UNIVERSITY OF CALIFORNIA
RIVERSIDE

Division Algebras, Supersymmetry and Higher Gauge Theory

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

John Gmerek Huerta

June 2011

Dissertation Committee:
    Professor John Baez, Chairperson
    Professor Vyjayanthi Chari
    Professor Stefano Vidussi

Acknowledgements

A dissertation is the capstone to a doctoral program, and the acknowledgements provide a useful place to thank the countless people who have helped out along the way, both personally and professionally.

First, of course, I thank my advisor, John Baez. It is hard to imagine a better advisor, and no one deserves more credit for my mathematical and professional growth during this program. "Thanks" does not seem sufficient, but it is all I have to give.

Also deserving special mention is John's collaborator, James Dolan. I am convinced there is no subject in mathematics for which Jim does not have some deep insight, and I thank him for sharing a few of these insights with me. Together, John and Jim are an unparalleled team: there are no two better people with whom to talk about mathematics, and no two people more awake to the joy of mathematics.

I would also like to thank Geoffrey Dixon, Tevian Dray, Robert Helling, Corinne Manogue, Chris Rogers, Hisham Sati, James Stasheff, and Riccardo Nicoletti for helpful conversations and correspondence. I especially thank Urs Schreiber for many discussions of higher gauge theory and L_∞-superalgebras, smooth ∞-groups, and supergeometry.

In addition, I thank Vyjayanthi Chari and Stefano Vidussi for serving on my committee. I thank Mike Stay for providing some beautiful diagrams formatted in TikZ. I thank An Huang, Theo Johnson-Freyd and David Speyer for catching some errors. This work was partially supported by the FQXi grant RFP2-08-04, and by a 2008–2009 Research Mentorship Fellowship from UC Riverside. I owe FQXi and UC Riverside thanks for their support.

Many of the people I name above are not just colleagues, but close personal friends. Besides those above, I would also like to thank Jack Bennett, Alex Hoffnung, Felix Marhuenda-Donate, Rob Niemeyer, Tim Ridenour and Jonathan Sarhad, among many others, for making graduate school so much fun. Finally, I would like to thank my girlfriend, Jen Crowley, for being a wonderful, loving companion.

For my parents, Eliseo and Marion Huerta.

ABSTRACT OF THE DISSERTATION

Division Algebras, Supersymmetry and Higher Gauge Theory

by

John Gmerek Huerta

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2011
Professor John Baez, Chairperson

From the four normed division algebras—the real numbers, complex numbers, quaternions and octonions, of dimension $k = 1$, 2, 4 and 8, respectively—a systematic procedure gives a 3-cocycle on the Poincaré superalgebra in dimensions $k + 2$=3, 4, 6 and 10, and a 4-cocycle on the Poincaré superalgebra in dimensions $k+3$=4, 5, 7 and 11. The existence of these cocycles follow from spinor identities that hold only in these dimensions, and which are closely related to the existence of the superstring in dimensions $k + 2$, and the super-2-brane in dimensions $k + 3$.

In general, an $(n + 1)$-cocycle on a Lie superalgebra yields a 'Lie n-superalgebra': that is, roughly, an n-term chain complex equipped with a bracket satisfying the axioms of a Lie superalgebra up to chain homotopy. We thus obtain Lie 2-superalgebras extending the Poincaré superalgebra in dimensions $k + 2$, and Lie 3-superalgebras extending the Poincaré superalgebra in dimensions $k + 3$. We present evidence, based on the work of Sati, Schreiber and Stasheff, that these Lie n-superalgebras describe infinitesimal 'higher symmetries' of the superstring and 2-brane.

Generically, integrating a Lie n-superalgebra to a Lie n-supergroup yields a 'Lie n-supergroup' that is hugely infinite-dimensional. However, when the Lie n-superalgebra is obtained from an $(n + 1)$-cocycle on a nilpotent Lie superalgebra, there is a geometric procedure to integrate the cocycle to one on the corresponding nilpotent Lie supergroup.

In general, a smooth $(n + 1)$-cocycle on a supergroup yields a 'Lie n-supergroup': that is, a weak n-group internal to supermanifolds. Using our geometric procedure to integrate the 3-cocycle in dimensions $k + 2$, we obtain a Lie 2-supergroup extending the Poincaré supergroup in those dimensions, and similarly integrating the 4-cocycle in dimensions $k + 3$, we obtain a Lie 3-supergroup extending the Poincaré supergroup in those dimensions.

Contents

List of Tables

Chapter 1

Introduction

1.1 Overview

There is a deep relationship between supersymmetry, division algebras, and higher gauge theory. In this thesis, we begin to tell this story: how division algebras give rise to higher infinitesimal symmetries of strings and membranes, modeled by a generalization of a Lie algebra called a 'Lie n-algebra', and how this infinitesimal picture can be integrated to global one, with higher symmetries modeled by a 'Lie n-group'. In this overview, we want to take the opportunity to explain the big picture, postponing references until the next section.

From a physical perspective, gauge theory is the geometric language which allows us to describe how point particles change as they move through spacetime. *Higher* gauge theory is a generalization which describes how strings and membranes change as they move through spacetime.

We can view higher gauge theory as a categorification of gauge theory, which is intuitively clear from the diagrams we use to describe higher categories: as a particle moves through spacetime from point x to point y, it sweeps out a worldline γ that we can view as a morphism from x to y in a certain category:

$$x \bullet \overset{\gamma}{\longrightarrow} \bullet y \, .$$

The job of a connection in gauge theory is to assign to γ an element $\mathrm{hol}(\gamma)$ in the gauge group which describes how the state of our particle changes as it moves along γ.

Boosting up a dimension, when a string moves through spacetime, it sweeps out a worldsheet Σ, which we can view as a 2-morphism in a certain 2-category:

$$\bullet \Downarrow \Sigma \bullet \, .$$
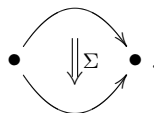
The job of a '2-connection' in 'higher gauge theory' is to assign to Σ an element $\mathrm{hol}(\Sigma)$ in the 'higher gauge group' which describes how the state of our string changes as it moves along Σ.

In practice, the strings and membranes of interest in physics are supersymmetric, so they are called superstrings and supermembranes. This also leads to higher gauge theory, but it goes through the normed division algebras. There is a mysterious connection between supersymmetry and the

four normed division algebras: the real numbers, complex numbers, quaternions and octonions. This can be seen in super-Yang–Mills theory, in superstring theory, and in theories of supermembranes and supergravity. Most simply, the connection is visible from the fact that the normed division algebras have dimensions 1, 2, 4 and 8, while classical superstring theories and minimal super-Yang–Mills theories live in spacetimes of dimension *two* higher: 3, 4, 6 and 10. The simplest classical super-2-brane theories make sense in spacetimes of dimensions *three* higher: 4, 5, 7 and 11. Classical supergravity makes sense in all of these dimensions, but the octonionic cases are the most important: in 10 dimensions supergravity is a low-energy limit of superstring theory, while in 11 dimensions it is believed to be a low-energy limit of 'M-theory', which incorporates the 2-brane.

These numerical relationships are far from coincidental. They arise because we can use the normed division algebras to construct the spacetimes in question, as well as their associated spinors. A certain spinor identity that holds in dimensions 3, 4, 6 and 10 is an easy consequence of this construction, as is a related identity that holds in dimensions 4, 5, 7 and 11. These identities are fundamental to the physical theories just listed.

Yet these identities have another interpretation: they are cocycle conditions in Lie superalgebra cohomology for suitably chosen Lie superalgebras. We can use them to categorify the infinitesimal symmetries of spacetime, or rather its supersymmetric analog, superspacetime. This gives rise to Lie 2-superalgebras and Lie 3-superalgebras.

Thanks to work by Hisham Sati, Urs Schreiber and Jim Stasheff, we expect that generalized connections valued in these Lie 2- and 3-algebras will incorporate fields of interest to string theory and supergravity. However, these generalized connections are described in terms of infinitesimal data, because Lie n-superalgebras are infinitesimal objects. We would like to know the global story, so we want to integrate these to Lie n-supergroups.

Given a Lie n-algebra, there is a general technique, due to Getzler and Henriques, to build a Lie n-group which integrates it. Usually, these are hugely infinite-dimensional. For instance, if $\mathfrak{g}$ is the finite-dimensional Lie algebra of a simply-connected, finite-dimensional Lie group G, applying the construction of Getzler and Henriques yields not G, but a simplicial Banach manifold which is infinite-dimensional at almost every level.

Fortuantely, our Lie n-algebras are special. The cocycles which define them are defined on nilpotent Lie subsuperalgebras, and these can be integrated using a geometric method to smooth cocycles on the corresponding Lie supergroups. So we obtain Lie n-supergroups which are finite-dimensional, and even algebraic. We expect that studying the higher gauge theory of these Lie n-supergroups will yield important results for physics.

1.2 Introduction

The relationship between division algebras and supersymmetry can be seen in super-Yang–Mills theory, in superstring theory, and in theories of supermembranes and supergravity. Most simply, the connection is visible from the fact that the normed division algebras have dimensions 1, 2, 4 and 8, while classical superstring theories and minimal super-Yang–Mills theories live in spacetimes of dimension *two* higher: 3, 4, 6 and 10. The simplest classical super-2-brane theories make sense in spacetimes of dimensions *three* higher: 4, 5, 7 and 11. Classical supergravity makes sense in all of these dimensions, but the octonionic cases are the most important: in 10 dimensions supergravity is a low-energy limit of superstring theory, while in 11 dimensions it is believed to be a low-energy limit of 'M-theory', which incorporates the 2-brane.

As we noted in our overview, these numerical relationships are far from coincidental. They arise because we can use the normed division algebras to construct the spacetimes in question, as well as their associated spinors. In a bit more detail, suppose $\mathbb{K}$ is a normed division algebra of dimension k. There are just four examples:

- the real numbers $\mathbb{R}$ ($k = 1$),

- the complex numbers $\mathbb{C}$ ($k = 2$),

- the quaternions $\mathbb{H}$ ($k = 4$),

- the octonions $\mathbb{O}$ ($k = 8$).

Then we can identify vectors in $(k + 2)$-dimensional Minkowski spacetime with 2×2 hermitian matrices having entries in $\mathbb{K}$. Similarly, we can identify spinors with elements of $\mathbb{K}^2$. Matrix multiplication then gives a way for vectors to act on spinors. There is also an operation that takes two spinors ψ and ϕ and forms a vector $[\psi, \phi]$. Using elementary properties of normed division algebras, we can prove that

$$[\psi, \psi]\psi = 0.$$

Following Schray [59], we call this identity the '3-ψ's rule'. This identity is an example of a 'Fierz identity'—roughly, an identity that allows one to reorder multilinear expressions made of spinors. This can be made more visible in the 3-ψ's rule if we polarize the above cubic form to extract a genuinely trilinear expression:

$$[\psi, \phi]\chi + [\phi, \chi]\psi + [\chi, \psi]\phi = 0.$$

In fact, the 3-ψ's rule holds *only* when Minkowski spacetime has dimension 3, 4, 6 or 10. Moreover, it is crucial for super-Yang–Mills theory and superstring theory in these dimensions. In minimal super-Yang–Mills theory, we need the 3-ψ's rule to check that the Lagrangian is supersymmetric, thanks to an argument we will review in Chapter 4. In superstring theory, we need it to check the supersymmetry of the Green–Schwarz Lagrangian [40, 39]. But the 3-ψ's rule also has a deeper significance, which we study here.

This deeper story involves not only the 3-ψ's rule but also the '4-Ψ's rule', a closely related Fierz identity required for super-2-brane theories in dimensions 4, 5, 7 and 11. To help the reader see the forest for the trees, we present a rough summary of this story in the form of a recipe:

1. Spinor identities that come from division algebras are *cocycle conditions*.

2. The corresponding cocycles allow us to extend the Poincaré Lie superalgebra to a higher structure, a *Lie n-superalgebra*.

3. Connections valued in these Lie n-superalgebras describe the *field content* of superstring and super-2-brane theories.

To begin our story in dimensions 3, 4, 6 and 10, let us first introduce some suggestive terminology: we shall call $[\psi, \phi]$ the **bracket of spinors**. This is because this function is symmetric, and it defines a Lie superalgebra structure on the supervector space

$$\mathcal{T} = V \oplus S$$

where the even subspace V is the vector representation of $\mathrm{Spin}(k+1,1)$, while the odd subspace S is a certain spinor representation. This Lie superalgebra is called the **supertranslation algebra**.

There is a cohomology theory for Lie superalgebras, sometimes called Chevalley–Eilenberg cohomology. The cohomology of $\mathcal{T}$ will play a central role in what follows. Why? First, because the 3-ψ's rule is really a cocycle condition, for a 3-cocycle α on $\mathcal{T}$ which eats two spinors and a vector and produces a number as follows:

$$\alpha(\psi, \phi, A) = \langle \psi, A\phi \rangle.$$

Here, $\langle -, - \rangle$ is a pairing between spinors. Since this 3-cocycle is Lorentz-invariant, it extends to a cocycle on the Poincaré superalgebra

$$\mathfrak{siso}(k+1,1) \cong \mathfrak{so}(k+1,1) \ltimes \mathcal{T}.$$

In fact, we obtain a nonzero element of the third cohomology of the Poincaré superalgebra this way.

Just as 2-cocycles on a Lie superalgebra give ways of extending it to larger Lie superalgebras, $(n+1)$-cocycles give extensions to *Lie n-superalgebras*. To understand this, we need to know a bit about L_∞-algebras [49, 57]. An L_∞-algebra is a chain complex equipped with a structure like that of a Lie algebra, but where the laws hold only 'up to d of something'. A Lie n-algebra is an L_∞-algebra in which only the first n terms are nonzero. All these ideas also have 'super' versions. In general, an $\mathfrak{h}$-valued $(n+1)$-cocycle ω on $\mathfrak{g}$ is a linear map:

$$\Lambda^{n+1}\mathfrak{g} \to \mathfrak{h}$$

satisfying a certain equation called a 'cocycle condition'. We can use an $\mathfrak{h}$-valued $(n+1)$-cocycle ω on a Lie superalgebra $\mathfrak{g}$ to extend $\mathfrak{g}$ to a Lie n-superalgebra of the following form:

$$\mathfrak{g} \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } \cdots \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } \mathfrak{h}.$$

Here, $\mathfrak{g}$ sits in degree 0 while $\mathfrak{h}$ sits in degree $n-1$. We call Lie n-superalgebras of this form 'slim Lie n-superalgebras', and denote them by $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$.

In particular, we can use the 3-cocycle α to extend $\mathfrak{siso}(k+1,1)$ to a slim Lie 2-superalgebra of the following form:

$$\mathfrak{siso}(k+1,1) \xleftarrow{\ d\ } \mathbb{R} \ .$$

We call this the 'superstring Lie 2-superalgebra', and denote it as $\mathfrak{superstring}(k+1,1)$. The superstring Lie 2-superalgebra is an extension of $\mathfrak{siso}(k+1,1)$ by $\mathfrak{b}\mathbb{R}$, the Lie 2-algebra with $\mathbb{R}$ in degree 1 and everything else trivial. By 'extension', we mean that there is a short exact sequence of Lie 2-superalgebras:

$$0 \to b\mathbb{R} \to \mathfrak{superstring}(k+1,1) \to \mathfrak{siso}(k+1,1) \to 0.$$

To see precisely what this means, let us expand it a bit. Lie 2-superalgebras are 2-term chain complexes, and writing these vertically, our short exact sequence looks like this:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathfrak{siso}(k+1,1) & \longrightarrow & \mathfrak{siso}(k+1,1) & \longrightarrow & 0
\end{array}
$$

In the middle, we see $\mathfrak{superstring}(k + 1, 1)$. This Lie 2-superalgebra is built from two pieces: $\mathfrak{siso}(k + 1, 1)$ in degree 0 and $\mathbb{R}$ in degree 1. But since the cocycle α is nontrivial, these two pieces still interact in a nontrivial way. Namely, the Jacobi identity for three 0-chains holds only up to d of a 1-chain. So, besides its Lie bracket, the Lie 2-superalgebra $\mathfrak{superstring}(k + 1, 1)$ also involves a map that takes three 0-chains and gives a 1-chain. This map is just α.

What is the superstring Lie 2-algebra good for? The answer lies in a feature of string theory called the 'Kalb–Ramond field', or 'B field'. The B field couples to strings just as the A field in electromagnetism couples to charged particles. The A field is described locally by a 1-form, so we can integrate it over a particle's worldline to get the interaction term in the Lagrangian for a charged particle. Similarly, the B field is described locally by a 2-form, which we can integrate over the worldsheet of a string.

Gauge theory has taught us that the A field has a beautiful geometric meaning: it is a connection on a U(1) bundle over spacetime. What is the corresponding meaning of the B field? It can be seen as a connection on a 'U(1) gerbe': a gadget like a U(1) bundle, but suitable for describing strings instead of point particles. Locally, connections on U(1) gerbes can be identified with 2-forms. But globally, they cannot. The idea that the B field is a U(1) gerbe connection is implicit in work going back at least to the 1986 paper by Gawedzki [37]. More recently, Freed and Witten [34] showed that the subtle difference between 2-forms and connections on U(1) gerbes is actually crucial for understanding anomaly cancellation. In fact, these authors used the language of 'Deligne cohomology' rather than gerbes. Later work made the role of gerbes explicit: see for example Carey, Johnson and Murray [19], and also Gawedzki and Reis [36].

More recently still, work on higher gauge theory has revealed that the B field can be viewed as part of a larger package. Just as gauge theory uses Lie groups, Lie algebras, and connections on bundles to describe the parallel transport of point particles, higher gauge theory generalizes all these concepts to describe parallel transport of extended objects such as strings and membranes [9, 11]. In particular, Schreiber, Sati and Stasheff [55] have developed a theory of 'n-connections' suitable for describing parallel transport of objects with n-dimensonal worldvolumes. In their theory, the Lie algebra of the gauge group is replaced by a Lie n-algebra—or in the supersymmetric context, a Lie n-superalgebra. Applying their ideas to $\mathfrak{superstring}(k + 1, 1)$, we get a 2-connection which can be described locally using the following fields:

$\mathfrak{superstring}(k + 1, 1)$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 2-form
$\downarrow$	
$\mathfrak{siso}(k + 1, 1)$	$\mathfrak{siso}(k + 1, 1)$-valued 1-form

The $\mathfrak{siso}(k+1, 1)$-valued 1-form consists of three fields which help define the background geometry on which a superstring propagates: the Levi-Civita connection A, the vielbein e, and the gravitino

ψ. But the $\mathbb{R}$-valued 2-form is equally important in the description of this background geometry: it is the B field!

Alas, this is only part of the story. Rather than building $\mathfrak{superstring}(k+1,1)$ with the cocycle α, quantum considerations indicate we should really use a certain linear combination of α and the canonical 3-cocycle on $\mathfrak{so}(k+1,1)$. This canonical 3-cocycle can be defined on any simple Lie algebra. It comes from combining the Killing form with the bracket:

$$j = \langle -, [-,-] \rangle.$$

To ensure that certain quantum 'anomalies' cancel, we need to replace α with the linear combination:

$$\frac{1}{32}j + \frac{1}{2}\alpha.$$

These coefficients come from careful analysis of the anomalies associated with superstring theory. See the paper by Bonora et al. and the references therein [15].

We choose, however, to focus on α. This simplifies our later work, and because Lie 2-algebras based on j have already been the subject of much scrutiny [5, 13, 42, 58], it should be possible to combine what we do here with the work of other authors to arrive at a more complete picture.

Next let us extend these ideas to Minkowski spacetimes one dimension higher: dimensions 4, 5, 7 and 11. In this case a certain subspace of 4×4 matrices with entries in $\mathbb{K}$ will form the vector representation of $\mathrm{Spin}(k+2,1)$, while $\mathbb{K}^4$ will form a spinor representation. As before, there is a 'bracket' operation that takes two spinors Ψ and Φ and gives a vector $[\Psi, \Phi]$. As before, there is an action of vectors on spinors. This time the 3-ψ's rule no longer holds:

$$[\Psi, \Psi]\Psi \neq 0.$$

However, we show that

$$[\Psi, [\Psi, \Psi]\Psi] = 0.$$

We call this the '4-Ψ's rule'. This identity plays a basic role for the super-2-brane, and related theories of supergravity. The explicit relationship between this identity and the division algebras goes back to Foot and Joshi [33].

Once again, the bracket of spinors defines a Lie superalgebra structure on the supervector space

$$\mathcal{T} = \mathcal{V} \oplus \mathcal{S}$$

where now $\mathcal{V}$ is the vector representation of $\mathrm{Spin}(k+2,1)$, while $\mathcal{S}$ is a certain spinor representation of this group. Once again, the cohomology of $\mathcal{T}$ plays a key role. The 4-Ψ's rule is a cocycle condition—but this time for a 4-cocycle β which eats two spinors and two vectors and produces a number as follows:

$$\beta(\Psi, \Phi, \mathcal{A}, \mathcal{B}) = \langle \Psi, (\mathcal{A} \wedge \mathcal{B})\Phi \rangle.$$

Here, $\langle -, - \rangle$ denotes the inner product of two spinors, and the bivector $\mathcal{A} \wedge \mathcal{B}$ acts on Φ via the usual Clifford action. Since β is Lorentz-invariant, we shall see that it extends to a 4-cocycle on the Poincaré superalgebra $\mathfrak{siso}(k+2,1)$.

We can use β to extend the Poincaré superalgebra to a Lie 3-superalgebra of the following form:

$$\mathfrak{siso}(k+2,1) \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } \mathbb{R} \ .$$

We call this the '2-brane Lie 3-superalgebra', and denote it as $2\text{-}\mathfrak{brane}(k+1,1)$. It is an extension of $\mathfrak{siso}(k+2,1)$ by $\mathfrak{b}^2\mathbb{R}$, the Lie 3-algebra with $\mathbb{R}$ in degree 2, and everything else trivial. In other words, there is a short exact sequence:

$$0 \to \mathfrak{b}^2\mathbb{R} \to 2\text{-}\mathfrak{brane}(k+2,1) \to \mathfrak{siso}(k+2,1) \to 0.$$

Again, to see what this means, let us expand it a bit. Lie 3-superalgebras are 3-term chain complexes. Writing out each of these vertically, our short exact sequence looks like this:

In the middle, we see $2\text{-}\mathfrak{brane}(k+2,1)$.

The most interesting Lie 3-algebra of this type, $2\text{-}\mathfrak{brane}(10,1)$, plays an important role in 11-dimensional supergravity. This idea goes back to the work of Castellani, D'Auria and Fré [20, 25]. These authors derived the field content of 11-dimensional supergravity starting from a differential graded-commutative algebra. Later, Sati, Schreiber and Stasheff [55] explained that these fields can be reinterpreted as a 3-connection valued in a Lie 3-algebra which they called '$\mathfrak{sugra}(10,1)$'. This is the Lie 3-algebra we are calling $2\text{-}\mathfrak{brane}(10,1)$. One of our messages here is that the all-important cocycle needed to construct this Lie 3-algebra arises naturally from the octonions, and has analogues for the other normed division algebras.

If we follow these authors and consider a 3-connection valued in $2\text{-}\mathfrak{brane}(10,1)$, we find it can be described locally by these fields:

$2\text{-}\mathfrak{brane}(k+2,1)$	Connection component
$\mathbb{R}$	$\mathbb{R}$-valued 3-form
$\downarrow$	
0	
$\downarrow$	
$\mathfrak{siso}(k+2,1)$	$\mathfrak{siso}(k+2,1)$-valued 1-form

Again, a $\mathfrak{siso}(k+2,1)$-valued 1-form contains familiar fields: the Levi-Civita connection, the vielbein, and the gravitino. But now we also see a 3-form, called the C field. This is again something we might expect on physical grounds, at least in dimension 11. While the case is less clear than in string theory, it seems that for the quantum theory of a 2-brane to be consistent, it must propagate in a background obeying the equations of 11-dimensional supergravity, in which the C field naturally shows up [66]. The work of Diaconescu, Freed, and Moore [28], as well as that of Aschieri and Jurco [2], is also relevant here.

So far, we have focused on Lie 2- and 3-algebras and generalized connections valued in them. This connection data is infinitesimal: it tells us how to parallel transport strings and 2-branes a little bit. Ultimately, we would like to understand this parallel transport globally, as we do with particles in ordinary gauge theory.

To achieve this global description, we will need 'Lie n-groups' rather than Lie n-algebras. Naively, one expects a Lie 2-supergroup $\mathrm{Superstring}(k+1,1)$ for which the Lie 2-superalgebra $\mathfrak{superstring}(k+1,1)$ is the infinitesimal approximation, and similarly a Lie 3-supergroup 2-$\mathrm{Brane}(k+2,1)$ for which the Lie 3-superalgebra 2-$\mathfrak{brane}(k+1,1)$ is an infinitesimal approximation. In fact, this is precisely what we will construct.

In order to 'integrate' Lie n-algebras to obtain Lie n-groups, we will have to overcome two obstacles: how does one define a Lie n-group? And, how does one integrate a Lie n-algebra to a Lie n-group? To answer the former question, we take a cue from Baez and Lauda's definition of Lie 2-group: it is a categorified Lie group, a 'weak 2-category' with one object with a manifold of weakly associative and weakly invertible morphisms, a manifold of strictly associative and strictly invertible 2-morphisms, and all structure maps smooth. While this definition is known to fall short in important ways, it has the virtue of being fairly simple. Ultimately, one should use an alternative definition, like that of Henriques [42] or Schommer-Pries [58], which weakens the notion of product on a group: rather than an algebraic operation in which there is a unique product of any two group elements, 'the' product is defined only up to equivalence.

So, roughly speaking, a Lie n-group should be a 'weak n-category' with one object, a manifold of weakly invertible morphisms, a manifold of weakly invertible 2-morphisms, and so on, up to a manifold of strictly invertible n-morphisms. To make this precise, however, we need to get very precise about what a 'weak n-category' is, which becomes more complicated as n gets larger. We therefore limit ourselves to the tractable cases of $n = 2$ and 3. We further limit ourselves to what we call a 'slim Lie n-group', at least for $n = 2$ and 3.

A 'slim Lie 2-group' is what Baez and Lauda call a 'special Lie 2-group': it is a skeletal bicategory with one object, a Lie group G of morphisms, a Lie group $G \ltimes H$ of 2-morphisms, and the group axioms hold strictly *except for associativity*—there is a nontrivial 2-morphism called the 'associator':

$$a(g_1, g_2, g_3) \colon (g_1 g_2) g_3 \Rightarrow g_1 (g_2 g_3).$$

The associator, in turn, satisfies the 'pentagon identity', which says the following pentagon commutes:

We shall see that this identity forces a to be a 3-cocycle on the Lie group G of morphisms. We denote the Lie 2-group of this from by $\mathrm{String}_a(G, H)$.

Likewise, a 'slim Lie 3-group' is a skeletal tricategory with one object, with a Lie group G of morphisms, trivial 2-morphisms, and a Lie group $G \ltimes H$ of 3-morphisms. The associator is necessarily trivial, because it is a 2-morphism:

$$a(g_1, g_2, g_3) \colon (g_1 g_2) g_3 \overset{1}{\Longrightarrow} g_1 (g_2 g_3),$$

However, it does not satisfy the pentagon identity! There is a nontrivial 3-morphism called the 'pentagonator':

This 3-morphism satisfies an identity of its own, called the 'pentagonator identity'. Similar to the case with the slim Lie 2-group $\mathrm{String}_a(G, H)$, the pentagonator identity forces π to be a Lie group 4-cocycle on G.

Moreover, we can generalize all of this to obtain Lie 2-supergroups and Lie 3-supergroups from 3- and 4-cocycles on Lie supergroups. In general, we expect that any supergroup $(n + 1)$-cocycle f gives rise to a slim n-supergroup, $\mathrm{Brane}_f(G, H)$, though this cannot be made precise without being more definite about n-categories for higher n.

Nonetheless, the precise examples of Lie 2- and 3-groups suggest a strong parallel to the way Lie algebra $(n + 1)$-cocycles give rise to Lie n-algebras. And this parallel suggests a naive scheme to integrate Lie n-algebras. Given a slim Lie n-superalgebra $\mathfrak{brane}_\omega(\mathfrak{g}, \mathfrak{h})$, we seek a slim Lie n-supergroup $\mathrm{Brane}_f(G, H)$ where:

- G is a Lie supergroup with Lie superalgebra $\mathfrak{g}$; i.e. it is a Lie supergroup integrating $\mathfrak{g}$,

- H is a Lie supergroup with Lie superalgebra $\mathfrak{h}$; i.e. it is a Lie supergroup integrating $\mathfrak{h}$,

- f is a Lie supergroup $(n + 1)$-cocycle on G that, in some suitable sense, integrates the Lie superalgebra $(n + 1)$-cocycle ω on $\mathfrak{g}$.

Admittedly, we only define $\mathrm{Brane}_f(G, H)$ precisely when $n = 2$ or 3, but that will suffice to handle our cases of interest, $\mathfrak{superstring}(k + 1, 1)$ and $\mathfrak{2\text{-}brane}(k + 2, 1)$.

Unfortunately, this naive scheme fails to work even for well-known examples of slim Lie 2-algebras, such as the the string Lie 2-algebra $\mathfrak{string}(n)$. In this case, we can:

- integrate $\mathfrak{so}(n)$ to $\mathrm{Spin}(n)$ or $\mathrm{SO}(n)$,

- integrate $\mathbb{R}$ to $\mathbb{R}$ or $\mathrm{U}(1)$,

- but there is no hope to integrate ω to a nontrivial $(n + 1)$-cocycle f on $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$, because compact Lie groups admit *no nontrivial smooth cocycles*.

Really, this failure is a symptom of the fact that our definition of Lie n-group is oversimplified. There are more sophisticated approaches to integrating the string Lie 2-algbera, like those due to Baez, Crans, Schreiber and Stevenson [13] or Schommer-Pries [58], and a general technique to integrate any Lie n-algebra due to Henriques [42], which Schreiber [61] has in turn generalized to handle Lie n-superalgebras and more. All three techniques involve generalizing the notion of Lie 2-group (or Lie n-group, for Henriques and Schreiber) away from the world of finite-dimensional manifolds, and the latter two generalize the notion of 2-group to one in which products are defined only up to equivalence.

Given this history, it is remarkable that the naive scheme we outlined for integration actually works for the Lie n-superalgebras we really care about—namely, the superstring Lie 2-algbera and the super-2-brane Lie 3-algebra. Moreover, this is not some weird quirk unique to these special cases, but the result of a beautiful geometric procedure for integrating Lie algebra cocycles defined on a *nilpotent* Lie algebra. Originally invented by Houard [43], we generalize this technique to the case of nilpotent Lie superalgebras and supergroups.

Finally, we mention another use for the cocycles α and β. These cocycles are also used to build Wess–Zumino–Witten terms for superstrings and 2-branes. For example, in the case of the string, one can extend the string's worldsheet to be the boundary of a three-dimensional manifold, and then integrate α over this manifold. This provides an additional term for the action of the superstring, a term that is required to give the action Siegel symmetry, balancing the number of bosonic and fermionic degrees of freedom. For the 2-brane, the Wess–Zumino–Witten term is constructed in complete analogy—we just 'add one' to all the dimensions in sight [1, 30].

Indeed, the network of relationships between supergravity, string and 2-brane theories, and cocycles constructed using normed division algebras is extremely tight. The Siegel symmetry of the string or 2-brane action constrains the background of the theory to be that of supergravity, at least in dimensions 10 and 11 [66], and without the WZW terms, there would be no Siegel symmetry. The WZW terms rely on the cocycles α and β. These cocycles also give rise to the Lie 2- and 3-superalgebras $\mathfrak{superstring}(9,1)$ and $\mathfrak{2\text{-}brane}(10,1)$. And these, in turn, describe the field content of supergravity in these dimensions!

As further grist for this mill, WZW terms can also be viewed in the context of higher gauge theory. In string theory, the WZW term is the holonomy of a connection on a $\mathrm{U}(1)$ gerbe [36]. Presumably the WZW term in a 2-brane theory is the holonomy of a connection on a $\mathrm{U}(1)$ 2-gerbe [64]. This is a tantalizing clue that we are at the beginning of a larger but ultimately simpler story.

1.3   Plan of the thesis

The focus of this thesis is not on the applications to physics that we sketched in the Introduction, but on constructing Lie n-superalgebras from division algebras, and integrating these Lie n-superalgebras to Lie n-supergroups. We organize the thesis as follows:

- In Chapter 2, we give a review of the needed facts about normed division algebras, and apply the division algebras to construct vectors and spinors in spacetimes of certain dimension. We conclude by using these constructions to prove certain spinor identities needed for supersymmetric physics.

- In Chapter 3, we introduce the algebra underlying supersymmetry: super vector spaces and Lie superalgebras. We construct some important examples of Lie superalgebras: the super-

translation algebras, $\mathcal{T}$, using division algebras. We give a well-known generalization of Chevalley–Eilenberg cohomology to Lie superalgebras, and prove that the supertranslation algebras admit nontrivial cocycles thanks to the spinor identities from the previous chapter.

- In Chapter 4, we take a break from the larger story to discuss super-Yang–Mills theory. We prove the supersymmetry of super-Yang–Mills theory in spacetime dimensions 3, 4, 6 and 10, using the division algebras.

- In Chapter 5, we describe how a Lie superalgebra $(n + 1)$-cocycle on $\mathfrak{g}$ gives rise to a Lie n-superalgebra which extends $\mathfrak{g}$. We use this general construction to build several important examples of Lie 2- and 3-superalgebras: the well-known string Lie 2-algebra $\mathfrak{string}(n)$ extending $\mathfrak{so}(n)$, the Heisenberg Lie 2-algebra extending the Heisenberg Lie algebra, the superstring Lie 2-algebra $\mathfrak{superstring}(k + 1, 1)$ and the super-2-brane Lie 3-algebra $2\text{-}\mathfrak{brane}(k + 2, 1)$, both extending the Poincaré superalgebras $\mathfrak{siso}(k + 1, 1)$ and $\mathfrak{siso}(k + 2, 1)$, respectively.

- In Chapter 6, we describe Lie group cohomology based on smooth group cochains. We define Lie n-groups for $n = 2$ and 3, using bicategories and tricategories internal to the category of smooth manifolds. We sketch how a Lie group $(n + 1)$-cocycle on G gives rise to a Lie n-group which extends G, and give a full construction for $n = 2$ and 3.

- In Chapter 7, we apply a little-known geometric technique to integrate nilpotent Lie n-algebras to Lie n-groups, by integrating Lie algebra $(n + 1)$-cocycles to Lie group $(n + 1)$-cocycles. We compute some examples for 2-step nilpotent Lie algebras, and conclude with by constructing the Heisenberg Lie 2-group from the Lie 2-algebra.

- In Chapter 8, we introduce a little supergeometry. Specifically, we sketch the definition of supermanifold, and discuss the functor of points approach to studying these spaces. We describe how to get a supermanifold from any super vector space, and show the corresponding functor of points is especially simple. We then describe how to integrate a nilpotent Lie superalgebra to a supergroup.

- In Chapter 9, we generalize everything from Chapter 6 to the super setting. We describe Lie supergroup cohomology, and we define Lie n-supergroups for $n = 2$ and 3, using bicategories and tricategories internal to the category of supermanifolds.

- In Chapter 10, we generalize everything from Chapter 7 to the super setting. We show how to integrate nilpotent Lie n-superalgebras to Lie n-supergroups, by integrating Lie superalgebra $(n+1)$-cocycles to Lie supergroup $(n+1)$-cocycles. This is done using the functor of points.

- Finally, in Chapter 11, we apply the results of the previous chapter to integrate $\mathfrak{superstring}(k + 1, 1)$ and $2\text{-}\mathfrak{brane}(k + 2, 1)$ to Lie n-supergroups, $\text{Superstring}(k + 1, 1)$ and $2\text{-}\text{Brane}(k + 2, 1)$. We conclude with some remarks about where these results could lead, and the next steps in this research program.

1.4   Prior work

Portions of this thesis are adapted from two papers coauthored with my advisor, John Baez, called "Supersymmetry and division algebras I and II" [7, 8]. Specifically, Sections 2.1, 2.2, 3.1, and

Chapter 4 are adapted from the first paper [7], Sections 1.2, 2.3, 3.2, the beginning of Chapter 5 and Section 5.1.4 are adapted from the second paper [8], and Section 2.4 combines related results from both papers.

Chapter 2

Spacetime geometry from division algebras

In this chapter, we begin to explore the relationship between:

- Normed division algebras of dimension $k = 1, 2, 4$ and 8.

- Superstring theories in spacetimes of dimension $k + 2 = 3, 4, 6$ and 10.

- Super-2-brane theories in spacetimes of dimension $k + 3 = 4, 5, 7$ and 11.

Physically, a supersymmetric theory requires the use of vector representations of the Lorentz group to describe its bosonic degrees of freedom, and the spinor representations of the Lorentz group to describe its fermionic degrees of freedom. In this chapter, we will show that a normed division algebra $\mathbb{K}$ of dimension k can be used to construct vectors and spinors in $k+2$ and $k+3$ dimensions.

First, let us describe the most general situation. Let V be a real vector space equipped with a nondegenerate quadratic form, $| \cdot |^2$. The group $\mathrm{Spin}(V)$, the double-cover of $\mathrm{SO}_0(V)$, acts on V as the symmetries of $| \cdot |^2$. We say that V is the **vector representation** of $\mathrm{Spin}(V)$, and call its elements **vectors**.

We can also construct representations $\mathrm{Spin}(V)$ by considering the Clifford algebra, $\mathrm{Cliff}(V)$. This is the associative algebra generated by V for which elements $A \in V$ square to their norm:

$$\mathrm{Cliff}(V) = TV/A^2 \sim |A|^2,$$

where TV denotes the tensor algebra on V. Because the **Clifford relation** $A^2 = |A|^2$ respects the parity of the number of vectors in any expression, the Clifford algebra is $\mathbb{Z}_2$-graded:

$$\mathrm{Cliff}(V) = \mathrm{Cliff}_0(V) \oplus \mathrm{Cliff}_1(V).$$

We call $\mathrm{Cliff}_0(V)$ and $\mathrm{Cliff}_1(V)$ the **even part** and **odd part** of $\mathrm{Cliff}(V)$, respectively. $\mathrm{Cliff}_0(V)$ is the subalgebra of $\mathrm{Cliff}(V)$ generated by products of pairs of vectors, while $\mathrm{Cliff}_1(V)$ is a mere subspace of $\mathrm{Cliff}(V)$, spanned by products of odd numbers of vectors.

It is well-known that $\mathrm{Spin}(V)$ lives inside $\mathrm{Cliff}_0(V)$. This is the group generated by products of pairs of **unit vectors**: vectors A for which $|A|^2 = \pm 1$. So, we can consider representations of

$\mathrm{Spin}(V)$ that come from modules of $\mathrm{Cliff}_0(V)$. Such a representation is called a **spinor representation** of $\mathrm{Spin}(V)$, and its elements are called **spinors**. The algebra $\mathrm{Cliff}_0(V)$ turns out to be either a matrix algebra or the sum of two matrix algebras, so there are either two irreducible spinor representations, S_+ and S_-, or just one, S. In this latter case, let us define $S_+ = S_- = S$, so that we may use uniform notation throughout. For a wonderfully clear introduction to Clifford algebras, including a complete classification, see the text of Porteous [53].

Since there are many different modules of $\mathrm{Cliff}_0(V)$, there are many different spinor representations. Physicists distinguish some of them with special names like 'Majorana spinors' or 'Weyl spinors', and we will see some examples of these below. We do not, however, need to define these terms precisely, because such distinctions are only important for comparing our work to the literature. Instead, we shall see how to handle all the vectors and spinors we need in a uniform way using normed division algebras.

So far, we have said nothing that depends on the dimension of the space of vectors, V. In some some dimensions, special phenomena occur, thanks to the existence of the normed division algebras. A **normed division algebra** is a real, possibly nonassociative algebra $\mathbb{K}$ with 1, equipped with a norm $|\cdot|$ satisfying

$$|ab| = |a||b|.$$

As with the complex numbers, this norm can be expressed using conjugation: $|a|^2 = aa^* = a^*a$, where $*\colon \mathbb{K} \to \mathbb{K}$ is a suitable involution. By a classic theorem of Hurwitz [44], there are only four finite-dimensional normed division algebras: the real numbers, $\mathbb{R}$, the complex numbers, $\mathbb{C}$, the quaternions, $\mathbb{H}$, and the octonions, $\mathbb{O}$. These algebras have dimension 1, 2, 4, and 8. Only the octonions are nonassociative, but mildly so: they are **alternative**, meaning that the subalgebra generated by any two elements is associative.

One can use the theory of Clifford algebras to prove that normed division algebras can only occur in these dimensions. This is a two-way street, however, and we will traverse it the other way, using the division algebras to better understand objects that are usually only studied with Clifford algebras: vectors and spinors. For a division algebra $\mathbb{K}$ of dimension k, we will mainly be interested in the vectors and spinors in Minkowski spacetime of dimension $k + 2$ or $k + 3$, but we can get a taste for how this works just by considering Euclidean space of dimension k.

In this case, something remarkable happens. Namely, we can identify the vector and irreducible spinor representations with the division algebra itself:

$$V = \mathbb{K}, \quad S_+ = \mathbb{K}, \quad S_- = \mathbb{K}.$$

Because each of these representations is just $\mathbb{K}$ in disguise, there is an obvious way for a vector to act on a spinor: multiplication! We define:

$$\begin{array}{rccc} \cdot\colon & V \otimes S_+ & \to & S_- \\ & A \otimes \psi & \mapsto & A\psi \end{array}$$

and

$$\begin{array}{rccc} \cdot\colon & V \otimes S_- & \to & S_+ \\ & A \otimes \phi & \mapsto & A^*\phi. \end{array}$$

Because the action of V swaps the spinor spaces, it preserves their direct sum, $S_+ \oplus S_-$. Acting on this latter space with the same vector twice, we get:

$$
\begin{aligned}
A \cdot A \cdot (\psi, \phi) &= A \cdot (A^* \phi, A\psi) \\
&= (A^* A \psi, AA^* \phi) \\
&= |A|^2 (\psi, \phi).
\end{aligned}
$$

Note that nonassociativity poses no problem for us in the above calculation, thanks to alternativity: everything in sight takes place in the subalgebra generated by only two elements, A and ψ.

Now, the above equation is the Clifford relation: acting twice by A is the same as multiplying by $|A|^2$. Thus the map $V \otimes (S_+ \oplus S_-) \to S_+ \oplus S_-$ induces a homomorphism:

$$
\mathrm{Cliff}(V) \to \mathrm{End}(S_+ \oplus S_-).
$$

In this way, $S_+ \oplus S_-$ becomes a module of $\mathrm{Cliff}(V)$. Because acting by vectors swaps S_+ and S_-, both of these subspaces are preserved by the subalgebra $\mathrm{Cliff}_0(V)$ generated by products of pairs of vectors, and in this way they become representations of $\mathrm{Spin}(V)$.

We thus see how the vectors and spinors in k-dimensional Euclidean space are both just elements in the division algebra $\mathbb{K}$, albeit with different actions of $\mathrm{Spin}(V)$. We can view this as a mathematical signpost that supersymmetry is possible: physically, vectors and spinors are used to describe bosons and fermions, so the fact that both vectors and spinors lie in division algebra in dimension k suggests there is a great deal of symmetry between bosons and fermions in dimension k. Such a symmetry is precisely what supersymmetry was invented to provide.

There is much more to this story even in Euclidean signature. But we are interested in physics, so having had a brief taste of Euclidean space, we now turn to Minkowski spacetime. First, in Section 2.1, we review the basic facts we need about normed division algebras. Then we develop vectors and spinors for $(k+2)$-dimensional spacetime in Section 2.2, and for $(k+3)$-dimensional spacetime in Section 2.3.

2.1 Normed division algebras

As we note above, in 1898 Hurwitz [44] proved there are only four finite-dimensional normed division algebras: the real numbers, $\mathbb{R}$, the complex numbers, $\mathbb{C}$, the quaternions, $\mathbb{H}$, and the octonions, $\mathbb{O}$, with dimensions 1, 2, 4, and 8, respectively. Decades later, in 1960, Urbanik and Wright [68] removed the finite-dimensionality condition from this result. For an overview of this subject, including a Clifford algebra proof of Hurwitz's theorem, see the review by Baez [4]. In this section, we focus on the tools we will need to study vectors and spinors with division algebras later in this chapter.

Recall, a **normed division algebra** $\mathbb{K}$ is a (possibly nonassociative) real algebra equipped with a multiplicative unit 1 and a norm $|\cdot|$ satisfying:

$$
|ab| = |a||b|
$$

for all $a, b \in \mathbb{K}$. Note this implies that $\mathbb{K}$ has no zero divisors. We will freely identify $\mathbb{R}1 \subseteq \mathbb{K}$ with $\mathbb{R}$.

In all cases, this norm can be defined using conjugation. Every normed division algebra has a **conjugation** operator—a linear operator $*\colon \mathbb{K} \to \mathbb{K}$ satisfying

$$a^{**} = a, \quad (ab)^* = b^* a^*$$

for all $a, b \in \mathbb{K}$. Conjugation lets us decompose each element of $\mathbb{K}$ into real and imaginary parts, as follows:

$$\mathrm{Re}(a) = \frac{a + a^*}{2}, \quad \mathrm{Im}(a) = \frac{a - a^*}{2}.$$

Conjugating changes the sign of the imaginary part and leaves the real part fixed. We can write the norm as

$$|a| = \sqrt{aa^*} = \sqrt{a^*a}.$$

This norm can be polarized to give an inner product on $\mathbb{K}$:

$$(a, b) = \mathrm{Re}(ab^*) = \mathrm{Re}(a^*b).$$

The algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are associative. The octonions $\mathbb{O}$ are not. Yet they come close: the subalgebra generated by any two octonions is associative. Another way to express this fact uses the **associator**:

$$[a, b, c] = (ab)c - a(bc),$$

a trilinear map $\mathbb{K} \otimes \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$. A theorem due to Artin [56] states that for any algebra, the subalgebra generated by any two elements is associative if and only if the associator is alternating (that is, completely antisymmetric in its three arguments). An algebra with this property is thus called **alternative**. The octonions $\mathbb{O}$ are alternative, and so of course are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$: for these three the associator simply vanishes!

In what follows, our calculations make heavy use of the fact that all four normed division algebras are alternative. Besides this, the properties we require are:

Proposition 2.1. *The associator changes sign when one of its entries is conjugated.*

Proof. Since the subalgebra generated by any two elements is associative, and real elements of $\mathbb{K}$ lie in every subalgebra, $[a, b, c] = 0$ if any one of a, b, c is real. It follows that $[a, b, c] = [\mathrm{Im}(a), \mathrm{Im}(b), \mathrm{Im}(c)]$, which yields the desired result. $\square$

Proposition 2.2. *The associator is purely imaginary.*

Proof. Since $(ab)^* = b^* a^*$, a calculation shows $[a, b, c]^* = -[c^*, b^*, a^*]$. By alternativity this equals $[a^*, b^*, c^*]$, which in turn equals $-[a, b, c]$ by the above proposition. So, $[a, b, c]$ is purely imaginary. $\square$

For any square matrix A with entries in $\mathbb{K}$, we define its **trace** $\mathrm{tr}(A)$ to be the sum of its diagonal entries. This trace lacks the usual cyclic property, because $\mathbb{K}$ is noncommutative, so in general $\mathrm{tr}(AB) \neq \mathrm{tr}(BA)$. Luckily, taking the real part restores this property:

Proposition 2.3. *Let a, b, and c be elements of $\mathbb{K}$. Then*

$$\mathrm{Re}((ab)c) = \mathrm{Re}(a(bc))$$

and this quantity is invariant under cyclic permutations of a, b, and c.

Proof. Proposition 2.2 implies that $\mathrm{Re}((ab)c) = \mathrm{Re}(a(bc))$. For the cyclic property, it then suffices to prove $\mathrm{Re}(ab) = \mathrm{Re}(ba)$. Since $(a,b) = (b,a)$ and the inner product is defined by $(a,b) = \mathrm{Re}(ab^*) = \mathrm{Re}(a^*b)$, we see:

$$\mathrm{Re}(ab^*) = \mathrm{Re}(b^*a).$$

The desired result follows upon substituting b^* for b. □

Proposition 2.4. *Let A, B, and C be $k \times \ell$, $\ell \times m$ and $m \times k$ matrices with entries in $\mathbb{K}$. Then*

$$\mathrm{Re}\ \mathrm{tr}((AB)C) = \mathrm{Re}\ \mathrm{tr}(A(BC))$$

and this quantity is invariant under cyclic permutations of A, B, and C. We call this quantity the **real trace** $\mathrm{Re}\ \mathrm{tr}(ABC)$.

Proof. This follows from the previous proposition and the definition of the trace. □

2.2 Spacetime geometry in *k+2* dimensions

We shall now see how to construct vectors and spinors for spacetimes of dimension $k + 2$ from a normed division algebra $\mathbb{K}$ of dimension k. Most of the material for the here is well-known [4, 23, 45, 50, 65]. We base our approach to it on the papers of Manogue and Schray [59, 60]. The key facts are that one can describe vectors in $(k + 2)$-dimensional Minkowski spacetime as 2×2 hermitian matrices with entries in $\mathbb{K}$, and spinors as elements of $\mathbb{K}^2$. In fact there are two representations of $\mathrm{Spin}(k+1,1)$ on $\mathbb{K}^2$, which we call S_+ and S_-. The nature of these representations depends on $\mathbb{K}$:

- When $\mathbb{K} = \mathbb{R}$, $S_+ \cong S_-$ is the Majorana spinor representation of $\mathrm{Spin}(2,1)$.

- When $\mathbb{K} = \mathbb{C}$, $S_+ \cong S_-$ is the Majorana spinor representation of $\mathrm{Spin}(3,1)$.

- When $\mathbb{K} = \mathbb{H}$, S_+ and S_- are the Weyl spinor representations of $\mathrm{Spin}(5,1)$.

- When $\mathbb{K} = \mathbb{O}$, S_+ and S_- are the Majorana–Weyl spinor representations of $\mathrm{Spin}(9,1)$.

Of course, these spinor representations are also representations of the even part of the relevant Clifford algebras:

Even parts of Clifford algebras		
$\mathrm{Cliff}_0(2,1)$	$\cong$	$\mathbb{R}[2]$
$\mathrm{Cliff}_0(3,1)$	$\cong$	$\mathbb{C}[2]$
$\mathrm{Cliff}_0(5,1)$	$\cong$	$\mathbb{H}[2] \oplus \mathbb{H}[2]$
$\mathrm{Cliff}_0(9,1)$	$\cong$	$\mathbb{R}[16] \oplus \mathbb{R}[16]$

Here we see $\mathbb{R}^2$, $\mathbb{C}^2$, $\mathbb{H}^2$ and $\mathbb{O}^2$ showing up as irreducible representations of these algebras, albeit with $\mathbb{O}^2$ masquerading as $\mathbb{R}^{16}$. The first two algebras have a unique irreducible representation. The last two both have two irreducible representations, which correspond to left-handed and right-handed spinors.

Our discussion so far has emphasized the differences between the 4 cases. But the wonderful thing about normed division algebras is that they allow a unified approach that treats all four cases simultaneously! They also give simple formulas for the basic $\mathrm{Spin}(k+1,1)$-equivariant operators involving vectors, spinors and scalars.

To begin, let $\mathbb{K}[m]$ denote the space of $m \times m$ matrices with entries in $\mathbb{K}$. Given $A \in \mathbb{K}[m]$, define its **hermitian adjoint** $A^\dagger$ to be its conjugate transpose:

$$A^\dagger = (A^*)^T.$$

We say such a matrix is **hermitian** if $A = A^\dagger$. Now take the 2×2 hermitian matrices:

$$\mathfrak{h}_2(\mathbb{K}) = \left\{ \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \; : \; t, x \in \mathbb{R}, \; y \in \mathbb{K} \right\}.$$

This is an $(k+2)$-dimensional real vector space. Moreover, the usual formula for the determinant of a matrix gives the Minkowski norm on this vector space:

$$-\det \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} = -t^2 + x^2 + |y|^2.$$

We insert a minus sign to obtain the signature $(k+1,1)$. Note this formula is unambiguous even if $\mathbb{K}$ is noncommutative or nonassociative.

It follows that the double cover of the Lorentz group, $\mathrm{Spin}(k+1,1)$, acts on $\mathfrak{h}_2(\mathbb{K})$ via determinant-preserving linear transformations. Since this is the 'vector' representation, we will often call $\mathfrak{h}_2(\mathbb{K})$ simply V. The Minkowski metric

$$g \colon V \otimes V \to \mathbb{R}$$

is given by

$$g(A, A) = -\det(A).$$

There is also a nice formula for the inner product of two different vectors. This involves the **trace reversal** of $A \in \mathfrak{h}_2(\mathbb{K})$, defined by

$$\tilde{A} = A - (\mathrm{tr}A)1.$$

Note we indeed have $\mathrm{tr}(\tilde{A}) = -\mathrm{tr}(A)$. Also note that

$$A = \begin{pmatrix} t+x & y \\ y^* & t-x \end{pmatrix} \qquad \Longrightarrow \qquad \tilde{A} = \begin{pmatrix} -t+x & y \\ y^* & -t-x \end{pmatrix},$$

so trace reversal is really *time reversal*. Moreover:

Proposition 2.5. *For any vectors* $A, B \in V = \mathfrak{h}_2(K)$, *we have*

$$A\tilde{A} = \tilde{A}A = -\det(A)1$$

and

$$\frac{1}{2}\mathrm{Re} \; \mathrm{tr}(A\tilde{B}) = \frac{1}{2}\mathrm{Re} \; \mathrm{tr}(\tilde{A}B) = g(A, B).$$

Proof. We check the first equation by a quick calculation. Taking the real trace and dividing by 2 gives

$$\frac{1}{2}\mathrm{Re}\,\mathrm{tr}(A\tilde{A}) = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}(\tilde{A}A) = -\det(A) = g(A, A).$$

Then we use the polarization identity, which says that two symmetric bilinear forms that give the same quadratic form must be equal. □

Next we consider spinors. As real vector spaces, the spinor representations S_+ and S_- are both just $\mathbb{K}^2$. However, they differ as representations of $\mathrm{Spin}(k + 1, 1)$. To construct these representations, we begin by defining ways for vectors to act on spinors:

$$\begin{array}{rrcl} \gamma\colon & V \otimes S_+ & \to & S_- \\ & A \otimes \psi & \mapsto & A\psi \end{array}$$

and

$$\begin{array}{rrcl} \tilde{\gamma}\colon & V \otimes S_- & \to & S_+ \\ & A \otimes \psi & \mapsto & \tilde{A}\psi. \end{array}$$

We have named these maps for definiteness, but we will also write the action of a vector on a spinor with a dot:

$$A \cdot \psi, \quad \psi \in S_\pm.$$

We can also think of γ and $\tilde{\gamma}$ as maps that send elements of V to linear operators:

$$\begin{array}{rrcl} \gamma\colon & V & \to & \mathrm{Hom}(S_+, S_-), \\ \tilde{\gamma}\colon & V & \to & \mathrm{Hom}(S_-, S_+). \end{array}$$

Here a word of caution is needed: since $\mathbb{K}$ may be nonassociative, 2×2 matrices with entries in $\mathbb{K}$ cannot be identified with linear operators on $\mathbb{K}^2$ in the usual way. They certainly induce linear operators via left multiplication:

$$L_A(\psi) = A\psi.$$

Indeed, this is how γ and $\tilde{\gamma}$ turn elements of V into linear operators:

$$\begin{array}{rcl} \gamma(A) & = & L_A, \\ \tilde{\gamma}(A) & = & L_{\tilde{A}}. \end{array}$$

However, because of nonassociativity, composing such linear operators is different from multiplying the matrices:

$$L_A L_B(\psi) = A(B\psi) \neq (AB)\psi = L_{AB}(\psi).$$

Since vectors act on elements of S_+ to give elements of S_- and vice versa, they map the space $S_+ \oplus S_-$ to itself. This gives rise to an action of the Clifford algebra $\mathrm{Cliff}(V)$ on $S_+ \oplus S_-$:

Proposition 2.6. *The vectors $V = \mathfrak{h}_2(\mathbb{K})$ act on the spinors $S_+ \oplus S_- = \mathbb{K}^2 \oplus \mathbb{K}^2$ via the map*

$$\Gamma\colon V \to \mathrm{End}(S_+ \oplus S_-)$$

given by

$$\Gamma(A)(\psi, \phi) = (\widetilde{A}\phi,\ A\psi).$$

Furthermore, $\Gamma(A)$ satisfies the Clifford algebra relation:

$$\Gamma(A)^2 = g(A, A)1$$

and so extends to a homomorphism $\Gamma \colon \mathrm{Cliff}(V) \to \mathrm{End}(S_+ \oplus S_-)$, i.e. a representation of the Clifford algebra $\mathrm{Cliff}(V)$ on $S_+ \oplus S_-$.

Proof. Suppose $A \in V$ and $\Psi = (\psi, \phi) \in S_+ \oplus S_-$. We need to check that

$$\Gamma(A)^2(\Psi) = -\det(A)\Psi.$$

Here we must be mindful of nonassociativity: we have

$$\Gamma(A)^2(\Psi) = (\tilde{A}(A\psi),\, A(\tilde{A}\phi)).$$

Yet it is easy to check that the expressions $\tilde{A}(A\psi)$ and $A(\tilde{A}\phi)$ involve multiplying at most two different nonreal elements of $\mathbb{K}$. These associate, since $\mathbb{K}$ is alternative, so in fact

$$\Gamma(A)^2(\Psi) = ((\tilde{A}A)\psi,\, (A\tilde{A})\phi).$$

To conclude, we use Proposition 2.5. $\qquad\square$

The action of a vector swaps S_+ and S_-, so acting by vectors twice sends S_+ to itself and S_- to itself. This means that while S_+ and S_- are *not* modules for the Clifford algebra $\mathrm{Cliff}(V)$, they are both modules for the even part of the Clifford algebra, generated by products of pairs of vectors. Recalling that $\mathrm{Spin}(k+1,1)$ lives in this even part, we see that S_+ and S_- are both representations of $\mathrm{Spin}(k+1,1)$.

Now that we have representations of $\mathrm{Spin}(k+1,1)$ on V, S_+ and S_-, we need to develop the $\mathrm{Spin}(k+1,1)$-equivariant maps that relate them. Ultimately, we need:

- An invariant pairing:
$$\langle -, - \rangle \colon S_+ \otimes S_- \to \mathbb{R}.$$

- An equivariant map that turns pairs of spinors into vectors:
$$[-, -] \colon S_\pm \otimes S_\pm \to V.$$

Another name for an equivariant map between group representations is an 'intertwining operator'. As a first step, we show that the action of vectors on spinors is itself an intertwining operator:

Proposition 2.7. *The maps*
$$\begin{array}{rrcl} \gamma \colon & V \otimes S_+ & \to & S_- \\ & A \otimes \psi & \mapsto & A\psi \end{array}$$

and
$$\begin{array}{rrcl} \tilde{\gamma} \colon & V \otimes S_- & \to & S_+ \\ & A \otimes \psi & \mapsto & \tilde{A}\psi \end{array}$$

are equivariant with respect to the action of $\mathrm{Spin}(k+1,1)$.

Proof. Both γ and $\tilde{\gamma}$ are restrictions of the map

$$\Gamma \colon V \otimes (S_+ \oplus S_-) \to S_+ \oplus S_-,$$

so it suffices to check that Γ is equivariant. Indeed, an element $g \in \mathrm{Spin}(k+1,1)$ acts on V by conjugation on $V \subseteq \mathrm{Cliff}(V)$, and it acts on $S_+ \oplus S_-$ by $\Gamma(g)$. Thus, we compute:

$$\Gamma(gAg^{-1})\Gamma(g)\Psi = \Gamma(g)(\Gamma(A)\Psi),$$

for any $\Psi \in S_+ \oplus S_-$. Here it is important to note that the conjugation gAg^{-1} is taking place in the associative algebra $\mathrm{Cliff}(V)$, not in the algebra of matrices. This equation says that Γ is indeed $\mathrm{Spin}(k+1,1)$-equivariant, as claimed. $\qquad\square$

Now we exhibit the key tool: the pairing between S_+ and S_-:

Proposition 2.8. *The pairing*

$$\langle -, - \rangle \colon \quad \begin{aligned} S_+ \otimes S_- &\to \mathbb{R} \\ \psi \otimes \phi &\mapsto \mathrm{Re}(\psi^\dagger \phi) \end{aligned}$$

is invariant under the action of $\mathrm{Spin}(k+1,1)$.

Proof. Given $A \in V$, we use the fact that the associator is purely imaginary to show that

$$\mathrm{Re}\left((\tilde{A}\phi)^\dagger(A\psi)\right) = \mathrm{Re}\left((\phi^\dagger \tilde{A})(A\psi)\right) = \mathrm{Re}\left(\phi^\dagger(\tilde{A}(A\psi))\right).$$

As in the proof of the Clifford relation, it is easy to check that the column vector $\tilde{A}(A\psi)$ involves at most two nonreal elements of $\mathbb{K}$ and equals $g(A,A)\psi$. So:

$$\langle \tilde{\gamma}(A)\phi, \gamma(A)\psi \rangle = g(A,A)\langle \psi, \phi \rangle.$$

In particular when A is a unit vector, acting by A swaps the order of ψ and ϕ and changes the sign at most. The pairing $\langle -, - \rangle$ is thus invariant under the group in $\mathrm{Cliff}(V)$ generated by products of *pairs* of unit vectors, which is $\mathrm{Spin}(k+1,1)$. $\qquad\square$

With this pairing in hand, there is a manifestly equivariant way to turn a pair of spinors into a vector. Given $\psi, \phi \in S_+$, there is a unique vector $[\psi, \phi]$ whose inner product with any vector A is given by

$$g([\psi, \phi], A) = \langle \psi, \gamma(A)\phi \rangle.$$

Similarly, given $\psi, \phi \in S_-$, we define $[\psi, \phi] \in V$ by demanding

$$g([\psi, \phi], A) = \langle \tilde{\gamma}(A)\psi, \phi \rangle$$

for all $A \in V$. This gives us maps

$$S_\pm \otimes S_\pm \to V$$

which are manifestly equivariant.

On the other hand, because $S_\pm = \mathbb{K}^2$ and $V = \mathfrak{h}_2(\mathbb{K})$, there is also a naive way to turn a pair of spinors into a vector using matrix operations: just multiply the column vector ψ by the row vector

29

$\phi^\dagger$ and then take the hermitian part:

$$\psi\phi^\dagger + \phi\psi^\dagger \in \mathfrak{h}_2(\mathbb{K}),$$

or perhaps its trace reversal:

$$\widetilde{\psi\phi^\dagger + \phi\psi^\dagger} \in \mathfrak{h}_2(\mathbb{K}).$$

In fact, these naive guesses match the manifestly equivariant approach described above:

Proposition 2.9. *The maps* $[-,-]\colon S_\pm \otimes S_\pm \to V$ *are given by:*

$$\begin{array}{rccc}
[-,-]\colon & S_+ \otimes S_+ & \to & V \\
& \psi \otimes \phi & \mapsto & \widetilde{\psi\phi^\dagger + \phi\psi^\dagger}
\end{array}$$

$$\begin{array}{rccc}
[-,-]\colon & S_- \otimes S_- & \to & V \\
& \psi \otimes \phi & \mapsto & \psi\phi^\dagger + \phi\psi^\dagger.
\end{array}$$

These maps are equivariant with respect to the action of $\mathrm{Spin}(k+1,1)$.

Proof. First suppose $\psi, \phi \in S_+$. We have already seen that the map $[-,-]\colon S_+ \otimes S_+ \to V$ is equivariant. We only need to show that this map has the desired form. We start by using some definitions:

$$g([\psi,\phi], A) = \langle \psi, \gamma(A)\phi \rangle = \mathrm{Re}(\psi^\dagger(A\phi)) = \mathrm{Re}\,\mathrm{tr}(\psi^\dagger A\phi).$$

We thus have

$$g([\psi,\phi], A) = \mathrm{Re}\,\mathrm{tr}(\psi^\dagger A\phi) = \mathrm{Re}\,\mathrm{tr}(\phi^\dagger A\psi),$$

where in the last step we took the adjoint of the inside. Applying the cyclic property of the real trace, we obtain

$$g([\psi,\phi], A) = \mathrm{Re}\,\mathrm{tr}(\phi\psi^\dagger A) = \mathrm{Re}\,\mathrm{tr}(\psi\phi^\dagger A).$$

Averaging gives

$$g([\psi,\phi], A) = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}((\psi\phi^\dagger + \phi\psi^\dagger)A).$$

On the other hand, Proposition 2.5 implies that

$$g([\psi,\phi], A) = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}(\widetilde{[\psi,\phi]}A).$$

Since both these equations hold for all A, we must have

$$\widetilde{[\psi,\phi]} = \psi\phi^\dagger + \phi\psi^\dagger.$$

Doing trace reversal twice gets us back where we started, so

$$[\psi,\phi] = \widetilde{\psi\phi^\dagger + \phi\psi^\dagger}$$

as desired. A similar calculation shows that if $\psi, \phi \in S_-$, then $[\psi,\phi] = \psi\phi^\dagger + \phi\psi^\dagger$. $\qquad\square$

Map	Division algebra notation	Index notation
$g\colon\ V \otimes V \to \mathbb{R}$	$\frac{1}{2}\mathrm{Re}\,\mathrm{tr}(A\tilde{B})$	$A^{\mu}B_{\mu}$
$\gamma\colon\ V \otimes S_{+} \to S_{-}$	$A\psi$	$A^{\mu}\gamma_{\mu}\psi$
$\tilde{\gamma}\colon\ V \otimes S_{-} \to S_{+}$	$\tilde{A}\psi$	$A^{\mu}\tilde{\gamma}_{\mu}\psi$
$[-,-]\colon\ S_{+} \otimes S_{+} \to V$	$\widetilde{\psi\phi^{\dagger} + \phi\psi^{\dagger}}$	$\overline{\psi}\gamma^{\mu}\phi$
$[-,-]\colon\ S_{-} \otimes S_{-} \to V$	$\psi\phi^{\dagger} + \phi\psi^{\dagger}$	$\overline{\psi}\tilde{\gamma}^{\mu}\phi$
$\langle-,-\rangle\colon\ S_{+} \otimes S_{-} \to \mathbb{R}$	$\mathrm{Re}(\psi^{\dagger}\phi)$	$\overline{\psi}\phi$

Table 2.1: Division algebra notation vs. index notation

We can summarize our work so far with a table of the basic bilinear maps involving vectors, spinors and scalars. Table 1 shows how to translate between division algebra notation and something more closely resembling standard physics notation. In this table the adjoint spinor $\overline{\psi}$ denotes the spinor dual to ψ under the pairing $\langle-,-\rangle$. The gamma matrix γ^{μ} denotes a Clifford algebra generator acting on S_{+}, while $\tilde{\gamma}^{\mu}$ denotes the same element acting on S_{-}. Of course $\tilde{\gamma}$ is not standard physics notation; the standard notation for this depends on which of the four cases we are considering: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

2.3   Spacetime geometry in *k+3* dimensions

In the last section we recalled how to describe spinors and vectors in $(k{+}2)$-dimensional Minkowski spacetime using a division algebra $\mathbb{K}$ of dimension k. Here we show how to boost this up one dimension, and give a division algebra description of vectors and spinors in $(k + 3)$-dimensional Minkowski spacetime.

We shall see that vectors in $(k + 3)$-dimensional Minkowski spacetime can be identified with 4×4 $\mathbb{K}$-valued matrices of this particular form:

$$\begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix}$$

where a is a real multiple of the 2×2 identity matrix and A is a 2×2 hermitian matrix with entries in $\mathbb{K}$. Moreover, $\mathrm{Spin}(k + 2, 1)$ has a representation on $\mathbb{K}^{4}$, which we call $\mathcal{S}$. Depending on $\mathbb{K}$, this gives the following types of spinors:

- When $\mathbb{K} = \mathbb{R}$, $\mathcal{S}$ is the Majorana spinor representation of $\mathrm{Spin}(3, 1)$.

- When $\mathbb{K} = \mathbb{C}$, $\mathcal{S}$ is the Dirac spinor representation of $\mathrm{Spin}(4, 1)$.

- When $\mathbb{K} = \mathbb{H}$, $\mathcal{S}$ is the Dirac spinor representation of $\mathrm{Spin}(6, 1)$.

- When $\mathbb{K} = \mathbb{O}$, $\mathcal{S}$ is the Majorana spinor representation of $\mathrm{Spin}(10, 1)$.

Again, these spinor representations are also representations of the even part of the relevant Clifford algebra:

Even parts of Clifford algebras		
$\mathrm{Cliff}_0(3,1)$	$\cong$	$\mathbb{C}[2]$
$\mathrm{Cliff}_0(4,1)$	$\cong$	$\mathbb{H}[2]$
$\mathrm{Cliff}_0(6,1)$	$\cong$	$\mathbb{H}[4]$
$\mathrm{Cliff}_0(10,1)$	$\cong$	$\mathbb{R}[32]$

These algebras have irreducible representations on $\mathbb{R}^4 \cong \mathbb{C}^2$, $\mathbb{C}^4 \cong \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{O}^4 \cong \mathbb{R}^{32}$, respectively.

The details can be described in a uniform way for all four cases. We take as our space of 'vectors' the following $(k+3)$-dimensional subspace of $\mathbb{K}[4]$:

$$\mathcal{V} = \left\{ \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix} : a \in \mathbb{R}, \quad A \in \mathfrak{h}_2(\mathbb{K}) \right\}.$$

In the last section, we defined vectors in $k+2$ dimensions to be $V = \mathfrak{h}_2(\mathbb{K})$. That space has an obvious embedding into $\mathcal{V}$, given by

$$\begin{array}{ccc} V & \hookrightarrow & \mathcal{V} \\ A & \mapsto & \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix}. \end{array}$$

The Minkowski metric

$$h \colon \mathcal{V} \otimes \mathcal{V} \to \mathbb{R}$$

is given by extending the Minkowski metric g on V:

$$h\left(\begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix}, \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix} \right) = g(A,A) + a^2.$$

From our formulas for g, we can derive formulas for h:

Proposition 2.10. *For any vectors $\mathcal{A}, \mathcal{B} \in \mathcal{V} \subseteq \mathbb{K}[4]$, we have*

$$\mathcal{A}^2 = h(\mathcal{A}, \mathcal{A})1$$

and

$$\frac{1}{4}\mathrm{Re}\,\mathrm{tr}(\mathcal{A}\mathcal{B}) = h(\mathcal{A}, \mathcal{B}).$$

Proof. For $\mathcal{A} = \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix}$, it is easy to check:

$$\mathcal{A}^2 = \begin{pmatrix} a^2 + \tilde{A}A & 0 \\ 0 & A\tilde{A} + a^2 \end{pmatrix}.$$

By Proposition 2.5, we have $A\tilde{A} = \tilde{A}A = g(A,A)1$, and substituting this in establishes the first formula. The second formula follows from polarizing and taking the real trace of both sides. $\square$

Define a space of 'spinors' by $\mathcal{S} = S_+ \oplus S_- = \mathbb{K}^4$. To distinguish elements of $\mathcal{V}$ from elements of $\mathfrak{h}_2(\mathbb{K})$, we will denote them with calligraphic letters such as $\mathcal{A}$ and $\mathcal{B}$. Similarly, to distinguish elements of $\mathcal{S}$ from $S_\pm$, we will denote them with capital Greek letters such as Ψ and Φ.

Elements of $\mathcal{V}$ act on $\mathcal{S}$ by left multiplication:

$$
\begin{aligned}
\mathcal{V} \otimes \mathcal{S} &\to \mathcal{S} \\
\mathcal{A} \otimes \Psi &\mapsto \mathcal{A}\Psi.
\end{aligned}
$$

We can dualize this to get a map:

$$
\begin{aligned}
\Gamma \colon \mathcal{V} &\to \operatorname{End}(\mathcal{S}) \\
\mathcal{A} &\mapsto L_{\mathcal{A}}.
\end{aligned}
$$

This induces the Clifford action of $\operatorname{Cliff}(\mathcal{V})$ on $\mathcal{S}$. Note that this Γ is the same as the map in Proposition 2.6 when we restrict to $V \subseteq \mathcal{V}$.

Proposition 2.11. *The vectors $\mathcal{V} \subseteq \mathbb{K}[4]$ act on the spinors $\mathcal{S} = \mathbb{K}^4$ via the map*

$$
\Gamma \colon \mathcal{V} \to \operatorname{End}(\mathcal{S})
$$

given by

$$
\Gamma(\mathcal{A})\Psi = \mathcal{A}\Psi.
$$

Furthermore, $\Gamma(\mathcal{A})$ satisfies the Clifford algebra relation:

$$
\Gamma(\mathcal{A})^2 = h(\mathcal{A}, \mathcal{A})1
$$

and so extends to a homomorphism $\Gamma \colon \operatorname{Cliff}(\mathcal{V}) \to \operatorname{End}(\mathcal{S})$, i.e. a representation of the Clifford algebra $\operatorname{Cliff}(\mathcal{V})$ on $\mathcal{S}$.

Proof. Here, we must be mindful of nonassociativity. For $\Psi = (\psi, \phi) \in \mathcal{S}$ and $\mathcal{A} = \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix} \in \mathcal{V}$, we have:

$$
\Gamma(\mathcal{A})^2 \Psi = \mathcal{A}(\mathcal{A}\Psi)
$$

which works out to be:

$$
\Gamma(\mathcal{A})^2 \Psi = \begin{pmatrix} a^2 \psi + \tilde{A}(A\psi) \\ A(\tilde{A}\phi) + a^2 \phi \end{pmatrix}.
$$

A quick calculation shows that the expressions $\tilde{A}(A\psi)$ and $A(\tilde{A}\phi)$ involve at most two nonreal elements of $\mathbb{K}$, so everything associates and we can write:

$$
\Gamma(\mathcal{A})^2 \Psi = \mathcal{A}^2 \Psi
$$

By Proposition 2.10, we are done. $\qquad\square$

This tells us how $\mathcal{S}$ is a module of $\operatorname{Cliff}(\mathcal{V})$, and thus a representation of $\operatorname{Spin}(\mathcal{V})$, the subgroup of $\operatorname{Cliff}(\mathcal{V})$ generated by products of pairs of unit vectors.

In the last section, we saw how to construct a $\operatorname{Spin}(V)$-invariant pairing

$$
\langle -, - \rangle \colon S_+ \otimes S_- \to \mathbb{R}.
$$

We can use this to build up to a $\mathrm{Spin}(\mathcal{V})$-invariant pairing on $\mathcal{S}$:

$$\langle (\psi, \phi), (\chi, \theta) \rangle = \langle \psi, \theta \rangle - \langle \chi, \phi \rangle$$

To see this, let

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then, because $\langle \psi, \phi \rangle = \mathrm{Re}(\psi^\dagger \phi)$, it is easy to check that:

$$\langle \psi, \theta \rangle - \langle \chi, \phi \rangle = \mathrm{Re}\left(\begin{pmatrix} \psi \\ \phi \end{pmatrix}^\dagger \Gamma^0 \begin{pmatrix} \chi \\ \theta \end{pmatrix} \right).$$

We can show this last expression is invariant by explicit calculation.

Proposition 2.12. *Define the nondegenerate skew-symmetric bilinear form*

$$\langle -, - \rangle \colon \mathcal{S} \otimes \mathcal{S} \to \mathbb{R}$$

by

$$\langle \Psi, \Phi \rangle = \mathrm{Re}(\Psi^\dagger \Gamma^0 \Phi).$$

This form is invariant under $\mathrm{Spin}(\mathcal{V})$.

Proof. It is easy to see that, for any spinors $\Psi, \Phi \in \mathcal{S}$ and vectors $\mathcal{A} \in \mathcal{V}$, we have

$$\langle \mathcal{A}\Psi, \mathcal{A}\Phi \rangle = \mathrm{Re}\left((\Psi^\dagger \mathcal{A}^\dagger) \Gamma^0 (\mathcal{A}\Phi) \right) = \mathrm{Re}\left(\Psi^\dagger (\mathcal{A}^\dagger \Gamma^0 (\mathcal{A}\Phi)) \right),$$

where in the last step we have used Proposition 2.4. Now, given that

$$\mathcal{A} = \begin{pmatrix} a & \tilde{A} \\ A & -a \end{pmatrix},$$

a quick calculation shows:

$$\mathcal{A}^\dagger \Gamma^0 = -\Gamma^0 \mathcal{A}.$$

So, this last expression becomes:

$$-\mathrm{Re}\left(\Psi^\dagger (\Gamma^0 \mathcal{A}(\mathcal{A}\Phi)) \right) = -\mathrm{Re}\left(\Psi^\dagger (\Gamma^0 \Gamma(\mathcal{A})^2 \Phi)) \right) = -|\mathcal{A}|^2 \mathrm{Re}\left(\Psi^\dagger \Gamma^0 \Phi \right),$$

where in the last step we have used the Clifford relation. Summing up, we have shown:

$$\langle \mathcal{A}\Psi, \mathcal{A}\Phi \rangle = -|\mathcal{A}|^2 \langle \Psi, \Phi \rangle.$$

In particular, when $\mathcal{A}$ is a unit vector, acting by $\mathcal{A}$ changes the sign at most. Thus, $\langle -, - \rangle$ is invariant under the group generated by products of pairs of unit vectors, which is $\mathrm{Spin}(\mathcal{V})$. It is easy to see that it is nondegenerate, and it is skew-symmetric because Γ^0 is. $\qquad \square$

With the form $\langle -, - \rangle$ in hand, there is a manifestly equivariant way to turn a pair of spinors into a vector. Given $\Psi, \Phi \in \mathcal{S}$, there is a unique vector $[\Psi, \Phi]$ whose inner product with any vector $\mathcal{A}$ is

given by

$$h([\Psi, \Phi], \mathcal{A}) = \langle \Psi, \Gamma(\mathcal{A})\Phi \rangle.$$

It will be useful to have an explicit formula for this operation:

Proposition 2.13. *Given* $\Psi = (\psi_1, \psi_2)$ *and* $\Phi = (\phi_1, \phi_2)$ *in* $\mathcal{S} = S_+ \oplus S_-$, *we have:*

$$[\Psi, \Phi] = \begin{pmatrix} \langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle & -\widetilde{[\psi_1, \psi_2]} + \widetilde{[\phi_1, \phi_2]} \\ -[\psi_1, \psi_2] + [\phi_1, \phi_2] & -\langle \psi_1, \phi_2 \rangle - \langle \phi_1, \psi_2 \rangle \end{pmatrix}.$$

Proof. Decompose $\mathcal{V}$ into orthogonal subspaces:

$$\mathcal{V} = \left\{ \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix} : A \in V \right\} \oplus \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{R} \right\}$$

The first of these is just a copy of V, a $(k+2)$-dimensional Minkowski spacetime. The second is the single extra spatial dimension in our $(k+3)$-dimensional Minkowski spacetime, $\mathcal{V}$.

Now, use the definition of $[\Psi, \Phi]$, but restricted to V. It is easy to see that, for any vector $A \in V$, we have:

$$h([\Psi, \Phi], A) = -\langle \psi_1, \gamma(A)\phi_1 \rangle + \langle \tilde{\gamma}(A)\psi_2, \phi_2 \rangle$$

Letting B be the component of $[\Psi, \Phi]$ which lies in V, this becomes:

$$g(B, A) = -\langle \psi_1, \gamma(A)\phi_1 \rangle + \langle \tilde{\gamma}(A)\psi_2, \phi_2 \rangle.$$

Note that we have switched to the metric g on V, to which h restricts. By definition, this is the same as:

$$g(B, A) = g(-[\psi_1, \phi_1] + [\psi_2, \phi_2], A).$$

Since this holds for all A, we must have $B = -[\psi_1, \phi_1] + [\psi_2, \phi_2]$.

It remains to find the component of $[\Psi, \Phi]$ orthogonal to B. Since $\left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{R} \right\}$ is 1-dimensional, this is merely a number. Specifically, it is the constant of proportionality in the expression:

$$h\left([\Psi, \Phi], \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\right) = a(\langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle).$$

Thus, this component is $\langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle$. Putting everything together, we get

$$[\Psi, \Phi] = \begin{pmatrix} \langle \psi_1, \phi_2 \rangle + \langle \phi_1, \psi_2 \rangle & -\widetilde{[\psi_1, \psi_2]} + \widetilde{[\phi_1, \phi_2]} \\ -[\psi_1, \psi_2] + [\phi_1, \phi_2] & -\langle \psi_1, \phi_2 \rangle - \langle \phi_1, \psi_2 \rangle \end{pmatrix}.$$

$\square$

2.4 The spinor identities

We now prove crucial identities involving spinors in spacetimes of dimension $k+2$ and $k+3$. In a sense, this solves our puzzle concerning how division algebras are related to string theory and 2-brane theory: the spinor identities allow the construction of WZW terms for these theories, thus guaranteeing they have Siegel symmetry. Siegel symmetry forces the bosonic and fermionic degrees

of freedom to match, so it is essential for supersymmetry. In dimensions 10 and 11, Siegel symmetry also constrains the background of spacetime to be that of supergravity. Yet, in solving the puzzle, we uncover new questions. What, for instance, is the significance of these spinor identities? We shall see, in the remainder of this thesis, that these identities lead the way to higher gauge theory.

The first identity we shall prove holds in spacetimes of dimension 3, 4, 6 and 10. This identity appears in several guises in the physics literature. Besides the role it plays in string theory, we shall see in Chapter 4 that it implies the supersymmetry of super-Yang–Mills theories in these dimensions.

Let us see the various forms this identity can take. In dimensions 3, 4, 6 and 10, we have what Schray [59] has dubbed the **3-ψ's rule**:

$$[\psi, \psi] \cdot \psi = 0,$$

for all spinors $\psi \in S_+$, where the bracket is the bilinear map defined in Proposition 2.13. That is, a spinor squared to a vector and then acting on itself vanishes. It is also common to see this cubic form polarized to obtain a trilinear in three spinors:

$$[\psi, \phi] \cdot \chi + [\chi, \psi] \cdot \phi + [\phi, \chi] \cdot \psi = 0.$$

A more geometric interpretation, emphasized by Deligne and Freed [27], is that spinors square to null vectors in these special dimensions:

$$|[\psi, \psi]|^2 = 0,$$

where $|A|^2 = g(A, A)$ is the quadratic form associated with the Minkowski inner product. On the other hand, physicists prefer to write all of these expressions using gamma matrices. Referring to Table 2.1, we write components of the vector $[\psi, \psi]$ as $\overline{\psi}\gamma^\mu\psi$ when $\psi \in S_+$. These identities then become, respectively:

$$(\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi = 0,$$

$$(\overline{\psi}\gamma^\mu\phi)\gamma_\mu\chi + (\overline{\chi}\gamma^\mu\psi)\gamma_\mu\phi + (\overline{\phi}\gamma^\mu\chi)\gamma_\mu\psi = 0,$$

and

$$(\overline{\psi}\gamma^\mu\psi)(\overline{\psi}\gamma_\mu\psi) = 0.$$

Finally, it also common for the spinors to be removed from the second identity, to obtain an equivalent expression in terms of gamma matrices alone. We now establish that these are all equivalent. In fact, this is a consequence of the following symmetries:

Proposition 2.14. *For any spinors $\psi, \phi, \chi, \theta \in S_+$, the 4-linear expression $\langle \theta, [\psi, \phi]\chi \rangle$ is symmetric under the exchange of the unbracketed spinors:*

$$\langle \theta, [\psi, \phi] \cdot \chi \rangle = \langle \chi, [\psi, \phi] \cdot \theta \rangle$$

and under the exchange of the bracketed and unbracketed spinors:

$$\langle \theta, [\psi, \phi] \cdot \chi \rangle = \langle \psi, [\theta, \chi] \cdot \phi \rangle$$

36

Similarly, for any spinors $\psi, \phi, \chi, \theta \in S_-$, the 4-linear expression $\langle[\psi, \phi] \cdot \theta, \chi\rangle$ is symmetric under the exchange of the unbracketed spinors:

$$\langle[\psi, \phi] \cdot \theta, \chi\rangle = \langle[\psi, \phi] \cdot \chi, \theta\rangle$$

and under the exchange of the bracketed and unbracketed spinors:

$$\langle[\psi, \phi] \cdot \theta, \chi\rangle = \langle[\theta, \chi] \cdot \psi, \phi\rangle$$

Proof. We shall prove this in the case where all spinors are in S_+, the case of S_- being almost identical. Using the definition of bracket, write:

$$\langle\theta, [\psi, \phi] \cdot \chi\rangle = g([\theta, \chi], [\psi, \phi]).$$

The first formula then follows from the symmetry of the bracket $[\theta, \chi]$. The second formula follows from the definition of the bracket $[\psi, \phi]$. □

Proposition 2.15. *The following are equivalent:*

1. $[\psi, \psi] \cdot \psi = 0$ *for all $\psi \in S_\pm$.*

2. $[\psi, \phi] \cdot \chi + [\chi, \psi] \cdot \phi + [\phi, \chi] \cdot \psi = 0$ *for all $\psi, \phi, \chi \in S_\pm$*

3. $|[\psi, \psi]|^2 = 0$ *for all $\psi \in S_\pm$.*

Proof. Again, we shall prove this in the case where all spinors are in S_+, the case of S_- being virtually identical. Because the bracket is symmetric, the trilinear expression

$$[\psi, \phi] \cdot \chi + [\chi, \psi] \cdot \phi + [\phi, \chi] \cdot \psi$$

is totally symmetric in its three arguments. Just as a symmetric bilinear vanishes if and only if the associated quadratic form vanishes, a symmetric trilinear vanishes if and only if the associated cubic form does. In this case, that cubic form is, up to a numerical factor:

$$[\psi, \psi] \cdot \psi.$$

So (1) holds if and only if (2) does.

On the other hand, we can use the symmetries of Proposition 2.14 to show that the following expression is symmetric in all four spinors:

$$\langle\theta, [\psi, \phi] \cdot \chi + [\chi, \psi] \cdot \phi + [\phi, \chi] \cdot \psi\rangle.$$

Statement (2) holds if and only if this expression vanishes for all θ, but this totally symmetric 4-linear vanishes if and only if the associated quartic form vanishes. In this case, that quartic form is, up to a multiplicative factor:

$$\langle\psi, [\psi, \psi] \cdot \psi\rangle = g([\psi, \psi], [\psi, \psi]).$$

Thus, (2) holds if and only if (3) holds. □

We now prove the 3-ψ's rule. Note that it is really the alternative law, rather than any division algebra axioms, that does the work. Our proof is based on an argument in the appendix to a paper by Dray, Janesky and Manogue [29].

Theorem 2.1. *Suppose $\psi \in S_\pm$. Then $[\psi, \psi] \cdot \psi = 0$. In other words, $[\psi, \psi]\psi = 0$ for $\psi \in S_+$, and $\widetilde{[\psi, \psi]}\psi = 0$ for $\psi \in S_-$.*

Proof. Let $\psi \in S_+$. By definition,

$$[\psi, \psi]\psi = 2(\widetilde{\psi\psi^\dagger})\psi = 2(\psi\psi^\dagger - \mathrm{tr}(\psi\psi^\dagger)1)\psi.$$

It is easy to check that $\mathrm{tr}(\psi\psi^\dagger) = \psi^\dagger\psi$, so

$$[\psi, \psi]\psi = 2((\psi\psi^\dagger)\psi - (\psi^\dagger\psi)\psi).$$

Since $\psi^\dagger\psi$ is a real number, it commutes with ψ:

$$[\psi, \psi]\psi = 2((\psi\psi^\dagger)\psi - \psi(\psi^\dagger\psi)).$$

Since $\mathbb{K}$ is alternative, every subalgebra of $\mathbb{K}$ generated by two elements is associative. The spinor $\psi \in \mathbb{K}^2$ is built from just two elements of $\mathbb{K}$, so the right-hand side vanishes. The proof for the second case is similar. $\square$

Similarly, spinors in dimension 4, 5, 7 and 11 satisfy a related identity, written in gamma matrix notation as follows:

$$\overline{\Psi}\Gamma_{ab}\Psi\overline{\Psi}\Gamma^b\Psi = 0$$

This identity shows up in two prominent places in the physics literature. First, it is required for the existence of 2-brane theories in these dimensions [1, 30]. This is because it allows the construction of a Wess–Zumino–Witten term for these theories, which give these theories Siegel symmetry.

Yet it is known that 2-branes in 11 dimensions are intimately connected to supergravity. Indeed, the Siegel symmetry imposed by the WZW term constrains the 2-brane background to be that of 11-dimensional supergravity [66]. So it should come as no surprise that this spinor identity also plays a crucial role in supergravity, most visibly in the work of D'Auria and Fré [25] and subsequent work by Sati, Schreiber and Stasheff [55].

This identity is equivalent to the **4-Ψ's rule**:

$$[\Psi, [\Psi, \Psi]\Psi] = 0.$$

To see this, note that we can turn a pair of spinors Ψ and Φ into a 2-form, $\Psi * \Phi$. This comes from the fact that we can embed bivectors inside the Clifford algebra $\mathrm{Cliff}(\mathcal{V})$ via the map

$$\mathcal{A} \wedge \mathcal{B} \mapsto \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \in \mathrm{Cliff}(\mathcal{V}).$$

These can then act on spinors using the Clifford action. Thus, define:

$$(\Psi * \Phi)(\mathcal{A}, \mathcal{B}) = \langle \Psi, (\mathcal{A} \wedge \mathcal{B})\Phi \rangle. \tag{2.1}$$

But when $\Psi = \Phi$, we can simplify this using the Clifford relation:

$$
\begin{aligned}
(\Psi * \Psi)(\mathcal{A}, \mathcal{B}) &= \langle \Psi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Psi \rangle \\
&= \langle \Psi, 2\mathcal{A}\mathcal{B}\Psi \rangle - \langle \Psi, \Psi \rangle h(\mathcal{A}, \mathcal{B}) \\
&= 2\langle \Psi, \mathcal{A}\mathcal{B}\Psi \rangle
\end{aligned}
$$

where we have used the skew-symmetry of the form. The index-ridden identity above merely says that inserting the vector $[\Psi, \Psi]$ into one slot of the 2-form $\Psi * \Psi$ is zero, no matter what goes into the other slot:

$$
(\Psi * \Psi)(\mathcal{A}, [\Psi, \Psi]) = 2\langle \Psi, \mathcal{A}[\Psi, \Psi]\Psi \rangle = 0
$$

for all $\mathcal{A}$. By the definition of the bracket, this is the same as

$$
2h([\Psi, [\Psi, \Psi]\Psi], \mathcal{A}) = 0
$$

for all $\mathcal{A}$. Thus, the index-ridden identity is equivalent to:

$$
[\Psi, [\Psi, \Psi]\Psi] = 0
$$

as required.

Now, let us prove this. As we shall see, we use both the 3-ψ's rule and the cyclic property of the real trace in the proof, both of which follow from our use of normed division algebras. For another derivation of the 4-Ψ's rule using division algebras, see the paper of Foot and Joshi [33].

Theorem 2.2. *Suppose $\Psi \in \mathcal{S}$. Then $[\Psi, [\Psi, \Psi]\Psi] = 0$.*

Proof. Let $\Psi = (\psi, \phi)$. By Proposition 2.13,

$$
[\Psi, \Psi] = \begin{pmatrix} 2\langle \psi, \phi \rangle & -\widetilde{[\psi, \psi]} + \widetilde{[\phi, \phi]} \\ -[\psi, \psi] + [\phi, \phi] & -2\langle \psi, \phi \rangle \end{pmatrix}
$$

and thus

$$
[\Psi, \Psi]\Psi = \begin{pmatrix} 2\langle \psi, \phi \rangle \psi - \widetilde{[\psi, \psi]}\phi + \widetilde{[\phi, \phi]}\phi \\ -[\psi, \psi]\psi + [\phi, \phi]\psi - 2\langle \psi, \phi \rangle \phi \end{pmatrix}.
$$

Both $[\psi, \psi]\psi = 0$ and $\widetilde{[\phi, \phi]}\phi = 0$ by the 3-ψ's rule, Theorem 2.1. So:

$$
[\Psi, \Psi]\Psi = \begin{pmatrix} 2\langle \psi, \phi \rangle \psi - \widetilde{[\psi, \psi]}\phi \\ [\phi, \phi]\psi - 2\langle \psi, \phi \rangle \phi \end{pmatrix}.
$$

The resulting matrix for $[\Psi, [\Psi, \Psi]\Psi]$ is large and unwieldy, so we shall avoid writing it out. Fortunately, all we really need is the $(1, 1)$ entry. Recall, this is the component of the vector $[\Psi, [\Psi, \Psi]\Psi]$ that is orthogonal to the subspace $V \subset \mathcal{V}$. Call this component a. A calculation shows:

$$
\begin{aligned}
a &= \langle \psi, [\phi, \phi]\psi \rangle - \langle \widetilde{[\psi, \psi]}\phi, \phi \rangle \\
&= \operatorname{Re} \operatorname{tr}(\psi^\dagger (2\phi\phi^\dagger)\psi) - \operatorname{Re} \operatorname{tr}(\phi^\dagger (2\psi\psi^\dagger)\phi) \\
&= 0,
\end{aligned}
$$

39

where the two terms cancel by the cyclic property of the real trace, Proposition 2.4. Thus, this component of the vector $[\Psi, [\Psi, \Psi]\Psi]$ vanishes. But since the map $\Psi \mapsto [\Psi, [\Psi, \Psi]\Psi]$ is equivariant with respect to the action of $\mathrm{Spin}(\mathcal{V})$, and $\mathcal{V}$ is an irreducible representation of this group, it follows that all components of this vector must vanish. $\qquad\square$

Chapter 3

Supertranslation algebras and their cohomology

3.1 Superalgebra

So far we have used normed division algebras to construct a number of algebraic structures: vectors as elements of $\mathfrak{h}_2(\mathbb{K})$ or $\mathbb{K}[4]$, spinors as elements of $\mathbb{K}^2$ or $\mathbb{K}^4$, and the various bilinear maps involving vectors, spinors, and scalars. However, to describe supersymmetry, we also need superalgebra. Specifically, we need anticommuting spinors. Physically, this is because spinors are fermions, so we need them to satisfy anticommutation relations. Mathematically, this means that we will do our algebra in the category of 'super vector spaces', SuperVect, rather than the category of vector spaces, Vect.

A **super vector space** is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ where V_0 is called the **even** or **bosonic** part, and V_1 is called the **odd** or **fermionic** part. Like Vect, SuperVect is a symmetric monoidal category [12]. It has:

- $\mathbb{Z}_2$-graded vector spaces as objects;

- Grade-preserving linear maps as morphisms;

- A tensor product $\otimes$ that has the following grading: if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, then $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$;

- A braiding
$$B_{V,W} \colon V \otimes W \to W \otimes V$$
defined as follows: $v \in V$ and $w \in W$ are of grade p and q, then
$$B_{V,W}(v \otimes w) = (-1)^{pq} w \otimes v.$$

The braiding encodes the 'the rule of signs': in any calculation, when two odd elements are interchanged, we introduce a minus sign.

There is an obvious notion of direct sums for super vector spaces, with

$$(V \oplus W)_0 = V_0 \oplus W_0, \qquad (V \oplus W)_1 = V_1 \oplus W_1$$

and also an obvious notion of duals, with

$$(V^*)_0 = (V_0)^*, \qquad (V^*)_1 = (V_1)^*.$$

We say a super vector space V is **even** if it equals its even part ($V = V_0$), and **odd** if it equals its odd part ($V = V_1$). Any subspace $U \subseteq V$ of an even (resp. odd) super vector space becomes a super vector space which is again even (resp. odd).

It is noteworthy that treating division algebras as *odd* is compatible with the physical applications of this thesis. For instance, this turns out to force the spinor representations $S_\pm$ to be odd and the vector representation V to be even, as follows.

We treat the spinor representations $S_\pm$ as super vector spaces using the fact that they are the direct sum of two copies of $\mathbb{K}$. Since $\mathbb{K}$ is odd, so are S_+ and S_-. Since $\mathbb{K}^2$ is odd, so is its dual. This in turn forces the space of linear maps from $\mathbb{K}^2$ to itself, $\mathrm{End}(\mathbb{K}^2) = \mathbb{K}^2 \otimes (\mathbb{K}^2)^*$, to be even. This even space contains the 2×2 matrices $\mathbb{K}[2]$ as the subspace of maps realized by left multiplication:

$$\begin{aligned} \mathbb{K}[2] &\hookrightarrow \mathrm{End}(\mathbb{K}^2) \\ A &\mapsto L_A. \end{aligned}$$

$\mathbb{K}[2]$ is thus even. Finally, this forces the subspace of hermitian 2×2 matrices, $\mathfrak{h}_2(\mathbb{K})$, to be even. So, the vector representation V is even. All this matches the usual rules in physics, where spinors are fermionic and vectors are bosonic.

3.2   Cohomology of Lie superalgebras

We now fuse the vectors and spinors we described with division algebras into a single structure. In any dimension, a symmetric bilinear intertwining operator that eats two spinors and spits out a vector gives rise to a 'super-Minkowski spacetime' [26]. The infinitesimal translation symmetries of this object form a Lie superalgebra, called the 'supertranslation algebra', $\mathcal{T}$. The cohomology of this Lie superalgebra is interesting and apparently rather subtle [17, 51]. We shall see that its 3rd cohomology is nontrivial in dimensions $k + 2 = 3, 4, 6$ and 10, thanks to the 3-ψ's rule. algebras. Similarly, its 4th cohomology is nontrivial in dimensions $k + 3 = 4, 5, 7$ and 11, thanks to the 4-Ψ's rule.

For arbitrary superspacetimes, the cohomology of $\mathcal{T}$ is not explicitly known. Techniques to compute it have been described by Brandt [17], who applied them in dimension 5 and below. Movshev, Schwarz and Xu [51] showed how to augment these techniques using the computer algebra system LiE [24], and fully describe the cohomology in dimension 6 and 10 in this way.

Based on the work of these authors, it seems likely that the 3rd and 4th cohomology of $\mathcal{T}$ is nontrivial in sufficiently large dimensions. We conjecture, however, that dimensions $k + 2$ and $k + 3$ are the only ones with *Lorentz invariant* 3- and 4-cocycles. Exploratory calculations with LiE bare this conjecture out, but the general answer appears to be unknown.

Now for some definitions. Briefly, a **Lie superalgebra** is a Lie algebra in the category of super vector spaces. More concretely, it is a super vector space $\mathfrak{g}$ equipped with a graded-antisymmetric bracket:

$$[-, -] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g},$$

that satisfies the Jacobi identity up to some signs:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]],$$

for homogeneous $X, Y, Z \in \mathfrak{g}$. Here, $\Lambda^2 \mathfrak{g}$ is the exterior square of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as a super vector space. As an ordinary vector space,

$$\Lambda^2 \mathfrak{g} \cong \Lambda^2 \mathfrak{g}_0 \oplus \mathfrak{g}_0 \otimes \mathfrak{g}_1 \oplus \mathrm{Sym}^2 \mathfrak{g}_1,$$

thanks to the rule of signs.

We will be concerned with several Lie superalgebras in this thesis. However, one of the most important is also one of the most simple. Take V to be the space of vectors in Minkowski spacetime in any dimension, and take S to be any spinor representation in this dimension. Suppose that there is a symmetric equivariant bilinear map:

$$[-, -] \colon S \otimes S \to V.$$

Form a super vector space $\mathcal{T}$ with

$$\mathcal{T}_0 = V, \qquad \mathcal{T}_1 = S.$$

We make $\mathcal{T}$ into a Lie superalgebra, the **supertranslation algebra**, by giving it a suitable bracket operation. This bracket will be zero except when we bracket a spinor with a spinor, in which case it is simply $[-, -]$. Since this is symmetric and spinors are odd, the bracket operation is graded-antisymmetric overall. Furthermore, the Jacobi identity holds trivially, thanks to the near triviality of the bracket. Thus $\mathcal{T}$ is indeed, a Lie superalgebra.

Despite the fact that $\mathcal{T}$ is nearly trivial, its cohomology is not. To see this, we must first recall how to generalize Chevalley–Eilenberg cohomology [3, 22] from Lie algebras to Lie superalgebras [48]. Suppose $\mathfrak{g}$ is a Lie superalgebra and R is a representation of $\mathfrak{g}$. That is, R is a supervector space equipped with a Lie superalgebra homomorphism $\rho \colon \mathfrak{g} \to \mathfrak{gl}(R)$. We now define the cohomology groups of $\mathfrak{g}$ with values in R.

First, of course, we need a cochain complex. We define the **Lie superalgebra cochain complex** $C^\bullet(\mathfrak{g}, R)$ to consist of graded-antisymmetric p-linear maps at level p:

$$C^p(\mathfrak{g}, R) = \{\omega \colon \Lambda^p \mathfrak{g} \to R\}.$$

In fact, the p-cochains $C^p(\mathfrak{g}, R)$ are a super vector space, in which grade-preserving elements are even, while grade-reversing elements are odd. When $R = \mathbb{R}$, the trivial representation, we typically omit it from the cochain complex and all associated groups, such as the cohomology groups. Thus, we write $C^\bullet(\mathfrak{g})$ for $C^\bullet(\mathfrak{g}, \mathbb{R})$.

Next, we define the coboundary operator $d: C^p(\mathfrak{g}, R) \to C^{p+1}(\mathfrak{g}, R)$. Let ω be a homogeneous p-cochain and let $X_1, \ldots, X_{p+1}$ be homogeneous elements of $\mathfrak{g}$. Now define:

$$d\omega(X_1, \ldots, X_{n+1}) =$$
$$\sum_{i=1}^{p+1} (-1)^{i+1}(-1)^{|X_i||\omega|}\epsilon_1^{i-1}(i)\rho(X_i)\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})$$
$$+ \sum_{i<j}(-1)^{i+j}(-1)^{|X_i||X_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j)\omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{p+1}).$$

Here, $\epsilon_i^j(k)$ is shorthand for the sign one obtains by moving X_k through $X_i, X_{i+1}, \ldots, X_j$. In other words,
$$\epsilon_i^j(k) = (-1)^{|X_k|(|X_i|+|X_{i+1}|+\cdots+|X_j|)}.$$

Following the usual argument for Lie algebras, one can check that:

Proposition 3.1. *The Lie superalgebra coboundary operator d satisfies $d^2 = 0$.*

We thus say a R-valued p-cochain ω on $\mathfrak{g}$ is an **p-cocycle** or **closed** when $d\omega = 0$, and an **p-coboundary** or **exact** if there exists an $(p-1)$-cochain θ such that $\omega = d\theta$. Every p-coboundary is an p-cocycle, and we say an p-cocycle is **trivial** if it is a coboundary. We denote the super vector spaces of p-cocycles and p-coboundaries by $Z^p(\mathfrak{g}, V)$ and $B^p(\mathfrak{g}, V)$ respectively. The pth **Lie superalgebra cohomology of $\mathfrak{g}$ with coefficients in R**, denoted $H^p(\mathfrak{g}, R)$ is defined by

$$H^p(\mathfrak{g}, R) = Z^p(\mathfrak{g}, R)/B^p(\mathfrak{g}, R).$$

This super vector space is nonzero if and only if there is a nontrivial p-cocycle. In what follows, we shall be especially concerned with the even part of this super vector space, which is nonzero if and only if there is a nontrivial even p-cocycle. Our motivation for looking for even cocycles is simple: these parity-preserving maps can regarded as morphisms in the category of super vector spaces, which is crucial for the construction in Theorem 5.1 and everything following it.

Now consider Minkowski spacetimes of dimensions 3, 4, 6 and 10. Here Minkowski spacetime can be written as $V = \mathfrak{h}_2(\mathbb{K})$, and we can take our spinors to be $S_+ = \mathbb{K}^2$. Since from Section 2.2 we know there is a symmetric bilinear intertwiner $[-, -]: S_+ \otimes S_+ \to V$, we obtain the supertranslation algebra $\mathcal{T} = V \oplus S_+$. We can decompose the space of n-cochains with on $\mathcal{T}$ into summands by counting how many of the arguments are vectors and how many are spinors:

$$C^n(\mathcal{T}) \cong \bigoplus_{p+q=n} (\Lambda^p(V) \otimes \text{Sym}^q(S_+))^*.$$

We call an element of $(\Lambda^p(V) \otimes \text{Sym}^q(S_+))^*$ a **(p, q)-form**. Since the bracket of two spinors is a vector, and all other brackets are zero, d of a (p, q)-form is a $(p-1, q+2)$-form.

Using the 3-ψ's rule we can show:

Theorem 3.1. *In dimensions 3, 4, 6 and 10, the supertranslation algebra $\mathcal{T}$ has a nontrivial, Lorentz-invariant even 3-cocycle taking values in the trivial representation $\mathbb{R}$, namely the unique $(1, 2)$-form with*

$$\alpha(\psi, \phi, A) = g([\psi, \phi], A)$$

for spinors $\psi, \phi \in S_+$ and vectors $A \in V$.

Proof. First, note that α has the right symmetry to be a linear map on $\Lambda^3(V \oplus S_+)$. Second, note that α is a $(1, 2)$-form, eating one vector and two spinors. Thus $d\alpha$ is a $(0, 4)$-form.

Because spinors are odd, $d\alpha$ is a symmetric function of four spinors. By the definition of d, $d\alpha(\psi, \phi, \chi, \theta)$ is the totally symmetric part of $\alpha([\psi, \phi], \chi, \theta) = \alpha(\chi, \theta, [\psi, \phi]) = g([\chi, \theta], [\psi, \phi])$. But any symmetric 4-linear form can be obtained from polarizing a quartic form. In this, we polarize $g([\psi, \psi], [\psi, \psi])$ to get $d\alpha$. Thus:

$$d\alpha(\psi, \psi, \psi, \psi) = g([\psi, \psi], [\psi, \psi]) = \langle \psi, [\psi, \psi]\psi \rangle$$

where we have used the definition of the bracket to obtain the last expression, which vanishes due to the 3-ψ rule. Thus α is closed.

It remains to show α is not exact. So suppose it is exact, and that

$$\alpha = d\omega.$$

By our remarks above we may assume ω is a $(2, 0)$-form: that is, an antisymmetric bilinear function of two vectors. By the definition of d, this last equation says:

$$g([\psi, \phi], A) = -\omega([\psi, \phi], A).$$

But since $S_+ \otimes S_+ \to V$ is onto, this implies

$$g = -\omega,$$

a contradiction, since g is symmetric while ω is antisymmetric. □

Next consider Minkowski spacetimes of dimensions 4, 5, 7 and 11. In this case Minkowski spacetime can be written as a subspace $\mathcal{V}$ of the 4×4 matrices valued in $\mathbb{K}$, and we can take our spinors to be $\mathcal{S} = \mathbb{K}^4$. Since from Section 2.3 we know there is a symmetric bilinear intertwiner $[-, -]\colon \mathcal{S} \otimes \mathcal{S} \to \mathcal{V}$, we obtain a supertranslation algebra $\mathcal{T} = \mathcal{V} \oplus \mathcal{S}$. As before, we can uniquely decompose any n-cochain in $C^n(\mathcal{T}, \mathbb{R})$ into a sum of (p, q)-forms, where now a **(p, q)-form** is an an element of $(\Lambda^p(\mathcal{V}) \otimes \mathrm{Sym}^q(\mathcal{S}))^*$. As before, d of a (p, q)-form is a $(p - 1, q + 2)$-form. And using the 4-Ψ's rule, we can show:

Theorem 3.2. *In dimensions 4, 5, 7 and 11, the supertranslation algebra $\mathcal{T}$ has a nontrivial, Lorentz-invariant even 4-cocycle, namely the unique $(2, 2)$-form with*

$$\beta(\Psi, \Phi, \mathcal{A}, \mathcal{B}) = \langle \Psi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Phi \rangle$$

for spinors $\Psi, \Phi \in \mathcal{S}$ and vectors $\mathcal{A}, \mathcal{B} \in \mathcal{V}$. Here the commutator $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is taken in the Clifford algebra of $\mathcal{V}$.

Proof. First, to see that β has the right symmetry to be a map on $\Lambda^4(\mathcal{V} \oplus \mathcal{S})$, we note that it is antisymmetric on vectors. Recalling from Proposition 2.12 that the skew-symmetric bilinear form $\langle -, - \rangle$ is defined by:

$$\langle \Psi, \Phi \rangle = \mathrm{Re}(\Psi^\dagger \Gamma^0 \Phi),$$

and that because
$$\Gamma^0 \mathcal{A} = -\mathcal{A}^\dagger \Gamma^0,$$

we have:
$$\Gamma^0 \mathcal{A}\mathcal{B} = \mathcal{A}^\dagger \mathcal{B}^\dagger \Gamma^0.$$

Thus:
$$\langle \Psi, \mathcal{A}\mathcal{B}\Phi \rangle = \langle \mathcal{B}\mathcal{A}\Psi, \Phi \rangle = -\langle \Phi, \mathcal{B}\mathcal{A}\Psi \rangle,$$

so we have:
$$\langle \Psi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Phi \rangle = \langle \Phi, (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Psi \rangle.$$

Thus, β is symmetric on spinors.

Next note that $d\beta$ is a $(1,4)$-form, symmetric on its four spinor inputs. It is thus proportional to the polarization of

$$\alpha(\Psi, \Psi, [\Psi, \Psi], \mathcal{A}) = \Psi * \Psi([\Psi, \Psi], \mathcal{A})$$

We encountered this object in Section 2.4, where we showed that it is proportional to

$$h([\Psi, [\Psi, \Psi]\Psi], \mathcal{A}).$$

Moreover, this last expression vanishes by the 4-Ψ's rule. So, β is closed.

Furthermore, β is not exact. To see this, consider the unit vector $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ orthogonal to $V \subseteq \mathcal{V}$. Taking the interior product of β with this vector, a quick calculation shows:

$$\beta(\Psi, \Phi, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right), \mathcal{A}) = 2\langle \psi_1, \gamma(A)\phi_1 \rangle + 2\langle \tilde{\gamma}(A)\psi_2, \phi_2 \rangle,$$

where we have decomposed $\Psi = (\psi_1, \psi_2)$ and $\Phi = (\phi_1, \phi_2)$ into their components in $\mathcal{S} = S_+ \oplus S_-$, and A is the component of $\mathcal{A}$ in V. Restricting to the subalgebra $V \oplus S_+ \subseteq \mathcal{V} \oplus \mathcal{S}$, we see this is just α, up to a factor.

So, it suffices to check that interior product with $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ preserves exactness. For then, if β were exact, it would contradict that fact that α is not. Indeed, let ω be an n-cochain on $\mathcal{T}$, and let $X_1, \ldots, X_n \in \mathcal{T}$. Then, by our formula for the coboundary operator, we have:

$$d\omega(X, X_1, \ldots, X_n) =$$
$$\sum_{i<j} -(-1)^{i+j}(-1)^{|X_i||X_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j)\omega(X, [X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_n)$$
$$+ \sum_{i=1}^{n} (-1)^{1+i}\epsilon_1^{i-1}(i)\omega([X, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_n),$$

where, taking care with signs, we have collected terms involving bracketing with X into the second summation. But X is a vector, so all brackets with it vanish, and the second summation is zero.

If we write $i_X\omega$ for the operation of taking the interior product of ω with X, we have just shown:

$$i_X d\omega = -d i_X \omega$$

for any ω. In particular, if $\omega = d\theta$ then $i_X\omega = d(-i_X\theta)$, and so interior product with X preserves exactness, as claimed. $\qquad \square$

Chapter 4

An application: super-Yang–Mills theory

For the moment, let us set aside our quest to understand division algebras, supersymmetry and higher gauge theory and focus on a special case: the connection between division algebras and super-Yang–Mills theories. Such theories are the low energy limit of superstring theories in a fixed background [39], so it is not surprising that they also occur only in spacetimes of dimension 3, 4, 6 and 10.

The minimal supersymmetric extension of pure Yang–Mills theory has the Lagrangian:

$$L = -\frac{1}{4}\langle F, F \rangle + \frac{1}{2}\langle \psi, \slashed{D}_A \psi \rangle.$$

Here A is a connection on a bundle with semisimple gauge group G, F is the curvature of A, ψ is a $\mathfrak{g}$-valued spinor field, and $\slashed{D}_A$ is the covariant Dirac operator associated with A. In the physics literature, it is well-known that this theory is supersymmetric if and only if the dimension of spacetime is $3, 4, 6$, or 10. Our goal in this section is to present a self-contained proof of the 'if' part of this result, based on the 3-ψ's rule. Along the way, we shall give a division algebra interpretation of the Lagrangian, L.

The proof that L is supersymmetric goes back to the work of Brink, Schwarz and Sherk [18], and others. The book by Green, Schwarz and Witten [39] contains a standard proof based on the properties of Clifford algebras in various dimensions. But Evans [32] has shown that the supersymmetry of L in dimension $k + 2$ implies the existence of a normed division algebra of dimension k. Conversely, Kugo and Townsend [45] showed how spinors in dimension 3, 4, 6 and 10 derive special properties from the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. They formulated a supersymmetric model in 6 dimensions using the quaternions, $\mathbb{H}$. They also speculated about a similar formalism in 10 dimensions using the octonions, $\mathbb{O}$.

Shortly after Kugo and Townsend's work, Sudbery [65] used division algebras to construct vectors, spinors and Lorentz groups in Minkowski spacetimes of dimensions 3, 4, 6 and 10. He then refined his construction with Chung [23], and with Manogue [50] he used these ideas to give an octonionic proof of the supersymmetry of the above Lagrangian in dimension 10. This proof was later simplified by Manogue, Dray and Janesky [29]. In the meantime, Schray [59] applied the same tools to the superparticle.

All this work has made it quite clear that normed division algebras explain why the above theory is supersymmetric in dimensions 3, 4, 6 and 10. Technically, what we need to check for supersymmetry is that δL is a total divergence with respect to the supersymmetry transformation

$$\begin{aligned} \delta A &= [\epsilon, \psi] \\ \delta \psi &= \tfrac{1}{2} F \epsilon \end{aligned}$$

for any constant spinor field ϵ. A calculation that works in any dimension shows that

$$\delta L = \operatorname{tri} \psi + \text{divergence}$$

where $\operatorname{tri} \psi$ is a certain expression depending in a trilinear way on ψ and linearly on ϵ.

So, the marvelous fact that needs to be understood is that $\operatorname{tri} \psi = 0$ in dimensions 3, 4, 6 and 10, thanks to special properties of the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Indeed, we shall show that it is a consequence of the 3-ψ's rule. Yet this rule is a direct consequence of the fact that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are alternative, so one could say that the vanishing of $\operatorname{tri} \psi$ is a direct consequence of the total antisymmetry of a simpler trilinear: the associator $[a, b, c]$.

Let's get to work. For simplicity, we shall work over Minkowski spacetime, M. This allows us to treat all bundles as trivial, sections as functions, and connections as $\mathfrak{g}$-valued 1-forms. At the outset, we fix an invariant inner product on $\mathfrak{g}$, the Lie algebra of a semisimple Lie group G. We shall use the following standard tools from differential geometry to construct L, none of which need involve spinors or division algebra technology:

- A connection A on a principal G-bundle over M. Since the bundle is trivial we think of this connection as a $\mathfrak{g}$-valued 1-form.

- The exterior covariant derivative $d_A = d + [A, -]$ on $\mathfrak{g}$-valued p-forms.

- The curvature $F = dA + \tfrac{1}{2}[A, A]$, which is a $\mathfrak{g}$-valued 2-form.

- The usual pointwise inner product $\langle F, F \rangle$ on $\mathfrak{g}$-valued 2-forms, defined using the Minkowski metric on M and the invariant inner product on $\mathfrak{g}$.

We also need the following spinorial tools. Because spinors describe fermions, we assume S_+ and S_- are odd objects in SuperVect. So, whenever we switch two spinors, we introduce a minus sign.

- A $\mathfrak{g}$-valued section ψ of a spin bundle over M. Note that this is, in fact, just a function:

$$\psi \colon M \to S_\pm \otimes \mathfrak{g}.$$

  We call the collection of all such functions $\Gamma(S_\pm \otimes \mathfrak{g})$.

- The covariant Dirac operator $\slashed{D}_A$ derived from D_A. Of course,

$$\slashed{D}_A \colon \Gamma(S_\pm \otimes \mathfrak{g}) \to \Gamma(S_\mp \otimes \mathfrak{g})$$

  and in fact,

$$\slashed{D}_A = \slashed{\partial} + A.$$

- A bilinear pairing

$$\langle -, - \rangle \colon \Gamma(S_+ \otimes \mathfrak{g}) \otimes \Gamma(S_- \otimes \mathfrak{g}) \to C^\infty(M)$$

built pointwise using our pairing

$$\langle -, - \rangle \colon S_+ \otimes S_- \to \mathbb{R}$$

and the invariant inner product on $\mathfrak{g}$.

The basic fields in our theory are a connection on a principal G-bundle, which we think of as a $\mathfrak{g}$-valued 1-form:

$$A \colon M \to V^* \otimes \mathfrak{g}.$$

and a $\mathfrak{g}$-valued spinor field, which we think of as a $S_+ \otimes \mathfrak{g}$-valued function on M:

$$\psi \colon M \to S_+ \otimes \mathfrak{g}.$$

All our arguments would work just as well with S_- replacing S_+.

To show that L is supersymmetric, we need to show δL is a total divergence when δ is the following supersymmetry transformation:

$$
\begin{aligned}
\delta A &= [\epsilon, \psi] \\
\delta \psi &= \frac{1}{2} F \epsilon
\end{aligned}
$$

where ϵ is an arbitrary constant spinor field, treated as odd, but not $\mathfrak{g}$-valued. By a **supersymmetry transformation** we mean that computationally we treat δ as a derivation on the algebra of functions on spacetime. So, it is linear:

$$\delta(\alpha f + \beta g) = \alpha \delta f + \beta \delta g$$

where $\alpha, \beta \in \mathbb{R}$, and it satisfies the product rule:

$$\delta(fg) = \delta(f)g + f\delta g.$$

For a more precise discussion of 'supersymmetry transformations', see Deligne and Freed [27].

The above equations require further explanation. The bracket $[\epsilon, \psi]$ denotes an operation that combines the spinor ϵ with the $\mathfrak{g}$-valued spinor ψ to produce a $\mathfrak{g}$-valued 1-form. We build this from our basic intertwiner

$$[-, -] \colon S_+ \otimes S_+ \to V.$$

We identify V with V^* using the Minkowski inner product g, obtaining

$$[-, -] \colon S_+ \otimes S_+ \to V^*.$$

Then we tensor both sides with $\mathfrak{g}$. This gives us a way to act by a spinor field on a $\mathfrak{g}$-valued spinor field to obtain a $\mathfrak{g}$-valued 1-form. We take the liberty of also denoting this with by the bracket:

$$[-, -] \colon \Gamma(S_+) \otimes \Gamma(S_+ \otimes \mathfrak{g}) \to \Omega^1(M, \mathfrak{g}).$$

We also need to explain how the 2-form F acts on the constant spinor field ϵ. Using the Minkowski metric, we can identify differential forms on M with sections of the Clifford algebra bundle over M:

$$\Omega^\bullet(M) \cong \mathrm{Cliff}(M).$$

Using this, differential forms act on spinor fields. Tensoring with $\mathfrak{g}$, we obtain a way for $\mathfrak{g}$-valued differential forms like F to act on spinor fields like ϵ to give $\mathfrak{g}$-valued spinor fields like $F\epsilon$.

Let us now apply the supersymmetry transformation to each term in the Lagrangian. First, the bosonic term:

Proposition 4.1. *The bosonic term has:*

$$\delta\langle F, F\rangle = 2(-1)^{k+1}\,\langle \psi, (\star d_A \star F)\epsilon\rangle + \text{divergence}.$$

Proof. By the symmetry of the inner product, we get:

$$\delta\langle F, F\rangle = 2\langle F, \delta F\rangle.$$

Using the handy formula $\delta F = d_A \delta A$, we have:

$$\langle F, \delta F\rangle = \langle F, d_A \delta A\rangle.$$

Now the adjoint of the operator d_A is $\star d_A \star$, up to a pesky sign: if ν is a $\mathfrak{g}$-valued $(p-1)$-form and μ is a $\mathfrak{g}$-valued p-form, we have

$$\langle \mu, d_A \nu\rangle = (-1)^{dp+d+1+s}\langle \star d_A \star \mu, \nu\rangle + \text{divergence}$$

where d is the dimension of spacetime and s is the signature, i.e., the number of minus signs in the diagonalized metric. It follows that

$$\langle F, \delta F\rangle = \langle F, d_A \delta A\rangle = (-1)^k\,\langle \star d_A \star F, \delta A\rangle + \text{divergence}$$

where k is the dimension of $\mathbb{K}$. By the definition of δA, we get

$$\langle \star d_A \star F, \delta A\rangle = \langle \star d_A \star F, [\epsilon, \psi]\rangle.$$

Now we can use division algebra technology to show:

$$\langle \star d_A \star F, [\epsilon, \psi]\rangle = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}\left((\star d_A \star F)(\epsilon\psi^\dagger + \psi\epsilon^\dagger)\right) = -\langle \psi, (\star d_A \star F)\epsilon\rangle,$$

using the cyclic property of the real trace in the last step, and introducing a minus sign in accordance with the sign rule. Putting everything together, we obtain the desired result. $\square$

Even though this proposition involved the bosonic term only, division algebra technology was still a useful tool in its proof. Ironically, division algebra technology is absent from the proof of the next proposition, which deals with the fermionic term:

Proposition 4.2. *The fermionic term has:*

$$\delta\langle \psi, \slashed{D}_A \psi\rangle = \langle \psi, \slashed{D}_A(F\epsilon)\rangle + \text{tri}\,\psi + \text{divergence}$$

where

$$\operatorname{tri} \psi = \langle \psi, [\epsilon, \psi]\psi \rangle.$$

Proof. It is easy to compute:

$$\delta \langle \psi, \slashed{D}_A \psi \rangle = \langle \delta\psi, \slashed{D}_A \psi \rangle + \langle \psi, \delta\slashed{D}_A \psi \rangle + \langle \psi, \slashed{D}_A \delta\psi \rangle.$$

Now we insert $\delta\slashed{D}_A = \delta A = [\epsilon, \psi]$, and thus see that the penultimate term is the trilinear one:

$$\operatorname{tri} \psi = \langle \psi, [\epsilon, \psi]\psi \rangle.$$

So, let us concern ourselves with the remaining terms:

$$\langle \delta\psi, \slashed{D}_A \psi \rangle + \langle \psi, \slashed{D}_A \delta\psi \rangle.$$

A computation using the product rule shows that the divergence of the 1-form $[\psi, \phi]$ is given by $-\langle \phi, \slashed{D}_A \psi \rangle + \langle \psi, \slashed{D}_A \phi \rangle$, where the minus sign on the first term arises from using the sign rule with these odd spinors. In the terms under consideration, we can use this identity to move $\slashed{D}_A$ onto $\delta\psi$:

$$\langle \delta\psi, \slashed{D}_A \psi \rangle + \langle \psi, \slashed{D}_A \delta\psi \rangle = 2\langle \psi, \slashed{D}_A \delta\psi \rangle + \text{divergence}.$$

Substituting $\delta\psi = \frac{1}{2}F\epsilon$, we obtain the desired result. $\qquad \square$

Using these two propositions, it is immediate that

$$
\begin{aligned}
\delta L &= -\frac{1}{4}\delta\langle F, F \rangle + \frac{1}{2}\delta\langle \psi, \slashed{D}_A \psi \rangle \\
&= \frac{1}{2}(-1)^k \langle \psi, (\star d_A \star F)\epsilon \rangle + \frac{1}{2}\langle \psi, \slashed{D}_A(F\epsilon) \rangle + \frac{1}{2}\operatorname{tri} \psi + \text{divergence}
\end{aligned}
$$

All that remains to show is that $\slashed{D}_A(F\epsilon) = (-1)^{k+1}(\star d_A \star F)\,\epsilon$. Indeed, Snygg shows (Eq. 7.6 in [63]) that for an ordinary, non-$\mathfrak{g}$-valued p-form F

$$\slashed{\partial}(F\epsilon) = (dF)\epsilon + (-1)^{d+dp+s}(\star d \star F)\epsilon$$

where d is the dimension of spacetime and s is the signature. This is easily generalized to covariant derivatives and $\mathfrak{g}$-valued p-forms:

$$\slashed{D}_A(F\epsilon) = (d_A F)\epsilon + (-1)^{d+dp+s}(\star d_A \star F)\epsilon.$$

In particular, when F is the curvature 2-form, the first term vanishes by the Bianchi identity $d_A F = 0$, and we are left with:

$$\slashed{D}_A(F\epsilon) = (-1)^{k+1}(\star d_A \star F)\epsilon$$

where k is the dimension of $\mathbb{K}$. We have thus shown:

Proposition 4.3. *Under supersymmetry transformations, the Lagrangian L has:*

$$\delta L = \frac{1}{2}\operatorname{tri} \psi + \text{divergence}.$$

The above result actually holds in every dimension, though our proof used division algebras and was thus adapted to the dimensions of interest: 3, 4, 6 and 10. The next result is where division algebra technology becomes really crucial:

Proposition 4.4. *For Minkowski spacetimes of dimensions 3, 4, 6 and 10,* $\mathrm{tri}\,\psi = 0$.

Proof. At each point of M, we can write

$$\psi = \sum \psi^a \otimes g_a,$$

where $\psi^a \in S_+$ and $g_a \in \mathfrak{g}$. When we insert this into $\mathrm{tri}\,\psi$, we see that

$$\mathrm{tri}\,\psi = \sum \langle \psi^a, [\epsilon, \psi^b]\psi^c \rangle \langle g_a, [g_b, g_c] \rangle.$$

Since $\langle g_a, [g_b, g_c] \rangle$ is totally antisymmetric, this implies $\mathrm{tri}\,\psi = 0$ for all ϵ if and only if the part of $\langle \psi^a, [\epsilon, \psi^b]\psi^c \rangle$ that is antisymmetric in a, b and c vanishes for all ϵ. Yet these spinors are odd; for even spinors, we require the part of $\langle \psi^a, [\epsilon, \psi^b]\psi^c \rangle$ that is *symmetric* in a, b and c to vanish for all ϵ.

Now let us remove our dependence on ϵ. While we do this, let us replace ψ^a with ψ, ψ^b with ϕ, and ψ^c with χ to lessen the clutter of indices. By the second formula in Proposition 2.14, we have:

$$\langle \psi, [\epsilon, \phi]\chi \rangle = \langle \epsilon, [\psi, \chi]\phi \rangle,$$

So, if we seek to show that the part of $\langle \psi, [\epsilon, \phi]\chi \rangle$ that is totally symmetric in ψ, ϕ and χ vanishes for all ϵ, it is equivalent to show the totally symmetric part of $[\phi, \chi]\psi$ vanishes. But this happens for all ψ, ϕ and χ in S_+ if and only if $[\psi, \psi]\psi = 0$ for all ψ in S_+. This is the 3-ψ's rule, Theorem 2.1, so we are done. $\square$

Chapter 5

Lie n-superalgebras from Lie superalgebra cohomology

In Section 3.2, we saw that the 3-ψ's and 4-Ψ's rules are cocycle conditions for the cocycles α and β. This sheds some light on the meaning of these rules, but it prompts an obvious followup question: what are these cocycles good for?

There is a very general answer to this question: a cocycle on a Lie superalgebra lets us extend it to an 'L_∞-superalgebra'. As we touched on in the Introduction, this is a chain complex equipped with structure like that of a Lie superalgebra, but where all the laws hold only 'up to chain homotopy'. We give the precise definition below.

It is well known that that the 2nd cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in some representation R classifies 'central extensions' of $\mathfrak{g}$ by R [3, 22]. These are short exact sequences of Lie algebras:

$$0 \to R \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where $\tilde{g}$ is arbitrary and R is treated as an abelian Lie algebra whose image lies in the center of $\tilde{g}$. The same sort of result is true for Lie superalgebras. But this is just a special case of an even more general fact.

Suppose $\mathfrak{g}$ is a Lie superalgebra with a representation on a supervector space R. Then we shall prove that an even R-valued $(n + 2)$-cocycle ω on $\mathfrak{g}$ lets us construct an L_∞-superalgebra, called $\mathfrak{brane}_\omega(\mathfrak{g}, R)$, of the following form:

$$\mathfrak{g} \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } \cdots \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } R.$$

where only the 0th and and nth grades are nonzero. Moreover, $\mathfrak{brane}_\omega(\mathfrak{g}, R)$ is an **extension** of $\mathfrak{g}$: there is a short exact sequence of L_∞-superalgebras

$$0 \to \mathfrak{b}^n R \to \mathfrak{brane}_\omega(\mathfrak{g}, R) \to \mathfrak{g} \to 0.$$

Here $\mathfrak{b}^n R$ is the abelian L_∞-superalgebra with R as its nth grade and all the rest zero:

$$0 \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } \cdots \xleftarrow{\ d\ } 0 \xleftarrow{\ d\ } R$$

Note that when $n = 0$ and our vector spaces are all purely even, we are back to the familiar construction of Lie algebra extensions from 2-cocycles.

Technically, we should be more general than this in defining extensions. Maps between L_∞-algebras admit homotopies among themselves, and this allows us to introduce a weakened notion of 'short exact sequence': a fibration sequence in the $(\infty, 1)$-category of L_∞-algebras. In general, these fibration sequences give the right concept of extension for L_∞-algebras. However, for the very special extensions we consider here, ordinary short exact sequences are all we need.

It is useful to have a special name for L_∞-superalgebras whose nonzero terms are all of degree $< n$: we call them **Lie n-superalgebras**. In this language, the 3-cocycle α defined in Theorem 3.1 gives rise to a Lie 2-superalgebra

$$\mathcal{T} \xleftarrow{d} \mathbb{R}$$

extending the supertranslation algebra $\mathcal{T}$ in dimensions 3, 4, 6 and 10. Similarly, the 4-cocycle β defined in Theorem 3.2 gives a Lie 3-superalgebra

$$\mathcal{T} \xleftarrow{d} 0 \xleftarrow{d} \mathbb{R}$$

extending the supertranslation algebra in dimensions $4, 5, 7$ and 11.

Now let us make all of these ideas precise. In what follows, we shall use **super chain complexes**, which are chain complexes in the category SuperVect of $\mathbb{Z}_2$-graded vector spaces:

$$V_0 \xleftarrow{d} V_1 \xleftarrow{d} V_2 \xleftarrow{d} \cdots$$

Thus each V_p is $\mathbb{Z}_2$-graded and d preserves this grading.

There are thus two gradings in play: the $\mathbb{Z}$-grading by **degree**, and the $\mathbb{Z}_2$-grading on each vector space, which we call the **parity**. We shall require a sign convention to establish how these gradings interact. If we consider an object of odd parity and odd degree, is it in fact even overall? By convention, we assume that it is. That is, whenever we interchange something of parity p and degree q with something of parity p' and degree q', we introduce the sign $(-1)^{(p+q)(p'+q')}$. We shall call the sum $p + q$ of parity and degree the **overall grade**, or when it will not cause confusion, simply the grade. We denote the overall grade of X by $|X|$.

We require a compressed notation for signs. If $x_1, \ldots, x_n$ are graded, $\sigma \in S_n$ a permutation, we define the **Koszul sign** $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ by

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \cdots x_{\sigma(n)},$$

the sign we would introduce in the free graded-commutative algebra generated by $x_1, \ldots, x_n$. Thus, $\epsilon(\sigma)$ encodes all the sign changes that arise from permuting graded elements. Now define:

$$\chi(\sigma) = \chi(\sigma; x_1, \ldots, x_n) := \text{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).$$

Thus, $\chi(\sigma)$ is the sign we would introduce in the free graded-anticommutative algebra generated by $x_1, \ldots, x_n$.

Yet we shall only be concerned with particular permutations. If n is a natural number and $1 \leq j \leq n - 1$ we say that $\sigma \in S_n$ is an $(j, n - j)$-**unshuffle** if

$$\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(j) \quad \text{and} \quad \sigma(j+1) \leq \sigma(j+2) \leq \cdots \leq \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses. A *shuffle* of two ordered sets (such as a deck of cards) is a permutation of the ordered union preserving the order of each of the given subsets. An *unshuffle* reverses this process. We denote the collection of all $(j, n - j)$ unshuffles by $S_{(j,n-j)}$.

The following definition of an L_∞-algebra was formulated by Schlessinger and Stasheff in 1985 [57]:

Definition 5.1. An $\boldsymbol{L_\infty}$**-superalgebra** is a graded vector space V equipped with a system $\{l_k | 1 \le k < \infty\}$ of linear maps $l_k \colon V^{\otimes k} \to V$ with $\deg(l_k) = k - 2$ which are totally antisymmetric in the sense that

$$l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \ldots, x_n) \tag{5.1}$$

for all $\sigma \in S_n$ and $x_1, \ldots, x_n \in V$, and, moreover, the following generalized form of the Jacobi identity holds for $0 \le n < \infty$:

$$\sum_{i+j=n+1} \sum_{\sigma \in S_{(i,n-i)}} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{5.2}$$

where the inner summation is taken over all $(i, n - i)$-unshuffles with $i \ge 1$.

The following result shows how to construct L_∞-superalgebras from Lie superalgebra cocycles. This is the 'super' version of a result due to Crans [5]. In this result, we require our cocycle to be even so we can consider it as a morphism in the category of super vector spaces.

Theorem 5.1. *There is a one-to-one correspondence between L_∞-superalgebras consisting of only two nonzero terms V_0 and V_n, with $d = 0$, and quadruples $(\mathfrak{g}, V, \rho, l_{n+2})$ where $\mathfrak{g}$ is a Lie superalgebra, V is a super vector space, ρ is a representation of $\mathfrak{g}$ on V, and l_{n+2} is an even $(n + 2)$-cocycle on $\mathfrak{g}$ with values in V.*

Proof. Given such an L_∞-superalgebra we set $\mathfrak{g} = V_0$. V_0 comes equipped with a bracket as part of the L_∞-structure, and since d is trivial, this bracket satisfies the Jacobi identity on the nose, making $\mathfrak{g}$ into a Lie superalgebra. We define $V = V_n$, and note that the bracket also gives a map $\rho \colon \mathfrak{g} \otimes V \to V$, defined by $\rho(x)f = [x, f]$ for $x \in \mathfrak{g}, f \in V$. We have

$$
\begin{aligned}
\rho([x,y])f &= [[x,y],f] \\
&= (-1)^{|y||f|}[[x,f],y] + [x,[y,f]] \quad \text{by (3) of Definition 5.1} \\
&= (-1)^{|f||y|}[\rho(x)f, y] + [x, \rho(y)f] \\
&= -(-1)^{|x||y|}\rho(y)\rho(x)f + \rho(x)\rho(y)f \\
&= [\rho(x), \rho(y)]f
\end{aligned}
$$

for all $x, y \in \mathfrak{g}$ and $f \in V$, so that ρ is indeed a representation. Finally, the L_∞ structure gives a map $l_{n+2} \colon \Lambda^{n+2}\mathfrak{g} \to V$ which is in fact an $(n + 2)$-cocycle. To see this, note that

$$0 = \sum_{i+j=n+4} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n+3)}),$$

55

where we sum over $(i, (n+3)-i)$-unshuffles $\sigma \in S_{n+3}$. However, the only choices for i and j that lead to nonzero l_i and l_j are $i = n+2, j = 2$ and $i = 2, j = n+2$. Thus, the above becomes, with σ a $(n+2, 1)$-unshuffle and τ a $(2, n+1)$-unshuffle:

$$
\begin{aligned}
0 &= \sum_{\sigma} \chi(\sigma)(-1)^{n+2}[l_{n+2}(x_{\sigma(1)}, \ldots, x_{\sigma(n+2)}), x_{\sigma(n+3)}] \\
&+ \sum_{\tau} \chi(\tau) l_{n+2}([x_{\tau(1)}, x_{\tau(2)}], x_{\tau(3)}, \ldots, x_{\tau(n+3)}) \\
&= \sum_{i=1}^{n+3} (-1)^{n+3-i}(-1)^{n+2} \epsilon_{i+1}^{n+2}(i)[l_{n+2}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+3}), x_i] \\
&+ \sum_{1 \le i < j \le n+3} (-1)^{i+j+1}(-1)^{|x_i||x_j|} \epsilon_1^{i-1}(i) \epsilon_1^{j-1}(j) l_{n+2}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+3}).
\end{aligned}
$$

On the second line, we have explicitly specified the unshuffles and unwrapped the signs encoded by χ. Since l_{n+2} is a morphism in SuperVect, it preserves parity, and thus the element

$$
l_{n+2}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+2})
$$

has parity $|x_1| + \cdots + |x_{i-1}| + |x_{i+1}| + \cdots + |x_{n+2}|$. So, we can reorder the bracket in the first term, at the cost of a sign:

$$
\begin{aligned}
0 &= \sum_{i=1}^{n+3} -(-1)^{i+1} \epsilon_1^{i-1}(i)[x_i, l_{n+2}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+3})] \\
&+ \sum_{1 \le i < j \le n+3} -(-1)^{i+j}(-1)^{|x_i||x_j|} \epsilon_1^{i-1}(i) \epsilon_1^{j-1}(j) l_{n+2}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+3}) \\
&= -dl_{n+2}.
\end{aligned}
$$

Here, we have used the fact that $\epsilon_{i+1}^{n+2}(i)(-1)^{|x_i|(|x_1|+\cdots+|x_{i-1}|+|x_{i+1}|+\cdots+|x_{n+2}|)} = \epsilon_1^{i-1}(i)$. Thus, l_{n+2} is indeed a cocycle.

Conversely, given a Lie superalgebra $\mathfrak{g}$, a representation ρ of $\mathfrak{g}$ on a vector space V, and an even $(n+2)$-cocycle l_{n+2} on $\mathfrak{g}$ with values in V, we define our L_∞-superalgebra V by setting $V_0 = \mathfrak{g}$, $V_n = V$, $V_i = \{0\}$ for $i \ne 0, n$, and $d = 0$. It remains to define the system of linear maps l_k, which we do as follows: Since $\mathfrak{g}$ is a Lie superalgebra, we have a bracket defined on V_0. We extend this bracket to define the map l_2, denoted by $[\cdot, \cdot]\colon V_i \otimes V_j \to V_{i+j}$ where $i, j = 0, n$, as follows:

$$
[x, f] = \rho(x)f, \quad [f, y] = (-1)^{|y||f|} \rho(y)f, \quad [f, g] = 0
$$

for $x, y \in V_0$ and $f, g \in V_n$. With this definition, the map $[\cdot, \cdot]$ satisfies condition (1) of Definition 5.1. We define $l_k = 0$ for $3 \le k \le n+1$ and $k > n+2$, and take l_{n+2} to be the given $(n+2)$ cocycle, which satisfies conditions (1) and (2) of Definition 5.1 by the cocycle condition. $\qquad \square$

This theorem tells us how to take a Lie superalgebra $(n+1)$-cocycle ω, and construct a Lie n-superalgebra with $d = 0$, concentrated in degrees 0 and $n-1$. We call such a Lie n-superalgebra a **slim Lie n-superalgebra**, and denote it by $\mathfrak{brane}_\omega(\mathfrak{g}, R)$. When $n = 2$, we will also write

$\mathfrak{string}_\omega(\mathfrak{g}, R)$ for the same object, and when R is the trivial representation $\mathbb{R}$, we omit it. In the next section, we give some examples of these objects.

5.1 Examples of slim Lie *n*-superalgebras

5.1.1 The string Lie 2-algebra

For $n \geq 3$, consider the Lie algebra $\mathfrak{so}(n)$ of infinitesimal rotations of n-dimensional Euclidean space. This matrix Lie algebra has Killing form given by the trace, $\langle X, Y \rangle = \mathrm{tr}(XY)$, and an easy calculation shows that

$$j = \langle -, [-, -] \rangle$$

is a 3-cocycle on $\mathfrak{so}(n)$. We call j the **canonical 3-cocycle** on $\mathfrak{so}(n)$. Using j, we get a Lie 2-algebra $\mathfrak{string}_j(\mathfrak{so}(n))$, which we denote simply by $\mathfrak{string}(n)$. We call this the **string Lie 2-algebra**. First defined by Baez–Crans [5], it is so-named because it turned out to be intimately related to the string group, $\mathrm{String}(n)$, the topological group obtained from $\mathrm{SO}(n)$ by killing the 1st and 3rd homotopy groups. For a description of this relationship, as well as the construction of Lie 2-groups which integrate $\mathfrak{string}(n)$, see the papers of Baez–Crans–Schreiber–Stevenson [13], Henriques [42], and Schommer-Pries [58].

5.1.2 The Heisenberg Lie 2-algebra

As we mentioned earlier, central extensions of Lie algebras are classified by second cohomology. A famous example of this is the 'Heisenberg Lie algebra', so named because it mimics the canonical commutation relations in quantum mechanics. Here we present a Lie 2-algebra generalization: the 'Heisenberg Lie 2-algebra'.

Consider the abelian Lie algebra of translations in position-momentum space:

$$\mathbb{R}^2 = \mathrm{span}(p, q).$$

Here, p and q are our names for the standard basis, the usual letters for momentum and position in physics. Up to rescaling, this Lie algebra has a single, nontrivial 2-cocycle:

$$p^* \wedge q^* \in \Lambda^2(\mathbb{R}^2),$$

where p^* and q^* comprise the dual basis. Thus it has a nontrivial central extension:

$$0 \to \mathbb{R} \to \mathfrak{H} \to \mathbb{R}^2 \to 0.$$

This central extension is called the **Heisenberg Lie algebra**. As a vector space, $\mathfrak{H} = \mathbb{R}^3$, and we call the basis vectors p, q and z, where z is central. When chosen with suitable normalization, they satisfy the relations:

$$[p, q] = z, \quad [p, z] = 0, \quad [q, z] = 0.$$

These are the same as the canonical commutation relations in quantum mechanics, except that the generator z would usually be a number, $-i\hbar$. It is from this parallel that the Heisenberg Lie algebra

57

derives its physical applications: a representation of $\mathfrak{H}$ is exactly a way of choosing linear operators p, q and z on a Hilbert space that satisfy the canonical commutation relations.

With Lie 2-algebras, we can repeat the process that yielded the Heisenberg Lie algebra to obtain a higher structure. Before we needed a 2-cocycle, but now we need a 3-cocycle. Indeed, letting p^*, q^* and z^* be the dual basis of $\mathfrak{H}^*$, it is easy to check that $\gamma = p^* \wedge q^* \wedge z^*$ is a nontrivial 3-cocycle on $\mathfrak{H}$. Thus there is a Lie 2-algebra $\mathfrak{string}_\gamma(\mathfrak{H})$, the **Heisenberg Lie 2-algebra**, which we denote by $\mathfrak{Heisenberg}$. Later, in Chapter 7, we will see how to integrate this Lie 2-algebra to a Lie 2-group.

We suspect the Heisenberg Lie 2-algebra, like its Lie algebra cousin, is also important for physics. We also suspect that the pattern continues: the Heisenberg Lie 2-algebra may admit a '4-cocycle', and a central extension to a Lie 3-algebra. However, since we have not defined the cohomology of Lie n-algebras [52], we do not pursue this here.

5.1.3   The supertranslation Lie *n*-superalgebras

Some exceptional cocycles arise on the supertranslation algebras in certain dimensions. Recall from Section 3.2 that a supertranslation algebra is a Lie superalgebra of the form:

$$\mathcal{T} = V \oplus S,$$

where the even part V is a vector space with a nondegerate quadratic form, the odd part S is a spinor representation of $\mathrm{Spin}(V)$, and the bracket comes from a symmetric, $\mathrm{Spin}(V)$-equivariant map that takes pairs of spinors to vectors:

$$[-,-] \colon \mathrm{Sym}^2 S \to V.$$

In spacetime dimensions 3, 4, 6 and 10, we proved in Theorem 3.1 that there is a 3-cocycle α, which is nonzero only when given two spinors and a vector:

$$\alpha \colon \quad \begin{aligned} \Lambda^3(\mathcal{T}) &\quad\to\quad \mathbb{R} \\ A \wedge \psi \wedge \phi &\quad\mapsto\quad \langle \psi, A\phi \rangle. \end{aligned}$$

There is thus a Lie 2-superalgebra, the **supertranslation Lie 2-superalgebra**, $\mathfrak{string}_\alpha(\mathcal{T})$.

Likewise, in spacetime dimensions 4, 5, 7 and 11, we proved in Theorem 3.2 that there is a 4-cocycle β, which is nonzero only when given two spinors and two vectors:

$$\beta \colon \quad \begin{aligned} \Lambda^4(\mathcal{T}) &\quad\to\quad \mathbb{R} \\ \mathcal{A} \wedge \mathcal{B} \wedge \Psi \wedge \Phi &\quad\mapsto\quad \langle \Psi, (\mathcal{A} \wedge \mathcal{B})\Phi \rangle. \end{aligned}$$

There is thus a Lie 3-superalgebra, the **supertranslation Lie 3-superalgebra**, $\mathfrak{brane}_\beta(\mathcal{T})$.

There is much more that one can do with the cocycles α and β, however. We can use them to extend not just the supertranslations $\mathcal{T}$ to a Lie n-superalgebra, but the full Poincaré superalgebra, $\mathfrak{so}(V) \ltimes \mathcal{T}$. We turn to this now.

5.1.4   Superstring Lie 2-superalgebras, 2-brane Lie 3-superalgebras

One of the principal themes of theoretical physics over the last century has been the search for the underlying symmetries of nature. This began with special relativity, which could be summarized as

the discovery that the laws of physics are invariant under the action of the Poincaré group:

$$\mathrm{ISO}(V) = \mathrm{Spin}(V) \ltimes V.$$

Here, V is the set of vectors in Minkowski spacetime and acts on Minkowski spacetime by translation, while $\mathrm{Spin}(V)$ is the **Lorentz group**: the double cover of $\mathrm{SO}_0(V)$, the connected component of the group of symmetries of the Minkowski norm. Much of the progress in physics since special relativity has been associated with the discovery of additional symmetries, like the $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ symmetries of the Standard Model of particle physics [6].

Today, 'supersymmetry' could be summarized as the hypothesis that the laws of physics are invariant under the 'Poincaré supergroup', which is larger than the Poincaré group:

$$\mathrm{SISO}(V) = \mathrm{Spin}(V) \ltimes T.$$

Here, V is again the set of vectors in Minkowski spacetime and $\mathrm{Spin}(V)$ is the Lorentz group, but T is the supergroup of translations on Minkowski 'superspacetime'. Though we have not yet learned enough supergeometry to talk about T precisely, we have already met its infinitesimal approximation: the superstranslation algebra, $\mathcal{T} = V \oplus S$. We think of the spinor representation S as giving extra, supersymmetric translations, or 'supersymmetries'.

In this thesis, we show how to further extend the Poincaré group to include higher symmetries, thanks to the normed division algebras. That is, we will show that in dimensions $k + 2 = 3$, 4, 6 and 10, one can extend the Poincaré supergroup $\mathrm{SISO}(k + 1, 1)$ to a 'Lie 2-supergroup' we call $\mathrm{Superstring}(k+1, 1)$. Similarly, in dimensions $k+3 = 4$, 5, 7 and 11, one can extend the Poincaré supergroup $\mathrm{SISO}(k + 2, 1)$ to a 'Lie 3-supergroup' we call $2\text{-}\mathrm{Brane}(k + 2, 1)$.

We begin this construction in this section by working at the infinitesimal level. We construct a Lie 2-superalgebra,

$$\mathfrak{superstring}(k + 1, 1),$$

which extends the Poincaré superalgebra in dimension $k + 2$:

$$\mathfrak{siso}(k + 1, 1 = \mathfrak{so}(k + 1, 1) \ltimes \mathcal{T}$$

Then we construct a Lie 3-superalgebra,

$$2\text{-}\mathfrak{brane}(k + 2, 1),$$

which extends the Poincaré superalgebra in dimension $k + 3$:

$$\mathfrak{siso}(k + 2, 1) = \mathfrak{so}(k + 2, 1) \ltimes \mathcal{T}.$$

We do this construction using the cocycles α and β. This is possible because both α and β are invariant under the action of the corresponding Lorentz algebra: $\mathfrak{so}(k + 1, 1)$ in the case of α, and $\mathfrak{so}(k + 2, 1)$ for β. This is manifestly true, because α and β are built from equivariant maps.

As we shall see, this invariance implies that α and β are cocycles, not merely on the supertranslations, but on the full Poincaré superalgebra—$\mathfrak{siso}(k + 1, 1)$ in the case of α, and $\mathfrak{siso}(k + 2, 1)$ in the case of β. We can extend α and β to these larger algebras in a trivial way: define the unique extension which vanishes unless all of its arguments come from $\mathcal{T}$. Doing this, α and β remain

cocycles, even though the Lie bracket (and thus d) has changed. Moreover, they remain nontrivial. All of this is contained in the following proposition:

Proposition 5.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie superalgebras such that $\mathfrak{g}$ acts on $\mathfrak{h}$, and let R be a representation of $\mathfrak{g} \ltimes \mathfrak{h}$. Given any R-valued n-cochain ω on $\mathfrak{h}$, we can uniquely extend it to an n-cochain $\tilde{\omega}$ on $\mathfrak{g} \ltimes \mathfrak{h}$ that takes the value of ω on $\mathfrak{h}$ and vanishes on $\mathfrak{g}$. When ω is even, we have:*

1. *$\tilde{\omega}$ is closed if and only if ω is closed and $\mathfrak{g}$-equivariant.*

2. *$\tilde{\omega}$ is exact if and only if $\omega = d\theta$, for θ a $\mathfrak{g}$-equivariant $(n-1)$-cochain on $\mathfrak{h}$.*

Proof. As a vector space, $\mathfrak{g} \ltimes \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{h}$, so that

$$\Lambda^n(\mathfrak{g} \ltimes \mathfrak{h}) \cong \bigoplus_{p+q=n} \Lambda^p \mathfrak{g} \otimes \Lambda^q \mathfrak{h},$$

as a vector space. Thanks to this decomposition, we can uniquely decompose n-cochains on $\mathfrak{g} \ltimes \mathfrak{h}$ by restricting to the summands. In keeping with our prior terminology, we call an n-cochain supported on $\Lambda^p \mathfrak{g} \otimes \Lambda^q \mathfrak{h}$ a (p,q)-form. Note that $\tilde{\omega}$ is just the n-cochain ω regarded as a $(0,n)$-form on $\mathfrak{g} \ltimes \mathfrak{h}$. We shall denote the space of (p,q)-forms by $C^{p,q}$.

We have two actions to distinguish: the action of $\mathfrak{g} \ltimes \mathfrak{h}$ on R, which we denote by ρ, and the action of $\mathfrak{g}$ on $\mathfrak{h}$, which we shall denote simply by the bracket, $[-,-]$. Inspecting the formula for the differential:

$$d\tilde{\omega}(X_1, \ldots, X_{n+1}) =$$
$$\sum_{i=1}^{n+1} (-1)^{i+1}(-1)^{|X_i||\tilde{\omega}|}\epsilon_1^{i-1}(i)\rho(X_i)\tilde{\omega}(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1})$$
$$+ \sum_{i<j} (-1)^{i+j}(-1)^{|X_i||X_j|}\epsilon_1^{i-1}(i)\epsilon_1^{j-1}(j)\tilde{\omega}([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{n+1})$$

it is easy to see that

$$d: C^{p,q} \to C^{p,q+1} \oplus C^{p+1,q}.$$

In particular:

$$d: C^{0,n} \to C^{0,n+1} \oplus C^{1,n}.$$

Given an n-cochain ω on $\mathfrak{h}$, it is easy to see that the part of $d\tilde{\omega}$ which lies in $C^{0,n+1}$ is just $\widetilde{d\omega}$, the extension of the $(n+1)$-cochain $d\omega$ to $\mathfrak{g} \ltimes \mathfrak{h}$.

Let $e\omega$ denote the $(1,n)$-form part of $d\tilde{\omega}$. To express this explicitly, choose $Y_1 \in \mathfrak{g}$ and $X_2, \ldots, X_{n+1} \in \mathfrak{h}$. By definition $e\omega(Y_1, X_2, \ldots, X_{n+1}) = d\tilde{\omega}(Y_1, X_2, \ldots, X_{n+1})$, and inspecting the formula for the differential once more, we see this consists of only two nonzero terms:

$$
\begin{aligned}
e\omega(Y_1, X_2, \ldots, X_{n+1}) &= (-1)^{|\tilde{\omega}||Y_1|}\rho(Y_1)\tilde{\omega}(X_2, \ldots, X_{n+1}) \\
&\quad + \sum_{i=2}^{n+1} (-1)^{i+1}\epsilon_2^{i-1}(i)\tilde{\omega}([Y_1, X_i], X_2, \ldots, \hat{X}_i, \ldots, X_{n+1}) \\
&= (-1)^{|\omega||Y_1|}\rho(Y_1)\omega(X_2 \wedge \cdots \wedge X_{n+1}) - \omega([Y_1, X_2 \wedge \cdots \wedge X_{n+1}]).
\end{aligned}
$$

In particular, note that for even ω, $e\omega = 0$ if and only if ω is $\mathfrak{g}$-equivariant.

To summarize, for any n-cochain ω, we have that

$$d\tilde{\omega} = \widetilde{d\omega} + e\omega,$$

where the first d is defined on $\mathfrak{g} \ltimes \mathfrak{h}$, while the second is only defined on $\mathfrak{h}$. The proof of 1 is now immediate: for even ω, $d\tilde{\omega} = 0$ if and only if $\widetilde{d\omega} = 0$ and $e\omega = 0$, which happens if and only if $d\omega = 0$ and ω is $\mathfrak{g}$-equivariant.

To prove 2, suppose ω is even. Assume $\tilde{\omega} = d\chi$, for some $(n-1)$-cochain χ on $\mathfrak{g} \ltimes \mathfrak{h}$. Because $d\chi$ is an even $(0, n)$-form, we may assume χ is an even $(0, n-1)$-form, as any other part of χ is closed and does not contribute to $d\chi$. Thus χ is the extension of an even $(n-1)$-cochain θ on $\mathfrak{h}$. By our prior formula, we have:

$$\tilde{\omega} = d\tilde{\theta} = \widetilde{d\theta} + e\theta$$

The left-hand side is a $(0, n)$-form, and thus the $(1, n-1)$-form part of the right-hand side, $e\theta$, vanishes. Thus θ is $\mathfrak{g}$-equivariant, and $\tilde{\omega} = \widetilde{d\theta}$, which implies $\omega = d\theta$. On the other hand, if $\omega = d\theta$ and θ is $\mathfrak{g}$-equivariant, then $e\theta = 0$ and thus $\tilde{\omega} = d\tilde{\theta}$. $\qquad\square$

Thus we can extend α and β to nonexact cocycles on the Poincaré Lie superalgebra, simply by defining α and β to vanish outside of the supertranslation algebra. Thanks to Theorem 5.1, we know that α lets us extend $\mathfrak{siso}(k+1, 1)$ to a Lie 2-superalgebra:

Theorem 5.2. *In dimensions 3, 4, 6 and 10, there exists a Lie 2-superalgebra formed by extending the Poincaré superalgebra $\mathfrak{siso}(k+1, 1)$ by the 3-cocycle α, which we call we the* **superstring Lie 2-superalgebra, $\mathfrak{superstring}(k+1, 1)$**.

Likewise, in dimensions one higher, β lets us extend $\mathfrak{siso}(k+2, 1)$ to a Lie 3-superalgebra. In the 11-dimensional case, this coincides with the Lie 3-superalgebra which Sati, Schreiber and Stasheff call $\mathfrak{sugra}(10, 1)$ [55], which is the Koszul dual of an algebra defined by D'Auria and Fré [25].

Theorem 5.3. *In dimensions 4, 5, 7 and 11, there exists a Lie 3-superalgebra formed by extending the Poincaré superalgebra $\mathfrak{siso}(k+2, 1)$ by the 4-cocycle β, which we call the* **2-brane Lie 3-superalgebra, $\mathfrak{2\text{-}brane}(k+2, 1)$**.

Chapter 6

Lie *n*-groups from group cohomology

Having constructed Lie n-algebras from Lie algebra $(n + 1)$-cocycles, we now turn to a parallel construction of Lie n-groups. Roughly speaking, an 'n-group' is a weak n-groupoid with one object—an n-category with one object in which all morphisms are weakly invertible, up to higher-dimensional morphisms. This definition is a rough one because there are many possible definitions to use for 'weak n-category', but despite this ambiguity, it can still serve to motivate us.

The richness of weak n-categories, no matter what definition we apply, makes n-groups a complicated subject. In the midst of this complexity, we seek to define a class of n-groups that have a simple description, and which are straightforward to internalize, so that we may easily construct Lie n-groups and Lie n-supergroups, as we shall do later in this thesis. The motivating example for this is what Baez and Lauda [10] call a 'special 2-group', which has a concrete description using group cohomology. Since Baez and Lauda prove that all 2-groups are equivalent to special ones, group cohomology also serves to classify 2-groups.

So, we will define 'slim Lie n-groups', at least for $n \leq 3$. This is an Lie n-group which is skeletal (every weakly isomorphic pair of objects are equal), and almost trivial: all k-morphisms are the identity for $1 < k < n$. Slim Lie n-groups are useful because they can be completely classified by Lie group cohomology. They are also easy to 'superize', and their super versions can be completely classified using Lie supergroup cohomology, as we shall see in Chapter 9. Finally, we note that we could equally well define 'slim n-groups', working in the category of sets rather than the category of smooth manifolds. The results in this section would hold in this case as well, but are of less use to us in this thesis.

We should stress that the definition of Lie n-group we sketch here (and make precise for $n \leq 3$), while it is good enough for our needs, is known to be too naive in some important respects. For instance, it does not seem possible to integrate every Lie n-algebra to a Lie n-group of this type, while Henriques's definition of Lie n-group does make this possible [42].

First we need to review the cohomology of Lie groups, as originally defined by van Est [70], who was working in parallel with the definition of group cohomology given by Eilenberg and MacLane. Fix a Lie group G, an abelian Lie group H, and a smooth action of G on H which respects addition in H. That is, for any $g \in G$ and $h, h' \in H$, we have:

$$g(h + h') = gh + gh'.$$

Then **the cohomology of G with coefficients in H** is given by the **Lie group cochain complex**, $C^\bullet(G, H)$. At level p, this consists of the smooth functions from G^p to H:

$$C^p(G, H) = \{f \colon G^p \to H\}\,.$$

We call elements of this set **H-valued p-cochains on G**. The boundary operator is the same as the one defined by Eilenberg–MacLane. On a p-cochain f, it is given by the formula:

$$
\begin{aligned}
df(g_1, \ldots, g_{p+1}) &= g_1 f(g_2, \ldots, g_{p+1}) \\
&+ \sum_{i=1}^{p} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{p+1}) \\
&+ (-1)^{p+1} f(g_1, \ldots, g_p).
\end{aligned}
$$

The proof that $d^2 = 0$ is routine. All the usual terminology applies: a p-cochain f for which $df = 0$ is called **closed**, or a **cocycle**, a p-cochain $f = dg$ for some $(p-1)$-cochain g is called **exact**, or a **coboundary**. A p-cochain is said to be **normalized** if it vanishes when any of its entries is 1. Every cohomology class can be represented by a normalized cocycle. Finally, when $H = \mathbb{R}$ with trivial G action, we omit it when writing the complex $C^\bullet(G)$, and we call real-valued cochains, cocycles, or coboundaries, simply cochains, cocycles or coboundaries, respectively.

This last choice, that $\mathbb{R}$ will be our default coefficient group, may seem innocuous, but there is another one-dimensional abelian Lie group we might have chosen: $\mathrm{U}(1)$, the group of phases. This would have been an equally valid choice, and perhaps better for some physical applications, but we have chosen $\mathbb{R}$ because it simplifies our formulas slightly.

We now sketch how to build a slim Lie n-group from an $(n+1)$-cocycle. In essence, given a normalized H-valued $(n+1)$-cocycle a on a Lie group G, we want to construct a Lie n-group $\mathrm{Brane}_a(G, H)$, which is the smooth, weak n-groupoid with:

- One object. We can depict this with a dot, or '0-cell': $\bullet$

- For each element $g \in G$, a 1-automorphism of the one object, which we depict as an arrow, or '1-cell':

$$\bullet \xrightarrow{\ g\ } \bullet\,, \quad g \in G.$$

  Composition corresponds to multiplication in the group:

$$\bullet \xrightarrow{\ g\ } \bullet \xrightarrow{\ g'\ } \bullet \;=\; \bullet \xrightarrow{\ gg'\ } \bullet\,.$$
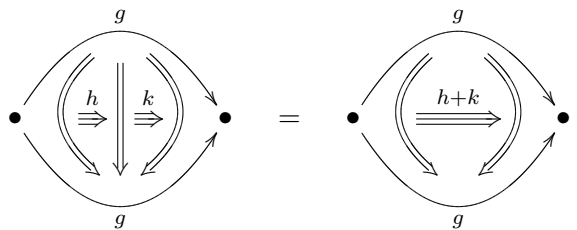
- Trivial k-morphisms for $1 < k < n$. If we depict 2-morphisms with 2-cells, 3-morphisms with 3-cells, then we are saying there is just one of each of these (the identity) up to level $n-1$:
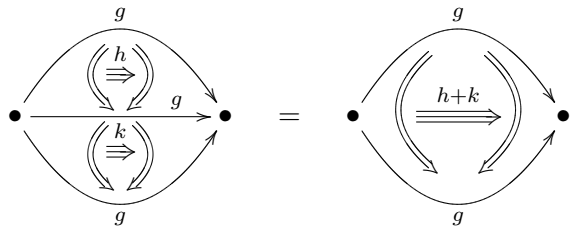

63
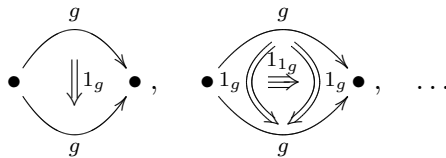
- For each element $h \in H$, an n-automorphism on the identity of the identity of ... the identity of the 1-morphism g, and no n-morphisms which are not n-automorphisms. For example, when $n = 3$, we have:

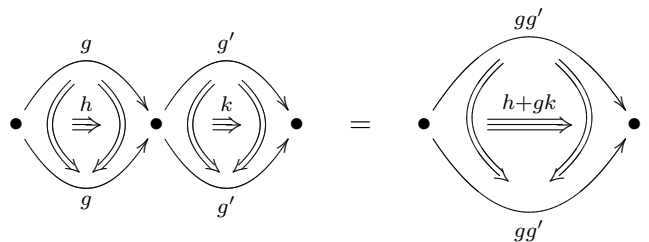$$\bullet \, 1_g \left(\overset{h}{\Longrightarrow} \right) 1_g \, \bullet \,, \quad h \in H.$$
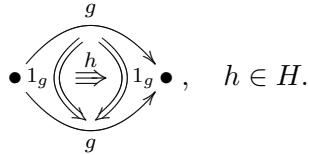
- There are n ways of composing n-morphisms, given by different ways of sticking n-cells together. For example, when $n = 3$, we can glue two 3-cells along a 2-cell, which should just correspond to addition in H:

$$\bullet \left(\overset{h}{\Longrightarrow} \; \overset{k}{\Longrightarrow} \right) \bullet \;\; = \;\; \bullet \left(\overset{h+k}{\Longrightarrow} \right) \bullet \,.$$

We also can glue two 3-cells along a 1-cell, which should again just be addition in H:

$$\bullet \left(\begin{array}{c} \overset{h}{\Longrightarrow} \\ \overset{k}{\Longrightarrow} \end{array} \right) \bullet \;\; = \;\; \bullet \left(\overset{h+k}{\Longrightarrow} \right) \bullet \,.$$

And finally, we can glue two 3-cells at the 0-cell, the object $\bullet$. This is the only composition of n-morphisms where the attached 1-morphisms can be distinct, which distinguishes it from the first two cases. It should be addition *twisted by the action of G*:

$$\bullet \left(\overset{h}{\Longrightarrow} \right) \bullet \left(\overset{k}{\Longrightarrow} \right) \bullet \;\; = \;\; \bullet \left(\overset{h+gk}{\Longrightarrow} \right) \bullet \,.$$

For arbitary n, we define all n compositions to be addition in H, except for gluing at the object, where it is addition twisted by the action.

- For any $(n+1)$-tuple of 1-morphisms, an n-automorphism $a(g_1, g_2, \ldots, g_{n+1})$ on the identity of the identity of ... the identity of the 1-morphism $g_1 g_2 \ldots g_{n+1}$. We call a the **n-associator**.

64

- a satisfies an equation corresponding to the n-dimensional associahedron, which is equivalent to the cocycle condition.

In principle, it should be possible to take a globular definition of n-category, such as that of Batanin or Trimble, and fill out this sketch to make it a real definition of an n-group. Doing this here, however, would lead us too far afield from our goal, for which we only need 2- and 3-groups. So let us flesh out these cases. The reader interested in learning more about the various definitions of n-categories should consult Leinster's survey [47] or Cheng and Lauda's guidebook [21].

6.1 Lie 2-groups

Speaking precisely, a **2-group** is a bicategory with one object in which all 1-morphisms and 2-morphisms are weakly invertible. Rather than plain 2-groups, we are interested in *Lie* 2-groups, where all the structure in sight is smooth. So, we really need a bicategory 'internal to the category of smooth manifolds', or a 'smooth bicategory'. To this end, we will give an especially long and unfamiliar definition of bicategory, isolating each operation and piece of data so that we can indicate its smoothness. Readers not familiar with bicategories are encouraged to read the introduction by Leinster [46].

Before we give this definition, let us review the idea of a 'bicategory', so that its basic simplicity is not obscured in technicalities. A bicategory has objects:

$$x \bullet ,$$

morphisms going between objects,

$$x \bullet \xrightarrow{\ f\ } \bullet y \ ,$$

and 2-morphisms going between morphisms:

Morphisms in a bicategory can be composed just as morphisms in a category:

$$x \xrightarrow{\ f\ } y \xrightarrow{\ g\ } z \quad = \quad x \xrightarrow{\ f \cdot g\ } z \ .$$

But there are two ways to compose 2-morphisms—vertically:

and horizontally:

Unlike a category, composition of morphisms need not be associative or have left and right units. The presence of 2-morphisms allow us to *weaken the axioms*. Rather than demanding $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, for composable morphisms f, g and h, the presence of 2-morphisms allow for the weaker condition that these two expressions are merely isomorphic:

$$a(f, g, h) \colon (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h),$$

where $a(f, g, h)$ is an 2-isomorphism called the **associator**. In the same vein, rather than demanding that:

$$1_x \cdot f = f = f \cdot 1_y,$$

for $f \colon x \to y$, and identities $1_x \colon x \to x$ and $1_y \colon y \to y$, the presence of 2-morphisms allow us to weaken these equations to isomorphisms:

$$l(f) \colon 1_x \cdot f \Rightarrow f, \quad r(f) \colon f \cdot 1_y \Rightarrow f.$$

Here, $l(f)$ and $r(f)$ are 2-isomorphisms called the **left and right unitors**.

Of course, these 2-isomorphisms obey rules of their own. The associator satisfies its own axiom, called the **pentagon identity**, which says that this pentagon commutes:

Finally, the associator and left and right unitors satisfy the **triangle identity**, which says the following triangle commutes:

A word of caution is needed here before we proceed: *in this chapter only*, we are bucking standard mathematical practice by writing the result of doing first α and then β as $\alpha \circ \beta$ rather than

$\beta \circ \alpha$, as one would do in most contexts where $\circ$ denotes composition of *functions*. This has the effect of changing how we read commutative diagrams. For instance, the commutative triangle:

$$f \xrightarrow{\alpha} g$$

(diagram: $f \xrightarrow{\alpha} g$, with γ from f to h and β from g to h)

reads $\gamma = \alpha \circ \beta$ rather than $\gamma = \beta \circ \alpha$.

We shall now give the full definition, not of a bicategory, but of a 'smooth bicategory'. To do this, we use the idea of internalization. Dating back to Ehresmann [31] in the 1960s, internalization has become a standard tool of the working category theorist. The idea is based on a familiar one: any mathematical structure that can be defined using sets, functions, and equations between functions can be defined in categories other than Set. For instance, a group in the category of smooth manifolds is a Lie group. To perform internalization, we apply this idea to the definition of category itself. We recall the essentials here to define 'smooth categories'. More generally, one can define a 'category in K' for many categories K, though here we will work exclusively with the example where K is the category of smooth manifolds. For a readable treatment of internalization, see Borceux's handbook [16].

Definition 6.1. A **smooth category** C consists of

- a **smooth manifold of objects** C_0;

- a **smooth manifold of morphisms** C_1;

together with

- smooth **source** and **target** maps $s, t \colon C_1 \to C_0$,

- a smooth **identity-assigning** map $i \colon C_0 \to C_1$,

- a smooth **composition** map $\circ \colon C_1 \times_{C_0} C_1 \to C_1$, where $C_1 \times_{C_0} C_1$ is the pullback of the source and target maps:

$$C_1 \times_{C_0} C_1 = \{(f, g) \in C_1 \times C_1 : t(f) = s(g)\},$$

and is assumed to be a smooth manifold.

such that the following diagrams commute, expressing the usual category laws:

- laws specifying the source and target of identity morphisms:

$$C_0 \xrightarrow{i} C_1 \qquad C_0 \xrightarrow{i} C_1$$

(diagrams: $C_0 \xrightarrow{i} C_1$ with 1 from C_0 and s from C_1 to C_0; $C_0 \xrightarrow{i} C_1$ with 1 from C_0 and t from C_1 to C_0)

- laws specifying the source and target of composite morphisms:

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\;\circ\;} & C_1 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle s} \\
C_1 & \xrightarrow{\;s\;} & C_0
\end{array}
\qquad
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\;\circ\;} & C_1 \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle t} \\
C_1 & \xrightarrow{\;t\;} & C_0
\end{array}
$$

- the associative law for composition of morphisms:

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\;\circ \times_{C_0} 1\;} & C_1 \times_{C_0} C_1 \\
\downarrow{\scriptstyle 1 \times_{C_0} \circ} & & \downarrow{\scriptstyle \circ} \\
C_1 \times_{C_0} C_1 & \xrightarrow{\;\circ\;} & C_1
\end{array}
$$

- the left and right unit laws for composition of morphisms:

$$
\begin{array}{ccccc}
C_0 \times_{C_0} C_1 & \xrightarrow{\;i \times 1\;} & C_1 \times_{C_0} C_1 & \xleftarrow{\;1 \times i\;} & C_1 \times_{C_0} C_0 \\
& {\scriptstyle p_2}\searrow & \downarrow{\scriptstyle \circ} & \swarrow{\scriptstyle p_1} & \\
& & C_1 & &
\end{array}
$$

The existence of pullbacks in the category of smooth manifolds is a delicate issue. When working with categories internal to some category K, it is customary to assume K contains all pullbacks, but this is merely a convenience. All the definitions still work as long as the existence of each required pullback is implicit.

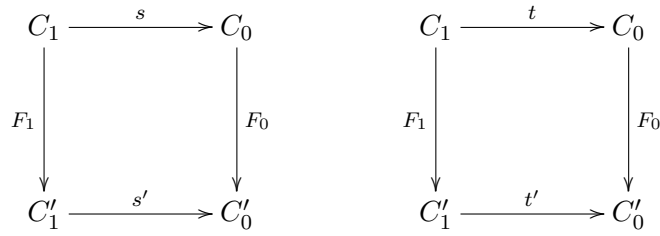To define smooth bicategories, we must first define smooth functors and natural transformations:

Definition 6.2. Given smooth categories C and C', a **smooth functor** $F\colon C \to C'$ consists of:
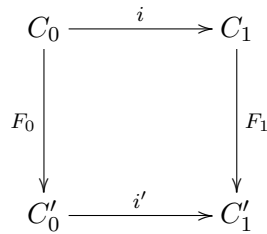
- a smooth map on objects, $F_0\colon C_0 \to C_0'$;

- a smooth map on morphisms, $F_1\colon C_1 \to C_1'$;

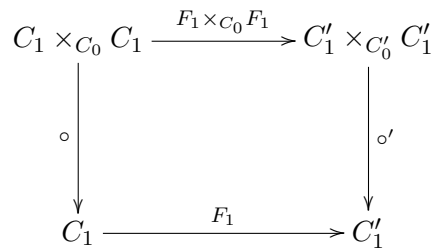such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:

- preservation of source and target:

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\ s\ } & C_0 \\
{\scriptstyle F_1}\big\downarrow & & \big\downarrow{\scriptstyle F_0} \\
C_1' & \xrightarrow{\ s'\ } & C_0'
\end{array}
\qquad\qquad
\begin{array}{ccc}
C_1 & \xrightarrow{\ t\ } & C_0 \\
{\scriptstyle F_1}\big\downarrow & & \big\downarrow{\scriptstyle F_0} \\
C_1' & \xrightarrow{\ t'\ } & C_0'
\end{array}
$$

- preservation of identity morphisms:

$$
\begin{array}{ccc}
C_0 & \xrightarrow{\ i\ } & C_1 \\
{\scriptstyle F_0}\big\downarrow & & \big\downarrow{\scriptstyle F_1} \\
C_0' & \xrightarrow{\ i'\ } & C_1'
\end{array}
$$

- preservation of composite morphisms:

$$
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\ F_1 \times_{C_0} F_1\ } & C_1' \times_{C_0'} C_1' \\
{\scriptstyle \circ}\big\downarrow & & \big\downarrow{\scriptstyle \circ'} \\
C_1 & \xrightarrow{\quad F_1 \quad} & C_1'
\end{array}
$$

Definition 6.3. Given categories smooth categories C and C', and smooth functors $F, G\colon C \to C'$, a **smooth natural transformation** $\theta\colon F \Rightarrow G$ is a smooth map $\theta\colon C_0 \to C_1'$ for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:

- laws specifying the source and target of the natural transformation:

$$
\begin{array}{ccc}
C_0 & & \\
{\scriptstyle \theta}\big\downarrow & \searrow^{F} & \\
C_1' & \xrightarrow{\ s\ } & C_0'
\end{array}
\qquad\qquad
\begin{array}{ccc}
C_0 & & \\
{\scriptstyle \theta}\big\downarrow & \searrow^{G} & \\
C_1' & \xrightarrow{\ t\ } & C_0'
\end{array}
$$
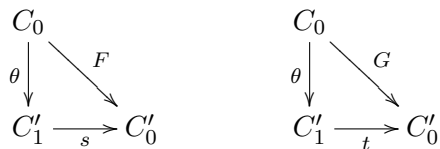
- the commutative square law:

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\ \Delta(s\theta\times G)\ } & C_1' \times_{C_0} C_1' \\
\Big\downarrow{\scriptstyle \Delta(F\times t\theta)} & & \Big\downarrow{\scriptstyle \circ'} \\
C_1' \times_{C_0} C_1' & \xrightarrow{\ \circ'\ } & C_1'
\end{array}
$$

Now we know enough about smooth category theory to bootstrap the definition of smooth bi-categories. We do this in a somewhat nonstandard way: we make use of the fact that the morphisms and 2-morphisms of a bicategory form an ordinary category under vertical composition. Generalizing this, the morphisms and 2-morphisms in a smooth bicategory should form, by themselves, a smooth category. We can then define horizontal composition as a smooth functor, and introduce the associator and left and right unitors as smooth natural transformations between certain functors. In detail:

Definition 6.4. A **smooth bicategory** B consists of

- a **manifold of objects** B_0;

- a **manifold of morphisms** B_1;

- a **manifold of 2-morphisms** B_2;

equipped with:

- a smooth category structure on $\underline{\mathrm{Mor}}B$, with

  - B_1 as the smooth manifold of objects;
  - B_2 as the smooth manifold of morphisms;

  The composition in $\underline{\mathrm{Mor}}B$ is called **vertical composition** and denoted $\circ$.

- smooth **source** and **target maps**:
$$s, t\colon B_1 \to B_0.$$

- a smooth **identity-assigning map**:
$$i\colon B_0 \to B_1.$$

- a smooth **horizontal composition** functor:

$$\cdot\colon \underline{\mathrm{Mor}}B \times_{B_0} \underline{\mathrm{Mor}}B \to \underline{\mathrm{Mor}}B.$$

  That is, a pair of smooth maps:

$$\cdot\colon B_1 \times_{B_0} B_1 \to B_1$$

$$\cdot\colon B_2 \times_{B_0} B_2 \to B_2,$$

  satisfying the axioms for a functor.

70

- a smooth natural transformation, the **associator**:

$$a(f, g, h) \colon (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h).$$

- smooth natural transformations, the **left** and **right unitors**, which are both trivial in the bicategories we consider:

$$l(f) \colon 1 \cdot f \Rightarrow f, \quad r(f) \colon f \cdot 1 \Rightarrow f.$$

such that the following diagrams commute, expressing the same laws regarding sources, targets and identities as with a smooth category, and two new laws expressing the compatibility of the various source and target maps:

- laws specifying the source and target of identity morphisms:

- laws specifying the source and target of the horizontal composite of 1-morphisms:

- laws expressing the compatibility of source and target maps:

Finally, associator and left and right unitors satisfy some laws of their own—the following diagrams commute:

- the **pentagon identity** for the associator:

$$
\begin{array}{ccc}
 & (fg)(hk) & \\
{}^{a(fg,h,k)}\nearrow & & \searrow {}^{a(f,g,hk)} \\
((fg)h)k & & f(g(hk)) \\
{}^{a(f,g,h)\cdot 1_k}\searrow & & \nearrow {}^{1_f\cdot a(g,h,k)} \\
(f(gh))k & \xrightarrow{\;a(f,gh,k)\;} & f((gh)k)
\end{array}
$$
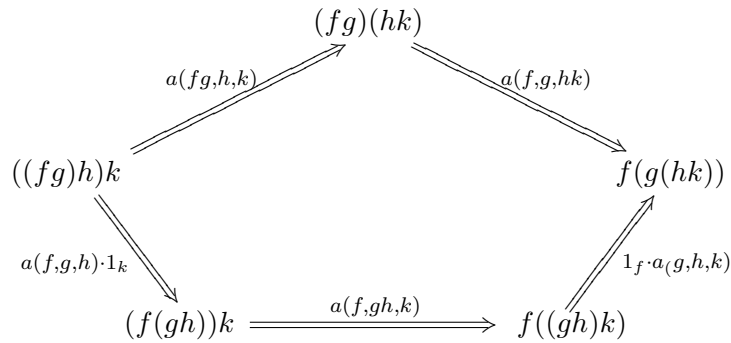
for any four composable morphisms f, g, h and k.

- the **triangle identity** for the left and right unit laws:

$$
\begin{array}{ccc}
(f1)g & \xrightarrow{\;a(f,1,g)\;} & f(1g) \\
{}_{r(f)\cdot 1_g}\searrow & & \swarrow {}_{1_f\cdot l(g)} \\
 & fg &
\end{array}
$$
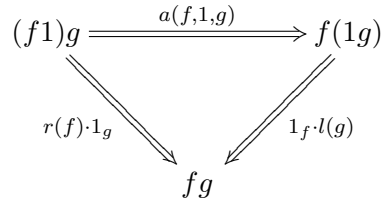
for any two composable morphisms f and g.

This definition of smooth bicategory may seem so long that checking it is utterly intimidating, but we shall see an example in a moment where this is easy. This will be an example of a **Lie 2-group**, a smooth bicategory with one object where all morphisms are weakly invertible, and all 2-morphisms are strictly invertible.

Secretly, the pentagon identity is a cocycle condition, as we shall now see. Given a normalized H-valued 3-cocycle a on a Lie group G, we can construct a Lie 2-group $\mathrm{String}_a(G,H)$ with:

- One object, $\bullet$, regarded as a manifold in the trivial way.

- For each element $g \in G$, an automorphism of the one object:

$$
\bullet \xrightarrow{\;g\;} \bullet.
$$

Horizontal composition given by multiplication in the group:

$$
\cdot : G \times G \to G.
$$

Note that source and target maps are necessarily trivial. The identity-assigning map takes the one object to $1 \in G$.

- For each $h \in H$, a 3-automorphism of the 2-morphism 1_g, and no 3-morphisms between distinct 2-morphisms:

$$h \in H.$$

Thus the space of all 2-morphisms is $G \times H$, and the source and target maps are projection onto the first factor. The identity-assigning map takes each element of G to $0 \in H$.

- Two kinds of composition of 2-morphisms: given a pair of 2-morphisms on the same morphism, vertical compostion is given by addition in H:

That is, vertical composition is just the map:

$$\circ = 1 \times + \colon G \times H \times H \to G \times H.$$

where we have used the fact that the pullback of 2-morphisms over the one object is trivially:

$$(G \times H) \times_{\bullet} (G \times H) \cong G \times H \times H.$$

Given a pair of 2-morphisms on different morphisms, horizontal composition is addition *twisted by the action of G*:

Or, in terms of a map, this is the multiplication on the semidirect product, $G \ltimes H$:

$$\cdot \colon (G \ltimes H) \times (G \ltimes H) \to G \ltimes H.$$

- For any triple of morphisms, a 2-isomorphism, the associator:

$$a(g_1, g_2, g_3) \colon g_1 g_2 g_3 \to g_1 g_2 g_3,$$

given by the 3-cocycle $a \colon G^3 \to H$, where by a slight abuse of definitions we think of this 2-isomorphism as living in H rather than $G \times H$, because the source (and target) are understood to be $g_1 g_2 g_3$.

- The left and right unitors are trivial.

A **slim Lie 2-group** is one of this form. When $H = \mathbb{R}$, we write simply $\mathrm{String}_a(G)$ for the above Lie 2-group. It remains to check that this is, in fact, a Lie 2-group:

Proposition 6.1. $\mathrm{String}_a(G, H)$ *is a Lie 2-group: a smooth bicategory with one object in which all 1-morphisms and 2-morphisms are weakly invertible.*

In brief, we prove this by showing that the 3-cocycle condition implies the one nontrivial axiom for this bicategory: the pentagon identity.

Proof. For $\mathrm{String}_a(G, H)$, the left and right unitors are the identity, and thus the triangle identity just says $a(g_1, 1, g_2) = 1$. Or, written additively, $a(g_1, 1, g_2) = 0$ Since a is normalized, this is automatic.

To check that $\mathrm{String}_a(G, H)$ is really a bicategory, it therefore remains to check the pentagon identity. This says that the following automorphisms of $g_1 g_2 g_3 g_4$ are equal:

$$a(g_1, g_2, g_3 g_4) \circ a(g_1 g_2, g_3, g_4) = (1_{g_1} \cdot a(g_2, g_3, g_4)) \circ a(g_1, g_2 g_3, g_4) \circ (a(g_1, g_2, g_3) \cdot 1_{g_4})$$

Or, using the definition of vertical composition:

$$a(g_1, g_2, g_3 g_4) + a(g_1 g_2, g_3, g_4) = (1_{g_1} \cdot a(g_2, g_3, g_4)) + a(g_1, g_2 g_3, g_4) + (a(g_1, g_2, g_3) \cdot 1_{g_4})$$

Finally, use the definition of the dot operation for 2-morphisms, as the semidirect product:

$$a(g_1, g_2, g_3 g_4) + a(g_1 g_2, g_3, g_4) = g_1 a(g_2, g_3, g_4)) + a(g_1, g_2 g_3, g_4) + a(g_1, g_2, g_3).$$

This is the 3-cocycle condition—it holds because a is a 3-cocycle.

So, $\mathrm{String}_a(G, H)$ is a bicategory. It is smooth because everything in sight is smooth: G, H, the source, target, identity-assigning, and composition maps, and the associator $a\colon G^3 \to H$. And it is a Lie 2-group: the morphisms in G and 2-morphisms in H are all strictly invertible, and thus of course they are weakly invertible. $\qquad\square$

In fact, we can say something a bit stronger about $\mathrm{String}_a(G, H)$, if we let a be any normalized H-valued 3-cochain, rather requiring it to be a cocycle. In this case, $\mathrm{String}_a(G, H)$ is a Lie 2-group if and only if a is a 3-cocycle, because a satisfies the pentagon identity if and only if it is a cocycle.

6.2 Lie 3-groups

We now sketch the construction of slim Lie 3-groups from a normalized 4-cocycle π. In a sense, this is a straightforward generalization of what we have done above, but the details must be checked against a specific definition of 3-category. We choose to use tricategories, originally defined by Gordon, Power and Street [35], but extensively studied by Gurski. We use the definition from his thesis [41].

We saw in the last section that a smooth bicategory B is consists of a smooth manifold of objects, B_0, a smooth manifold of morphisms, B_1, and a smooth manifold of 2-morphisms, B_2, such that:

- B_1 and B_2 fit together to form a smooth category;

- horizontal composition is a smooth functor;

- satisfying associativity and left and right unit laws up to natural transformations, the associator and left and right unitors;

- satisfying the pentagon and triangle identities.

Here, we will define a 'smooth tricategory' T to consist of a smooth manifold of objects, T_0, a smooth manifold of morphisms, T_1, a smooth manifold of 2-morphisms, T_2, and a smooth manifold of 3-morphisms, T_3, such that:

- T_1, T_2 and T_3 fit together to form a smooth bicategory;

- horizontal composition is a smooth 'pseudofunctor';

- satisfying associativity and left and right unit laws up to smooth 'pseudonatural transformations', the associator and left and right unitors;

- satisfying the pentagon and triangle identities up to smooth 'modifications';

- satisfying some identities of their own.

Each of the above quoted terms—pseudofunctor, pseudonatural transformations, modification—would usually need to be defined completely in order to understand tricategories. But we really only need modifications, because our functors and natural transformations will not be 'pseudo'. Nonetheless, so it is clear what we leave out, let us discuss each of these terms briefly.

- 'Pseudofunctor' is to 'bicategory' as 'functor' is to 'category': it is a map $F\colon B \to B'$ between bicategories B and B', preserving all structure in sight *except* horizontal composition and identities, which are only preserved up to specified 2-isomorphisms:

$$F(f \cdot g) \Rightarrow F(f) \cdot F(g), \quad F(1_x) \Rightarrow 1_{F(x)}.$$

  For the tricategories we construct, all pseudofunctors will be strict: the above 2-isomorphisms are identities.

- 'Pseudonatural transformation' is to 'pseudofunctor' as 'natural transformation' is to 'functor': given two pseudofunctors
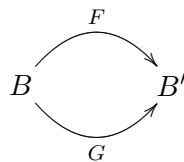
a pseudonatural transformation is a map:
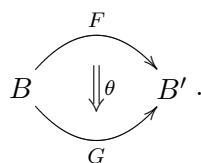
Like a natural transformation, this consists of a morphism for each object x in B:

$$\theta(x)\colon F(x) \to G(x).$$

Unlike a natural transformation, it is only natural up to a specified 2-isomorphism. That is, the naturality square:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle\theta(x)}\big\downarrow & & \big\downarrow{\scriptstyle\theta(y)} \\
G(x) & \xrightarrow{G(f)} & G(y)
\end{array}
$$

does *not* commute. It is replaced with a 2-isomorphism:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle\theta(x)}\big\downarrow & {\scriptstyle\theta(f)} & \big\downarrow{\scriptstyle\theta(y)} \\
G(x) & \xrightarrow[G(f)]{} & G(y)
\end{array}
$$

that satisfies some equations of its own. For the tricategories we construct, all pseudonatural transformations will be strict: the 2-isomorphism above is the identity.

- Finally, a 'modification' is something new: it is a map between pseudonatural transformations. Given two pseudonatural transformations:

a modification Γ is a map:

Just as a pseudonatural transformation consists of a morphism for each object x in B, a modification consists of a 2-morphism for each object x in B:

These 2-morphisms satisfy an equation that will hold trivially in the tricategories we consider, so we omit it. See Leinster [46] for more details.

With these preliminaries in mind, we can now sketch the definition of a smooth tricategory.

Definition 6.5. A **smooth tricategory** T consists of:

- a **manifold of objects**, T_0;

- a **manifold of morphisms**, T_1;

- a **manifold of 2-morphisms**, T_2;

- a **manifold of 3-morphisms**, T_3;

equipped with:

- a smooth bicategory structure on $\underline{\mathrm{Mor}}\, T$, with

  – T_1 as the smooth manifold of objects;
  – T_2 as the smooth manifold of morphisms;
  – T_3 as the smooth manifold of 2-morphisms;

  We call the vertical composition in $\underline{\mathrm{Mor}}\, T$ **composition at a 2-cell**, and the horizontal composition in $\underline{\mathrm{Mor}}\, T$ **composition at a 1-cell**.

- smooth **source** and **target maps**:
  $$s, t \colon T_1 \to T_0.$$

- a smooth **identity-assigning map**:
  $$i \colon T_0 \to T_1.$$

- a smooth **composition** pseudofunctor, called **composition at a 0-cell**, which is strict in the tricategories we consider:
  $$\cdot \colon \underline{\mathrm{Mor}}\, T \times_{T_0} \underline{\mathrm{Mor}}\, T \to \underline{\mathrm{Mor}}\, T.$$

  That is, three smooth maps:
  $$\cdot \colon T_1 \times_{T_0} T_1 \to T_1$$
  $$\cdot \colon T_2 \times_{T_0} T_2 \to T_2$$
  $$\cdot \colon T_3 \times_{T_0} T_3 \to T_3$$

  satisfying the axioms of a strict functor.

- a smooth pseudonatural transformation, the **associator**, which is trivial in the tricategories we consider:
  $$a(f, g, h) \colon (f \cdot g) \cdot h \Rightarrow f \cdot (g \cdot h).$$

- smooth pseudonatural transformations, the **left** and **right unitors**, all trivial in the tricategories we consider:
  $$l(f) \colon 1 \cdot f \Rightarrow f, \quad r(f) \colon f \cdot 1 \Rightarrow f.$$

- a smooth modification called the **pentagonator**:

$$
\begin{array}{c}
(fg)(hk) \\
\end{array}
$$

$$
((fg)h)k \xrightarrow{a(fg,h,k)} (fg)(hk) \xrightarrow{a(f,g,hk)} f(g(hk))
$$

$$\pi(f,g,h,k)$$

$$
a(f,g,h)\cdot 1_k \qquad 1_f\cdot a(g,h,k)
$$

$$
(f(gh))k \xrightarrow{a(f,gh,k)} f((gh)k)
$$
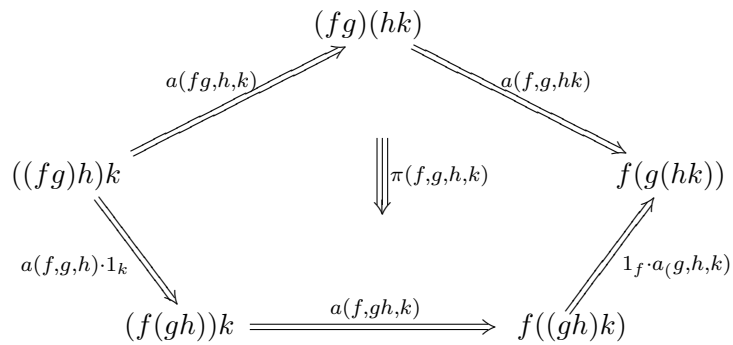
- smooth modifications called the **middle, left** and **right triangulators**, all trivial in the tricategories we consider:

$$
(f1)g \xrightarrow{\ a\ } f(1g)
$$

$$
r^*\cdot 1_g \qquad \Downarrow\ \mu \qquad 1_f\cdot l
$$

$$
fg \xrightarrow{1_f\cdot 1_g} fg
$$

$$
(1f)g \xrightarrow{\ a\ } 1(fg)
$$

$$
l\cdot 1_g \qquad \Rightarrow \lambda
$$

$$
fg \xleftarrow{\ l\ }
$$

$$
f(g1) \xrightarrow{\ a^*\ } (fg)1
$$

$$
1_f\cdot r \qquad \Rightarrow \rho
$$

$$
fg \xleftarrow{\ r\ }
$$
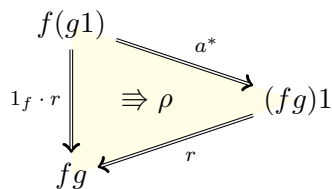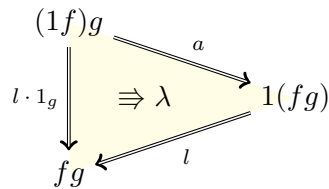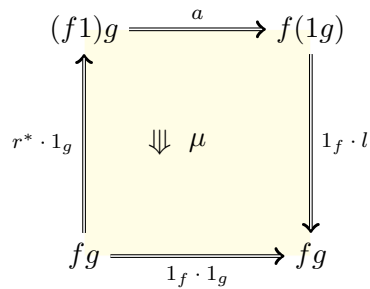
These smooth modifications all satisfy their own axioms. When λ, ρ and μ are trivial, their axioms boil down to the statement that π is trivial whenever one its arguments is trivial. We therefore omit them. The one axiom we need to consider is the **pentagonator identity**:

$$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$$

$$a$$

$$(f(g(hk)))p$$

$$1_f(1_g a)$$

$$f(g(h(kp)))$$

$$(1_f a)1_p$$

$$a1_p \qquad \Downarrow \ \pi$$

$$a$$

$$(f((gh)k))p \qquad \cong$$

$$\Rrightarrow \pi \cdot 1_{1_p}$$

$$((fg)(hk))p \xrightarrow{\quad a \quad} (fg((hk)p))$$

$$a1_p \qquad a1_p$$

$$((f(gh))k)p \qquad a$$

$$(a1_k)1_p$$

$$(((fg)h)k)p \qquad \Downarrow \ \pi \qquad (1_f 1_g)a$$

$$(fg)(h(kp))$$

$$a \qquad a$$

$$((fg)h)(kp)$$

$$=$$

$$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$$

$$a$$

$$(f(g(hk)))p$$

$$1_f(1_g a)$$

$$f(g(h(kp)))$$

$$(1_f a)1_p \qquad \cong \qquad 1_f(a1_p) \qquad \Downarrow \ 1_{1_f} \cdot \pi$$

$$f(((gh)k)p) \xrightarrow{1_f a} f((gh)(kp))$$

$$1_f a$$

$$(f((gh)k))p \qquad a$$

$$a1_p \qquad \Downarrow \ \pi$$

$$((f(gh))k)p \qquad a \qquad a \qquad \Downarrow \ \pi$$

$$(a1_k)1_p \qquad \cong$$

$$(((fg)h)k)p \qquad (f(gh))(kp)$$

$$(fg)(h(kp))$$

$$a \qquad a(1_k 1_p) \qquad a$$

$$((fg)h)(kp)$$
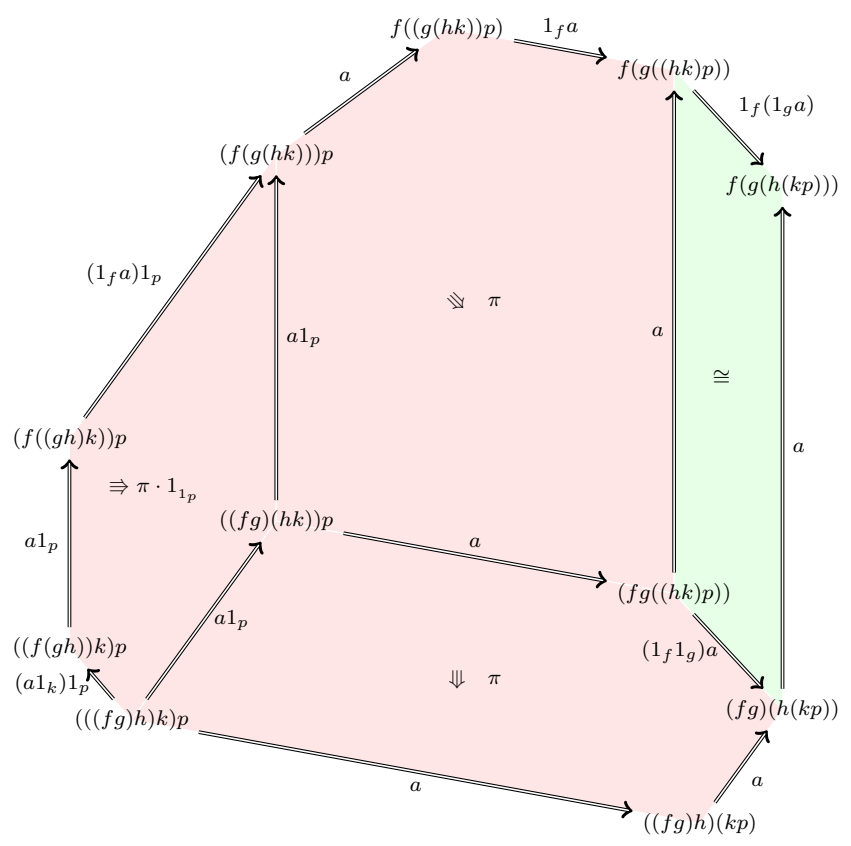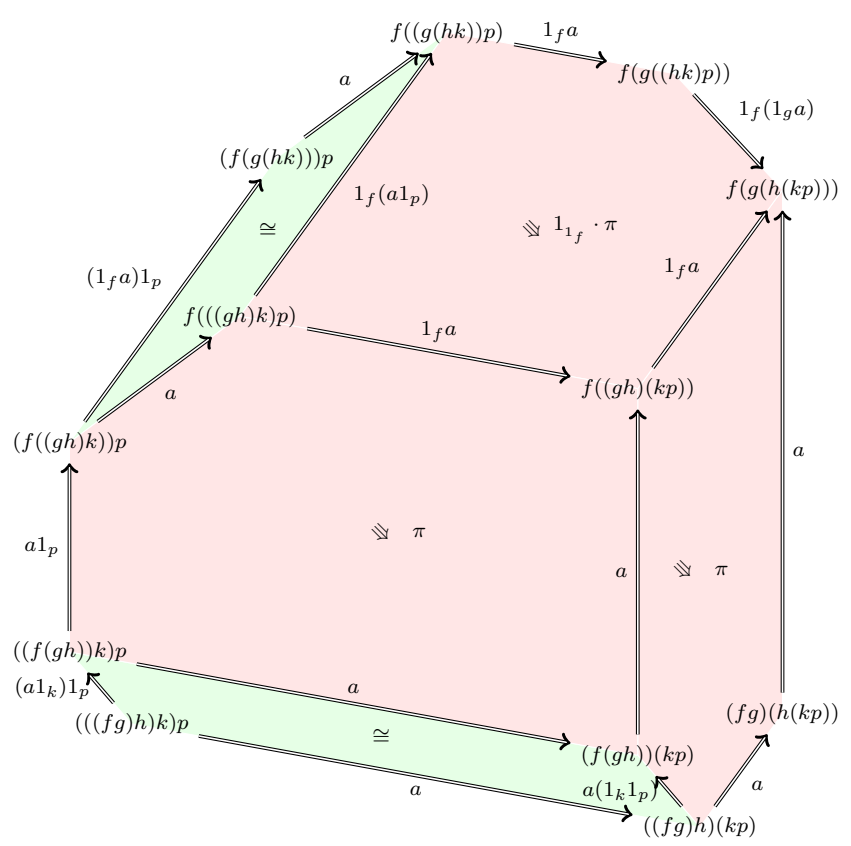
In the pentagonator identity, we have omitted $\cdot$ everywhere except the faces to save space. This identity comes from a 3-dimensional solid called the **associahedron**. This is the polyhedron where:

- vertices are parenthesized lists of five morphisms, e.g. $(((fg)h)k)p$;

- edges connect any two vertices related by an application of the associator, e.g.

$$(((fg)h)k)p \Rightarrow ((fg)(hk))p.$$

In fact, the pentagonator identity gives us a picture of the associahedron. Regarding the left-hand side of the equation as the back and the right-hand side as the front, we assemble the following polyhedron:
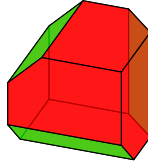
Identifying the vertices, edges and faces of this polyhedron with the corresponding morphisms, 2-morphisms and 3-morphisms from the pentagonator identity, we see the identity just says that the associahedron commutes.

A **Lie 3-group** is a smooth tricategory with one object where all morphisms, 2-morphisms and 3-morphisms are weakly invertible. Though it looks quite complex, the pentagonator identity is secretly a cocycle condition. In fact, given a normalized H-valued 4-cocycle π on a Lie group G, we can construct a Lie 3-group $\mathrm{Brane}_\pi(G, H)$ with:

- One object, $\bullet$, regarded as a manifold in the trivial way.

- For each element $g \in G$, an automorphism of the one object:

$$\bullet \xrightarrow{\;g\;} \bullet$$

  Composition at a 0-cell given by multiplication in the group:

$$\cdot : G \times G \to G.$$

  The source and target maps are trivial, and identity-assigning map takes the one object to $1 \in G$.

- Only the identity 2-morphism on any 1-morphism, and no 2-morphisms between distinct 1-morphisms:

  So the space of 2-morphisms is also G. The source, target and identity-assigning maps are all the identity on G. Composition at a 1-cell is trivial, while composition at a 0-cell is again multiplication in G.

- For each $h \in H$, a 3-automorphism of the 2-morphism 1_g, and no 3-morphisms between distinct 2-morphisms:

$$\bullet \, 1_g \left(\!\! \begin{array}{c} g \\ \overset{h}{\Longrightarrow} \\ g \end{array} \!\!\right) 1_g \, \bullet \, , \quad h \in H.$$
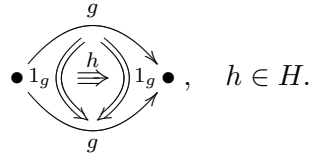
Thus the space of 3-morphisms is $G \times H$. The source and target maps are projection onto G, and the identity assigning map takes 1_g to $0 \in H$, for all $g \in G$.

- Three kinds of composition of 3-morphisms: given a pair of 3-morphisms on the same 2-morphism, we can compose them at at a 2-cell, which we take to be addition in H:

$$\bullet \left(\!\! \begin{array}{c} g \\ \overset{h}{\Longrightarrow} \Big\| \overset{h'}{\Longrightarrow} \\ g \end{array} \!\!\right) \bullet \quad = \quad \bullet \left(\!\! \begin{array}{c} g \\ \overset{h+h'}{\Longrightarrow} \\ g \end{array} \!\!\right) \bullet .$$
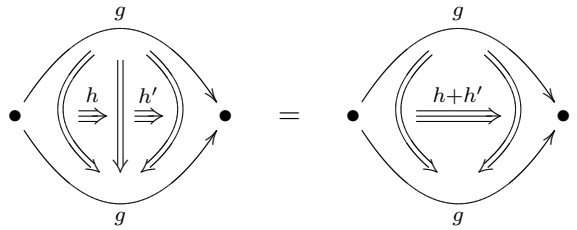
We can also compose two 3-morphisms at a 1-cell, which we again take to be composition in H:

$$\bullet \left(\!\! \begin{array}{c} g \\ \overset{h}{\Longrightarrow} \\ g \\ \overset{h'}{\Longrightarrow} \\ g \end{array} \!\!\right) \bullet \quad = \quad \bullet \left(\!\! \begin{array}{c} g \\ \overset{h+h'}{\Longrightarrow} \\ g \end{array} \!\!\right) \bullet .$$
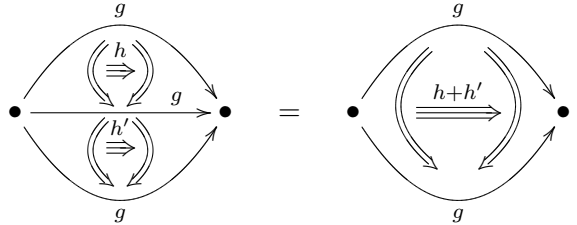
In terms of maps, both of these compositions are just:

$$1 \times + \colon G \times H \times H \to G \times H.$$

And finally, we can glue two 3-cells at the 0-cell, the object. We call this **composition at a 0-cell**, and define it to be addition *twisted by the action of G*:

$$\bullet \left(\!\! \begin{array}{c} g \\ \overset{h}{\Longrightarrow} \\ g \end{array} \!\!\right) \bullet \left(\!\! \begin{array}{c} g' \\ \overset{h'}{\Longrightarrow} \\ g' \end{array} \!\!\right) \bullet \quad = \quad \bullet \left(\!\! \begin{array}{c} gg' \\ \overset{h+gh'}{\Longrightarrow} \\ gg' \end{array} \!\!\right) \bullet .$$
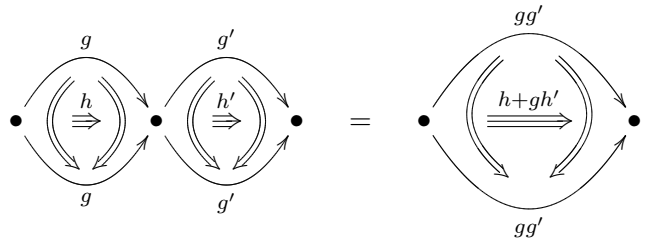
In terms of a map, $\cdot$ is just given by multiplication on the semidirect product:

$$\cdot \colon (G \ltimes H) \times (G \ltimes H) \to G \ltimes H.$$

- The associator, left and right unitors are automatically trivial, because all 2-morphisms are trivial.

- For each quadruple of 1-morphisms, a specified 3-isomorphism, the **2-associator** or **pentagonator**:
$$\pi(g_1, g_2, g_3, g_4) \colon 1_{g_1 g_2 g_3 g_4} \to 1_{g_1 g_2 g_3 g_4}.$$
given by the 4-cocycle $\pi \colon G^4 \to H$, which we think of as element of H because the source (and target) are understood to be $1_{g_1 g_2 g_3 g_4}$.

- The three other specified 3-isomorphisms are trivial.

A **slim Lie 3-group** is one of this form. As before, it remains to check that it is, in fact, a Lie 3-group. We claim:

Proposition 6.2. $\mathrm{Brane}_\pi(G, H)$ *is a Lie 3-group: a smooth tricategory with one object and all morphisms weakly invertible.*

Once again, we prove this by showing that the 4-cocycle condition implies the one nontrivial axiom for this tricategory: the pentagonator identity.

Proof. As noted above, the triangulators λ, ρ and μ are trivial. The axioms they satisfy are automatic because π is normalized.

So to check that $\mathrm{Brane}_\pi(G, H)$ is a tricategory, it remains to check that π satisfies the pentagonator identity. Since the 3-cells of the pentagonator identity commute (they represent elements of H), and all the faces not involving a π are trivial, the first half reads:

$$\pi(g_1, g_2, g_3, g_4) \cdot 0_{g_5} + \pi(g_1 g_2, g_3, g_4, g_5) + \pi(g_1, g_2, g_3 g_4, g_5).$$

Here, we write 0_{g_5} to denote 3-morphism which is the identity on the identity of the 1-morphism g_5. We do not need to be worried about order of terms, since composition of 3-morphisms is addition in H. The second half of the pentagonator identity reads:

$$0_{g_1} \cdot \pi(g_2, g_3, g_4, g_5) + \pi(g_1, g_2 g_3, g_4, g_5) + \pi(g_1, g_2, g_3, g_4 g_5).$$

Here, we write 0_{g_1} for to denote the 3-morphism which is the identity on the identity of the 1-morphism g_1. Applying the definition of $\cdot$ as the semidirect product, we see the equality of the first half with the second half is just the cocycle condition on π:

$$\pi(g_1, g_2, g_3, g_4) + \pi(g_1 g_2, g_3, g_4, g_5) + \pi(g_1, g_2, g_3 g_4, g_5)$$
$$= g_1 \pi(g_2, g_3, g_4, g_5) + \pi(g_1, g_2 g_3, g_4, g_5) + \pi(g_1, g_2, g_3, g_4 g_5).$$

So, $\mathrm{Brane}_\pi(G, H)$ is a tricategory. It is smooth because everything in sight is smooth: G, H, and the map $\pi \colon G^4 \to H$. And it is a Lie 3-group: the 1-morphisms G, the trivial 2-morphisms, and the 3-morphisms H are all strictly invertible, and thus of course they are weakly invertible. $\square$

Once again, we can say something a bit stronger about $\mathrm{Brane}_\pi(G, H)$, if we let π be any normalized H-valued 4-cochain, rather requiring it to be a cocycle. In this case, $\mathrm{Brane}_\pi(G, H)$ is a Lie 3-group if and only if π is a 4-cocycle, because π satisfies the pentagon identity if and only if it is a cocycle.

Chapter 7

Integrating nilpotent Lie *n*-algebras

Any mathematician worth her salt knows that we can easily construct Lie algebras as the infinitesimal versions of Lie groups, and that a more challenging inverse construction exists: we can 'integrate' Lie algebras to get Lie groups. By analogy, we expect that the same is true of Lie n-algebras and Lie n-groups: that we can construct Lie n-algebras as the infinitesimal versions of Lie n-groups, and we can 'integrate' Lie n-algebras to obtain Lie n-groups.

In fact, it is easy to see how to obtain slim Lie n-algebras from slim Lie n-groups. As we saw in Chapter 5, slim Lie n-algebras are built from $(n + 1)$-cocycles in Lie algebra cohomology. Remember, p-cochains on the Lie algebra $\mathfrak{g}$ are linear maps:

$$C^p(\mathfrak{g}, \mathfrak{h}) = \{\omega \colon \Lambda^p \mathfrak{g} \to \mathfrak{h}\},$$

where $\mathfrak{h}$ is a representation of $\mathfrak{g}$, though we shall restrict ourselves to the trivial representation $\mathfrak{h} = \mathbb{R}$ in this chapter.

On the other hand, in Chapter 6, we saw that slim Lie n-groups are built from $(n + 1)$-cocycles in Lie group cohomology, at least for $n = 2$ and 3. Remember, p-cochains on G are smooth maps:

$$C^p(G, H) = \{f \colon G^p \to H\},$$

where H is an abelian group on which G acts by automorphism, though we shall restrict ourselves to $H = \mathbb{R}$ with trivial action in this chapter.

Thus, to derive a Lie n-algebra from a Lie n-group, just differentiate the defining Lie group $(n + 1)$-cocycle at the identity to obtain a Lie algebra $(n + 1)$-cocycle. In other words, for every Lie group G with Lie algebra $\mathfrak{g}$, there is a cochain map:

$$D \colon C^\bullet(G) \to C^\bullet(\mathfrak{g}),$$

given by differentiation. Here, we have omitted reference to the coefficients H and $\mathfrak{h}$ because both are assumed to be $\mathbb{R}$. We continue this practice for the rest of the chapter.

Going the other way, however, is challenging—integrating a Lie n-algebra is harder, even when the Lie n-algebra in question is slim. Nonetheless, this challenge has been met. Building on the earlier work of Getzler [38] on integrating nilpotent Lie n-algebras, Henriques [42] has shown that any Lie n-algebra can be integrated to a 'Lie n-group', which Henriques defines as a sort of smooth Kan complex in the category of Banach manifolds. More recently, Schreiber [61] has

generalized this integration procedure to a setting much more general than that of Banach manifolds, including both supermanifolds and manifolds with infinitesimals. For both Henriques and Schreiber, the definition of Lie n-group is weaker than the one we sketched in Chapter 6—it weakens the notion of multiplication so that the product of two group 'elements' is only defined up to equivalence. This level of generality seems essential for the construction to work for *every* Lie n-algebra.

However, for *some* Lie n-algebras, we can integrate them using the more naive idea of Lie n-group we prefer in this thesis: a smooth n-category with one object in which every k-morphism is weakly invertible, for all $1 \leq k \leq n$. We shall see that, for some slim Lie n-algebras, we can integrate the defining Lie algebra $(n+1)$-cocycle to obtain a Lie group $(n+1)$-cocycle. In other words, for certain Lie groups G with Lie algebra $\mathfrak{g}$, there is a cochain map:

$$\textstyle\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G).$$

which is a chain homotopy inverse to differentiation.

When is this possible? We can always differentiate Lie group cochains to obtain Lie algebra cochains, but if we can also integrate Lie algebra cochains to obtain Lie group cochains, the cohomology of the Lie group and its Lie algebra will coincide:

$$H^\bullet(\mathfrak{g}) \cong H^\bullet(G).$$

By a theorem of van Est [70], this happens when all the homology groups of G, as a topological space, vanish.

Thus, we should look to Lie groups with vanishing homology for our examples. How bad can things be when the Lie group is not homologically trivial? To get a sense for this, recall that any semisimple Lie group G is diffeomorphic to the product of its maximal compact subgroup K and a contractible space C:

$$G \approx K \times C.$$

When K is a point, G is contractible, and certainly has vanishing homology. At the other extreme, when C is a point, G is compact. And indeed, in this case there is no hope of obtaining a nontrivial cochain map from Lie algebra cochains to Lie group cochains:

$$\textstyle\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

because *every smooth cochain on a compact group is trivial*.

This fact provided an obstacle to early attempts to integrate Lie 2-algebras. For instance, consider the string Lie 2-algebra $\mathfrak{string}(n)$ we described in Section 5.1.1. Recall that it is the slim Lie 2-algebra $\mathfrak{string}_j(\mathfrak{so}(n))$, where j is the canonical 3-cocycle on $\mathfrak{so}(n)$, given by combining the Killing form with the bracket:

$$j = \langle -, [-, -] \rangle.$$

One could attempt to integrate $\mathfrak{string}(n)$ to a slim Lie 2-group $\mathrm{String}_{\int j}(\mathrm{SO}(n))$, where $\int j$ is a Lie group 3-cocycle on $\mathrm{SO}(n)$ which somehow integrates j, but because the compact group $\mathrm{SO}(n)$ admits no nontrivial smooth Lie group cocycles, this idea fails.

The real lesson of the string Lie 2-algebra is that, once again, our notion of Lie 2-group is not general enough. By generalizing the concept of Lie 2-group, various authors, like Baez–Crans–

Schreiber–Stevenson [13], Henriques [42], and Schommer-Pries [58], were successful in integrating $\mathfrak{string}(n)$.

Nonetheless, there is a large class of Lie n-algebras for which our Lie n-groups *are* general enough. In particular, when G is an 'exponential' Lie group, the story is completely different. A Lie group or Lie algebra is called **exponential** if the exponential map

$$\exp \colon \mathfrak{g} \to G$$

is a diffeomorphism. For instance, all simply-connected nilpotent Lie groups are exponential, though the reverse is not true. Certainly, all exponential Lie groups have vanishing homology, because $\mathfrak{g}$ is contractible. We caution the reader that some authors use the term 'exponential' merely to indicate that $\exp$ is surjective.

When G is an exponential Lie group with Lie algebra $\mathfrak{g}$, we can use a geometric technique developed by Houard [43] to construct a cochain map:

$$\smallint \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G).$$

The basic idea behind this construction is simple, a natural outgrowth of a familiar concept from the cohomology of Lie algebras. Because a Lie algebra p-cochain is a linear map:

$$\omega \colon \Lambda^p \mathfrak{g} \to \mathbb{R},$$

using left translation, we can view ω as defining a p-form on the Lie group G. So, we can integrate this p-form over p-simplices in G. Thus we can define a smooth function:

$$\smallint \omega \colon G^p \to \mathbb{R},$$

by viewing the integral of ω as a function of the vertices of a p-simplex:

$$\smallint \omega(g_1, g_2, \ldots, g_p) = \int_{[1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_p]} \omega.$$

For the right-hand side to truly be a function of the p-tuple $(g_1, g_2, \ldots, g_p)$, we will need a standard way to 'fill out' the p-simplex $[1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_p]$, based only on its vertices. It is here that the fact that G is exponential is key: in an exponential group, we can use the exponential map to define a unique path from the identity 1 to any group element. We think of this path as giving a 1-simplex, $[1, g]$, and we can extend this idea to higher dimensional p-simplices.

Therefore, when G is exponential, we can construct $\smallint$. Using this cochain map, it is possible to integrate the slim Lie n-algebra $\mathfrak{brane}_\omega(\mathfrak{g})$ to the slim Lie n-group $\mathrm{Brane}_{\smallint \omega}(G)$.

We proceed as follows. In Section 7.1, we construct $\smallint$ and show that, along with D, it gives a homotopy equivalence between the complexes $C^\bullet(\mathfrak{g})$ and $C^\bullet(G)$. In Section 7.2, we give explicit formulas for the p-cochain $\smallint \omega$ in terms of ω, for $p = 0, 1, 2$, and 3. Finally, in Section 7.3, we use $\smallint$ to integrate the Heisenberg Lie 2-algebra of Section 5.1.2. Later, in Chapter 10, we shall see that this construction can be 'superized', and integrate Lie n-superalgebras to n-supergroups.

7.1 Integrating Lie algebra cochains

In what follows, we shall see that for an exponential Lie group G, we can construct simplices in G that get along with the action of G on itself. Since we can treat any p-cochain ω on $\mathfrak{g}$ as a left-invariant p-form on G, we can integrate ω over a p-simplex in S in G. Regarding $\int_S \omega$ as a function of the vertices of S, we will see that it defines a Lie group p-cochain. The fact that this is cochain map is purely geometric: it follows automatically from Stokes' theorem.

Let us begin by replacing the cohomology of $\mathfrak{g}$ with the cohomology of left-invariant differential forms on G. Recall that the cohomology of the Lie algebra $\mathfrak{g}$ is given by the Lie algebra cochain complex, $C^\bullet(\mathfrak{g})$, which at level p consists of p-linear maps from $\mathfrak{g}$ to $\mathbb{R}$:

$$C^p(\mathfrak{g}) = \{\omega \colon \Lambda^n \mathfrak{g} \to \mathbb{R}\}\,.$$

We already defined this for Lie superalgebras in Section 3.2. In that section, we saw that the coboundary map d on this complex is usually defined by a rather lengthy formula, but here we shall substitute an equivalent, more geometric definition. Since we can think of the Lie algebra $\mathfrak{g}$ as the tangent space $T_1 G$, we can think of a p-cochain on $\mathfrak{g}$ as giving a p-form on this tangent space. Using left translation on the group, we can translate this p-form over G to define a p-form on all of G. This p-form is left-invariant, and it is easy to see that any left-invariant p-form on G can be defined in this way.

So, in fact, we could just as well define

$$C^p(\mathfrak{g}) = \{\text{left-invariant } p\text{-forms on } G\}\,.$$

It is well-known that the de Rham differential of a left-invariant p-form ω is again left-invariant, and remarkably, the formula for $d\omega_1$ involves only the Lie bracket on $\mathfrak{g}$. This formula is Chevalley and Eilenberg's original definition of d [22], the one we gave in Section 3.2, albeit adapted for Lie superalgebras. In this section, we may forget about this messy formula, and use the de Rham differential instead.

The cohomology of the Lie group G is given by the Lie group cochain complex, $C^\bullet(G)$, which at level p is given by the set of smooth functions from G^p to $\mathbb{R}$:

$$C^p(G) = \{f \colon G^p \to \mathbb{R}\}\,.$$

We have already discussed this in Chapter 6. The coboundary map d on this complex is usually defined by a complicated formula we gave in that chapter, but we can give it a more geometric description just as we did in the case of Lie algebras.

Since we are going to construct a cochain map by integrating p-forms over p-simplices, it would be best to view Lie group cohomology in terms of simplices now. To this end, let us define a **combinatorial p-simplex** in the group G to be an $(p+1)$-tuple of elements of G, which we call the vertices in this context. Of course, G acts on the set of combinatorial p-simplices by left multiplication of the vertices.

Now, we would like to think of Lie group p-cochains as 'smooth, homogeneous, $\mathbb{R}$-valued cochains' on the free abelian group on combinatorial p-simplices. Of course, we need to say what this means. We say an $\mathbb{R}$-valued p-cochain F is **homogeneous** if it is invariant under the action of

G, and that it is **smooth** if the corresponding map

$$F\colon G^{p+1} \to \mathbb{R}$$

is smooth. Now if

$$C_H^p(G) = \{\text{smooth homogeneous } p\text{-cochains}\}\,.$$

denotes the abelian group of all smooth, homogeneous p-cochains, there is a standard way to make $C_H^\bullet(G)$ into a cochain complex. Just take the coboundary operator to be:

$$dF = F \circ \partial,$$

where ∂ is the usual boundary operator on p-chains. It is automatic that $d^2 = 0$.

In fact, this cochain complex is isomorphic to the original one, which we distinguish as the **inhomogeneous cochains**:

$$C_I^p(G) = \{f\colon G^p \to \mathbb{R}\}\,.$$

To see this, note that any inhomogeneous cochain:

$$f\colon G^p \to \mathbb{R}$$

gives rise to a unique, smooth, homogeneous p-cochain F, by defining:

$$F(g_0, \ldots, g_p) = f(g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{p-1}^{-1}g_p)$$

for each combinatorial p-simplex $(g_0, \ldots, g_p)$. Conversely, every smooth, homogeneous p-cochain F gives a unique inhomogeneous p-cochain $f\colon G^p \to \mathbb{R}$, by defining:

$$f(g_1, \ldots, g_p) = F(1, g_1, g_1g_2, \ldots, g_1g_2 \ldots g_p).$$

Finally, note that these isomorphisms commute with the coboundary operators on $C_H^\bullet(G)$ and $C_I^\bullet(G)$. Henceforth, we will write $C^\bullet(G)$ to mean either complex.

These simplicial notions will permit us to define a cochain map from the Lie algebra complex to the Lie group complex:

$$C^\bullet(\mathfrak{g}) \to C^\bullet(G).$$

For $\omega \in C^p(\mathfrak{g})$, the idea is to define an element $\int \omega \in C^p(G)$ by *integrating* the left-invariant p-form ω over a p-simplex S in the group G. In other words, the value which $\int \omega$ assigns to S is defined to be:

$$(\textstyle\int \omega)(S) = \int_S \omega.$$

This is nice because Stokes' theorem will tell us it is a cochain map:

$$(\textstyle\int d\omega)(S) = \int_S d\omega = \int_{\partial S} \omega = d(\textstyle\int \omega)(S)$$

The only hard part is defining p-simplices in G in such a way that $\int \omega$ is actually a smooth, homogeneous p-cochain. It is here that the fact that G is *exponential* is key.

Note that, up until this point, we have only discussed combinatorial p-simplices, which have no relationship to the Lie group structure of G—they are just $(p+1)$-tuples of vertices. We now wish to 'fill out' the combinatorial simplices. That is, we want to create a rule that to any $(p+1)$-tuple $(g_0, \ldots, g_p)$ of vertices in G assigns a filled p-simplex in G, which we denote

$$[g_0, \ldots, g_p].$$

In order to prove that $\int \omega$ is smooth, we need smoothness conditions for this rule, and in order to prove $\int \omega$ is homogeneous, we shall require the left-translate of a p-simplex to again be a p-simplex. In other words, we need:

$$g[g_0, \ldots, g_p] = [gg_0, \ldots, gg_p].$$

We make this precise as follows.

Definition 7.1. Let Δ^p denote $\{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} : \sum x_i = 1, x_i \geq 0\}$, the standard p-simplex in $\mathbb{R}^{p+1}$. Given a collection of smooth maps

$$\varphi_p \colon \Delta^p \times G^{p+1} \to G$$

for each $p \geq 0$, we say this collection defines a **left-invariant notion of simplices** in G if it satisfies:

1. **The vertex property.** For any $(p+1)$-tuple, the restriction

$$\varphi_p \colon \Delta^p \times \{(g_0, \ldots, g_p)\} \to G$$

   sends the vertices of Δ^p to $g_0, \ldots, g_p$, in that order. We denote this restriction by

$$[g_0, \ldots, g_p].$$

   We call this map a **p-simplex**, and regard it as a map from Δ^p to G.

2. **Left-invariance.** For any p-simplex $[g_0, \ldots, g_p]$ and any $g \in G$, we have:

$$g[g_0, \ldots, g_p] = [gg_0, \ldots, gg_p].$$

3. **The face property.** For any p-simplex

$$[g_0, \ldots, g_p] \colon \Delta^p \to G$$

   the restriction to a face of Δ^p is a $(p-1)$-simplex.

   Note that the second condition just says that the map

$$\varphi_p \colon \Delta^p \times G^{p+1} \to G$$

is equivariant with respect to the left action of G, where we take G to act trivially on Δ^p.

On any group equipped with a left-invariant notion of simplex, we have the following result:

Proposition 7.1. *Let G be a Lie group equipped with a left-invariant notion of simplices, and let $\mathfrak{g}$ be its Lie algebra. Then there is a cochain map from the Lie algebra cochain complex to the Lie*

group cochain complex

$$\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

given by integration—that is, if ω is a left-invariant p-form on G, and S is a p-simplex in G, then define:

$$(\textstyle\int \omega)(S) = \int_S \omega.$$

Proof. Let $\omega \in C^p(\mathfrak{g})$. We have already noted that Stokes' theorem

$$\int_S d\omega = \int_{\partial S} \omega$$

implies that this map is a cochain map. We only need to check that $\int \omega$ really lands in $C^p(G)$. That is, that it is smooth and homogeneous. Because G acts trivially on the coefficient group $\mathbb{R}$, homogeneity means that $(\int \omega)(S)$ is invariant of the left action of G on S.

Indeed, note that we can pull the smooth, left-invariant p-form ω back along

$$\varphi_p \colon \Delta^p \times G^{p+1} \to G.$$

The result, $\varphi_p^* \omega$, is a smooth p-form on $\Delta^p \times G^{p+1}$, still invariant under the action of G. Integrating out the dependence on Δ^p, we see this results in a smooth, invariant map:

$$\textstyle\int \omega \colon G^{p+1} \to \mathbb{R},$$

which is precisely what we wanted to prove. $\qquad\square$

We would now like to show that any exponential Lie group G comes with a left-invariant notion of simplices. Our essential tool for this is our ability to use the exponential map to connect any element of G to the identity by a uniquely-defined path. If $h = \exp(X) \in G$ is such an element, we can then define the 'based' 1-simplex $[1, h]$ to be swept out by the path $\exp(tX)$, left translate this to define the general 1-simplex $[g, gh]$ as that swept out by the path $g \exp(tX)$, and proceed to define higher-dimensional simplices with the help of the exponential map and induction, using what we call the **apex-base construction**: given a definition of $(p-1)$-simplex, we define the p-simplex

$$[1, g_1, \ldots, g_p]$$

by using the exponential map to sweep out a path from 1, the **apex**, to each point of the already defined $(p-1)$-simplex, the **base**:

$$[g_1, \ldots, g_p].$$

Having done this, we can then use left translation to define the general p-simplex:

$$[g_0, g_1, \ldots, g_p] = g_0[1, g_0^{-1} g_1, \ldots, g_0^{-1} g_p].$$

In fact, this construction also covers the 1-simplex case. All we need to kick off our induction is to define 0-simplices to be points in G.

To make all this precise, we must use it to define smooth maps

$$\varphi_p \colon \Delta^p \times G^{p+1} \to G,$$

for each $p \geq 0$. To overcome some analytic technicalities in constructing φ_p, we will also need to fix a smooth increasing function:

$$\ell \colon [0,1] \to [0,1]$$

which is 0 on a *neighborhood* of 0, and then monotonically increases to 1 at 1. We shall call ℓ the **smoothing factor**. We shall see latter that our choice of smoothing factor is immaterial: φ_p depends on ℓ, but integrals over simplices do not.

Let us begin by defining 0-simplices as points. That is, we define

$$\varphi_0 \colon \Delta^0 \times G \to G$$

as the obvious projection.

Now, assume that we have defined $(p-1)$-simplices, so we have:

$$\varphi_{p-1} \colon \Delta^{p-1} \times G^p \to G.$$

Using this, we wish to define:

$$\varphi_p \colon \Delta^p \times G^{p+1} \to G.$$

But since we want this to be G-equivariant, we might as well define it for **based p-simplices**: a simplex whose first vertex is 1. So first, we will give a map:

$$\tilde{f}_p \colon \Delta^p \times G^p \to G$$

which we think of as giving us the based p-simplex

$$[1, g_1, \ldots, g_p]$$

for any p-tuple. We do this using the apex-base construction. First, the map $\varphi_{p-1} \colon \Delta^{p-1} \times G^p \to G$ can be extended to a map

$$f_p \colon [0,1] \times \Delta^{p-1} \times G^p \to G$$

by defining f_p to be φ_{p-1} on $\{1\} \times \Delta^{p-1} \times G^p$, to be 1 on $\{0\} \times \Delta^{p-1} \times G^p$, and using the exponential map to interpolate in between. Since $[0,1] \times \Delta^p$ is a kind of generalized prism, we take the liberty of calling $\{0\} \times \Delta^p$ the **0 face**, and $\{1\} \times \Delta^p$ the **1 face**.

Here, the requirement for smoothness complicates things slightly, because we shall actually need f_p to be 1 on a *neighborhood* of the 0 face. So, to be precise, for $(t, x, g_1, \ldots, g_p)) \in [0,1] \times \Delta^{p-1} \times G^p$, we have that $\varphi_{p-1}(x, g_0, \ldots, g_p)$ is a point of G, say $\exp(X)$. Define:

$$f_p(t, x, g_0, \ldots, g_p) = \exp(\ell(t)X).$$

where ℓ is the smoothing factor we mention above: a smooth increasing function which is 0 on a *neighborhood* of 0, and then monotonically increases to 1 at 1. This guarantees f_p will be 1 on a neighborhood of the 0 face, and will match φ_{p-1} on the 1 face.

Since f_p is smooth and is constant on a neighborhood of the 0 face of the prism, $[0,1] \times \Delta^{p-1}$, we can quotient by this face and obtain a smooth map:

$$\tilde{f}_p \colon \Delta^p \times G^p \to G.$$

For definiteness, we can use the smooth quotient map defined by:

$$q_p\colon \quad \begin{aligned} [0,1] \times \Delta^{p-1} &\to & \Delta^p \\ (t,x) &\mapsto & (1-t, tx) \end{aligned}$$

which we note sends the 0 face to the 0th vertex of Δ^p, and sends the vertices of Δ^{p-1} to the remaining vertices of Δ^p, in order. Finally, to define the nonbased p-simplices, we extend by the left action of G—for any $g \in G$ and any $(x, g_1, \ldots, g_p) \in \Delta^p \times G^p$, set:

$$\varphi_p(x, g, gg_1, \ldots, gg_p) = g\tilde{f}_p(x, g_1, \ldots, g_p).$$

This defines

$$\varphi_p\colon \Delta^p \times G^{p+1} \to G.$$

It just remains to check that:

Proposition 7.2. *This defines a left-invariant notion of simplices on G, which we call the* **standard left-invariant notion of simplices with smoothing factor ℓ.**

Proof. By construction, the φ_p are all smooth and G-equivariant, so we only need to check the vertex property and the face property. We do this inductively.

For 0-simplices, the vertex property is trivial. Assume it holds for $(p-1)$-simplices. In particular, the map

$$[g_1, \ldots, g_p]\colon \Delta^{p-1} \to G$$

sends the vertices of Δ^{p-1} to $g_1, \ldots, g_p$, in that order. By construction, the based p-simplex

$$[1, g_1, \ldots, g_p]\colon \Delta^p \to G$$

sends the 0th vertex to 1 and the rest of the vertices to $g_1, \ldots, g_p$, since the $(p-1)$-simplex $[g_1, \ldots, g_p]$ has the vertex property and is defined to be the base of this p-simplex in the apex-base construction. By G-equivariance, this extends to all p-simplices.

For 0-simplices, the face property holds vacuously, and for 1-simplices it is the same as the vertex property. Now take $p \geq 2$, and assume the face property holds for all k-simplices with $k < p$. By G-equivariance, the face property will hold for all p-simplices as long as it holds for all based p-simplices, for instance:

$$[1, g_1, \ldots, g_p].$$

By the apex-base construction, the $(p-1)$-simplex $[g_1, \ldots, g_p]$ is the 0th face of $[1, g_1, \ldots, g_p]$, since it was chosen as the base. For any other face, say the ith face, the apex-base construction gives the $(p-1)$-simplex

$$[1, g_1, \ldots, \hat{g}_i, \ldots, g_p]\colon \Delta^{p-1} \to G$$

with 1 as apex, and the $(p-2)$-simplex $[g_1, \ldots, \hat{g}_i, \ldots, g_p]$ as base. Thus, the face property holds for the p-simplex $[1, g_1, \ldots, g_p]$. $\qquad\square$

While the existence of *any* left-invariant notion of simplices in G suffices to integrate Lie algebra cochains, we have found an almost overwhelming wealth of these notions—one for each smoothing factor ℓ. In fact, for the moment we will indicate the dependence of the standard notion of left-

invariant simplices on ℓ with a superscript:

$$\varphi_p^\ell \colon \Delta^p \times G^{p+1} \to G.$$

Of course, the dependence of φ_p^ℓ on ℓ passes to the individual simplices, so we give them a superscript as well:

$$[g_0, \ldots, g_p]^\ell \colon \Delta^p \to G.$$

Fortunately, however, the cochain map:

$$\textstyle\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

is *independent* of ℓ. That is, if ℓ' is another smoothing factor, we have:

$$\int_{[g_0,\ldots,g_p]^\ell} \omega = \int_{[g_0,\ldots,g_p]^{\ell'}} \omega,$$

for any left-invariant p-form ω.

We shall prove this not by comparing the integrals for two smoothing factors, but rather computing the integral in a way that is manifestly independent of smoothing factor. We do this by showing that the role of the smoothing factor is basically to allow us to smoothly quotient the p-dimensional cube $[0,1]^p$ down to the standard p-simplex Δ^p. Had we parameterized our p-simplices with cubes to begin with, we would have had no need for a smoothing factor. As a trade off, however, our proof that integration gives a chain map would have required more care when analyzing the boundary.

Now we get to work. Rather than parameterizing the p-simplex on the domain Δ^p:

$$[g_0, g_1, \ldots, g_p]^\ell \colon \Delta^p \to G,$$

we shall show how to parameterize it on the p-dimensional cube:

$$\langle g_0, g_1, \ldots, g_p \rangle \colon [0,1]^p \to G,$$

That is, these two functions have the same images—a p-simplex in G with vertices $g_0, \ldots, g_p \in G$, they induce the same orientations on their images, and both traverse it the image precisely once. So, as we shall prove, the integral over either simplex is the same. But, as we shall also see, the latter parameterization does not depend on the smoothing factor ℓ.

How do we discover the parameterization $\langle g_0, \ldots, g_p \rangle$? We just repeat the apex-base construction, but we avoid quotienting to down to Δ^p! Begin by defining the 0-simplices to map the 0-dimensional cube to the indicated vertex:

$$\langle g_0 \rangle \colon \{0\} \to G$$

Define a 1-simplex by using the exponential map to sweep out a path from g_0 to g_1:

$$\langle g_0, g_1 \rangle \colon [0,1] \to G,$$

by defining:

$$\langle g_0, g_1 \rangle(t_1) = g_0 \exp(t_1 X_1), \quad t_1 \in [0,1].$$

where $g_0^{-1}g_1 = \exp(X_1)$. Now, define a 2-simplex using the exponential map to sweep out paths from g_0 to the 1-simplex $\langle g_1, g_2 \rangle$. That is, define:

$$\langle g_0, g_1, g_2 \rangle \colon [0,1]^2 \to G,$$

to be given by:

$$\langle g_0, g_1, g_2 \rangle(t_1, t_2) = g_0 \exp(t_1 Z(X_1, t_2 X_2)),$$

where $g_0^{-1}g_1 = \exp(X_1)$, $g_1^{-1}g_2 = \exp(X_2)$, and Z denotes the Baker–Campbell–Hausdorff series:

$$g_1 g_2 = \exp(Z(X_1, X_2) = \exp(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \cdots).$$

Continuing in this manner, with a bit of work one can see that the p-simplex:

$$\langle g_0, g_1, g_2, \ldots, g_{p-1}, g_p \rangle \colon [0,1]^p \to G$$

is given by the horrendous formula:

$$\langle g_0, \ldots, g_p \rangle(t_1, \ldots, t_p) = g_0 \exp(t_1 Z(X_1, t_2 Z(X_2, \ldots, t_{p-1} Z(X_{p-1}, t_p X_p) \ldots))),$$

where $g_0^{-1}g_1 = \exp(X_1)$, $g_1^{-1}g_2 = \exp(X_2)$, ..., $g_{p-1}^{-1}g_p = \exp(X_p)$. While horrendous, this formula is at least independent of the smoothing factor ℓ, and this forms the basis of the following proposition:

Proposition 7.3. *Let G be an exponential Lie group with Lie algebra $\mathfrak{g}$, let ℓ be a smoothing factor, and equip G with the standard left-invariant notion of simplices with smoothing factor ℓ. For any p-simplex*

$$[g_0, \ldots, g_p]^\ell \colon \Delta^p \to G,$$

depending on ℓ and parameterized on the domain Δ^p, there is a p-simplex:

$$\langle g_0, \ldots, g_p \rangle \colon [0,1]^p \to G$$

given by the formula:

$$\langle g_0, \ldots, g_p \rangle(t_1, \ldots, t_p) = g_0 \exp(t_1 Z(X_1, t_2 Z(X_2, \ldots, t_{p-1} Z(X_{p-1}, t_p X_p) \ldots))),$$

where $g_0^{-1}g_1 = \exp(X_1)$, $g_1^{-1}g_2 = \exp(X_2)$, ..., $g_{p-1}^{-1}g_p = \exp(X_p)$. Then $\langle g_0, \ldots, g_p \rangle$ is independent of ℓ, parameterized on the domain $[0,1]^p$, and has the same image and orientation as $[g_0, \ldots, g_p]^\ell$. Furthermore, for any p-form ω on G, the integral of ω is the same over either simplex:

$$\int_{[g_0, \ldots, g_p]^\ell} \omega = \int_{\langle g_0, \ldots, g_p \rangle} \omega.$$

Sketch of proof. Equality of images and orientations follows from the apex-base construction, and equality of the integrals follows from reparameterization invariance—specifically the change of variables formula for multiple integrals, for which the monotonicity of ℓ becomes crucial. $\qquad \square$

Corollary 7.1. *Let G be an exponential Lie group with Lie algebra $\mathfrak{g}$, let ℓ be a smoothing factor, and equip G with the standard left-invariant notion of simplices with smoothing factor ℓ. Let*

$$\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

be the cochain map from Lie algebra cochains to Lie group cochains given by integration over simplices. Then the cochain map $\int$ is independent of the smoothing factor ℓ.

Proof. Recall that if ω is a left-invariant p-form on G, and $[g_0, \ldots, g_p]^\ell$ is a p-simplex in G, the cochain map $\int$ is defined by:

$$(\int \omega)(g_0, \ldots, g_p) = \int_{[g_0, \ldots, g_p]^\ell} \omega$$

By the previous proposition, this integral is equal to

$$\int_{\langle g_0, \ldots, g_p \rangle} \omega,$$

where $\langle g_0, \ldots, g_p \rangle \colon [0,1]^p \to G$ is given as above, and is independent of ℓ. Thus, $\int$ is also independent of ℓ. $\qquad\square$

Having proven that the cochain map $\int$ is independent of smoothing factor, we will now allow the smoothing factor to recede into the background. Henceforth, we abuse terminology somewhat and speak of *the* standard left-invariant notion of simplices to mean the standard notion with some implicit choice of smoothing factor.

The hard work of integrating Lie algebra cochains is now done. We would now like to go the other way, and show how to get a Lie algebra cochain from a Lie group cochain. This direction is much easier: in essence, we differentiate the Lie group cochain at the identity, and antisymmetrize the result. To do this, we make use of the fact that any element of the Lie algebra can be viewed as a directional derivative at the identity. The following result, due to van Est (c.f. [70], Formula 46) just says this map defines a cochain map:

Proposition 7.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there is a cochain map from the van Est complex to the Chevalley–Eilenberg complex:*

$$D \colon C^\bullet(G) \to C^\bullet(\mathfrak{g})$$

given by differentiation—that is, if F is a homogeneous p-cochain on G, and $X_1, \ldots, X_p \in \mathfrak{g}$, then we can define:

$$DF(X_1, \ldots, X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) X^1_{\sigma(1)} \ldots X^p_{\sigma(p)} F(1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_p),$$

where by X_i^j we indicate that the operator X_i differentiates only the jth variable, g_j.

Proof. See Houard [43], p. 224, Lemma 1. $\qquad\square$

Having now defined cochain maps

$$\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(G)$$

and

$$D \colon C^\bullet(G) \to C^\bullet(\mathfrak{g}),$$

the obvious next question is whether or not this defines a homotopy equivalence of cochain complexes. Indeed, as proved by Houard, they do:

Theorem 7.1. *Let G be a Lie group equipped with a left-invariant notion of simplices, and $\mathfrak{g}$ its Lie algebra. The cochain map*

$$D\int \colon C^\bullet(\mathfrak{g}) \to C^\bullet(\mathfrak{g}),$$

is the identity, whereas the cochain map

$$\int D \colon C^\bullet(G) \to C^\bullet(G)$$

is cochain-homotopic to the identity. Therefore the Lie algebra cochain complex $C^\bullet(\mathfrak{g})$ and the Lie group cochain complex $C^\bullet(G)$ are homotopy equivalent and thus have isomorphic cohomology.

Proof. See Houard [43], p. 234, Proposition 2. □

7.2 Examples: Explicitly integrating 0-, 1-, 2- and 3-cochains

In this section, in order to get a feel for the integration procedure given in Proposition 7.1, we shall explicitly calculate some Lie group cochains from Lie algebra cochains. The resulting formulas are polynomials on the Lie group, at least in the nilpotent case. It is important to note, however, that you do not need to know these explicit formulas in what follows. It is enough to understand that they exist, and have the properties described in Section 7.1. We nonetheless suspect that explicit formulas will prove useful in future work, so we collect some here.

To facilitate this calculation, we shall also have to explicitly construct some low-dimensional left-invariant simplices. For 0-cochains and 1-cochains, we will find the task very easy—we only need our Lie group G to be exponential. On the other hand, for 2- and 3-cochains, the construction gets much harder. This complexity shows just how powerful the abstract approach of the previous section actually is—imagine having to prove Proposition 7.1 through an explicit integration such as those we present here!

So, for 2- and 3-cochains, we simplify the problem by assuming our Lie algebra $\mathfrak{g}$ to be **2-step nilpotent**: all brackets of brackets are zero. This allows us to use a simplified form of the Baker–Campbell–Hausdorff formula:

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y])$$

and the Zassenhaus formula:

$$\exp(X + Y) = \exp(X)\exp(Y)\exp(-\frac{1}{2}[X,Y]) = \exp(X)\exp(Y - \frac{1}{2}[X,Y]). \tag{7.1}$$

Partially, this nilpotency assumption just makes our calculations tenable, but secretly it is because our main application of these ideas will be to 2-step nilpotent Lie *superalgebras*.

0-cochains

Let ω be a Lie algebra 0-cochain: that is, a real number. Then $\int \omega = \omega$ is a Lie group 0-cochain. We can view it as the integral of ω over the 0-simplex $[1]$.

1-cochains

Let ω be a Lie algebra 1-cochain: that is, a linear map

$$\omega \colon \mathfrak{g} \to \mathbb{R},$$

which we extend to a 1-form on G by left translation. We define a Lie group 1-cochain $\int \omega$ by integrating ω over 1-simplices in G. In particular

$$\int \omega(g) = \int_{[1,g]} \omega.$$

Since G is exponential, it has a standard left-invariant notion of 1-simplex, given by exponentiation. So, if $g = \exp(X)$, then the 1-simplex $[1, g]$ is given by

$$[1, g](t) = \exp(tX), \quad 0 \le t \le 1.$$

We denote this map by φ for brevity. So:

$$\int \omega(g) = \int_0^1 \omega(\dot{\varphi}(t)) \, dt$$

Noting that the derivative of φ is

$$\dot{\varphi}(t) = \exp(tX)X$$

we have

$$\int \omega(g) = \int_0^1 \omega(\exp(tX)X) \, dt = \int_0^1 \omega(X) \, dt = \omega(X),$$

where we have used the left invariance of ω. In summary,

$$\int \omega(g) = \omega(X),$$

for $g = \exp(X)$.

As a check on this, note that because we have proved $\int$ is a cochain map, $\int \omega$ should be a cocycle whenever ω is. So let us verify this. Assume $d\omega = 0$. That is, for all X and $Y \in \mathfrak{g}$, we have:

$$d\omega(X, Y) = -\omega([X, Y]) = 0.$$

So the cocycle condition merely says that ω must vanish on brackets.

Now compute the coboundary of $\int \omega$. As a function of a single group element, $\int \omega$ is an inhomogeneous Lie group 2-cochain. So we must use the coboundary formula from Chapter 6, which we recall here for convenience: when $f\colon G^p \to \mathbb{R}$ is an inhomogeneous Lie group p-cochain, its coboundary is:

$$df(g_1, \ldots, g_{p+1}) \;=\; \sum_{i=1}^{p}(-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{p+1})$$
$$+(-1)^{p+1} f(g_1, \ldots, g_p).$$

In the case of $\int \omega$, this becomes simply:

$$d\int \omega(g, h) = \int \omega(h) - \int \omega(gh) + \int \omega(g).$$

Finally, check that this coboundary is zero when ω is a cocycle, and hence that $\int \omega$ is a cocycle whenever ω is. If $g = \exp(X)$ and $h = \exp(Y)$, we have

$$gh = \exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \cdots)$$

by the Baker–Campbell–Hausdorff formula, and thus:

$$d\int \omega(g, h) = \omega(Y) - \omega(X + Y + \frac{1}{2}[X, Y] + \cdots) + \omega(X) = 0$$

where we have used ω's linearity along with the cocycle condition that ω must vanish on brackets.

2-cochains

As we have just seen, 0-cochains and 1-cochains are easily integrated on any exponential Lie group, and the result is always a polynomial Lie group cochain. Unfortunately, even for 2-cochains, the integration is much more complicated, and no longer polynomial unless $\mathfrak{g}$ is nilpotent. So, at this point, we will simplify matters by assuming $\mathfrak{g}$ to be 2-step nilpotent. To hint at this with our notation, we will now call our Lie algebra $\mathfrak{n}$ and the corresponding simply-connected Lie group N.

Let ω be a Lie algebra 2-cochain: that is, a left-invariant 2-form. We define a Lie group 2-cochain $\int \omega$ by integrating ω over 2-simplices in N. In particular:

$$\int \omega(g, h) = \int_{[1, g, gh]} \omega.$$

Now suppose $g = \exp(X)$ and $h = \exp(Y)$. Recall we that obtain the 2-simplex $[1, g, gh]$ using the apex-base construction: we connect each point of the base $[g, gh] = g[1, h]$ to 1 by the exponential map. Since $[1, h](t) = \exp(tY)$, the base is parameterized by

$$[g, gh](t) = g\exp(tY) = \exp(X + tY + \frac{t}{2}[X, Y])$$

97

by the Baker–Campbell–Hausdorff formula. Now let us construct $[1, g, gh]$ by first constructing a map from the square

$$\varphi \colon [0, 1] \times [0, 1] \to N$$

given by

$$\varphi(s, t) = \exp(s(X + tY + \frac{t}{2}[X, Y])).$$

At this stage in our general construction, since this map is 1 on the $\{0\} \times [0, 1]$ edge of the square, we would typically quotient the square out by this edge to obtain a map from the standard 2-simplex. But in practice, we do not need to do this. Since the integral $\int_{[1,g,gh]} \omega$ is invariant under reparameterization, we might as well parameterize our 2-simplex $[1, g, gh]$ with φ and integrate over the square to obtain:

$$\int \omega(g, h) = \int_0^1 \int_0^1 \omega(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}) \, ds \, dt.$$

Our task has essentially been reduced to computing the partial derivatives of φ. Thanks to the left invariance of ω, we may as well left translate these partials back to 1 once we have them, since:

$$\omega(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}) = \omega(\varphi^{-1}\frac{\partial \varphi}{\partial s}, \varphi^{-1}\frac{\partial \varphi}{\partial t}).$$

Let us begin with $\frac{\partial \varphi}{\partial s}$. Since the exponent of $\varphi(s, t) = \exp(s(X + tY + \frac{t}{2}[X, Y]))$ is linear in s, this is simply:

$$\frac{\partial \varphi}{\varphi s}(s, t) = \varphi(s, t)(X + tY\frac{t}{2}[X, Y]).$$

This is a tangent vector at $\varphi(s, t)$. We can left translate it back to 1 to obtain:

$$\varphi^{-1}\frac{\partial \varphi}{\partial s} = X + tY\frac{t}{2}[X, Y]).$$

The partial with respect to t is slightly harder, because the exponent is not linear in t. To compute this, we need the Zassenhaus formula, Formula 7.1, to separate the terms linear in t from those that are not. Applying this, we obtain

$$\varphi(s, t) = \exp(sX) \exp(stY + \frac{st}{2}[X, Y] - \frac{s^2 t}{2}[X, Y]).$$

Differentiating this with respect to t and left translating the result to 1, we get:

$$\varphi^{-1}\frac{\partial \varphi}{\partial t} = sY + \frac{s - s^2}{2}[X, Y].$$

Substituting these partial derivatives into the integral, our problem becomes:

$$\int \omega(g, h) = \int_0^1 \int_0^1 \omega(X + tY + \frac{t}{2}[X, Y], \, sY + \frac{s - s^2}{2}[X, Y]) \, ds \, dt.$$

It is now easy enough, using ω's bilinearity and antisymmetry, to bring all the polynomial coefficients out and integrate them, obtaining an expression which is the sum of three terms:

$$\textstyle\int \omega(g,h) = \frac{1}{2}\omega(X,Y) + \frac{1}{12}\omega(X,[X,Y]) - \frac{1}{12}\omega(Y,[X,Y]).$$

Nevertheless, we would like to do this calculation explicitly. In essence, we use ω's bilinearity and antisymmetry to our advantage, to write these coefficients as integrals of various determinants. To wit, the coefficent of $\omega(X,Y)$ is the integral of the determinant

$$\begin{vmatrix} 1 & t \\ 0 & s \end{vmatrix} = s,$$

which we obtain from reading off the coefficients of X and Y in the integrand:

$$\omega(X + tY + \frac{t}{2}[X,Y],\ sY + \frac{s - s^2}{2}[X,Y]).$$

So the coefficient of $\omega(X,Y)$ is $\int_0^1 \int_0^1 s\, ds\, dt = \frac{1}{2}$. We can use this idea to obtain the other two coefficients as well—the coefficient of $\omega(X,[X,Y])$ is the integral of the determinant

$$\begin{vmatrix} 1 & \frac{t}{2} \\ 0 & \frac{s-s^2}{2} \end{vmatrix} = \frac{s - s^2}{2},$$

which is $\frac{1}{12}$, and the coefficient of $\omega(Y,[X,Y])$ is the integral of the determinant

$$\begin{vmatrix} t & \frac{t}{2} \\ s & \frac{s-s^2}{2} \end{vmatrix} = -\frac{s^2 t}{2},$$

which is $-\frac{1}{12}$.

As a final check on this calculation, let us again show that when ω is a cocycle, so is $\int \omega$. We know this must be true by Proposition 7.1, of course, but when checking it explicitly the cocycle condition seems almost miraculous. Since this final computation is a bit of a workout, we tuck it into the proof of the following proposition. It is only a check, and understanding the calculation is not necessary for what follows.

Proposition 7.5. *Let N be a simply-connected Lie group whose Lie algebra $\mathfrak{n}$ is 2-step nilpotent. If ω is a Lie algebra 2-cocycle on $\mathfrak{n}$, then the Lie group 2-cochain on N defined by*

$$\textstyle\int \omega(g,h) = \frac{1}{2}\omega(X,Y) + \frac{1}{12}\omega(X - Y,[X,Y]),$$

where $g = \exp(X)$ and $h = \exp(Y)$, is also a cocycle.

Proof. As already noted, this fact is immediate from Proposition 7.1, but we want to ignore this and check it explicitly. To do this, we repeatedly use the Baker–Campbell–Hausdorff formula, the assumption that $\mathfrak{n}$ is 2-step nilpotent, and the cocycle condition on ω. This latter condition reads:

$$d\omega(X,Y,Z) = -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) = 0.$$

99

Note how this resembles the Jacobi identity. We prefer to write it as follows:

$$\omega(X, [Y, Z]) = \omega([X, Y], Z) + \omega(Y, [X, Z]).$$

To begin, the coboundary of the inhomogeneous Lie group 3-cochain $\int \omega$ is given by:

$$d\int \omega(g, h, k) = \int \omega(h, k) - \int \omega(gh, k) + \int \omega(g, hk) - \int \omega(g, h).$$

Let us assume that

$$g = \exp(X), \quad h = \exp(Y), \quad k = \exp(Z),$$

so that

$$gh = \exp(X + Y + \frac{1}{2}[X, Y]), \quad hk = \exp(Y + Z + \frac{1}{2}[Y, Z]).$$

Now we repeatedly insert the expression for our Lie group 2-cochain, so the coboundary of $\int \omega$ becomes:

$$
\begin{aligned}
d\int \omega(g, h, k) \; = \; & \frac{1}{2}\omega(Y, Z) + \frac{1}{12}\omega(Y - Z, [Y, Z]) \\
& - \frac{1}{2}\omega(X + Y + \frac{1}{2}[X, Y], Z) - \frac{1}{12}\omega(X + Y + \frac{1}{2}[X, Y] - Z, [X + Y, Z]) \\
& + \frac{1}{2}\omega(X, Y + Z + \frac{1}{2}[Y, Z]) + \frac{1}{12}\omega(X - Y - Z - \frac{1}{2}[Y, Z], [X, Y + Z]) \\
& - \frac{1}{2}\omega(X, Y) - \frac{1}{12}\omega(X - Y, [X, Y]),
\end{aligned}
$$

Note that the cocycle condition combined with nilpotency implies that any term in which ω eats two brackets vanishes. In general,

$$\omega([X, Y], [Z, W]) = \omega([[X, Y], Z], W) + \omega(Z, [[X, Y], W]) = 0,$$

thanks to the fact that brackets of brackets vanish. So, in the expression for $d\int \omega$, we can simplify the fourth term:

$$
\begin{aligned}
\omega(X + Y + \frac{1}{2}[X, Y] - Z, [X + Y, Z]) \; & = \; \omega(X + Y - Z, [X + Y, Z]) + \frac{1}{2}\omega([X, Y], [X + Y, Z]) \\
& = \; \omega(X + Y - Z, [X + Y, Z]).
\end{aligned}
$$

Similarly for the sixth term:

$$\omega(X - Y - Z - \frac{1}{2}[Y, Z], [X, Y + Z]) = \omega(X - Y - Z, [X, Y + Z]).$$

This leaves us with:

$$
\begin{aligned}
d \textstyle\int \omega(g, h, k) &= \tfrac{1}{2}\omega(Y, Z) + \tfrac{1}{12}\omega(Y - Z, [Y, Z]) \\
&\quad - \tfrac{1}{2}\omega(X + Y + \tfrac{1}{2}[X, Y], Z) - \tfrac{1}{12}\omega(X + Y - Z, [X + Y, Z]) \\
&\quad + \tfrac{1}{2}\omega(X, Y + Z + \tfrac{1}{2}[Y, Z]) + \tfrac{1}{12}\omega(X - Y - Z, [X, Y + Z]) \\
&\quad - \tfrac{1}{2}\omega(X, Y) - \tfrac{1}{12}\omega(X - Y, [X, Y]),
\end{aligned}
$$

Expanding this using bilinearity, we obtain, after many cancellations:

$$
\begin{aligned}
d \textstyle\int \omega(g, h, k) &= -\tfrac{1}{4}\omega([X, Y], Z) - \tfrac{1}{12}\omega(X, [Y, Z]) - \tfrac{1}{12}\omega(Y, [X, Z]) \\
&\quad + \tfrac{1}{4}\omega(X, [Y, Z]) - \tfrac{1}{12}\omega(Y, [X, Z]) - \tfrac{1}{12}\omega(Z, [X, Y]).
\end{aligned}
$$

We combine the two terms with coefficient $1/4$ using the cocycle condition:

$$
-\omega([X, Y], Z) + \omega(X, [Y, Z]) = \omega([Y, X], Z) + \omega(X, [Y, Z]) = \omega(Y, [X, Z]).
$$

Similarly, for the first and fourth terms with coefficent $1/12$, we apply the cocycle condition to get:

$$
\omega(X, [Y, Z]) + \omega(Z, [X, Y]) = \omega(Y, [X, Z]).
$$

So, substituting these in, we finally obtain:

$$
d \textstyle\int \omega(g, h, k) = \tfrac{1}{4}\omega(Y, [X, Z]) - \tfrac{1}{12}\omega(Y, [X, Z]) - \tfrac{1}{12}\omega(Y, [X, Z]) - \tfrac{1}{12}\omega(Y, [X, Z]) = 0,
$$

as desired.

$\square$

As a corollary, note that we could equally well have said:

Corollary 7.2. *Let N be a simply-connected Lie group whose Lie algebra $\mathfrak{n}$ is 2-step nilpotent. If ω is a Lie algebra 2-cocycle on $\mathfrak{n}$, then the Lie group 2-cochain on N defined by*

$$
\textstyle\int \omega(g, h) = \int_0^1 \int_0^1 \omega(X + tY + \tfrac{t}{2}[X, Y], \, sY + \tfrac{s - s^2}{2}[X, Y]) \, ds \, dt,
$$

where $g = \exp(X)$ and $h = \exp(Y)$, is also a cocycle.

Proof. By our calculation in this section,

$$
\textstyle\int \omega(g, h) = \tfrac{1}{2}\omega(X, Y) + \tfrac{1}{12}\omega(X - Y, [X, Y]),
$$

so the result is immediate.

$\square$

101

3-cochains

Let ω be a 3-cochain on the Lie algebra: that is, a left-invariant 3-form. Judging by our experience in the last section, the complexity of integrating ω to a Lie group 3-cochain may be quite high. Indeed, we shall ultimately avoid writing down $\int \omega$, except as an integral. Nonetheless, we can make this integral quite explicit.

We define the Lie group 3-cochain $\int \omega$ to be the integral of ω over a 3-simplex in N. In particular:

$$\int \omega(g, h, k) = \int_{[1,g,gh,ghk]} \omega.$$

Now assume that $g = \exp(X)$, $h = \exp(Y)$ and $k = \exp(Z)$. Recall we that obtain the 3-simplex $[1, g, gh, ghk]$ using the apex-base construction: we connect each point of the base $[g, gh, ghk] = g[1, h, hk]$ to 1 by the exponential map. In the last section, we saw that $[1, h, hk](t, u) = \exp(t(Y + uZ + \frac{u}{2}[Y, Z]))$, so the base is parameterized by

$$
\begin{aligned}
[g, gh, ghk](t, u) &= g \exp(t(Y + uZ + \frac{u}{2}[Y, Z]) \\
&= \exp(X + tY + tuZ + \frac{tu}{2}[Y, Z] + \frac{1}{2}[X, tY + tuZ]),
\end{aligned}
$$

by the Baker–Campbell–Hausdorff formula. Now let us construct $[1, g, gh, ghk]$ by first constructing a map from the cube

$$\varphi \colon [0, 1] \times [0, 1] \times [0, 1] \to N$$

given by

$$
\begin{aligned}
\varphi(s, t, u) &= \exp(s(X + tY + tuZ + \frac{tu}{2}[Y, Z] + \frac{1}{2}[X, tY + tuZ])) \\
&= \exp(sX + stY + stuZ + \frac{st}{2}[X, Y] + \frac{stu}{2}[Y, Z] + \frac{stu}{2}[X, Z]).
\end{aligned}
$$

At this stage in our general construction, since this map is 1 on the $\{0\} \times [0, 1] \times [0, 1]$ face of the cube and on the lines $\{s\} \times \{0\} \times [0, 1]$ of constant s on the $[0, 1] \times 0 \times [0, 1]$ face of the cube, we could quotient the cube out by these sets to obtain a map from the standard 3-simplex. But in practice, we do not need to do this. Since the integral $\int_{[1,g,gh,ghk]} \omega$ is invariant under reparameterization, we might as well parameterize our 3-simplex $[1, g, gh, ghk]$ with φ and integrate over the cube to obtain:

$$\int \omega(g, h, k) = \int_0^1 \int_0^1 \int_0^1 \omega(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial u}) \, ds \, dt \, du.$$

Once again, our task has essentially reduced to computing the partial derivatives of φ, and once again, thanks to the left invariance of φ, we may as well left translate these partials back to 1 once we have them, since:

$$\omega(\frac{\partial \varphi}{\partial s}, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial u}) = \omega(\varphi^{-1} \frac{\partial \varphi}{\partial s}, \varphi^{-1} \frac{\partial \varphi}{\partial t}, \varphi^{-1} \frac{\partial \varphi}{\partial u}).$$

Let us begin with $\frac{\partial \varphi}{\partial s}$. Since the exponent of $\varphi(s, t, u)$ is linear in s, this is simply:

$$\frac{\partial \varphi}{\varphi s}(s, t, u) = \varphi(s, t, u)(X + tY + tuZ + \frac{t}{2}[X, Y] + \frac{tu}{2}[Y, Z] + \frac{tu}{2}[X, Z]).$$

This is a tangent vector at $\varphi(s, t, u)$. We can left translate it back to 1 to obtain:

$$\varphi^{-1}\frac{\partial \varphi}{\varphi s} = X + tY + tuZ + \frac{t}{2}[X, Y] + \frac{tu}{2}[Y, Z] + \frac{tu}{2}[X, Z].$$

The partial with respect to t is slightly harder, because the exponent is not linear in t. To compute this, we again need the Zassenhaus formula, Formula 7.1, to separate the terms linear in t from those that are not. Applying this, we obtain

$$\varphi(s, t, u) = \exp(sX)\exp(stY + stuZ + \frac{st}{2}[X, Y] + \frac{stu}{2}[Y, Z] + \frac{stu}{2}[X, Z] - \frac{1}{2}[sX, stY + stuZ]).$$

Differentiating this with respect to t and left translating the result to 1, we get:

$$\varphi^{-1}\frac{\partial \varphi}{\partial t} = sY + suZ + \frac{s}{2}[X, Y] + \frac{su}{2}[Y, Z] + \frac{su}{2}[X, Z] - \frac{1}{2}[sX, sY + suZ],$$

which we can simplify by combining like terms:

$$\varphi^{-1}\frac{\partial \varphi}{\partial t} = sY + suZ + \frac{s - s^2}{2}[X, Y] + \frac{su}{2}[Y, Z] + \frac{su - s^2u}{2}[X, Z].$$

Finally, the partial with respect to u requires that we separate out the terms linear in u, again using the Zassenhaus formula:

$$\varphi(s, t, u) = \exp(sX + stY + \frac{st}{2}[X, Y])\exp(stuZ + \frac{stu}{2}[Y, Z] + \frac{stu}{2}[X, Z] - \frac{1}{2}[sX + stY, stuZ]).$$

Differentiating this with respect to u and left translating the result to 1, we get:

$$\varphi^{-1}\frac{\partial \varphi}{\partial u} = stZ + \frac{st}{2}[Y, Z] + \frac{st}{2}[X, Z] - \frac{1}{2}[sX + stY, stZ],$$

which we can again simplify by combining like terms:

$$\varphi^{-1}\frac{\partial \varphi}{\partial u} = stZ + \frac{st - s^2t^2}{2}[Y, Z] + \frac{st - s^2t}{2}[X, Z].$$

Substituting these partial derivatives into the integral, our problem becomes:

$$
\begin{aligned}
\int \omega(g, h, k) \quad = \quad & \int_0^1 \int_0^1 \int_0^1 \omega(\quad X + tY + tuZ + \frac{t}{2}[X, Y] + \frac{tu}{2}[Y, Z] + \frac{tu}{2}[X, Z], \\
& sY + suZ + \frac{s - s^2}{2}[X, Y] + \frac{su}{2}[Y, Z] + \frac{su - s^2u}{2}[X, Z], \\
& stZ + \frac{st - s^2t^2}{2}[Y, Z] + \frac{st - s^2t}{2}[X, Z] \quad) \, ds \, dt \, du.
\end{aligned}
$$

This integral is bad enough. Further evaluating this integral is quite a chore (the answer involves 17 nonzero terms!), so we stop here. We would only like to give a hint as to how the evaluation could be done. As in Section 7.2, thanks to ω's trilinearity and antisymmetry, the coefficients of the terms in $\int \omega(g, h, k)$ are integrals of various determinants. For instance, the coefficient of $\omega(X, Y, Z)$ is the integral of the 3×3 determinant

$$
\begin{vmatrix}
1 & t & tu \\
0 & s & su \\
0 & 0 & st
\end{vmatrix} = s^2 t,
$$

which we obtain from reading off the coefficients of X, Y and Z in the integrand. So the coefficient of $\omega(X, Y, Z)$ in $\int \omega(g, h, k)$ is $\int_0^1 \int_0^1 \int_0^1 s^2 t \, ds \, dt \, du = \frac{1}{6}$. The other terms may be computed similarly.

Just as we shall not attempt to evaluate the integral for $\int \omega(g, h, k)$, we also do not attempt to demonstrate that it gives a Lie group cocycle when ω is a Lie algebra cocycle. After all, Proposition 7.1 does this for us:

Proposition 7.6. *Let N be a simply-connected Lie group whose Lie algebra $\mathfrak{n}$ is 2-step nilpotent. If ω is a Lie algebra 3-cocycle on $\mathfrak{n}$, then the Lie group 3-cochain on N given by*

$$
\begin{aligned}
\int \omega(g, h, k) \;=\; \int_0^1 \int_0^1 \int_0^1 \omega(\quad & X + tY + tuZ + \frac{t}{2}[X, Y] + \frac{tu}{2}[Y, Z] + \frac{tu}{2}[X, Z], \\
& sY + suZ + \frac{s - s^2}{2}[X, Y] + \frac{su}{2}[Y, Z] + \frac{su - s^2 u}{2}[X, Z], \\
& stZ + \frac{st - s^2 t^2}{2}[Y, Z] + \frac{st - s^2 t}{2}[X, Z] \quad) \, ds \, dt \, du,
\end{aligned}
$$

where $g = \exp(X)$, $h = \exp(Y)$ and $k = \exp(Z)$, is also a cocycle.

Proof. This is immediate upon combining Proposition 7.1 with the above discussion. $\quad\square$

7.3 The Heisenberg Lie 2-group

In Section 5.1.2, we met the Heisenberg Lie algebra, $\mathfrak{H} = \mathrm{span}(p, q, z)$. This is the 3-dimensional Lie algebra where the generators p, q and z satisfy relations which mimic the canonical commutation relations from quantum mechanics:

$$
[p, q] = z, \quad [p, z] = 0, \quad [q, z] = 0.
$$

As one can see from the above relations, $\mathfrak{H}$ is 2-step nilpotent: brackets of brackets are zero.

We then met the Lie 2-algebra generalization, the Heisenberg Lie 2-algebra:

$$
\mathfrak{Heisenberg} = \mathfrak{string}_\gamma(\mathfrak{H}),
$$

built by extending $\mathfrak{H}$ with the 3-cocycle $\gamma = p^* \wedge q^* \wedge z^*$, where p^*, q^*, and z^* is the basis dual to p, q and z.

It is easy to construct a Lie group H with Lie algebra $\mathfrak{H}$. Just take the group of 3×3 upper triangular matrices with units down the diagonal:

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

This is an exponential Lie group:

$$\begin{array}{ccc} \exp: & \mathfrak{H} & \to & H \\ & ap + cq + bz & \mapsto & \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}. \end{array}$$

So we can apply Proposition 7.2 to construct the standard left-invariant notion of simplices in H, and Proposition 7.1 to integrate the Lie algebra 3-cocycle γ to a Lie group 3-cocycle $\int \gamma$. We therefore get a Lie 2-group, the **Heisenberg Lie 2-group**:

$$\text{Heisenberg} = \text{String}_{\int \gamma}(H).$$

Chapter 8

Supergeometry and supergroups

We would now like to generalize our work from Lie algebras and Lie groups to Lie superalgebras and supergroups. Of course, this means that we need a way to talk about Lie supergroups, their underlying supermanifolds, and the maps between supermanifolds. This task is made easier because we do not need the full machinery of supermanifold theory. Because our supergroups will be exponential, we only need to work with supermanifolds that are diffeomorphic to super vector spaces. Nonetheless, let us begin with a sketch of supermanifold theory from the perspective that suits us best, which could loosely be called the 'functor of points' approach.

The rough geometric picture one should have of a supermanifold M is that of an ordinary manifold with infinitesimal 'superfuzz', or 'superdirections', around each point. At the infinitesimal level, an ordinary manifold is merely a vector space—its tangent space at a point. In contrast, the tangent space to M has a $\mathbb{Z}_2$-grading: tangent vectors which point along the underlying manifold of M are taken to be even, while tangent vectors which point along the superdirections are taken to be odd.

At least infinitesimally, then, all supermanifolds look like super vector spaces,

$$\mathbb{R}^{p|q} := \mathbb{R}^p \oplus \mathbb{R}^q,$$

where $\mathbb{R}^p$ is even and $\mathbb{R}^q$ is odd. And indeed, just as ordinary manifolds are locally modeled on ordinary vector spaces, $\mathbb{R}^n$, supermanifolds are locally modeled on super vector spaces, $\mathbb{R}^{p|q}$. But before we sketch how this works, let us introduce our main tool: the so-called 'functor of points'.

The basis for the functor of points is the Yoneda Lemma, a very general and fundamental fact from category theory:

Yoneda Lemma. *Let C be a category. The functor*

$$
\begin{aligned}
C &\to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}) \\
x &\mapsto \mathrm{Hom}(-, x)
\end{aligned}
$$

is a full and faithful embedding of C into the category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ of contravariant functors from C to Set. *This embedding is called the* **Yoneda embedding**.

The upshot of this lemma is that, without losing any information, we can replace an object x by a functor $\mathrm{Hom}(-, x)$, and a morphism $f \colon x \to y$ by a natural transformation

$$\mathrm{Hom}(-, f) \colon \mathrm{Hom}(-, x) \Rightarrow \mathrm{Hom}(-, y)$$

of functors. Each component of this natural transformation is the 'obvious' thing: for an object z, the function

$$\mathrm{Hom}(z, f) \colon \mathrm{Hom}(z, x) \to \mathrm{Hom}(z, y)$$

just takes the morphism $g \colon z \to x$ to the morphism $fg \colon z \to y$.

On a more intuitive level, the functor of points tells us how to reconstruct a 'space' $x \in C$ by probing it with *every other space* $z \in C$—that is, by looking at all the ways in which z maps into x, which forms the set $\mathrm{Hom}(z, x)$. The true power of the functor of points, however, arises when we can reconstruct x without having to probe it will *every* z, but with z from a manageable subcategory of C. And while it deviates slightly from the spirit of the Yoneda Lemma, we can shrink this subcategory still further if we allow $\mathrm{Hom}(z, x)$ to have more structure than that of a mere set. In fact, when M is a supermanifold, we will consider probes z for which $\mathrm{Hom}(z, M)$ is an ordinary manifold.

For what z is $\mathrm{Hom}(z, M)$ a manifold? One clue is that when M is an ordinary manifold, there is a manifold of ways to map a point into M:

$$M \cong \mathrm{Hom}(\mathbb{R}^0, M),$$

but the space of maps from any higher-dimensional manifold to M is generally not a finite-dimensional manifold in its own right. Similarly, when M is a supermanifold, there is an ordinary manifold of ways to map a point into M:

$$M_{\mathbb{R}^{0|0}} = \mathrm{Hom}(\mathbb{R}^{0|0}, M).$$

One should think of this as the ordinary manifold one gets from M by forgetting about the superdirections. But thanks to the superdirections, we now we have more ways of obtaining a manifold of maps to M: there is an ordinary manifold of ways to map a point with q superdirections into M:

$$M_{\mathbb{R}^{0|q}} = \mathrm{Hom}(\mathbb{R}^{0|q}, M).$$

So, for every supermanifold M, we get a functor:

$$\mathrm{Hom}(-, M) \colon \quad \begin{array}{ccc} \mathrm{SuperPoints}^{\mathrm{op}} & \to & \mathrm{Man} \\ \mathbb{R}^{0|q} & \mapsto & \mathrm{Hom}(\mathbb{R}^{0|q}, M) \end{array}$$

where $\mathrm{SuperPoints}$ is the category consisting of supermanifolds of the form $\mathbb{R}^{0|q}$ and smooth maps between them. Of course, we have not yet said what this category is precisely, but one should think of $\mathbb{R}^{0|q}$ as a supermanifold whose underlying manifold consists of one point, with q infinitesimal superdirections—a 'superpoint'. Because this lets us probe the superdirections of M, this functor has enough information to completely reconstruct M. We will go further, however, and sketch how to define M as a certain kind of functor from $\mathrm{SuperPoints}^{\mathrm{op}}$ to Man.

This approach goes back to Schwarz [62] and Voronov [69], who used it to formalize the idea of 'anticommuting coordinates' used in the physics literature. Since Schwarz, a number of other

authors have developed the functor of points approach to supermanifolds, most recently Sachse [54] and Balduzzi, Carmeli and Fioresi [14]. We will follow Sachse, who defines supermanifolds entirely in terms of their functors of points, rather than using sheaves.

8.1 Supermanifolds

8.1.1 Super vector spaces as supermanifolds

Let us now dive into supermathematics. Our main need is to define smooth maps between super vector spaces, but we will sketch the full definition of supermanifolds and the smooth maps between them. Just as an ordinary manifold is a space that is locally modeled on a vector space, a supermanifold is locally modeled on a super vector space. Since we will define a supermanifold M as a functor

$$M \colon \mathrm{SuperPoints}^{\mathrm{op}} \to \mathrm{Man},$$

we first need to say how to think of the simplest kind of supermanifold, a super vector space V, as such a functor:

$$V \colon \mathrm{SuperPoints}^{\mathrm{op}} \to \mathrm{Man}.$$

But first we owe the reader a definition of the category of superpoints.

Recall from Section 3.1 that a **super vector space** is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ where V_0 is called the **even** part, and V_1 is called the **odd** part. There is a symmetric monoidal category SuperVect which has:

- $\mathbb{Z}_2$-graded vector spaces as objects;

- Grade-preserving linear maps as morphisms;

- A tensor product $\otimes$ that has the following grading: if $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$, then $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$;

- A braiding
  $$B_{V,W} \colon V \otimes W \to W \otimes V$$
  defined as follows: $v \in V$ and $w \in W$ are of grade $|v|$ and $|w|$, then
  $$B_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

The braiding encodes the 'the rule of signs': in any calculation, when two odd elements are interchanged, we introduce a minus sign. We write $\mathbb{R}^{p|q}$ for the super vector space with even part $\mathbb{R}^p$ and odd part $\mathbb{R}^q$.

We define a **supercommutative superalgebra** to be a commutative algebra A in the category SuperVect. More concretely, it is a real, associative algebra A with unit which is $\mathbb{Z}_2$-graded:

$$A = A_0 \oplus A_1,$$

and is graded-commutative. That is:

$$ab = (-1)^{|a||b|} ba,$$

for all homogeneous elements $a, b \in A$, as required by the rule of signs. We define a **homomorphism of superalgebras** $f \colon A \to B$ to be an algebra homomorphism that respects the grading. So, there is a category $\mathrm{SuperAlg}$ with supercommutative superalgebras as objects, and homomorphisms of superalgebras as morphisms. Henceforth, we will assume all our superalgebras to be supercommutative unless otherwise indicated.

A particularly important example of a supercommutative superalgebra is a **Grassmann algebra**: a finite-dimensional exterior algebra

$$A = \Lambda \mathbb{R}^n,$$

equipped with the grading:

$$A_0 = \Lambda^0 \mathbb{R}^n \oplus \Lambda^2 \mathbb{R}^n \oplus \cdots, \quad A_1 = \Lambda^1 \mathbb{R}^n \oplus \Lambda^3 \mathbb{R}^n \oplus \cdots.$$

Let us write GrAlg for the category with Grassmann algebras as objects and homomorphisms of superalgebras as morphisms.

In fact, the Grassmann algebras are essential for our approach to supermanifold theory, because:

$$\mathrm{GrAlg} = \mathrm{SuperPoints}^{\mathrm{op}}$$

so rather than thinking of a supermanifold M as a contravariant functor from $\mathrm{SuperPoints}$ to Man, we can view a supermanifold as a covariant functor:

$$M \colon \mathrm{GrAlg} \to \mathrm{Man}$$

To see why this is sensible, recall that a smooth map between ordinary manifolds

$$\varphi \colon M \to N$$

is the same as a homomorphism between their algebras of smooth functions which goes the other way:

$$\varphi^* \colon C^\infty(N) \to C^\infty(M)$$

By analogy, we expect something similar to hold for supermanifolds. In particular, a smooth map from a superpoint:

$$\varphi \colon \mathbb{R}^{0|q} \to M$$

ought to be to the same as a homomorphism of their 'superalgebras of smooth functions' which points the other way:

$$\varphi^* \colon C^\infty(M) \to C^\infty(\mathbb{R}^{0|q}).$$

But since $\mathbb{R}^{0|q}$ is a purely odd super vector space, we *define* its algebra of smooth functions to be $\Lambda(\mathbb{R}^q)^*$. Intuitively, this is because $\mathbb{R}^{0|q}$ is a supermanifold with q 'odd, anticommuting coordinates', given by the standard projections:

$$\theta^1, \ldots, \theta^q \colon \mathbb{R}^q \to \mathbb{R},$$

so a 'smooth function' f on $\mathbb{R}^{0|q}$ should have a 'power series expansion' that looks like:

$$f = \sum_{i_1 < i_2 < \cdots < i_k} f_{i_1 i_2 \ldots i_k} \theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_k}.$$

where the coefficients $f_{i_1 i_2 \ldots i_k}$ are real. Thus f is precisely an element of $\Lambda(\mathbb{R}^q)^*$. Thus, we *define*

$$\mathrm{Hom}(\mathbb{R}^{0|q}, M) = \mathrm{Hom}(C^\infty(M), \Lambda(\mathbb{R}^q)^*).$$

In this way, rather than thinking of M as a functor:

$$\mathrm{Hom}(-, M)\colon \quad \mathrm{SuperPoints}^{\mathrm{op}} \quad \to \quad \mathrm{Man}$$
$$\mathbb{R}^{0|q} \quad \mapsto \quad \mathrm{Hom}(\mathbb{R}^{0|q}, M)$$

where Hom is in the category of supermanifolds (though we have not defined this), we think of M as a functor:

$$\mathrm{Hom}(C^\infty(M), -)\colon \quad \mathrm{GrAlg} \quad \to \quad \mathrm{Man}$$
$$A \quad \mapsto \quad \mathrm{Hom}(C^\infty(M), A)$$

where Hom is in the category of superalgebras (which we have defined, though we have not defined $C^\infty(M)$).

Since we have just given a slew of definitions, let us bring the discussion back down to earth with a concise summary:

- Every supermanifold is a functor:

$$M\colon \mathrm{GrAlg} \to \mathrm{Man},$$

  though not every such functor is a supermanifold.

- Every smooth map of supermanifolds is a natural transformation:

$$\varphi\colon M \to N,$$

  though not every such natural transformation is a smooth map of supermanifolds.

Next, let us introduce some concise notation:

- Let us write M_A for the value of M on the Grassmann algebra A, and call this the **A-points** of M.

- Let us write $M_f\colon M_A \to M_B$ for the smooth map induced by a homomorphism $f\colon A \to B$.

- Finally, we write $\varphi_A\colon M_A \to N_A$ for the smooth map which the natural transformation φ gives between the A-points. We call φ_A a **component** of the natural transformation φ.

With this background, we can now build up the theory of supermanifolds in perfect analogy to the theory of manifolds. First, we need to say how to think of our model spaces, the super vector spaces, as supermanifolds.

Indeed, given a finite-dimensional super vector space V, the **supermanifold associated to V**, or just the **supermanifold V** to be the functor:

$$V \colon \mathrm{GrAlg} \to \mathrm{Man}$$

which takes:

- each Grassmann algebra A to the vector space:

$$V_A = (A \otimes V)_0 = A_0 \otimes V_0 \oplus A_1 \otimes V_1$$

  regarded as a manifold in the usual way;

- each homomorphism $f \colon A \to B$ of Grassmann algebras to the linear map $V_f \colon V_A \to V_B$ that is the identity on V and f on A:

$$V_f = (f \otimes 1)_0 \colon (A \otimes V)_0 \to (B \otimes V)_0.$$

  This map, being linear, is also smooth.

We take this definition because, roughly speaking, the set of A-points is the set of homomorphisms of superalgebras, $\mathrm{Hom}(C^\infty(V), A)$. By analogy with the ordinary manifold case, we expect that any such homomorphism is determined by its restriction to the 'dense subalgebra' of polynomials:

$$\mathrm{Hom}(C^\infty(V), A) \cong \mathrm{Hom}(\mathrm{Sym}(V^*), A),$$

though here we are being very rough, because we have not assumed any topology on our superalgebras, so the term 'dense subalgebra' is not meaningful. Since $\mathrm{Sym}(V^*)$ is the free supercommutative superalgebra on V^*, a homomorphism out of it is the same as a linear map of super vector spaces:

$$\mathrm{Hom}(\mathrm{Sym}(V^*), A) \cong \mathrm{Hom}(V^*, A),$$

where the first Hom is in $\mathrm{SuperAlg}$ and the second Hom is in $\mathrm{SuperVect}$. Finally, because V is finite-dimensional and linear maps of super vector spaces preserve grading, this last Hom is just:

$$\mathrm{Hom}(V^*, A) \cong V_0 \otimes A_0 \oplus V_1 \otimes A_1.$$

which, up to a change of order in the factors, is how we defined V_A. This last set is a manifold in an obvious way: it is an ordinary, finite-dimensional, real vector space. In fact, it is just the even part of the super vector space $A \otimes V$:

$$V_A = (A \otimes V)_0,$$

as we have noted in our definition.

In fact, $V_A = A_0 \otimes V_0 \oplus A_1 \otimes V_1$ is more than a mere vector space—it is an A_0-module. Moreover, given any linear map of super vector spaces:

$$L \colon V \to W$$

we get an A_0-module map between the A-points in a natural way:

$$L_A = (1 \otimes L)_0 \colon (A \otimes V)_0 \to (A \otimes W)_0.$$

Indeed, L induces a natural transformation between the supermanifold V and the supermanifold W. That is, given any homomorphism $f \colon A \to B$ of Grassmann algebras, the following square commutes:

$$\begin{array}{ccc} V_A & \xrightarrow{\;L_A\;} & W_A \\ {\scriptstyle V_f}\downarrow & & \downarrow{\scriptstyle W_f} \\ V_B & \xrightarrow[\;L_B\;]{} & W_B \end{array}$$

We therefore have a functor

$$\mathrm{SuperVect} \to \mathrm{Fun}(\mathrm{GrAlg}, \mathrm{Man})$$

which takes super vector spaces to their associated supermanifolds, and linear transformations to natural transformations between supermanifolds. For future reference, we note this fact in a proposition:

Proposition 8.1. *There is a faithful functor:*

$$\mathrm{SuperVect} \to \mathrm{Fun}(\mathrm{GrAlg}, \mathrm{Man})$$

that takes a super vector space V to the supermanifold V whose A-points are:

$$V_A = (A \otimes V)_0,$$

and takes a linear map of super vector spaces:

$$L \colon V \to W$$

to the natural transformation whose components are:

$$L_A = (1 \otimes L)_0 \colon (A \otimes V)_0 \to (A \otimes W)_0.$$

In the above, A is a Grassmann algebra and the tensor product takes place in $\mathrm{SuperVect}$.

Proof. It is easy to check that this defines a functor. Faithfulness follows from a more general result in Sachse [54], c.f. Proposition 3.1. $\square$

While this functor is faithful, it is far from full; in particular, it misses all of the 'smooth maps' between super vector spaces which do not come from a linear map. We define these additional maps now.

Infinitesimally, all smooth maps should be like a linear map $L \colon V \to W$, so given two finite-dimensional super vector spaces V and W, we define a **smooth map between super vector spaces**:

$$\varphi \colon V \to W,$$

to be a natural transformation between the supermanifolds V and W such that the derivative

$$(\varphi_A)_* \colon T_x V_A \to T_{\varphi_A(x)} W_A$$

is A_0-linear at each A-point $x \in V_A$, where the A_0-module structure on each tangent space comes from the canonical identification of a vector space with its tangent space:

$$T_x V_A \cong V_A, \quad T_{\varphi(x)} W_A \cong W_A.$$

Note that each component $\varphi_A \colon V_A \to W_A$ is smooth in the ordinary sense, because by virtue of living in the category of smooth manifolds. We say that a smooth map $\varphi_A \colon V_A \to W_A$ whose derivative is A_0-linear at each point is $\mathbf{A_0}$**-smooth** for short.

Finally, note that there is a supermanifold:

$$1 \colon \mathrm{GrAlg} \to \mathrm{Man},$$

which takes each Grassmann algebra to the one-point manifold. We call this the **one-point super-manifold**, and note that it is the supermanifold associated to the super vector space $\mathbb{R}^{0|0}$. Just as $\mathbb{R}^{0|0}$ is the terminal object in $\mathrm{SuperVect}$, 1 is the terminal object in $\mathrm{Fun}(\mathrm{GrAlg}, \mathrm{Man})$, as well as the smaller category of supermanifolds, whose definition we now describe.

8.1.2 Supermanifolds in general

The last section treated the special kind of a supermanifold of greatest interest to us: the supermanifold associated to a super vector space.

Nonetheless, we now sketch how to define a general supermanifold, M. Since M will be locally isomorphic to a super vector space V, it helps to have local pieces of V to play the same role that open subsets of $\mathbb{R}^n$ play for ordinary manifolds. So, fix a super vector space V, and let $U \subseteq V_0$ be open. The **superdomain** over U is the functor:

$$\mathcal{U} \colon \mathrm{GrAlg} \to \mathrm{Man}$$

that takes each Grassmann algebra A to

$$\mathcal{U}_A = V_{\epsilon_A}^{-1}(U)$$

where $\epsilon_A \colon A \to \mathbb{R}$ the projection of the Grassmann algebra A that kills all nilpotent elements. We say that $\mathcal{U}$ is a **superdomain in V**, and write $\mathcal{U} \subseteq V$.

If $\mathcal{U} \subseteq V$ and $\mathcal{U}' \subseteq W$ are two superdomains in super vector spaces V and W, a **smooth map of superdomains** is a natural transformation:

$$\varphi \colon \mathcal{U} \to \mathcal{U}'$$

such that for each Grassmann algebra A, the component on A-points is smooth:

$$\varphi_A \colon \mathcal{U}_A \to \mathcal{U}'_A.$$

and the derivative:
$$(\varphi_A)_* \colon T_x \mathcal{U}_A \to T_{\varphi_A(x)} \mathcal{U}'_A$$
is A_0-linear at each A-point $x \in \mathcal{U}_A$, where the A_0-module structure on each tangent space comes from the canonical identification with the ambient vector spaces:

$$T_x \mathcal{U}_A \cong V_A, \quad T_{\varphi(x)} \mathcal{U}'_A \cong W_A.$$

Again, we say that a smooth map $\varphi_A \colon \mathcal{U}_A \to \mathcal{U}'_A$ whose derivative is A_0-linear at each point is **A_0-smooth** for short.

At long last, a **supermanifold** is a functor

$$M \colon \mathrm{GrAlg} \to \mathrm{Man}$$

equipped with an atlas

$$(\mathcal{U}_\alpha, \varphi_\alpha \colon \mathcal{U} \to M),$$

where each $\mathcal{U}_\alpha$ is a superdomain, each φ_α is a natural transformation, and one can define transition functions that are smooth maps of superdomains.

Finally, a **smooth map of supermanifolds** is a natural transformation:

$$\psi \colon M \to N$$

which induces smooth maps between the superdomains in the atlases. Equivalently, each component

$$\varphi_A \colon M_A \to N_A$$

is **A_0-smooth**: it is smooth and its derivative

$$(\varphi_A)_* \colon T_x M_A \to T_{\varphi_A(x)} N_A$$

is A_0-linear at each A-point $x \in M_A$, where the A_0-module structure on each tangent space comes from the superdomains in the atlases. Thus, there is a category $\mathrm{SuperMan}$ of supermanifolds. See Sachse [54] for more details.

8.2 Supergroups from nilpotent Lie superalgebras

We now describe a procedure to integrate a nilpotent Lie superalgebra to a Lie supergroup. This is a partial generalization of Lie's Third Theorem, which describes how any Lie algebra can be integrated to a Lie group. In fact, the full theorem generalizes to Lie supergroups [67], but we do not need it here.

Recall from Section 3.2 that a **Lie superalgebra** $\mathfrak{g}$ is a Lie algebra in the category of super vector spaces. More concretely, it is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, equipped with a graded-antisymmetric bracket:

$$[-,-] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g},$$

which satisfies the Jacobi identity up to signs:
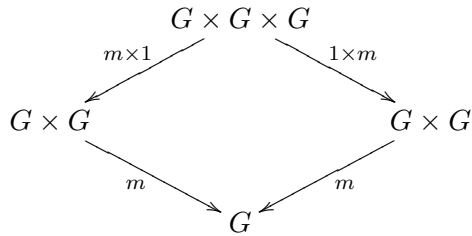
$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]].$$

for all homogeneous $X, Y, Z \in \mathfrak{g}$. A Lie superalgebra $\mathfrak{n}$ is called **k-step nilpotent** is any k nested brackets vanish, and it is called **nilpotent** if it is k-step nilpotent for some k. Nilpotent Lie superalgebras can be integrated to a unique supergroup N defined on the same underlying super vector space $\mathfrak{n}$.

A **Lie supergroup**, or **supergroup**, is a group object in the category of supermanifolds. That is, it is a supermanifold G equipped with the following maps of supermanifolds:
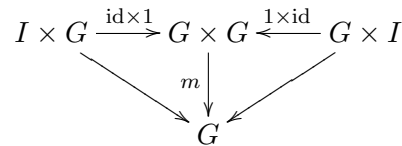
- **multiplication**, $m \colon G \times G \to G$;

- **inverse**, $\mathrm{inv} \colon G \to G$;

- **identity**, $\mathrm{id} \colon 1 \to G$, where 1 is the one-point supermanifold;

such that the following diagrams commute, encoding the usual group axioms:

- the associative law:

$$
\begin{array}{ccc}
& G \times G \times G & \\
{\scriptstyle m \times 1} \swarrow & & \searrow {\scriptstyle 1 \times m} \\
G \times G & & G \times G \\
{\scriptstyle m} \searrow & & \swarrow {\scriptstyle m} \\
& G &
\end{array}
$$

- the right and left unit laws:

$$
\begin{array}{ccccc}
I \times G & \xrightarrow{\ \mathrm{id} \times 1\ } & G \times G & \xleftarrow{\ 1 \times \mathrm{id}\ } & G \times I \\
& \searrow & \downarrow {\scriptstyle m} & \swarrow & \\
& & G & &
\end{array}
$$

- the right and left inverse laws:

$$
\begin{array}{cc}
G \times G \xrightarrow{\ 1 \times \mathrm{inv}\ } G \times G \\
{\scriptstyle \Delta} \nearrow \qquad\qquad \searrow {\scriptstyle m} \\
G \qquad\qquad\qquad\quad G \\
\searrow \qquad \nearrow {\scriptstyle \mathrm{id}} \\
1
\end{array}
\qquad
\begin{array}{cc}
G \times G \xrightarrow{\ \mathrm{inv} \times 1\ } G \times G \\
{\scriptstyle \Delta} \nearrow \qquad\qquad \searrow {\scriptstyle m} \\
G \qquad\qquad\qquad\quad G \\
\searrow \qquad \nearrow {\scriptstyle \mathrm{id}} \\
1
\end{array}
$$
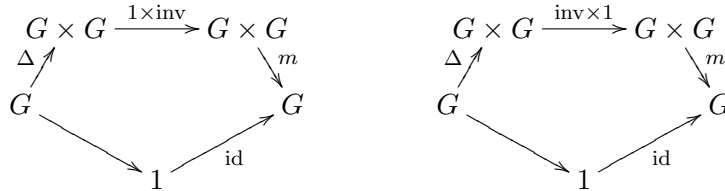
where $\Delta \colon G \to G \times G$ is the diagonal map. In addition, a supergroup is **abelian** if the following diagram commutes:

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\ \tau\ } & G \times G \\
{\scriptstyle m} \searrow & & \downarrow {\scriptstyle m} \\
& & G
\end{array}
$$
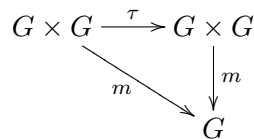
where $\tau\colon G \times G \to G \times G$ is the **twist map**. Using A-points, it is defined to be:

$$\tau_A(x, y) = (y, x),$$

for $(x, y) \in G_A \times G_A$.

Examples of supergroups arise easily from Lie groups: if G is an Lie group, it is also a Lie group defined on the supermanifold whose A-points are:

$$G_A = \mathrm{Hom}(C^\infty(G), A),$$

where $C^\infty(G)$ is the ordinary algebra of smooth functions on G, regarded as a purely even superalgebra. In this way, any classical Lie group, such as $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, becomes a supergroup.

To obtain more interesting examples, we will integrate a nilpotent Lie superalgebra, $\mathfrak{n}$ to a supergroup N. For any Grassmann algebra A, the bracket

$$[-,-]\colon \Lambda^2\mathfrak{n} \to \mathfrak{n}$$

induces an A_0-linear map between the A-points:

$$[-,-]_A\colon \Lambda^2\mathfrak{n}_A \to \mathfrak{n}_A,$$

where $\Lambda^2\mathfrak{n}_A$ denotes the exterior square of the A_0-module $\mathfrak{n}_A$. Thus $[-,-]_A$ is antisymmetric, and it easy to check that it makes $\mathfrak{n}_A$ into a Lie algebra which is also nilpotent.

On each such A_0-module $\mathfrak{n}_A$, we can thus define a Lie group N_A where the multplication is given by the Baker–Campbell–Hausdorff formula, inversion by negation, and the identity is 0. Because we want to write the group N_A multiplicatively, we write $\exp_A\colon \mathfrak{n}_A \to N_A$ for the identity map, and then define the multiplication, inverse and identity maps:

$$m_A\colon N_A \times N_A \to N_A, \quad \mathrm{inv}_A\colon N_A \to N_A, \quad \mathrm{id}_A\colon 1_A \to N_A,$$

as follows:

$$m_A(\exp_A(X), \exp_A(Y)) = \exp_A(X)\exp_A(Y) = \exp_A(X + Y + \frac{1}{2}[X, Y]_A + \cdots)$$

$$\mathrm{inv}_A(\exp_A(X)) = \exp_A(-X) = \exp_A(X)^{-1},$$

$$\mathrm{id}_A(1) = \exp_A(0) = 1,$$

for any A-points $X, Y \in \mathfrak{n}_A$, and the first 1 in the last equation refers to the single element of 1_A. But it is clear that all of these maps are natural in A. Furthermore, they are all A_0-smooth, because as polynomials with coefficients in A_0, they are smooth with derivatives that are A_0-linear. They thus define smooth maps of supermanifolds:

$$m\colon N \times N \to N, \quad \mathrm{inv}\colon N \to N, \quad \mathrm{id}\colon 1 \to N,$$

where N is the supermanifold $\mathfrak{n}$. And because each of the N_A is a group, N is a supergroup. We have thus proved:

Proposition 8.2. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra. Then there is a supergroup N defined on the supermanifold $\mathfrak{n}$, obtained by integrating the nilpotent Lie algebra $\mathfrak{n}_A$ with the Baker–Campbell–Hausdorff formula for all Grassmann algebras A. More precisely, we define the maps:*

$$m\colon N \times N \to N, \quad \mathrm{inv}\colon N \to N, \quad \mathrm{id}\colon 1 \to N,$$

by defining them on A-points as follows:

$$m_A(\exp_A(X), \exp_A(Y)) = \exp_A(Z(X, Y)),$$

$$\mathrm{inv}_A(\exp_A(X)) = \exp_A(-X),$$

$$\mathrm{id}_A(1) = \exp_A(0),$$

where

$$\exp\colon \mathfrak{n} \to N$$

is the identity map of supermanifolds, and:

$$Z(X, Y) = X + Y + \frac{1}{2}[X, Y]_A + \cdots$$

denotes the Baker–Campbell–Hausdorff series on $\mathfrak{n}_A$, which terminates because $\mathfrak{n}_A$ is nilpotent.

Experience with ordinary Lie theory suggests that, in general, there will be more than one supergroup which has Lie superalgebra $\mathfrak{n}$. To distinguish the one above, we call N the **exponential supergroup** of $\mathfrak{n}$.

Chapter 9

Lie n-supergroups from supergroup cohomology

We saw in Chapter 6 that 3-cocycles in Lie group cohomology allow us to construct Lie 2-groups. We now generalize this to supergroups. The most significant barrier is that we now work internally to the category of supermanifolds instead of the much more familiar category of smooth manifolds. Our task is to show that this change of categories does not present a problem. The main obstacle is that the category of supermanifolds is not a concrete category: morphisms are determined not by their value on the underlying set of a supermanifold, but by their value on A-points for all Grassmann algebras A.

The most common approach is to define morphisms without reference to elements, and to define equations between morphisms using commutative diagrams. This is how we gave the definition of smooth bicategory, except that we found it convenient to state the pentagon and triangle identities using elements. As an alternative to commutative diagrams, for supermanifolds, one can use A-points to define morphisms and specify equations between them. This tends to make equations look friendlier, because they look like equations between functions. We shall use this approach.

First, let us define the cohomology of a supergroup G with coefficients in an abelian supergroup H, on which G **acts by automorphism**. This means that we have a morphism of supermanifolds:

$$\alpha \colon G \times H \to H,$$

which, for any Grassmann algebra A, induces an action of the group G_A on the abelian group H_A:

$$\alpha_A \colon G_A \times H_A \to H_A.$$

For this action to be by automorphism, we require:

$$\alpha_A(g)(h + h') = \alpha_A(g)(h) + \alpha_A(g)(h'),$$

for all A-points $g \in G_A$ and $h, h' \in H_A$.

We define supergroup cohomology using **the supergroup cochain complex**, $C^\bullet(G, H)$, which at level p just consists of the set of maps from G^p to H as supermanifolds:

$$C^p(G, H) = \{f \colon G^p \to H\}.$$

Addition on H makes $C^p(G, H)$ into an abelian group for all p. The differential is given by the usual formula, but using A-points:

$$
\begin{aligned}
df_A(g_1, \ldots, g_{p+1}) \;=\; & g_1 f_A(g_2, \ldots, g_{p+1}) \\
& + \sum_{i=1}^{p} (-1)^i f_A(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) \\
& + (-1)^{p+1} f_A(g_1, \ldots, g_p),
\end{aligned}
$$

where $g_1, \ldots, g_{p+1} \in G_A$ and the action of g_1 is given by α_A. Noting that f_A, α_A, multiplication and $+$ are all:

- natural in A;

- A_0-smooth: smooth with derivatives which are A_0-linear;

we see that df_A is:

- natural in A;

- A_0-smooth: smooth with a derivative which is A_0-linear;

so it indeed defines a map of supermanifolds:

$$
df \colon G^{p+1} \to H.
$$

Furthermore, it is immediate that:
$$
d^2 f_A = 0
$$
for all A, and thus
$$
d^2 f = 0.
$$

So $C^\bullet(G, H)$ is truly a cochain complex. Its cohomology $H^\bullet(G, H)$ is the **supergroup cohomology of G with coefficients in H**. Of course, if $df = 0$, f is called a **cocycle**, and f is **normalized** if
$$
f_A(g_1, \ldots, g_p) = 0
$$
for any Grassmann algebra A, whenever one of the A-points $g_1, \ldots, g_p$ is 1. When $H = \mathbb{R}$, we omit reference to it, and write $C^\bullet(G, \mathbb{R})$ as $C^\bullet(G)$.

A **super bicategory** B has

- a **supermanifold of objects** B_0;

- a **supermanifold of morphisms** B_1;

- a **supermanifold of 2-morphisms** B_2;

equipped with maps of supermanifolds as described in Definition 6.4: source, target, identity-assigning, horizontal composition, vertical composition, associator and left and right unitors all maps of supermanifolds, and satisfying the same axioms as a smooth bicategory. The associator

satisfies the pentagon identity, which we state in terms of A-points: the following pentagon commutes:

$$
\begin{array}{ccc}
& (fg)(hk) & \\
{}^{a(fg,h,k)}\nearrow & & \searrow{}^{a(f,g,hk)} \\
((fg)h)k & & f(g(hk)) \\
{}_{a(f,g,h)\cdot 1_k}\searrow & & \nearrow{}_{1_f\cdot a(g,h,k)} \\
(f(gh))k & \xrightarrow{\ a(f,gh,k)\ } & f((gh)k)
\end{array}
$$

for any 'composable quadruple of morphisms':

$$(f, g, h, k) \in (B_1 \times_{B_0} B_1 \times_{B_0} B_1 \times_{B_0} B_1)_A.$$

Similarly, the associator and left and right unitors satisfy the triangle identity, which we state in terms of A-points: the following triangle commutes:

$$
\begin{array}{ccc}
(f1)g & \xrightarrow{\ a(f,1,g)\ } & f(1g) \\
{}_{r(f)\cdot 1_g}\searrow & & \swarrow{}_{1_f\cdot l(g)} \\
& fg &
\end{array}
$$

for any 'composable pair of morphisms':

$$(f, g) \in (B_1 \times_{B_0} B_1)_A.$$

A **2-supergroup** is a super bicategory with one object (more precisely, the one-point supermanifold), and all morphisms and 2-morphisms weakly invertible. Given a normalized H-valued 3-cocycle a on G, we can construct a 2-supergroup $\text{String}_a(G, H)$ in the same way we constructed the Lie 2-group $\text{String}_a(G, H)$ when G and H were Lie groups, by just deleting every reference to elements of G or H:

- The supermanifold of objects is the one-point supermanifold, 1.

- The supermanifold of morphisms is the supergroup G, with composition given by the multiplication:
$$\cdot \colon G \times G \to G.$$

The source and target maps are the unique maps to the one-point supermanifold. The identity-assigning map is the identity-assigning map for G:

$$\text{id} \colon 1 \to G.$$

- The supermanifold of 2-morphisms is $G \times H$. The source and target maps are both the projection map to G. The identity assigning map comes from the identity-assigning map for H:

$$1 \times \mathrm{id} \colon G \times 1 \to G \times H.$$

- Vertical composition of 2-morphisms is given by addition in H:

$$1 \times + \colon G \times H \times H \to G \times H,$$

where we have used the fact that the pullback of 2-morphisms over objects is trivially:

$$(G \times H) \times_1 (G \times H) \cong G \times H \times H.$$

Horizontal composition, $\cdot$, given by the multiplication on the semidirect product:

$$\cdot \colon (G \ltimes H) \times (G \ltimes H) \to G \ltimes H.$$

- The left and right unitors are trivial.

- The associator is given by the 3-cocycle $a \colon G^3 \to H$, where the source (and target) is understood to come from multiplication on G.

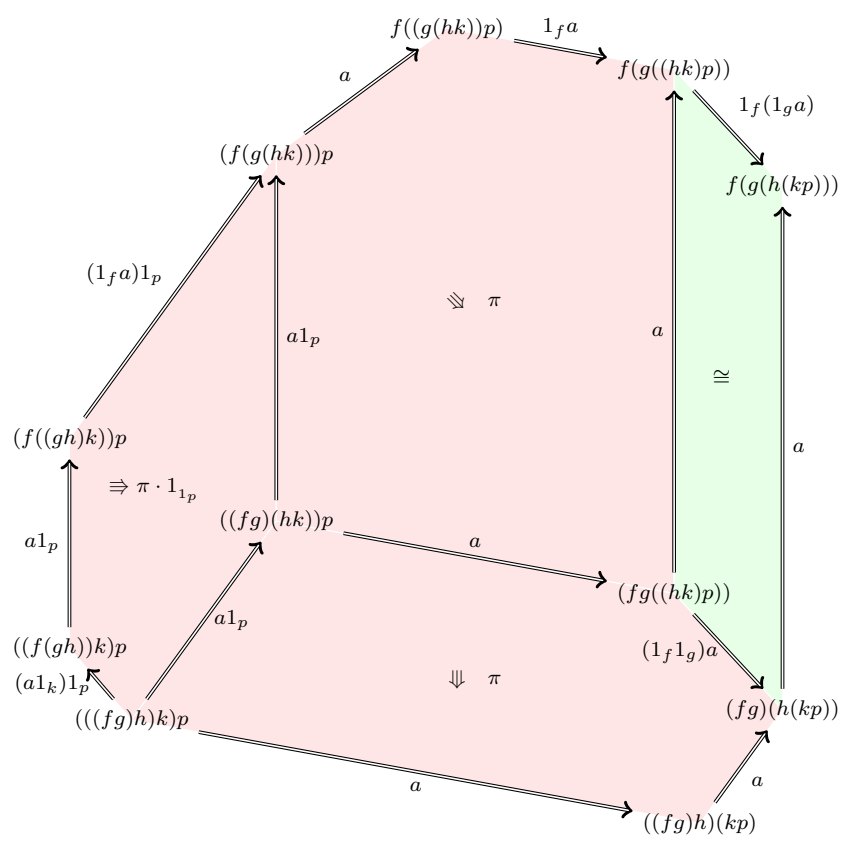A **slim 2-supergroup** is one of this form. It remains to check that it is, indeed, a 2-supergroup.

Proposition 9.1. $\mathrm{String}_a(G, H)$ *is a 2-supergroup: a super bicategory with one object and all morphisms and 2-morphisms weakly invertible.*

Proof. This proof is a duplicate of the proof of Proposition 6.1, but with A-points instead of elements. $\qquad\square$

In a similar way, we can generalize our construction of Lie 3-groups to '3-supergroups'. A **super tricategory** T has

- a **supermanifold of objects** T_0;

- a **supermanifold of morphisms** T_1;

- a **supermanifold of 2-morphisms** T_2;

- a **supermanifold of 3-morphisms** T_3;

equipped with maps of supermanifolds as described in Definition 6.5: source, target, identity-assigning, composition at 0-cells, 1-cells and 2-cells, associator and left and right unitors, pentagonator and triangulators all maps of supermanifolds, and satisfying the same axioms as a smooth tricategory. As in the case of the pentagon identity above, we express the pentagonator identity in terms of A-points: the following equation holds:

$$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$$

a $1_f(1_g a)$

$(f(g(hk)))p$ $f(g(h(kp)))$

$(1_f a)1_p$ $\Downarrow\ \pi$ a $\cong$

$a1_p$ a

$(f((gh)k))p$ a

$\Rrightarrow \pi \cdot 1_{1_p}$ $((fg)(hk))p \xrightarrow{\quad a \quad} (fg((hk)p))$ a

$a1_p$ $a1_p$ $(1_f 1_g)a$

$((f(gh))k)p$ $(fg)(h(kp))$

$(a1_k)1_p$ $\Downarrow\ \pi$

$(((fg)h)k)p \xrightarrow{\quad a \quad} ((fg)h)(kp)$ a

$=$

$$f((g(hk))p) \xrightarrow{1_f a} f(g((hk)p))$$

a $1_f(1_g a)$

$(f(g(hk)))p$ $f(g(h(kp)))$

$\cong$ $1_f(a1_p)$ $\Downarrow\ 1_{1_f}\cdot\pi$

$(1_f a)1_p$ $1_f a$

$f(((gh)k)p) \xrightarrow{\quad 1_f a \quad} f((gh)(kp))$ $1_f a$

a

$(f((gh)k))p$ a

$a1_p$ $\Downarrow\ \pi$ a $\Downarrow\ \pi$

$((f(gh))k)p$ $(fg)(h(kp))$

$(a1_k)1_p$ a $(f(gh))(kp)$ a

$(((fg)h)k)p \xrightarrow{\quad a \quad} (f(gh))(kp)$ $\cong$

a $a(1_k 1_p)$ $((fg)h)(kp)$
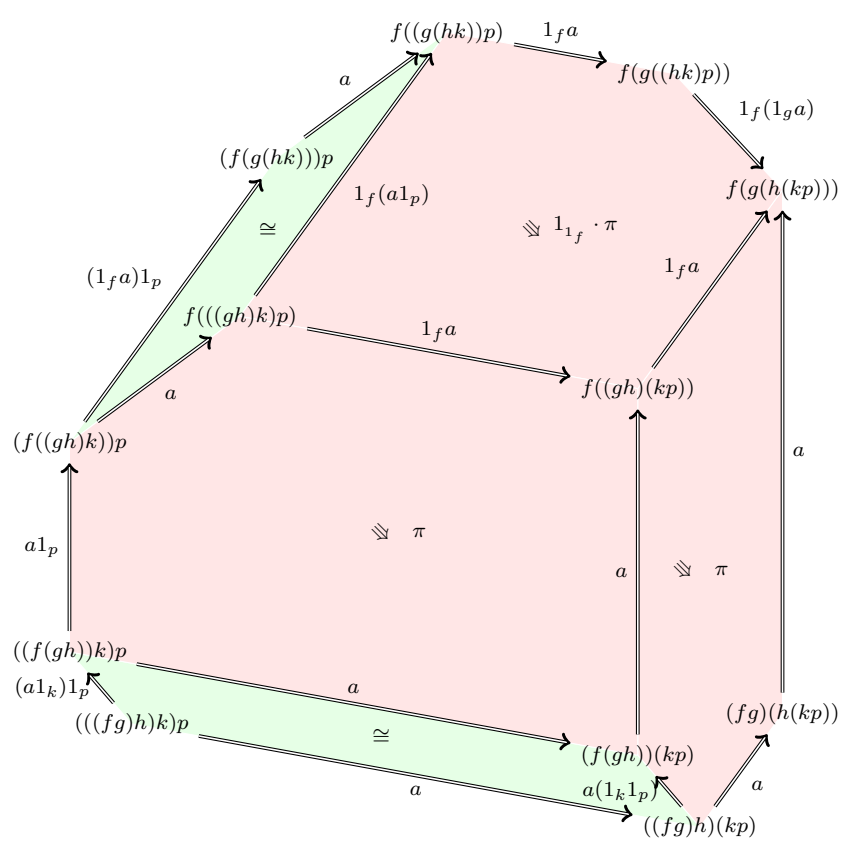
for any 'composable quintet of morphisms':

$$(f, g, h, k, p) \in (T_1 \times_{T_0} T_1 \times_{T_0} T_1 \times_{T_0} T_1 \times_{T_0} T_1)_A$$

A **3-supergroup** is a super tricategory with one object (more precisely, the one-point superman-ifold) and all morphisms, 2-morphisms and 3-morphisms weakly invertible. Given a normalized H-valued 4-cocycle π on G, we can construct a 3-supergroup $\mathrm{Brane}_\pi(G, H)$ in the same way we constructed the Lie 3-group $\mathrm{Brane}_\pi(G, H)$ when G and H were Lie groups, but deleting every reference to elements of G or H:

- The supermanifold of objects is the one-point supermanifold, 1.

- The supermanifold of morphisms is the supergroup G. Composition at a 0-cell is given by multiplication in the group:
  $$\cdot\colon G \times G \to G.$$
  The source and target maps are the unique maps to 1. The identity-assigning map is the identity-assigning map for G:
  $$\mathrm{id}\colon 1 \to G.$$

- The supermanifold of 2-morphisms is again G. The source, target and identity-assigning maps are all the identity on G. Composition at a 1-cell is the identity on G, while composition at a 0-cell is again multiplication in G. This encodes the idea that all 2-morphisms are trivial.

- The supermanifold of 3-morphisms is $G \times H$. The source and target maps are projection onto G. The identity-assigning map is the inclusion:
  $$G \to G \times H$$
  that takes A-points $g \in G_A$ to $(g, 0) \in G_A \times H_A$, for all A.

- Three kinds of composition of 3-morphisms: composition at a 2-cell and at a 3-cell are both given by addition on H:
  $$1 \times +\colon G \times H \times H \to G \times H.$$
  While composition at a 0-cell is just given by multiplication on the semidirect product:
  $$\cdot\colon (G \ltimes H) \times (G \ltimes H) \to G \ltimes H.$$

- The associator, left and right unitors are automatically trivial, because all 2-morphisms are trivial.

- The triangulators are trivial.

- The **2-associator** or **pentagonator** is given by the 4-cocycle $\pi\colon G^4 \to H$, where the source (and target) is understood to come from multiplication on G.

A **slim 3-supergroup** is one of this form. It remains to check that it is, indeed, a 3-supergroup.

Proposition 9.2. $\mathrm{Brane}_a(G, H)$ *is a 3-supergroup: a super tricategory with one object and all morphisms, 2-morphisms and 3-morphisms weakly invertible.*

Proof. This proof is a duplicate of the proof of Proposition 6.2, but with A-points instead of elements. □

Chapter 10

Integrating nilpotent Lie n-superalgebras

We now generalize our technique for integrating cocycles from nilpotent Lie algebras to nilpotent Lie *superalgebras*. Those familiar with supermanifold theory may find it surprising that this is possible—the theory of differential forms is very different for supermanifolds than for manifolds, and integrating differential forms on a manifold was crucial to our method in Section 7.1. But we can sidestep this issue on a supergroup N by considering A-points for any Grassmann algebra A. Then N_A is a manifold, so the usual theory of differential forms applies.

Here is how we will proceed. Fixing a nilpotent Lie superalgebra $\mathfrak{n}$ with exponential supergroup N, we can use Proposition 8.1 turn any Lie superalgebra cochain ω on $\mathfrak{n}$ into a Lie algebra cochain ω_A on $\mathfrak{n}_A$. We then use the techniques in Section 7.1 to turn ω_A into a Lie group cochain $\int \omega_A$ on N_A. Checking that $\int \omega_A$ is natural in A and A_0-smooth, this defines a supergroup cochain $\int \omega$ on N.

As we saw in Proposition 8.1, any map of super vector spaces becomes an A_0-linear map on A-points. We have already touched on the way this interacts with symmetry: for a Lie superalgebra $\mathfrak{g}$, the graded-antisymmetric bracket

$$[-,-] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g}$$

becomes an honest antisymmetric bracket on A-points:

$$[-,-]_A \colon \Lambda^2 \mathfrak{g}_A \to \mathfrak{g}_A.$$

More generally, we have:

Lemma 10.1. *Graded-symmetric maps of super vector spaces:*

$$f \colon \mathrm{Sym}^p V \to W$$

induce symmetric maps on A-points:

$$f_A \colon \mathrm{Sym}^p V_A \to W_A,$$

defined by:
$$f_A(a_1 v_1, \ldots, a_p v_p) = a_p \cdots a_1 f(v_1, \ldots, v_p),$$

where $\mathrm{Sym}^p V_A$ *is the symmetric power of* V_A *as an* A_0*-module and* $a_i \in A$, $v_i \in V$ *are of matching parity. Similarly, graded-antisymmetric maps of super vector spaces:*

$$f \colon \Lambda^p V \to W$$

induce antisymmetric maps on A*-points:*

$$f_A \colon \Lambda^p V_A \to W_A,$$

defined by:

$$f_A(a_1 v_1, \ldots, a_p v_p) = a_p \cdots a_1 f(v_1, \ldots, v_p),$$

where $\Lambda^p V_A$ *is the exterior power* V_A *as an* A_0*-module and* $a_i \in A$, $v_i \in V$ *are of matching parity.*

Proof. This is straightforward and we leave it to the reader. $\qquad\square$

Next, we need to show that Lie superalgebra cochains ω on $\mathfrak{n}$ give rise to Lie algebra cochains ω_A on the A-points $\mathfrak{n}_A$. In fact, this works for any Lie superalgebra, but there is one twist: because $\mathfrak{n}_A$ is an A_0-module, $\omega \colon \Lambda^p \mathfrak{n} \to \mathbb{R}$ gives rise to an A_0-linear map:

$$\omega_A \colon \Lambda^p \mathfrak{n}_A \to A_0,$$

using the fact that $\mathbb{R}_A = A_0$. So, we need to say how to do Lie algebra cohomology with coefficients in A_0. It is just a straightforward generalization of cohomology with coefficients in $\mathbb{R}$.

Indeed, any Lie superalgebra $\mathfrak{g}$ induces a Lie algebra structure on $\mathfrak{g}_A$ where the bracket is A_0-bilinear. We say that $\mathfrak{g}_A$ is an **A_0-Lie algebra**. Given any A_0-Lie algebra $\mathfrak{g}_A$, we define its cohomology with the **A_0-Lie algebra cochain complex**, which at level p consists of antisymmetric A_0-multilinear maps:

$$C^p(\mathfrak{g}_A) = \{\omega \colon \Lambda^p \mathfrak{g}_A \to A_0\}.$$

We define d on this complex in exactly the same way we define d for $\mathbb{R}$-valued Lie algebra cochains. This makes $C^\bullet(\mathfrak{g}_A)$ into a cochain complex, and the **cohomology of an A_0-Lie algebra with coefficients in A_0** is the cohomology of this complex.

Proposition 10.1. *Let* $\mathfrak{g}$ *be a Lie superalgebra, and let* $\mathfrak{g}_A$ *be the* A_0*-Lie algebra of its A-points. Then there is a cochain map:*
$$C^\bullet(\mathfrak{g}) \to C^\bullet(\mathfrak{g}_A)$$

given by taking the p-cochain ω
$$\omega \colon \Lambda^p \mathfrak{g} \to \mathbb{R}$$

to the induced A_0*-linear map* ω_A:
$$\omega_A \colon \Lambda^p \mathfrak{g}_A \to A_0,$$

where $\Lambda^p \mathfrak{g}_A$ *denotes the pth exterior power of* $\mathfrak{g}_A$ *as an* A_0*-module.*

Proof. We need to show:
$$d(\omega_A) = (d\omega)_A.$$

Since these are both linear maps on $\Lambda^{p+1}(\mathfrak{g}_A)$, it suffices to check that they agree on generators, which are of the form:

$$a_1 X_1 \wedge a_2 X_2 \wedge \cdots \wedge a_{p+1} X_{p+1}$$

for $a_i \in A$ and $X_i \in \mathfrak{g}$ of matching parity. By definition:

$$(d\omega)_A(a_1 X_1 \wedge a_2 X_2 \wedge \cdots \wedge a_{p+1} X_{p+1}) = a_{p+1} a_p \cdots a_1 d\omega(X_1 \wedge X_2 \wedge \cdots \wedge X_{p+1}).$$

On the other hand, to compute $d(\omega_A)$, we need to apply the formula for d to obtain the intimidating expression:

$$
\begin{aligned}
&d(\omega_A)(a_1 X_1, \ldots, a_{p+1} X_{p+1}) \\
&= \sum_{i<j} (-1)^{i+j} \omega_A([a_i X_i, a_j X_j]_A, a_1 X_1, \ldots, \widehat{a_i X_i}, \ldots, \widehat{a_j X_j}, \ldots, a_{p+1} X_{p+1}) \\
&= \sum_{i<j} (-1)^{i+j} a_{p+1} \cdots \hat{a}_j \cdots \hat{a}_i \cdots a_1 a_j a_i \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).
\end{aligned}
$$

If we reorder the each of the coefficients $a_{p+1} \cdots \hat{a}_j \cdots \hat{a}_i \cdots a_1 a_j a_i$ to $a_{p+1} \cdots a_2 a_1$ at the cost of introducing still more signs, we can factor all of the a_is out of the summation to obtain:

$$
\begin{aligned}
&a_{p+1} \cdots a_2 a_1 \\
&\times \sum_{i<j} (-1)^{i+j} (-1)^{|X_i||X_j|} \epsilon_1^{i-1}(i) \epsilon_1^{j-1}(j) \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}) \\
&= a_{p+1} \cdots a_2 a_1 d\omega(X_1 \wedge X_2 \wedge \cdots \wedge X_{p+1}).
\end{aligned}
$$

Note that the first two lines are a single quantity, the product of $a_{p+1} \cdots a_1$ and a large summation. The last line is $(d\omega)_A(a_1 X_1 \wedge \cdots \wedge a_{p+1} X_{p+1})$, as desired. $\quad\square$

This proposition says that from any Lie superalgebra cocycle on $\mathfrak{n}$ we obtain a Lie algebra cocycle on $\mathfrak{n}_A$, albeit now valued in A_0. Since N_A is an exponential Lie group with Lie algebra $\mathfrak{n}_A$, we can apply the techniques we developed in Section 7.1 to integrate ω_A to a group cocycle, $\int \omega_A$, on N_A.

First, however, we must pause to give some preliminary definitions concerning calculus on N_A, which is diffeomorphic to the A_0-module $\mathfrak{n}_A$. Recall from Section 8.1 that a map

$$\varphi \colon V \to W$$

between two A_0-modules said to be $\boldsymbol{A_0}$**-smooth** if it is smooth in the ordinary sense and its derivative

$$\varphi_* \colon T_x V \to T_{\varphi(x)} W$$

is A_0-linear at each point $x \in V$. Here, the A_0-module structure on each tangent space comes from the canonical identification with the ambient vector space:

$$T_x V \cong V, \quad T_{\varphi(x)} W \cong W.$$

It is clear that the identity is A_0-smooth and the composite of any two A_0-smooth maps is A_0-smooth. A vector field X on V is $\boldsymbol{A_0}$**-smooth** if Xf is an A_0-smooth function for all $f\colon V \to A_0$ that are A_0-smooth. An A_0-valued differential p-form ω on V is $\boldsymbol{A_0}$**-smooth** if $\omega(X_1, \ldots, X_p)$ is an A_0-smooth function for all A_0-smooth vector fields $X_1, \ldots, X_p$.

Now, we return to integrating ω. As a first step, because $\mathfrak{n}_A = T_1 N_A$, we can view ω_A as an A_0-valued p-form on $T_1 N_A$. Using left translation, we can extend this to a left-invariant A_0-valued p-form on N_A. Indeed, we can do this for any A_0-valued p-cochain on $\mathfrak{n}_A$:

$$C^p(\mathfrak{n}_A) \cong \{\text{left-invariant } A_0\text{-valued } p\text{-forms on } N_A\}.$$

Note that any left-invariant A_0-valued form on N_A is automatically A_0-smooth: multiplication on N_A is given the A_0-smooth operation induced from multiplication on N, and so left translation on N_A is A_0-smooth. We can differentiate and integrate A_0-valued p-forms in just the same way as we would real-valued p-forms, and the de Rham differential d of left-invariant p-forms coincides with the usual differential of Lie algebra p-cochains.

As before, we need a notion of simplices in N. Since N is a supermanifold, the vertices of a simplex should not be points of N, but rather A-points for arbitrary Grassmann algebras A. This means that for any $(p+1)$-tuple of A-points, we want to get a p-simplex:

$$[n_0, n_1, \ldots, n_p]\colon \Delta^p \to N_A,$$

where, once again, Δ^p is the standard p-simplex in $\mathbb{R}^{p+1}$, and this map is required to be smooth. But this only defines a p-simplex in N_A. To really get our hands on a p-simplex in N, we need it to depend functorially on the choice of Grassmann algebra A we use to probe N. So if $f\colon A \to B$ is a homomorphism between Grassmann algebras and $N_f\colon N_A \to N_B$ is the induced map between A-points and B-points, we require:

$$N_f \circ [n_0, n_1, \ldots, n_p] = [N_f(n_0), N_f(n_1), \ldots, N_f(n_p)]$$

Thus given a collection of maps:

$$(\varphi_p)_A\colon \Delta^p \times (N_A)^{p+1} \to N_A$$

for all A and $p \geq 0$, we say this collection defines a **left-invariant notion of simplices** in N if

- each $(\varphi_p)_A$ is smooth, and for each $x \in \Delta^p$, the restriction:

$$(\varphi_p)_A\colon \{x\} \times N_A^{p+1} \to N_A$$

  is A_0-smooth;

- it defines a left-invariant notion of simplices in N_A for each A, as in Definition 7.1;

- the following diagram commutes for all homomorphisms $f \colon A \to B$:

$$
\begin{array}{ccc}
\Delta^p \times N_A^{p+1} & \xrightarrow{(\varphi_p)_A} & N_A \\
{\scriptstyle 1 \times N_f^{p+1}} \downarrow & & \downarrow {\scriptstyle N_f} \\
\Delta^p \times N_B^{p+1} & \xrightarrow[(\varphi_p)_B]{} & N_B
\end{array}
$$

We can use a left-invariant notion of simplices to define a cochain map $\int \colon C^\bullet(\mathfrak{n}) \to C^\bullet(N)$:

Proposition 10.2. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra, and let N be the exponential supergroup which integrates $\mathfrak{n}$. If N is equipped with a left-invariant notion of simplices, then there is a cochain map:*

$$
\int \colon C^\bullet(\mathfrak{n}) \to C^\bullet(N)
$$

which sends the Lie superalgebra p-cochain ω to the supergroup p-cochain $\int \omega$, given on A-points by:

$$
\left(\int \omega \right)_A (n_1, \ldots, n_p) = \int_{[1, n_1, n_1 n_2, \ldots, n_1 n_2 \ldots n_p]} \omega_A
$$

for $n_1, \ldots, n_p \in N_A$.

Proof. First, we must check that $\int \omega_A \colon N_A^p \to A_0$ is natural in A and A_0-smooth, and hence defines a map of supermanifolds:

$$
\int \omega \colon N^p \to \mathbb{R}.
$$

Smoothness is clear, so we check naturality and the A_0-linearity of the derivative.

To check naturality, let $f \colon A \to B$ be a homomorphism, and $N_f \colon N_A \to N_B$ be the induced map from A-points to B-points. We wish to show the following square commutes:

$$
\begin{array}{ccc}
N_A^p & \xrightarrow{\int \omega_A} & A_0 \\
{\scriptstyle N_f^p} \downarrow & & \downarrow {\scriptstyle f_0} \\
N_B^p & \xrightarrow[\int \omega_B]{} & B_0
\end{array}
$$

For A-points $n_1, \ldots, n_p \in N_A$, we have:

$$
f_0 \int_{[1, n_1, n_1 n_2, \ldots, n_1 n_2 \ldots n_p]} \omega_A = \int_{[1, n_1, n_1 n_2, \ldots, n_1 \ldots n_p]} f_0 \omega_A.
$$

Since $\omega_A \colon \Lambda^p \mathfrak{n}_A \to A_0$ is natural itself, we have:

$$
f_0 \omega_A(X_1, \ldots, X_p) = \omega_B(\mathfrak{n}_f(X_1), \ldots, \mathfrak{n}_f(X_p)),
$$

for all $X_1, \ldots, X_p \in \mathfrak{n}_A$. Now, under the identification $\mathfrak{n}_A \cong T_1 N_A$, the linear map:

$$
\mathfrak{n}_f \colon \mathfrak{n}_A \to \mathfrak{n}_B
$$

129

is the derivative of the linear map $N_f \colon N_A \to N_B$, so we get the pullback of ω_A along N_f:

$$\omega_B(\mathfrak{n}_f(X_1), \ldots, \mathfrak{n}_f(X_p)) = \omega_B((N_f)_*(X_1), \ldots, (N_f)_*(X_p)) = N_f^* \omega_B(X_1, \ldots, X_p).$$

Finally:

$$
\begin{aligned}
f_0 \int_{[1, n_1, n_1 n_2, \ldots, n_1 n_2 \ldots n_p]} \omega_A &= \int_{[1, n_1, n_1 n_2, \ldots, n_1 n_2 \ldots n_p]} N_f^* \omega_B \\
&= \int_{N_f \circ [1, n_1, n_1 n_2, \ldots, n_1 n_2 \ldots n_p]} \omega_B \\
&= \int_{[1, N_f(n_1), N_f(n_1) N_f(n_2), \ldots, N_f(n_1) N_f(n_2) \ldots N_f(n_p)]} \omega_B
\end{aligned}
$$

where in the last step we have used the fact that $[1, n_1, n_1 n_2, \ldots, n_1 \ldots n_p]$ is a left-invariant simplex in N, as well as the fact that N_f is a group homomorphism. But this says exactly that $\int \omega_A$ is natural in A.

Next, we check that $\int \omega_A$ has a derivative that is A_0-linear. Briefly, this holds because the derivative of $(\varphi_p)_A$ with respect to N_A is A_0-linear. The A_0-linearity of the derivative of $\int \omega_A$ then follows from the elementary analytic fact that integration with respect to one variable and differentiation respect to another commute with each other, at least when the integration is performed over a compact set.

In detail, let us write ψ for the function $\int \omega_A \colon N_A^p \to A_0$. Let $v \in T_n N_A^p$ be a tangent vector, and let $a \in A_0$. Tedious as it may seem, we will show directly that the derivative of ψ at n is A_0-linear by computing its value on av. Denoting the derivative of ψ at n by ψ_*, we want to show that:

$$\psi_*(av) = a\psi_*(v).$$

Take γ to be a path through n with tangent v:

$$\gamma(0) = n, \quad \dot{\gamma}(0) = v,$$

and δ to be a path through n with tangent av:

$$\delta(0) = n, \quad \dot{\delta}(0) = av,$$

With wish to show that:

$$\frac{d}{dt} \psi(\delta(t)) \mid_{t=0} = a \frac{d}{dt} \psi(\gamma(t)) \mid_{t=0}.$$

Now, by definition,

$$\psi(n) = \int_{\Delta^p} g(x, n) \, dx,$$

where $n \in N_A^p$, and g denotes the pullback of ω_A along the function:

$$
\begin{aligned}
\Delta^p \times N_A^p &\to & N_A \\
(x, n_1, \ldots, n_p) &\mapsto & (\varphi_p)_A(x, 1, n_1, n_1 n_2, \ldots, n_1 n_2 \cdots n_p),
\end{aligned}
$$

where $x \in \Delta^p$ and $n_1, \ldots, n_p \in N_A$. So:

$$\psi(\delta(t)) = \int_{\Delta^p} g(x, \delta(t)) \, dx, \quad \psi(\gamma(t)) = \int_{\Delta^p} g(x, \gamma(t)) \, dx.$$

And thus, by differentiating and commuting with integration, we get:

$$\psi_*(av) = \int_{\Delta^p} \frac{\partial}{\partial t} g(x, \delta(t)) \mid_{t=0} \, dx, \quad \psi_*(v) = \int_{\Delta^p} \frac{\partial}{\partial t} g(x, \gamma(t)) \mid_{t=0} \, dx$$

By hypothesis, $(\varphi_p)_A$ and ω_A are A_0-smooth in N_A, therefore so is the pullback h. Thus:

$$\frac{\partial}{\partial t} h(x, \delta(t)) \mid_{t=0} = a \frac{\partial}{\partial t} h(x, \gamma(t)) \mid_{t=0} .$$

Using the A_0-linearity of integration, it follows that:

$$\psi_*(av) = a\psi_*(v)$$

as desired.

Thus, $\int \omega \colon N_A^p \to A_0$, being natural in A and A_0-smooth, defines a map of supermanifolds:

$$\int \omega \colon N^p \to \mathbb{R}.$$

We therefore have a map:

$$\int \colon C^\bullet(\mathfrak{n}) \to C^\bullet(N)$$

It remains check that it is a cochain map. Indeed, $\int \omega_A$ is the composite of the cochain maps:

$$\omega \mapsto \omega_A \mapsto \int \omega_A,$$

So on A-points, $\int d\omega$ is:

$$
\begin{aligned}
(\int d\omega)_A &= \int (d\omega)_A \\
&= d(\int \omega_A) \\
&= (d \int \omega)_A,
\end{aligned}
$$

where in the last step we have used the fact that $d(f_A) = (df)_A$ by definition of d. In brief, for every Grassmann algebra A:

$$(\int d\omega)_A = (d \int \omega)_A.$$

We therefore conclude:

$$\int d\omega = d \int \omega,$$

and so $\int$ is a cochain map.

$\square$

Finally, we shall prove that there is a left-invariant notion of simplices with which we can equip N. For a fixed Grassmann algebra A, the Lie group N_A is exponential. We shall show that if we

take:

$$(\varphi_p)_A \colon \Delta^p \times N_A^{p+1} \to N_A$$

to be the standard notion of left-invariant simplices in Proposition 7.2, then this defines a left-invariant notion of simplices in N. The key is to note that each stage of the inductive definition of $(\varphi_p)_A$ we get maps that are natural in A.

Proposition 10.3. *Let N be the exponential supergroup of the nilpotent Lie superalgebra $\mathfrak{n}$. Fix a smoothing factor $\ell \colon [0,1] \to [0,1]$. For each Grassmann algebra A and $p \geq 0$, define:*

$$(\varphi_p)_A \colon \Delta^p \times N_A^{p+1} \to N_A$$

to be the standard left-invariant notion of simplices with smoothing factor ℓ. Then this defines a left-invariant notion of simplices in N.

Proof. Fix Grassmann algebras A and B and a map $f \colon A \to B$. We proceed by induction on p. For $p = 0$, the maps:

$$(\varphi_0)_A \colon \Delta^0 \times N_A \to N_A,$$

$$(\varphi_0)_B \colon \Delta^0 \times N_B \to N_B,$$

are the obvious projections. The fact that:

$$
\begin{array}{ccc}
\Delta^0 \times N_A & \xrightarrow{(\varphi_0)_A} & N_A \\
{\scriptstyle 1 \times N_f} \downarrow & & \downarrow {\scriptstyle N_f} \\
\Delta^0 \times N_B & \xrightarrow[(\varphi_0)_B]{} & N_B
\end{array}
$$

commutes is then automatic.

For arbitrary p, suppose that the following square commutes:

$$
\begin{array}{ccc}
\Delta^{p-1} \times N_A^p & \xrightarrow{(\varphi_{p-1})_A} & N_A \\
{\scriptstyle 1 \times N_f^p} \downarrow & & \downarrow {\scriptstyle N_f} \\
\Delta^{p-1} \times N_B^p & \xrightarrow[(\varphi_{p-1})_B]{} & N_B
\end{array}
$$

and that $(\varphi_{p-1})_A$ and $(\varphi_{p-1})_B$ are A_0- and B_0-smooth. In other words, the above square says that for any p-tuple of A-points, we have:

$$N_f \circ [n_1, \ldots, n_p] = [N_f(n_1), \ldots, N_f(n_p)].$$

We construct $(\varphi_p)_A$ and $(\varphi_p)_B$ from $(\varphi_{p-1})_A$ and $(\varphi_{p-1})_B$, respectively, using the apex-base construction. That is, given the $(p-1)$-simplex $[n_1, \ldots, n_p]$ given by $(\varphi_{p-1})_A$ for the A-points $n_1, \ldots, n_p \in N_A$, we define the based p-simplex:

$$[1, n_1, \ldots, n_p]$$

in N_A by using the exponential map $\exp_A$ to sweep out a path from the apex 1 to each point of the base $[n_1, \ldots, n_p]$. Similarly, we define the based p-simplex:

$$[1, N_f(n_1), \ldots, N_f(n_p)]$$

in N_B by using the exponential map $\exp_B$ to sweep out a path from the apex 1 to each point of the base $[N_f(n_1), \ldots, N_f(n_p)]$. From the naturality of $\exp$, we will establish that:

$$N_f \circ [1, n_1, \ldots, n_p] = [1, N_f(n_1), \ldots, N_f[n_p]].$$

To verify this claim, let

$$\exp_A(X) = [n_1, \ldots, n_p](x), \text{ for some } x \in \Delta^{p-1}$$

be a point of the base in N_A. By the inductive hypothesis, $N_f(\exp_A(X)) = \exp_B(\mathfrak{n}_f(X))$ is the corresponding point of the base in N_B. We wish to see that points of the path $\exp_A(\ell(t)X)$ connecting 1 to $\exp_A(X)$ in N_A correspond via N_f to points on the path $\exp_B(\ell(t)\mathfrak{n}_f(X))$ connecting 1 to $\exp_B(\mathfrak{n}_f(X))$ in N_B. But this is automatic, because:

$$N_f(\exp_A(\ell(t)X) = \exp_B(\mathfrak{n}_f(\ell(t)X)) = \exp_B(\ell(t)\mathfrak{n}_f(X)),$$

where in the last step we use the fact that $\mathfrak{n}_f \colon \mathfrak{n}_A \to \mathfrak{n}_B$ is linear. Thus, it is true that:

$$N_f \circ [1, n_1, \ldots, n_p] = [1, N_f(n_1), \ldots, N_f[n_p]],$$

for based p-simplices.

Using left translation, we can show that:

$$N_f \circ [n_0, n_1, \ldots, n_p] = [N_f(n_0), N_f(n_1), \ldots, N_f[n_p]].$$

for all p-simplices. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
\Delta^p \times N_A^{p+1} & \xrightarrow{(\varphi_p)_A} & N_A \\
{\scriptstyle 1 \times N_f^{p+1}} \downarrow & & \downarrow {\scriptstyle N_f} \\
\Delta^p \times N_B^{p+1} & \xrightarrow[(\varphi_p)_B]{} & N_B
\end{array}
$$

Because each step in the apex-base construction preserves A_0- or B_0-smoothness, we note that $(\varphi_p)_A$ and $(\varphi_p)_B$ are A_0- and B_0-smooth, respectively. The result now follows for all p by induction. $\qquad \square$

Chapter 11

Superstring Lie 2-supergroups, 2-brane Lie 3-supergroups

We are now ready to unveil the Lie n-supergroups which integrate our favorite Lie n-superalgebras, $\mathfrak{superstring}(k+1,1)$ and $\mathfrak{2\text{-}brane}(k+2,1)$. Remember, these are the Lie n-superalgebras which occur only in the dimensions for which string theory and 2-brane theory make sense. They are *not* nilpotent, simply because the Poincaré superalgebras $\mathfrak{siso}(k+1,1)$ and $\mathfrak{siso}(k+2,1)$ that form degree 0 of $\mathfrak{superstring}(k+1,1)$ and $\mathfrak{2\text{-}brane}(k+2,1)$ are not nilpotent. Nonetheless, we are equipped to integrate them using only the tools we have built to perform this task for nilpotent Lie n-superalgebras.

The road to this result has been a long one, and there is yet some ground to cover before we are finished. So, let us take stock of our progress before we move ahead:

- In spacetime dimensions $k + 2 = 3$, 4, 6 and 10, we used division algebras to construct a 3-cocycle α on the supertranslation algebra:

$$\mathcal{T} = V \oplus S$$

  which is nonzero only when it eats a vector and two spinors:

$$\alpha(A, \psi, \phi) = \langle \psi, A\phi \rangle.$$

- In spacetime dimensions $k + 3 = 4$, 5, 7 and 11, we used division algebras to construct a 4-cocycle α on the supertranslation algebra:

$$\mathcal{T} = \mathcal{V} \oplus \mathcal{S}$$

  which is nonzero only when it eats two vectors and two spinors:

$$\beta(\mathcal{A}, \mathcal{B}, \Psi, \Phi) = \langle \Psi, (\mathcal{AB} - \mathcal{BA})\Phi \rangle.$$

- Because α is invariant under the action of $\mathfrak{so}(k+1,1)$, it can be extended to a 3-cocycle on the Poincaré superalgebra:

$$\mathfrak{siso}(k+1,1) = \mathfrak{so}(k+1,1) \ltimes \mathcal{T}.$$

  The extension is just defined to vanish outside of $\mathcal{T}$, and we call it α as well.

- Because β is invariant under the action of $\mathfrak{so}(k+2,1)$, it can be extended to a 3-cocycle on the Poincaré superalgebra:

$$\mathfrak{siso}(k+2,1) = \mathfrak{so}(k+2,1) \ltimes \mathcal{T}.$$

  The extension is just defined to vanish outside of $\mathcal{T}$, and we call it β as well.

- Therefore, in spacetime dimensions $k+2$, we get a Lie 2-superalgebra $\mathfrak{superstring}(k+1,1)$ by extending $\mathfrak{siso}(k+1,1)$ by the 3-cocycle α.

- Likewise, in spacetime dimensions $k+3$, we get a Lie 3-superalgebra $2\text{-}\mathfrak{brane}(k+2,1)$ by extending $\mathfrak{siso}(k+2,1)$ by the 4-cocycle β.

In the last chapter, we built the technology necessary to integrate Lie superalgebra cocycles to supergroup cocycles, *provided* the Lie superalgebra in question is nilpotent. This allows us to integrate nilpotent Lie n-superalgebras to n-supergroups. But $\mathfrak{superstring}(k+1,1)$ and $2\text{-}\mathfrak{brane}(k+2,1)$ are not nilpotent, so we cannot use this directly here.

However, the cocycles α and β are supported on a nilpotent subalgebra: the supertranslation algebra, $\mathcal{T}$, for the appropriate dimension. This saves the day: we can integrate α and β as cocycles on $\mathcal{T}$. This gives us cocycles $\int \alpha$ and $\int \beta$ on the supertranslation supergroup, T, for the appropriate dimension. We will then be able to extend these cocycles to the Poincaré supergroup, thanks to their invariance under Lorentz transformations.

The following proposition helps us to accomplish this, but takes its most beautiful form when we work with 'homogeneous supergroup cochains', which we have not actually defined. Rest assured—they are exactly what you expect. If G is a supergroup that acts on the abelian supergroup M by automorphism, a **homogeneous M-valued p-cochain** on G is a smooth map:

$$F \colon G^{p+1} \to M$$

such that, for any Grassmann algebra A and A-points $g, g_0, \ldots, g_p \in G_A$:

$$F_A(gg_0, gg_1, \ldots, gg_p) = gF_A(g_1, \ldots, g_p).$$

We can define the supergroup cohomology of G using homogeneous or inhomogeneous cochains, just as was the case with Lie group cohomology.

Proposition 11.1. *Let G and H be Lie supergroups such that G acts on H, and let M be an abelian supergroup on which $G \ltimes H$ acts by automorphism. Given a homogeneous M-valued p-cochain F on H:*

$$F \colon H^{p+1} \to M,$$

we can extend it to a map of supermanifolds:

$$\tilde{F}\colon (G \ltimes H)^{p+1} \to M$$

by pulling back along the projection $(G \ltimes H)^{p+1} \to H^{p+1}$. In terms of A-points

$$(g_0, h_0), \ldots, (g_p, h_p) \in G_A \ltimes H_A,$$

this means $\tilde{F}$ is defined by:

$$\tilde{F}_A((g_0, h_0), \ldots, (g_p, h_p)) = F_A(h_0, \ldots, h_p),$$

Then $\tilde{F}$ is a homogeneous p-cochain on $G \ltimes H$ if and only if F is G-equivariant, and in this case $d\tilde{F} = \widetilde{dF}$.

Proof. We work over A-points, $G_A \ltimes H_A$. Denoting the action of $g \in G_A$ on $h \in H_A$ by h^g, recall that multiplication in the semidirect product $G_A \ltimes H_A$ is given by:

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2^{g_1}).$$

Now suppose $\tilde{F}$ is homogeneous. By definition of homogeneity, we have:

$$\tilde{F}_A((g,h)(g_0, h_0), \ldots, (g,h)(g_p, h_p)) = (g,h)\tilde{F}_A((g_0, h_0), \ldots, (g_p, h_p)).$$

Multiplying out each pair on the left and using the definition of $\tilde{F}$ on both sides, we get:

$$F_A(hh_0^g, \ldots, hh_p^g) = (g,h)F_A(h_0, \ldots, h_p).$$

Writing (g, h) as $(1, h)(g, 1)$, and pulling h out on the left-hand side, we now obtain:

$$(1, h)F_A(h_0^g, \ldots, h_p^g) = (1, h)(g, 1)F_A(h_0, \ldots, h_p).$$

Cancelling $(1, h)$ from both sides, this last equation just says that F_A is G_A-equivariant. The converse follows from reversing this calculation. Since this holds for all A-points, we conclude that $\tilde{F}$ is homogeneous if and only if F is G-equivariant.

When F is G-equivariant, it is easy to see that dF is also, and that $d\tilde{F} = \widetilde{dF}$, so we are done. $\square$

Now, at long last, we are ready to integrate α and β. In the following proposition, T denotes the **supertranslation group**, the exponential supergroup of the supertranslation algebra $\mathcal{T}$.

Proposition 11.2. *In dimensions 3, 4, 6 and 10, the Lie supergroup 3-cocycle $\int \alpha$ on the supertranslation group T is invariant under the action of $\mathrm{Spin}(k + 1, 1)$. Similarly, in dimensions 4, 5, 7 and 11, the Lie supergroup 4-cocycle $\int \beta$ on the supertranslation group T is invariant under the action of $\mathrm{Spin}(k + 2, 1)$.*

This is an immediate consequence of the following:

Proposition 11.3. *Let H be a nilpotent Lie supergroup with Lie superalgebra $\mathfrak{h}$. Assume H is equipped with its standard left-invariant notion of simplices, and let G be a Lie supergroup that*

acts on H by automorphism. If $\omega \in C^p(\mathfrak{h})$ is an even Lie superalgebra p-cochain which is invariant under the induced action of G on $\mathfrak{h}$, then $\int \omega \in C^p(H)$ is a Lie supergroup p-cochain which is invariant under the action of G on H.

Proof. Fixing a Grassmann algebra A, we must prove that

$$
\int_{[h_0^g, h_1^g, \ldots, h_p^g]} \omega_A = \int_{[h_0, h_1, \ldots, h_p]} \omega_A,
$$

for all A-points $g \in G_A$ and $h_0, h_1, \ldots, h_p \in H_A$. We shall see this follows from the fact that the p-simplices in H are themselves G-equivariant, in the sense that:

$$
[h_0^g, h_1^g, \ldots, h_p^g] = [h_0, h_1, \ldots, h_p]^g.
$$

Assuming this for the moment, let us check that our result follows. Indeed, applying the above equation, we get:

$$
\begin{aligned}
\int_{[h_0^g, h_1^g, \ldots, h_p^g]} \omega_A &= \int_{[h_0, h_1, \ldots, h_p]^g} \omega_A \\
&= \int_{[h_0, h_1, \ldots, h_p]} \mathrm{Ad}(g)^* \omega_A \\
&= \int_{[h_0, h_1, \ldots, h_p]} \omega_A,
\end{aligned}
$$

where the final step uses $\mathrm{Ad}(g)^* \omega_A = \omega_A$, which is just the G-invariance of ω.

It therefore remains to prove the equation $[h_0^g, h_1^g, \ldots, h_p^g] = [h_0, h_1, \ldots, h_p]^g$ actually holds. Note that this is the same as saying that the map

$$
(\varphi_p)_A \colon \Delta^p \times H_A^{p+1} \to H_A
$$

is G_A-equivariant. We check it by induction on p.

For $p = 0$, the map:

$$
(\varphi_0)_A \colon \Delta^0 \times H_A \to H_A
$$

is just the projection, and G_A-equivariance is obvious. So fix some $p \geq 0$ and suppose that $(\varphi_{p-1})_A$ is G_A-equivariant. We now construct $(\varphi_p)_A$ from $(\varphi_{p-1})_A$ using the apex-base construction, and show that equivariance is preserved.

So, given the $(p-1)$-simplex $[h_1, \ldots, h_p]$ given by $(\varphi_{p-1})_A$ for the A-points $h_1, \ldots, h_p \in H_A$, we define the based p-simplex:

$$
[1, h_1, \ldots, h_p]
$$

in H_A by using the exponential map to sweep out a path from the apex 1 to each point of the base $[h_1, \ldots, h_p]$. In a similar way, we define the based p-simplex:

$$
[1, h_1^g, \ldots, h_p^g]
$$

By hypothesis, $[h_1^g, \ldots, h_p^g] = [h_1, \ldots, h_p]^g$, and since the exponential map $\exp\colon \mathfrak{h}_A \to H_A$ is itself G_A-equivariant, it follows for based p-simplices that:

$$[1, h_1^g, \ldots, h_p^g] = [1, h_1, \ldots, h_p]^g.$$

The result now follows for all p-simplices by left translation. This completes the proof. $\qquad\square$

It thus follows that in dimensions 3, 4, 6 and 10, the cocycle $\int \alpha$ on the supertranslations can be extended to a 3-cocycle on the full Poincaré supergroup:

$$\mathrm{SISO}(k + 1, 1) = \mathrm{Spin}(k + 1, 1) \ltimes T,$$

while in dimensions 4, 5, 7 and 11, the cocycle $\int \beta$ can be extended to the Poincaré supergroup:

$$\mathrm{SISO}(k + 2, 1) = \mathrm{Spin}(k + 2, 1) \ltimes T.$$

By a slight abuse of notation, we continue to denote these extensions by $\int \alpha$ and $\int \beta$, respectively. As an immediate consequence, we have:

Theorem 11.1. *In dimensions 3, 4, 6 and 10, there exists a slim Lie 2-supergroup formed by extending the Poincaré supergroup* $\mathrm{SISO}(k+1, 1)$ *by the 3-cocycle $\int \alpha$, which we call we the* **superstring Lie 2-supergroup**, $\mathbf{Superstring(k + 1, 1)}$.

Theorem 11.2. *In dimensions 4, 5, 7 and 11, there exists a slim Lie 3-supergroup formed by extending the Poincaré supergroup* $\mathrm{SISO}(k + 2, 1)$ *by the 4-cocycle $\int \beta$, which we call the* **2-brane Lie 3-supergroup**, $\mathbf{2\text{-}Brane(k + 2, 1)}$.

11.1 Outlook

In this thesis we have seen a number of clues that a categorified geometry is relevant to superstrings, M-theory, and supergravity. Categorifying gauge theory to obtain higher gauge theory boosts the dimension of objects which we can parallel transport. The special identities which make supersymmetry work allow us to categorify the spacetime symmetries of superstrings. We propose there is a simple underlying reason for all of this: strings are extended objects, not point particles, so we need a geometry in which extended objects can play a role as fundamental as points. It is precisely this kind of geometry which we are now ready to explore, now that we have the n-supergroups $\mathrm{Superstring}(k + 1, 1)$ and $2\text{-}\mathrm{Brane}(k + 2, 1)$ in our hands.

Bibliography

[1] A. Achúcarro, J. M. Evans, P. K. Townsend and Wiltshire, Super p-branes, *Phys. Lett.* **B198** (1987), 441–446.

[2] P. Aschieri and B. Jurco, Gerbes, M5-brane anomalies and E_8 gauge theory. Also available as `arXiv:hep-th/0409200`.

[3] J. Azcárraga and J. Izquierdo, *Lie Groups, Lie Algebras, Cohomology and Some Applications to Physics*, Cambridge U. Press, Cambridge, 1995, pp. 230–236.

[4] J. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145–205. Also available as `arXiv:math/0105155`.

[5] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *TAC* **12** (2004), 492-528. Available at `http://www.tac.mta.ca/tac/volumes/12/15/12-15abs.html`. Also available as `arXiv:math.QA/0307263`.

[6] J. Baez and J. Huerta, The algebra of grand unified theories, *Bull. Amer. Math. Soc.* **47** (2010), 483–552. Also available as `arXiv:0904.1556`.

[7] J. Baez and J. Huerta, Divsion algebras and supersymmetry I, in *Superstrings, Geometry, Topology, and C^*-algebras*, eds. R. Doran, G. Friedman, J. Rosenberg, Proc. Symp. Pure Math. 81, AMS, Providence, 2010, pp. 65–80. Also available as `arXiv:0909.0551`.

[8] J. Baez and J. Huerta, Division algebras and supersymmetry II. Available as `arXiv:1003.3436`.

[9] J. Baez and J. Huerta, An invitation to higher gauge theory, to appear in *General Relativity and Gravitation*. Available as `arXiv:1003.4485`.

[10] J. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups, *TAC* **12** (2004), 423–491. Available at `http://www.tac.mta.ca/tac/volumes/12/14/12-14abs.html`. Also available as `arXiv:math/0307200`.

[11] J. Baez and U. Schreiber, Higher gauge theory, in *Categories in Algebra, Geometry and Mathematical Physics*, eds. A. Davydov et al, Contemp. Math. 431, AMS, Providence, 2007, pp. 7–30. Also available as `arXiv:math/0511710`.

[12] J. Baez and M. Stay, Physics, topology, logic and computation: a Rosetta Stone, to appear in *New Structures For Physics*, ed. Bob Coecke. Also available as `arXiv:0903.0340`.

[13] J. Baez, D. Stevenson, A. Crans, and U. Schreiber, From loop groups to 2-groups, *Homology Homotopy Appl.* **9** (2007), 101–135. Available as `arXiv:math/0504123`.

[14] L. Balduzzi, C. Carmeli, R. Fioresi, A comparison of the functors of points of supermanifolds. Available as `arXiv:0902.1824`.

[15] L. Bonora, M. Bregola, R. DAuria, P. Fr K. Lechner, P. Pasti, I. Pesando, M. Raciti, F. Riva, M. Tonin and D. Zanon, Some remarks on the supersymmetrization of the Lorentz Chern-Simons form in $D = 10$ $N = 1$ supergravity theories, *Phys. Lett. B* **277**, 306–312. Also available at `http://ncatlab.org/nlab/files/BonoraSuperGS.pdf`.

[16] F. Borceux, *Handbook of Categorical Algebra 1: Basic Category Theory*, Cambridge U. Press, Cambridge, 1994.

[17] F. Brandt, Supersymmetry algebra cohomology I: Definition and general structure, *J. Math. Phys.* **51**, 122302, 2010. Also available as `arXiv:0911.2118`. Supersymmetry algebra cohomology II: Primitive elements in 2 and 3 dimensions, *J. Math. Phys.* **51** 112303, 2010. Also available as `arXiv:1004.2978`. Supersymmetry algebra cohomology III: Primitive elements in four and five dimensions. Available as `arXiv:1005.2102`.

[18] L. Brink, J. Schwarz and J. Scherk, Supersymmetric Yang–Mills theory, *Nucl. Phys.* **B121** (1977), 77–92.

[19] A. Carey, S. Johnson, and M. Murray, Holonomy on D-branes, *J. Geom. Phys.* **52** (2004), no. 2, 186–216. Also available as `arXiv:hep-th/0204199v3`.

[20] L. Castellani, R. D'Auria and P. Fré, *Supergravity and Superstrings: A Geometric Perspective*, World Scientific, Singapore, 1991.

[21] E. Cheng and A. Lauda, Higher-dimensional categories: an illustrated guidebook. Available at `http://www.cheng.staff.shef.ac.uk/guidebook/index.html`.

[22] C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[23] K.-W. Chung and A. Sudbery, Octonions and the Lorentz and conformal groups of ten-dimensional space-time, *Phys. Lett. B* **198** (1987), 161–164.

[24] A. M. Cohen, B. Lisser, M. A. A. van Leeuwen, et al., *LiE, A computer algebra package for Lie group computations*, version 2.2.2, the computer algebra group, Centre for Mathematics and Computer Science in Amsterdam. Available at `http://www-math.univ-poitiers.fr/~maavl/LiE/`.

[25] R. D'Auria and P. Fré, Geometric supergravity in $D = 11$ and its hidden supergroup, *Nucl. Phys.* **B201** (1982), pp. 101–140.

[26] P. Deligne *et al*, eds., *Quantum Fields and Strings: A Course for Mathematicians*, Volume 1, Amer. Math. Soc., Providence, Rhode Island, 1999.

[27] P. Deligne and D. Freed, Supersolutions, in *Quantum Fields and Strings: A Course for Mathematicians*, P. Deligne *et al*, eds., Volume 1, Amer. Math. Soc., Providence, Rhode Island, 1999.

[28] E. Diaconescu, D. Freed, and G. Moore, The M theory 3-form and E_8 gauge theory. Available as `arXiv:hep-th/0312069`.

[29] T. Dray, J. Janesky and C. Manogue, Octonionic hermitian matrices with non-real eigenvalues, *Adv. Appl. Clifford Algebras* **10** (2000), 193–216. Also available as `arXiv:math/0006069`.

[30] M. J. Duff, Supermembranes: the first fifteen weeks, *Class. Quantum Grav.* **5** (1988), 189–205. Also available at `http://ccdb4fs.kek.jp/cgi-bin/img_index?8708425`.

[31] C. Ehresmann, Categories structurees, *Ann. Ec. Normale Sup.* **80** (1963).

C. Ehresmann, Categories structurees III: Quintettes et applications covariantes, *Cahiers Top. et DG V* (1963).

C. Ehresmann, Introduction to the theory of structured categories, *Technical Report Univ. of Kansas at Lawrence* (1966).

[32] J. M. Evans, Supersymmetric Yang–Mills theories and division algebras, *Nucl. Phys.* **B298** (1988), 92–108. Also available as `http://ccdb4fs.kek.jp/cgi-bin/img_index?8801412`.

[33] R. Foot and G. C. Joshi, On a certain supersymmetric identity and the division algebras, *Mod. Phys. Lett. A***3** (1988), 999–1004.

[34] D. S. Freed and E. Witten, Anomalies in string theory with D-branes, *Asian J. Math.* **3** (1999), 819. Also available as `arXiv:hep-th/9907189`.

[35] R. Gordon, A. J. Power, and R. Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. 117 (1995), no. 558.

[36] K. Gawedzki and N. Reis, WZW branes and gerbes, *Rev. Math. Phys.* **14** (2002), 1281–1334. Also available as `arXiv:hep-th/0205233`.

[37] K. Gawedzki, Topological actions in two-dimensional quantum field theories, in *Nonperturbative Quantum Field Theory*, eds. G. t'Hooft, A. Jaffe, G. Mack, P. K. Mitter and R. Stora, Plenum, New York, 1988, pp. 101–141.

[38] E. Getzler, Lie theory for nilpotent L_∞-algebras, *Ann. Math.* **170** (2009), 271–301. Available as `arXiv:math.AT/0404003`.

[39] M. Green, J. Schwarz and E. Witten, *Superstring Theory*, Volume 1, Cambridge U. Press, Cambridge, 1987. Appendix 4.A: Super Yang–Mills theories, pp. 244–247. Section 5.1.2: The supersymmetric string action, pp. 253–255.

[40] M. Green and J. Schwarz, Covariant description of superstrings, *Phys. Lett.* **B136** (1984), 367–370.

[41] N. Gurski, *An algebraic theory of tricategories*, Ph.D. thesis, Department of Mathematics, University of Chicago, 2006. Available at `http://www.math.yale.edu/~mg622/tricats.pdf`.

[42] A. Henriques, Integrating L_∞-algebras, *Compos. Math.* **144** (2008), 1017–1045. Available as `arXiv:math/0603563`.

[43] J. Houard, An integral formula for cocycles of Lie groups, *Annales de l'I.H.P.* **32** (1980), 221–247. Available at `http://archive.numdam.org/article/AIHPA_1980__32_3_221_0.pdf`.

[44] A. Hurwitz, Über die Composition der quadratischen Formen von beliebig vielen Variabeln, *Nachr. Ges. Wiss. Göttingen* (1898), 309–316.

[45] T. Kugo and P. Townsend, Supersymmetry and the division algebras, *Nucl. Phys.* **B221** (1983), 357–380. Also available at `http://ccdb4fs.kek.jp/cgi-bin/img_index?198301032`.

[46] T. Leinster, Basic bicategories. Available as `arXiv:math/9810017`.

[47] T. Leinster, A survey of definitions of n-category, *TAC* **10**, 1–70. Available at `http://www.tac.mta.ca/tac/volumes/10/1/10-01abs.html`. Also available as `arXiv:math/0107188`.

[48] D. Leites, Cohomologies of Lie superalgebras, *Funct. Anal. Appl.* **9**, 340–341.

[49] M. Markl, S. Schnider and J. Stasheff, *Operads in Algebra, Topology and Physics*, AMS, Providence, Rhode Island, 2002.

[50] C. Manogue and A. Sudbery, General solutions of the covariant superstring equations of motion, *Phys. Rev. D* **40** (1989), 4073–4077.

[51] M. Movshev, A. Schwarz and R. Xu, Homology of Lie algebra of supersymmetries. Available as `arXiv:1011.4731`.

[52] M. Penkava, L_∞ algebras and their cohomology. Available as `arXiv:q-alg/9512014`.

[53] I. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge University Press, Cambridge, 1995.

[54] C. Sachse, A categorical formulation of superalgebra and supergeometry. Available as `arXiv:0802.4067`.

[55] H. Sati, U. Schreiber, J. Stasheff, L_∞-algebras and applications to string– and Chern–Simons n-transport. Available as `arXiv:0801.3480`.

[56] R. D. Schafer, *Introduction to Non-Associative Algebras*, Dover, New York, 1995.

[57] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, *Jour. Pure App. Alg.* **38** (1985), 313–322.

[58] C. Schommer-Pries, A central extension of smooth 2-groups and a finite-dimensional string 2-group. To appear in *Geom. Top.* Available as `arXiv:0911.2483`.

[59] J. Schray, The general classical solution of the superparticle, *Class. Quant. Grav.* **13** (1996), 27–38. Also available as `arXiv:hep-th/9407045`.

[60] J. Schray and C. Manogue, Octonionic representations of Clifford algebras and triality, *Found. Phys.* **26** (1996), 17–70. Also available as `arXiv:hep-th/9407179`.

[61] U. Schreiber, *Differential cohomology in a cohesive topos*. Habilitation thesis. Available at `http://nlab.mathforge.org/schreiber/show/differential+cohomology+in+a+cohesive+topos`.

[62] A. S. Schwarz, On the definition of superspace, *Teoret. Mat. Fiz.* **60** (1984), 37–42 (in Russian). English translation in *Th. Math. Phys.* **60**, 657–660.

[63] J. Snygg, *Clifford Algebra: a Computational Tool for Physicists*, Oxford U. Press, Oxford, 1997.

[64] D. Stevenson, *The Geometry of Bundle Gerbes*, Ph.D. thesis, Department of Mathematics, University of Adelaide, 2000. Also available as `arXiv:math/0004117`.

[65] A. Sudbery, Division algebras, (pseudo)orthogonal groups and spinors, *Jour. Phys.* **A17** (1984), 939–955.

[66] Y. Tanii, Introduction to supergravities in diverse dimensions. Available as `arXiv:hep-th/9802138`.

[67] G. Tuynman, *Supermanifolds and Supergroups: Basic Theory*, Kluwer Academic Publishers, Dordrecht, 2004.

[68] K. Urbanik and F. B. Wright, Absolute-valued algebras, *Proc. Amer. Math. Soc.* **11** (1960), 861–866. Available at `http://www.ams.org/journals/proc/1960-011-06/S0002-9939-1960-0120264-6/S0002-9939-1960-0120264-6.pdf`.

[69] A. A. Voronov, Mappings of supermanifolds, *Teoret. Mat. Fiz.* **60** (1984), 43–48 (in Russian). English translation in *Th. Math. Phys.* **60**, 660–664.

[70] W. T. van Est, Group cohomology and Lie algebra cohomology in Lie groups, I and II, *Indag. Math.* **15** (1953), 484–504.