3/2$. Values of the computed Legendre coefficients along with the absolute error are given in Table 1.

The increment of performances, with respect to the computation of the Legendre coefficients c_n by ordinary quadrature, has been verified in terms of speed of computation at (nearly) equality of precision. All accuracies have been determined by comparing the results of the algorithm with the reference values of c_n, computed with 20 significant figures by standard quadrature with Mathematica [11]. For these tests we used various functions (many of them have been already used in previous works), including polynomials, exponential/hyperbolic functions, rational functions (e.g., $f(x) = \frac{1+x}{\gamma^2+x^2}$ with $\gamma = $ constant). All the results have confirmed the enormous increase of computational speed (the expected improvement ratio being proportional to $N/\log N$). Such an increase of performances will become even more crucial for the efficient evaluation of multivariate Legendre transform [2] and expansions in Gegenbauer (alias Ultraspherical Legendre) polynomials, which will be the subject of a forthcoming paper.

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