

# Noncommutative spectral geometry, algebra doubling and the seeds of quantization

Mairi Sakellariadou\*

*Department of Physics, King's College, University of London, Strand WC2R 2LS, London, U.K.*

Antonio Stabile<sup>†</sup> and Giuseppe Vitiello<sup>‡</sup>

*Facoltà di Scienze and Gruppo Collegato I.N.F.N., Università di Salerno, I-84100 Salerno, Italy*

A physical interpretation of the two-sheeted space, the most fundamental ingredient of noncommutative spectral geometry proposed by Connes as an approach to unification, is presented. It is shown that the doubling of the algebra is strictly related to dissipation. As a consequence, the doubling of the algebra is intimately related to the gauge structure of the theory. In a regime of completely deterministic dynamics, dissipation seems also to play a key rôle in the quantization of the theory, following 't Hooft's conjecture. It is thus argued that Connes' classical construction carries implicit in its feature of the doubling of the algebra the seeds of quantization.

PACS numbers: 02.40.Gh, 03.65.Yz, 11.15.-q, 12.10-g

## I. INTRODUCTION AND MOTIVATION

Noncommutative Spectral Geometry [1, 2] is a rich mathematical theory which combines notions of Non-Commutative Geometry (NCG) with spectral triples, a mathematical tool conceived by Alain Connes. Within this context, Connes and collaborators built a model which offers a purely geometric explanation for the Standard Model (SM) of electroweak and strong interactions — the most successful model of particle physics today at hand — compatible with right-handed neutrinos and neutrino masses [3]. This model succeeds at finding a way to merge the diffeomorphism invariance which governs General Relativity, with the local gauge invariance which governs Gauge Theories upon which the SM is based. The noncommutative spectral geometry model has also been used to derive supersymmetric extensions to the SM [4].

This unification model lives by construction at high energy scales (namely at unification scale), thus providing a natural environment to address unresolved issues of early universe cosmology [5–12]. Various criticisms have however been raised. One may for instance argue that since the model is at present purely classical, strictly speaking one cannot employ it within the context of the early universe since then the energy scales were so high that quantum corrections could no longer be neglected. Or one may oppose that since the action functional is obtained through a perturbative approach in inverse powers of the cut-off scale, it ceases to be valid at lower energy scales relevant for astrophysical studies. Note that the original approach may *a priori* also be treated nonperturbatively, however it is very difficult to compute exactly the spec-

tral action in its nonperturbative form. Another criticism that a physicist may have is that while the model is naturally developed in Euclidean signature, any physical studies must be performed in Lorentzian signature.

The aim of this paper is twofold: firstly, to address the intimate connection between gauge theories and the algebra doubling and offer a simple physical insight to this rich mathematical theory; secondly, to reply to some of the above mentioned criticisms.

In what follows, after a short introduction to noncommutative spectral geometry in Section II, we show in Section III how the algebra doubling, which is a crucial mathematical feature of Alain Connes' construction, is intimately related to the gauge structure of the theory. We introduce the notion of dissipation within this context, which in Section IV will lead to the quantum aspect of the noncommutative spectral geometry and the notion of temperature. In our discussion we will resort to the proposal by 't Hooft [13–15], according which quantum features and behaviors in a theory would result from a more fundamental deterministic scenario due to a process of information loss. In other words, according to 't Hooft's proposal, quantum mechanics emerges from an underlying deterministic classical dynamics acting at an energy scale much higher than the one of our observations, provided dissipation has occurred. This means that Alain Connes' "classical" construction, holding at high energy scales, may carry in itself the seeds for quantum behavior, provided in the same construction there is room for dissipation. In the following, we argue that this is indeed the case, since, as we show, the characterizing feature of the algebra doubling is strictly related to dissipation, which in turn can be described in terms of gauge fields. Thus the two-sheeted space selected by Alain Connes is related to gauge theories, as well as to dissipation and to quantization. We summarize our physical interpretation of Alain Connes' purely gravitational approach to unification in our conclusions.

\*mairi.sakellariadou@kcl.ac.uk

<sup>†</sup>anstabile@gmail.com

<sup>‡</sup>vitiello@sa.infn.it

## II. ELEMENTS OF NONCOMMUTATIVE SPECTRAL GEOMETRY

For the reader's convenience we summarize very briefly some of the basic features of NCG in the present section.

In the context of NCG, one is not interested in the properties of a set of points, but on the algebraic properties of functions on the spaces, instead. To capture the effect of the SM on the continuous four-dimensional manifold, Alain Connes considered a model of a two-sheeted space, made from the product of a four-dimensional smooth compact Riemannian manifold  $\mathcal{M}$  with a fixed spin structure, by a discrete noncommutative space  $\mathcal{F}$  composed by only two points. In Alain Connes' approach, the SM of electroweak and strong interactions is seen as a phenomenological model, which however specifies the geometry of space-time in such a way so that the Maxwell-Dirac action functional leads to the SM action. Following this proposal, the geometric space is defined as the tensor product of a continuous geometry  $\mathcal{M}$  for space-time by an internal geometry  $\mathcal{F}$  for the SM.

A main difference between noncommutative spectral geometry and other approaches of quantizing gravity is that here one is searching for a hidden signature of space-time geometry within the functional of gravity coupled to SM at present energy scales, instead of postulating the geometry around the Planck scale which necessitates an extrapolation by many orders of magnitude.

The noncommutative nature of the discrete space  $\mathcal{F}$  is given by a spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ , where  $\mathcal{A}$  is an involution of operators on the finite-dimensional Hilbert space  $\mathcal{H}$  of Euclidean fermions, and  $\mathcal{D}$  is a self-adjoint unbounded operator in  $\mathcal{H}$ . Spectral triples are analogous to Fourier transform in commutative spaces and are introduced in order to create a link with experimental data, which are all of a spectral nature. It is worth noting that the spectral nature approach is intrinsic to the noncommutative spectral geometry. In the product noncommutative space  $\mathcal{M} \times \mathcal{F}$ , the algebra of smooth functions is abelian, whereas the derivative in the discrete direction is a finite difference quotient.

The Hilbert space  $\mathcal{H}$  is the Hilbert space  $L^2(\mathcal{M}, S)$  of square integrable spinors  $S$  on  $\mathcal{M}$  and the algebra  $\mathcal{A}$  is the algebra  $\mathcal{A} = C^\infty(\mathcal{M})$  of smooth functions on  $\mathcal{M}$  and acts in  $\mathcal{H}$  by multiplication operators. The algebra  $\mathcal{A}$ , related to the gauge group of local gauge transformations, is the algebra of coordinates. Within NCG all information about space is encoded in the algebra of coordinates  $\mathcal{A}$ . The operator  $D$  is the Dirac operator  $\not{D}_{\mathcal{M}} = \sqrt{-1}\gamma^\mu \nabla_\mu^s$  (where  $\nabla_s$  is the spin connection) on the spin Riemannian manifold  $\mathcal{M}$ . The operator  $D$  corresponds to the inverse of the Euclidean propagator of fermions, and is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi-Maskawa mixing parameters.

The product geometry is specified by the rules:

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad (1)$$

and hence for  $\mathcal{M} \times \mathcal{F}$  the rules read:

$$\begin{aligned} \mathcal{A} &= C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\mathcal{F}} = C^\infty(\mathcal{M}, \mathcal{A}_{\mathcal{F}}), \\ \mathcal{H} &= L^2(\mathcal{M}, S) \otimes \mathcal{H}_{\mathcal{F}} = L^2(\mathcal{M}, S \otimes \mathcal{H}_{\mathcal{F}}), \\ D &= \not{D}_{\mathcal{M}} \otimes 1 + \gamma_5 \otimes D_{\mathcal{F}}; \end{aligned} \quad (2)$$

$\gamma_5$  is the chirality operator in the four-dimensional case.

Geometry is described by focusing on the Dirac operator  $D$ , instead of the metric tensor  $g_{\mu\nu}$  used for spaces with commuting coordinates. The familiar geodesic formula

$$d(x, y) = \inf_{\gamma} \int_{\gamma} ds, \quad (3)$$

where the infimum is taken over all possible paths connecting  $x$  to  $y$ , used to determine the distance  $d(x, y)$  between two points  $x$  and  $y$  within Riemannian geometry, is then replaced by

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\}. \quad (4)$$

Within the noncommutative spectral geometry  $D$  plays the rôle of the inverse of the line element  $ds$ .

Assuming the algebra  $\mathcal{A}$  constructed in the geometry  $\mathcal{M} \times \mathcal{F}$  is symplectic-unitary, it must be of the form [16]

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}), \quad (5)$$

with  $k = 2a$  and  $\mathbb{H}$  being the algebra of quaternions. The field of quaternions  $\mathbb{H}$  plays an important rôle in this construction and its choice remains to be explained. To obtain the SM one assumes quaternion linearity. The first possible value for the even number  $k$  is 2, corresponding to a Hilbert space of four fermions, but this choice is ruled out from the existence of quarks. The next possible value is  $k = 4$  leading to the correct number of  $k^2 = 16$  fermions in each of the three generations. Note that if new particles are discovered at the Large Hadron Collider (LHC), one may be able to accommodate them by considering a higher value for the even number  $k$ .

Let us just mention that another basic ingredient of Alain Connes' approach is to consider the Dixmier trace — a noncommutative analogue of integration on a compact  $n$ -dimensional Riemannian spin manifold — as the fundamental functional to define the action of the theory. Alain Connes connected the Dixmier trace with residues of zeta functions.

The noncommutative spectral geometry model is based upon the spectral action principle stating that, within the context of a product noncommutative geometry, the bare bosonic Euclidean action is given by the trace of the heat kernel associated with the square of the noncommutative Dirac operator and is of the form

$$\text{Tr}(f(D/\Lambda)), \quad (6)$$

where  $f$  is a cut-off function and  $\Lambda$  fixes the energy scale;  $D$  and  $\Lambda$  have physical dimensions of a mass and there is no absolute scale on which they can be measured. This

action can be seen *à la* Wilson as the bare action at the mass scale  $\Lambda$ . The fermionic term can be included in the action functional by adding  $(1/2)\langle J\psi, D\psi\rangle$ , where  $J$  is the real structure on the spectral triple and  $\psi$  is a spinor in the Hilbert space  $\mathcal{H}$  of the quarks and leptons.

In this approach, the fermions of the Standard Model provide the Hilbert space of a spectral triple for the algebra, while the bosons (including the Higgs boson) are obtained through inner fluctuations of the Dirac operator of the product geometry.

For the four-dimensional Riemannian geometry, the trace  $\text{Tr}(f(D/\Lambda))$  is expressed perturbatively in terms of the geometrical Seeley-deWitt coefficients  $a_n$ , which are known for any second order elliptic differential operator, as [17–20]

$$\begin{aligned} \text{Tr}(f(D/\Lambda)) \sim & 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \dots \\ & + \Lambda^{-2k} f_{-2k} a_{4+2k} + \dots, \end{aligned} \quad (7)$$

where the smooth even cut-off function  $f$ , which decays fast at infinity, appears through its momenta  $f_k$  given by:

$$\begin{aligned} f_0 &\equiv f(0) \\ f_k &\equiv \int_0^\infty f(u) u^{k-1} du, \quad \text{for } k > 0, \\ f_{-2k} &= (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0). \end{aligned}$$

Moreover, since its Taylor expansion at zero vanishes, the asymptotic expansion Eq. (7) reduces to

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4. \quad (8)$$

In conclusion, the cut-off function  $f$  plays a rôle only through its three momenta  $f_0, f_2, f_4$ , which are three real parameters in the model; they are intimately related to the coupling constants at unification, the gravitational constant, and the cosmological constant. In this four-dimensional Riemannian manifold (one brane of the two-sheeted space), the term in  $\Lambda^4$  gives a cosmological term, the term in  $\Lambda^2$  gives the Einstein-Hilbert action functional, and the  $\Lambda$ -independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric.

The computation of the asymptotic expression for the spectral action functional results to the full Lagrangian for the Standard Model minimally coupled to gravity, with neutrino mixing and Majorana mass terms. Thus, this approach leads to a geometric explanation of the SM; in particular, the vacuum expectation value of the Higgs field is related to the noncommutative distance between the two sheets. The Higgs field is found to be conformally coupled to the Ricci scalar. The generalized Einstein-Hilbert action contains in addition a minimally coupled massless scalar field  $\sigma$  related to the distance  $d$  between the two sheets by  $d \propto e^{-\sigma(y)}$ , with  $y$  an element of the product noncommutative space.

### III. NONCOMMUTATIVE SPECTRAL GEOMETRY, THE ALGEBRA DOUBLING AND THE GAUGE STRUCTURE

In Ref. [1] Alain Connes considers the work of Heisenberg establishing, in the early years of Quantum Mechanics (QM), the *matrix mechanics* — where physical quantities are governed by noncommutative algebra — and he discusses how close such a discovery is to experimental reality and how strict is its relation to the observed discretization of the energy of the atomic levels and of angular momentum. In this section our aim is twofold: Firstly, we show that one central ingredient in NCG, namely the “doubling” of the algebra  $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$  acting on the “doubled” space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  (cf., Eq. (1)), is also present in the standard QM formalism of the density matrix and the associated Wigner function. We then show that the doubling of the algebra is implicit even in the classical theory when considering the Brownian motion of a particle and it is strictly related to dissipation. Secondly, in Subsection III A we show that the doubling of the algebra and the dissipation are related to the gauge structure of the theory.

For the first part of our discussion, let us start by observing that in the formalism of the density matrix and the Wigner function, the coordinate  $x(t)$  of a quantum particle is split into two coordinates  $x_+(t)$  (going forward in time) and  $x_-(t)$  (going backward in time). Indeed, the Wigner function has the standard expression [21],

$$\begin{aligned} W(p, x, t) & \quad (9) \\ &= \frac{1}{2\pi\hbar} \int \psi^* \left( x - \frac{1}{2}y, t \right) \psi \left( x + \frac{1}{2}y, t \right) e^{-i\frac{xy}{\hbar}} dy, \end{aligned}$$

where

$$x_\pm = x \pm \frac{1}{2}y. \quad (10)$$

The associated density matrix is

$$W(x_+, x_-, t) \equiv \langle x_+ | \rho(t) | x_- \rangle = \psi^*(x_-, t) \psi(x_+, t), \quad (11)$$

and the mean value of a quantum operator  $A$ , computed using the density matrix, is given by

$$\begin{aligned} \bar{A}(t) &= \langle \psi(t) | A | \psi(t) \rangle \\ &= \iint \psi^*(x_-, t) \langle x_- | A | x_+ \rangle \psi(x_+, t) dx_+ dx_- \\ &= \iint \langle x_+ | \rho(t) | x_- \rangle \langle x_- | A | x_+ \rangle dx_+ dx_-. \end{aligned} \quad (12)$$

The forward and the backward in time evolution of the density matrix  $W(x_+, x_-, t)$  is then described by “two copies” of the Schrödinger equation:

$$i\hbar \frac{\partial \psi(x_+, t)}{\partial t} = H_+ \psi(x_+, t), \quad (13)$$

$$-i\hbar \frac{\partial \psi^*(x_-, t)}{\partial t} = H_- \psi^*(x_-, t), \quad (14)$$

respectively, i.e.,

$$i\hbar \frac{\partial \langle x_+ | \rho(t) | x_- \rangle}{\partial t} = H \langle x_+ | \rho(t) | x_- \rangle , \quad (15)$$

where  $H$  is given in terms of the two Hamiltonian operators  $H_{\pm}$  as

$$H = H_+ - H_- . \quad (16)$$

The connection with Alain Connes' discussion of spectroscopic experiments, noncommutative matrix algebra, energy level discretization and the algebra doubling is thus evident: the density matrix and the Wigner function *require* the introduction of a "doubled" set of coordinates  $(x_{\pm}, p_{\pm})$  (or  $(x, p_x)$  and  $(y, p_y)$ ) and of their respective algebras. Using the two copies of the Hamiltonian  $H_{\pm}$  operating on the outer product of two Hilbert spaces  $\mathcal{H}_+ \otimes \mathcal{H}_-$  has been implicitly required in QM since the very beginning of the theory. Use of Eqs. (15), (16) shows immediately that the eigenvalues of  $H$  are directly the Bohr transition frequencies  $h\nu_{nm} = E_n - E_m$ , which was the first hint towards an explanation of spectroscopic structure.

We now show that the need to double the degrees of freedom is implicit even in the classical theory when considering the Brownian motion. We closely follow Ref. [22] where the results here summarized are derived.

We recall that in the classical Brownian theory one has the equation of motion

$$m\ddot{x}(t) + \gamma\dot{x}(t) = f(t) , \quad (17)$$

where  $f(t)$  is a random (Gaussian distributed) force obeying

$$\langle f(t)f(t') \rangle_{\text{noise}} = 2\gamma k_B T \delta(t-t') . \quad (18)$$

Equation (17) can be derived from a Lagrangian in a canonical procedure by employing a delta functional classical constraint representation as a functional integral. By averaging over the fluctuating force  $f$ , one indeed obtains [22]

$$\begin{aligned} & \langle \delta[m\ddot{x} + \gamma\dot{x} - f] \rangle_{\text{noise}} \\ &= \int \mathcal{D}y \langle \exp\left[\frac{i}{\hbar} \int dt L_f(\dot{x}, \dot{y}, x, y)\right] \rangle_{\text{noise}} , \end{aligned} \quad (19)$$

where

$$L_f(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) + fy . \quad (20)$$

Note that  $\hbar$  is introduced solely for dimensional reasons. We thus see that the constraint condition at the classical level introduced a new coordinate  $y$ , and the standard Euler-Lagrange equations are obtained, namely

$$\frac{d}{dt} \frac{\partial L_f}{\partial \dot{y}} = \frac{\partial L_f}{\partial y} ; \quad \frac{d}{dt} \frac{\partial L_f}{\partial \dot{x}} = \frac{\partial L_f}{\partial x} , \quad (21)$$

i.e.,

$$m\ddot{x} + \gamma\dot{x} = f , \quad m\ddot{y} - \gamma\dot{y} = 0 . \quad (22)$$

We remark that the Lagrangian system Eqs. (20)-(22) were obtained in a completely classical context in the search aimed to build up a canonical formalism for dissipative system [23–25]. The  $x$ -system is an *open* system. In order to set up the canonical formalism it is required to *close* the system; this is the rôle of the  $y$ -system. The  $\{x - y\}$  system appears as a closed system (in the  $x$ -system there is damping, in the  $y$  system there is amplification: the  $y$ -system is the time-reversed copy of the  $x$ -system). We thus see that doubling of the algebra and dissipation are intimately related.

We also remark that the exact expression for the imaginary part of the action reads [26, 27]

$$\text{Im}\mathcal{S}[x, y] = \frac{1}{2\hbar} \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s) y(t) y(s) , \quad (23)$$

where  $N(t-s)$  denotes the quantum noise in the fluctuating random force given by the Nyquist theorem [27].

The meaning of Eq. (23) is that nonzero  $y$  yields an "unlikely process" in the classical limit " $\hbar \rightarrow 0$ ", in view of the large imaginary part of the action. At quantum level, instead, nonzero  $y$  may allow quantum noise effects arising from the imaginary part of the action [27]. This sheds some light on the rôle played by the doubled degrees of freedom in the interplay between classical and quantum. We thus see that the second sheet cannot be neglected: in the perturbative approach one may drop higher order terms in the action functional expansion, since they correspond to unlikely processes at the classical level. However, these terms may be responsible for quantum corrections (dissipation) and therefore, in order to not preclude the quantization effects, one should keep them.

## A. The gauge structure

Let us now show how the doubling of the degrees of freedom is strictly related to the gauge structure of the theory. Our subsequent discussion will thus unveil the intimate relation between the two-sheeted space in Connes' construction and the gauge structure of the theory.

We consider the equation of the classical one-dimensional damped harmonic oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = 0 , \quad (24)$$

with time independent  $m$ ,  $\gamma$  and  $k$ , which is a simple prototype of dissipative systems.

As we have seen, to set up the canonical formalism for dissipative systems, the doubling of the degrees of freedom is required in such a way as to complement the given dissipative system with its time-reversed image, thus obtaining a globally closed system for which the Lagrangian

formalism is well defined. The doubling of the  $x$  degree of freedom leads to consider the damped harmonic oscillator in the *doubled*  $y$  coordinate

$$m\ddot{y} - \gamma\dot{y} + ky = 0 . \quad (25)$$

The system of damped harmonic oscillator Eq. (24) and its time-reversed ( $\gamma \rightarrow -\gamma$ ) image Eq. (25) is then a closed system described by the Lagrangian density Eq. (20) where we put  $f = kx$ . The canonically conjugate momenta  $p_x$  and  $p_y$  can now be introduced as customary in the Lagrangian formalism:

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}y \quad , \quad p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}x \quad , \quad (26)$$

and the dynamical variables  $\{x, p_x; y, p_y\}$  span the new phase-space.

It is convenient to use the coordinates  $x_1(t)$  and  $x_2(t)$  obtained through the (canonical) transformation

$$x_1(t) = \frac{x(t) + y(t)}{\sqrt{2}} \quad , \quad x_2(t) = \frac{x(t) - y(t)}{\sqrt{2}} \quad , \quad (27)$$

in terms of which the motion equations can be rewritten as

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0 \quad , \quad (28a)$$

$$m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0 \quad , \quad (28b)$$

and  $p_1 = m\dot{x}_1 + (1/2)\gamma x_2$ ;  $p_2 = -m\dot{x}_2 - (1/2)\gamma x_1$ . The Hamiltonian is then found to be

$$\begin{aligned} H &= H_1 - H_2 \\ &= \frac{1}{2m}(p_1 - \frac{\gamma}{2}x_2)^2 + \frac{k}{2}x_1^2 \\ &\quad - \frac{1}{2m}(p_2 + \frac{\gamma}{2}x_1)^2 - \frac{k}{2}x_2^2 . \end{aligned} \quad (29)$$

Following Refs. [28–31] we can now introduce the vector potential as

$$A_i = \frac{B}{2}\epsilon_{ij}x_j \quad (i, j = 1, 2) \quad , \quad (30)$$

with

$$B \equiv \frac{c}{e}\gamma \quad , \quad \epsilon_{ii} = 0 \quad , \quad \epsilon_{12} = -\epsilon_{21} = 1 . \quad (31)$$

We realize that  $H_i$  (with  $i = 1, 2$ ) in Eq. (29) describe two particles with opposite charges  $e_1 = -e_2 = e$  in the (oscillator) potential  $\Phi \equiv (k/2/e)(x_1^2 - x_2^2) \equiv \Phi_1 - \Phi_2$  with  $\Phi_i \equiv (k/2/e)x_i^2$  and in the constant magnetic field  $\mathbf{B}$  defined as  $\mathbf{B} = \nabla \times \mathbf{A} = -B\hat{\mathbf{z}}$ , namely:

$$\begin{aligned} H &= H_1 - H_2 \\ &= \frac{1}{2m}(p_1 - \frac{e_1}{c}A_1)^2 + e_1\Phi_1 \\ &\quad - \frac{1}{2m}(p_2 + \frac{e_2}{c}A_2)^2 + e_2\Phi_2 . \end{aligned} \quad (32)$$

Using Eq. (30) the Lagrangian of the system can be written in the familiar form

$$\begin{aligned} L &= \frac{1}{2m}(m\dot{x}_1 + \frac{e_1}{c}A_1)^2 - \frac{1}{2m}(m\dot{x}_2 + \frac{e_2}{c}A_2)^2 \\ &\quad - \frac{e^2}{2mc^2}(A_1^2 + A_2^2) - e\Phi \\ &= \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{e}{c}(\dot{x}_1A_1 + \dot{x}_2A_2) - e\Phi . \end{aligned} \quad (33)$$

Remarkably, we have the Lorentzian-like (pseudoeuclidean) metric in Eq. (33) (cf. also Eqs. (16), (32) and (41) below). The “minus” sign, not imposed by hand, but required by the doubling of the degrees of freedom, is crucial in our derivation (and in Connes’ construction).

In conclusion, the doubled coordinate, e.g.,  $x_2$  acts as the gauge field component  $A_1$  to which the  $x_1$  coordinate is coupled, and *vice versa*. The interpretation is recovered of the gauge field as the bath or reservoir in which the system is embedded. The gauge structure appears intrinsic to the doubling procedure. Let us see then how such a conclusion can be also reached in the case of a fermion field.

For brevity we discuss the simple case of the massless fermion<sup>1</sup> and the U(1) local gauge transformation group. We will see how in this case the doubling of the algebra  $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$  acting on the outer product space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  is related with the gauge structure of the theory.

We consider the classical (pre-quantum) theory. The Lagrangian of the massless free Dirac field is:

$$L = -\bar{\psi}\gamma^\mu\partial_\mu\psi . \quad (34)$$

Under the U(1) local gauge transformation,

$$\psi(x) \rightarrow \exp[ig\alpha(x)]\psi(x) \quad , \quad (35)$$

$L$  transforms as

$$L \rightarrow L' = L - ig\partial^\mu\alpha(x)\bar{\psi}(x)\gamma_\mu\psi(x) . \quad (36)$$

It is well known that in order to make  $L$  invariant under the local gauge transformation Eq. (35), the coupling of the current  $j_\mu = i\bar{\psi}\gamma_\mu\psi$  with the gauge vector field  $A_\mu$  has to be introduced in  $L$  in such a way that, when  $\psi(x)$  transforms as in Eq. (35),  $j^\mu(x)A_\mu(x)$  transforms as

$$j^\mu(x)A_\mu(x) \rightarrow j^\mu(x)A_\mu(x) + j^\mu(x)\partial_\mu\alpha(x) \quad , \quad (37)$$

i.e.,

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\alpha(x) . \quad (38)$$

The Lagrangian  $L$  modified by the coupling  $gj^\mu A_\mu$  leads to the lagrangian  $L_g$  defined as

$$L_g = -\bar{\psi}\gamma^\mu\partial_\mu\psi + ig\bar{\psi}\gamma^\mu\psi A_\mu \quad , \quad (39)$$

<sup>1</sup> Extension to the massive fermion case, the boson case and non-Abelian gauge transformation groups is possible, see Refs. [30, 31].

which is by construction invariant under the U(1) local gauge transformations Eqs. (35), (38), namely  $L_g \rightarrow L'_g = L_g$ . As usual, in order for  $A_\mu$  to be a dynamical field, the term  $-(1/4)F^{\mu\nu}F_{\mu\nu}$  has to be added to the modified Lagrangian  $L_g$ . Moreover, the Lorentz gauge condition

$$\partial^\mu A_\mu(x) = 0 , \quad (40)$$

has to be adopted in order to ensure that only transverse modes of the  $A_\mu$  field enter physical states. As said in the Introduction, these are represented by square integrable (spinor) functions in the Hilbert space  $\mathcal{H} = L^2(\mathcal{M}, S)$  where the algebra acts by multiplication operators. Equation (40) expresses the restriction to the physical states in  $\mathcal{H}$  where the gauge constraint is satisfied, which we will denote by  $\langle \partial^\mu A_\mu(x) \rangle = 0$ , where  $\langle \dots \rangle$  stands for expectation values in the physical states  $\langle \text{phys} | \dots | \text{phys} \rangle$ .

Now, let us go back to the Lagrangian Eq. (34) for a classical fermion field and show how the doubling of the fermion degrees of freedom is related, under convenient constraints, to the local gauge invariance.

The field algebra is doubled by introducing the fermion tilde-field  $\tilde{\psi}(x)$ . The tilde-system is a ‘‘copy’’ (with the same spectrum and couplings) of the  $\psi$ -system. The Lagrangian is written now as

$$\hat{L} = L - \tilde{L} = -\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\tilde{\psi}}\gamma^\mu\partial_\mu\tilde{\psi} . \quad (41)$$

We assume, for simplicity, that in  $\hat{L}$  there is no coupling term of the field  $\psi(x)$  with the tilde field  $\tilde{\psi}(x)$ . The Hamiltonian for the system is of the form  $\hat{H} = H - \tilde{H}$  (to be compared with Eq. (16)), which in terms of creation and annihilation operators of the  $\psi(x)$  and  $\tilde{\psi}(x)$  fields is given by  $\hat{H} = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}})$ . Let the zero energy eigenstate of  $\hat{H}$  be denoted by  $|0(\theta)\rangle$ <sup>2</sup>. The space of states  $\hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}}$  is constructed out of  $|0(\theta)\rangle$  by repeated applications of creation operators of  $\psi(\theta; x)$  and  $\tilde{\psi}(\theta; x)$  and is called the  $\theta$ -representation  $\{|0(\theta)\rangle\}$  [26, 30, 31].

In the following we consider the subspace  $\mathcal{H}_\theta \subset \{|0(\theta)\rangle\}$  made of all the states  $|a\rangle_\theta$ , including  $|0(\theta)\rangle$ , such that the  $\theta$ -state condition

$$[a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}}] |a\rangle_\theta = 0 , \quad \text{for any } \mathbf{k} , \quad (42)$$

holds for any  $|a\rangle_\theta$  in  $\mathcal{H}_\theta$ . This condition can be shown to be the realization in  $\mathcal{H}_\theta$  of the Lorentz gauge condition Eq. (40)<sup>3</sup>. We have

$$\langle j_\mu(x) \rangle_\theta = \langle \tilde{j}_\mu(x) \rangle_\theta , \quad (43)$$

<sup>2</sup> In other words,  $|0(\theta)\rangle$  is the vacuum with respect to the fields  $\psi(\theta; x)$  and  $\tilde{\psi}(\theta; x)$  obtained from  $\psi(x)$  and  $\tilde{\psi}(x)$ , respectively, by means of the Bogoliubov transformation [26, 32].

<sup>3</sup> Equation (42) turns out to be equivalent to the Gupta-Bleuler condition in quantum electrodynamics [26, 30, 31].

where  $\langle \dots \rangle_\theta$  denotes matrix elements in  $\mathcal{H}_\theta$ . We will denote equalities between matrix elements in  $\mathcal{H}_\theta$ , say  $\langle A \rangle_\theta = \langle B \rangle_\theta$ , by  $A \cong B$  and call them  $\theta$ -weak equalities. Since they are equalities among c-numbers, they are classical equalities.

Now, the key point is that, due to Eq. (43), the matrix elements in  $\mathcal{H}_\theta$  of the Lagrangian Eq. (41) (as well as of a more general Lagrangian than the simple one presently considered) are invariant under the simultaneous local gauge transformations of  $\psi$  and  $\tilde{\psi}$  given by Eq. (35) and

$$\tilde{\psi}(x) \rightarrow \exp[ig\alpha(x)] \tilde{\psi}(x) , \quad (44)$$

respectively, i.e.,

$$\langle \hat{L} \rangle_\theta \rightarrow \langle \hat{L}' \rangle_\theta = \langle \hat{L} \rangle_\theta , \quad \text{in } \mathcal{H}_\theta , \quad (45)$$

under the gauge transformations Eqs. (35), (44).

We thus realize that a crucial rôle in the  $\theta$ -weak gauge invariance of  $\hat{L}$  under Eqs. (35), (44) is played by the tilde term  $\bar{\tilde{\psi}}\gamma^\mu\partial_\mu\tilde{\psi}$  since it transforms in such a way as to compensate the local gauge transformation of the  $\psi$  kinematic term, i.e.,

$$\bar{\tilde{\psi}}(x)\gamma^\mu\partial_\mu\tilde{\psi}(x) \rightarrow \bar{\tilde{\psi}}(x)\gamma^\mu\partial_\mu\tilde{\psi}(x) + g\partial^\mu\alpha(x)\tilde{j}_\mu(x) . \quad (46)$$

This suggests to introduce the vector field  $A'_\mu$  by

$$g\tilde{j}^{\bar{\mu}}(x)A'_{\bar{\mu}}(x) \cong \bar{\tilde{\psi}}(x)\gamma^{\bar{\mu}}\partial_{\bar{\mu}}\tilde{\psi}(x) , \quad \bar{\mu} = 0, 1, 2, 3 . \quad (47)$$

Here and in the following, the bar over  $\mu$  means no summation over repeated indices. Equation (46) implies that  $A'_\mu$  transforms as

$$A'_\mu(x) \rightarrow A'_\mu(x) + \partial_\mu\alpha(x) , \quad (48)$$

when Eqs. (35), (44) are implemented. This suggests to identify, in  $\mathcal{H}_\theta$ ,  $A'_\mu$  with the conventional U(1) gauge vector field and to introduce it in the original Lagrangian through the usual coupling term  $ig\bar{\psi}\gamma^\mu\psi A'_\mu$ .

We remark that provided we restrict ourselves to the  $\theta$ -weak equalities, i.e., to matrix elements in  $\mathcal{H}_\theta$ , matrix elements of physical observables, which are solely functions of the  $\psi(x)$  field, are not changed by Eq. (47). Moreover, observables turn out to be invariant under gauge transformations. Next, one can show that the conservation laws derivable from  $\hat{L}$ , namely in the simple case of Eq. (41) the current conservation laws:

$$\partial^\mu j_\mu(x) = 0 , \quad \partial^\mu \tilde{j}_\mu(x) = 0 , \quad (49)$$

are also preserved as  $\theta$ -weak equalities when Eq. (47) is adopted. One may also show that

$$\partial^\nu F'_{\mu\nu}(x) \cong -gj_\mu(x) , \quad \partial^\nu F'_{\mu\nu}(x) \cong -g\tilde{j}_\mu(x) , \quad (50)$$

in  $\mathcal{H}_\theta$ . In the Lorentz gauge, from Eq. (50) we also obtain the  $\theta$ -weak relations

$$\begin{aligned} \partial^\mu A'_\mu(x) &\cong 0 , \\ \partial^2 A'_\mu(x) &\cong gj_\mu(x) . \end{aligned} \quad (51)$$

In conclusion, our discussion shows the intrinsic gauge properties of the “doubling” procedure: we have obtained that in  $\mathcal{H}_\theta$  the “doubled algebra” Lagrangian Eq. (41) for the field  $\psi$  and its “double”  $\tilde{\psi}$  can be substituted by the Lagrangian:

$$\hat{L}_g \cong -\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu} - \bar{\psi}\gamma^\mu\partial_\mu\psi + ig\bar{\psi}\gamma^\mu\psi A'_\mu, \quad (52)$$

which is indeed the standard  $U(1)$  local gauge invariant Lagrangian for the fermion field  $\psi$ . Remarkably, the tilde-kinematical term is replaced, in a  $\theta$ -weak sense, by the gauge field-current coupling. The second equation in Eq. (50), shows that the variations of the gauge field tensor  $F'_{\mu\nu}$  have their source in the current  $\tilde{j}_\mu$ , which suggests that the tilde field plays the rôle of a “reservoir”. Such an interpretation in terms of a reservoir, may thus be extended also to the gauge field  $A'_\mu$ , which indeed acts in a way to “compensate” the changes in the matter field configurations due to the local gauge freedom.

Finally, in the case an interaction term is present in the Lagrangian Eq. (41),  $\hat{L}_{\text{tot}} = \hat{L} + \hat{L}_I$ ,  $\hat{L}_I = L_I - \tilde{L}_I$ , the above conclusions still hold provided  $\mathcal{H}_\theta$  is an invariant subspace under the dynamics described by  $\hat{L}_{\text{tot}}$ .

The state  $|0(\theta)\rangle$  can be shown to be a finite temperature state, which means that the algebra doubling leads to a thermal field theory [25, 26]. It can be also shown that in the formalism of the algebra doubling a relevant rôle is played by the noncommutative  $q$ -deformed Hopf algebra [33], pointing to a deep physical meaning of the noncommutativity in Alain Connes’ construction.

#### IV. ALGEBRA DOUBLING, DISSIPATION AND QUANTIZATION

We have considered till now the doubling of the algebra such as the one occurring in Alain Connes’ NCG construction and have shown that such a doubling is related to the gauge structure of the theory. We have done this by considering essentially classical systems and have mentioned in several points features of such systems at a quantum level. We have also stressed that the doubling of the system degrees of freedom, say  $x$ , amounts to consider the fact that the system is embedded in some environment which is indeed described by the doubled  $y$  coordinate. The doubling of the algebra thus underlies the *openness* of the system under study. As a specific case we have considered the classical damped harmonic oscillator, which is a simple prototype of *open* systems whose evolution is characterized by energy dissipation.

In a series of papers [13–15] ’t Hooft has discussed classical, deterministic, dissipative models and has conjectured that, provided some specific energy conditions are met and some constraints are imposed, loss of information might lead to a quantum evolution. In this section, following Refs. [34, 35], we show that in agreement with ’t Hooft’s conjecture, dissipation in a regime

of completely deterministic dynamics appears to be responsible of the system’s quantum mechanical evolution. Our conjecture is then that Alain Connes’ classical construction carries *implicit* in its feature of the doubling of the algebra the seeds of quantization.

In order to be specific, we consider the classical damped harmonic  $x$ -oscillator described by Eq. (24) and its time-reversed image, the  $y$ -oscillator Eq. (25). It is also convenient to put [29]  $x_1 = r \cosh u$ ,  $x_2 = r \sinh u$ , and

$$\mathcal{C} = \frac{1}{4\Omega m} [(p_1^2 - p_2^2) + m^2\Omega^2(x_1^2 - x_2^2)], \quad (53)$$

$$J_2 = \frac{m}{2} [(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \Gamma r^2], \quad (54)$$

where  $\mathcal{C}$  is taken to be positive and

$$\Gamma = \frac{\gamma}{2m}, \quad \Omega = \sqrt{\frac{1}{m}(\kappa - \frac{\gamma^2}{4m})}, \quad \text{with } \kappa > \frac{\gamma^2}{4m}.$$

Using  $z = r^2$  and the canonical transformation:

$$\begin{aligned} q_1 &= \int \frac{dz}{\sqrt{4J_2^2 + 4m\Omega C z - m^2\Omega^2 z^2}} \frac{m\Omega}{z}, \\ q_2 &= 2u + \int \frac{dz}{z} \frac{2J_2}{\sqrt{4J_2^2 + 4m\Omega C z - m^2\Omega^2 z^2}}, \\ p_1 &= \mathcal{C}, \\ p_2 &= J_2, \end{aligned} \quad (55)$$

the system’s Hamiltonian Eq. (29) can be rewritten as

$$H = \sum_{i=1}^2 p_i f_i(q), \quad (56)$$

with  $f_1(q) = 2\Omega$ ,  $f_2(q) = -2\Gamma$ . Note that  $\{q_i, p_i\} = 1$ , and the other Poisson brackets are vanishing.

The Hamiltonian Eq. (56) belongs to the class of Hamiltonians considered by ’t Hooft. There, the  $f_i(q)$  are nonsingular functions of the canonical coordinates  $q_i$  and the equations for the  $q$ ’s, namely  $\dot{q}_i = \{q_i, H\} = f_i(q)$ , are decoupled from the conjugate momenta  $p_i$ . It then exists a complete set of observables, called *beables*, which Poisson commute at all times. The meaning of this is that the system admits a deterministic description even when expressed in terms of operators acting on some functional space of states  $|\psi\rangle$ , such as the Hilbert space [14]. We stress that such a description in terms of operators and Hilbert space, does not imply *per se* quantization of the system. As we will see, quantization is achieved only as a consequence of the dissipation of information.

Thus we see that  $J_2$  and  $\mathcal{C}$  are beables (it can be seen from the Hamiltonian Eq. (56) that  $q_1$  and  $q_2$  are also beables). Next we put  $H = H_1 - H_\pi$ , with

$$H_1 = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J_2)^2, \quad H_\pi = \frac{\Gamma^2}{2\Omega\mathcal{C}}J_2^2 \quad (57)$$

and impose the constraint

$$J_2|\psi\rangle = 0, \quad (58)$$

which defines physical states and guaranties that  $H$  is bounded from below.

Due to the constraint Eq. (58) we can then write

$$H|\psi\rangle = H_1|\psi\rangle = 2\Omega\mathcal{C}|\psi\rangle = \left(\frac{1}{2m}p_r^2 + \frac{K}{2}r^2\right)|\psi\rangle, \quad (59)$$

with  $K \equiv m\Omega^2$ . We thus realize that  $H_1$  reduces to the Hamiltonian for the two-dimensional ‘‘isotropic’’ (or ‘‘radial’’) harmonic oscillator  $\ddot{r} + \Omega^2 r = 0$ .

The physical states are invariant under time-reversal ( $|\psi(t)\rangle = |\psi(-t)\rangle$ ) and periodical with period  $\tau = 2\pi/\Omega$ .

The generic state  $|\psi(t)\rangle_H$  can be written as

$$|\psi(t)\rangle_H = \hat{T} \left[ \exp \left( \frac{i}{\hbar} \int_{t_0}^t 2\Gamma J_2 dt' \right) \right] |\psi(t)\rangle_{H_1}, \quad (60)$$

where  $\hat{T}$  denotes time-ordering and the constant  $\hbar$ , with dimension of an action, is needed for dimensional reasons. The states  $|\psi(t)\rangle_H$  and  $|\psi(t)\rangle_{H_1}$  satisfy the equations:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_H = H |\psi(t)\rangle_H, \quad (61)$$

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_{H_1} = 2\Omega\mathcal{C} |\psi(t)\rangle_{H_1}. \quad (62)$$

Note that  $H_1 = 2\Omega\mathcal{C}$  has the spectrum  $\mathcal{H}_1^n = \hbar\Omega n$ ,  $n = 0, \pm 1, \pm 2, \dots$ ; since our choice has been that  $\mathcal{C}$  is positive, only positive values of  $n$  will be considered.

Let us now exploit the periodicity of the physical states  $|\psi\rangle$ . Following Ref. [36], one may generally write

$$\begin{aligned} |\psi(\tau)\rangle &= \exp \left( i\phi - \frac{i}{\hbar} \int_0^\tau \langle \psi(t) | H | \psi(t) \rangle dt \right) |\psi(0)\rangle \\ &= \exp(-i2\pi n) |\psi(0)\rangle, \end{aligned} \quad (63)$$

i.e.,

$$\frac{\langle \psi(\tau) | H | \psi(\tau) \rangle}{\hbar} \tau - \phi = 2\pi n, \quad n = 0, 1, 2, \dots$$

Using  $\tau = 2\pi/\Omega$  and  $\phi = \alpha\pi$  leads to

$$\mathcal{H}_{1,\text{eff}}^n \equiv \langle \psi_n(\tau) | H | \psi_n(\tau) \rangle = \hbar\Omega \left( n + \frac{\alpha}{2} \right). \quad (64)$$

The index  $n$  has been introduced to exhibit the  $n$  dependence of the state and the corresponding energy. We see that  $\mathcal{H}_{1,\text{eff}}^n$  gives the effective  $n$ th energy level of the physical system, namely the energy given by  $\mathcal{H}_1^n$  corrected by its interaction with the environment. We conclude that the dissipation term  $J_2$  of the Hamiltonian is responsible for the zero point ( $n = 0$ ) energy:  $E_0 = (\hbar/2)\Omega\alpha$ .

We remark that in Quantum Mechanics the zero point energy is formally due to the nonzero commutator of the canonically conjugate  $q$  and  $p$  operators: the zero point energy is the ‘‘signature’’ of quantization. Our discussion thus shows that dissipation manifests itself as ‘‘quantization’’. In other words, the (zero point) ‘‘quantum contribution’’  $E_0$  to the spectrum of physical states signals the underlying dissipative dynamics.

Let us consider further the dynamical rôle of  $J_2$ . Using  $u(t) = -\Gamma t$ , Eq. (60) can be rewritten as

$$|\psi(t)\rangle_H = \hat{T} \left[ \exp \left( i \frac{1}{\hbar} \int_{u(t_0)}^{u(t)} 2J_2 du' \right) \right] |\psi(t)\rangle_{H_1}, \quad (65)$$

and we have that

$$-i\hbar \frac{\partial}{\partial u} |\psi(t)\rangle_H = 2J_2 |\psi(t)\rangle_H. \quad (66)$$

Thus,  $2J_2$  induces translations in the  $u$  variable and in operatorial notation one can write  $p_u = -i\hbar(\partial/\partial u)$ . Equation (58) thus defines families of physical states, representing stable, periodic trajectories. Note that  $2J_2$  implements transitions from family to family, according to Eq. (66). Equation (61) can be then rewritten as

$$i\hbar \frac{d}{dt} |\psi(t)\rangle_H = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_H + i\hbar \frac{du}{dt} \frac{\partial}{\partial u} |\psi(t)\rangle_H. \quad (67)$$

The contribution to the energy due to dissipation is thus described by ‘‘translations’’ in the  $u$  variable.

Consider the defining relation for temperature in thermodynamics (with  $k_B = 1$ )

$$\frac{\partial S}{\partial U} = \frac{1}{T}. \quad (68)$$

Using  $S \equiv (2J_2/\hbar)$  and  $U \equiv 2\Omega\mathcal{C}$ , Eq. (56) gives  $T = \hbar\Gamma$ . Provided  $S$  is identified with the entropy,  $\hbar\Gamma$  can be regarded as the temperature. Thus, the ‘‘full Hamiltonian’’ Eq. (56) plays the rôle of the free energy  $\mathcal{F}$ , and  $2\Gamma J_2$  represents the heat contribution in  $H$  (or  $\mathcal{F}$ ). Note that the statement that  $2J_2/\hbar$  behaves as the entropy is not surprising since it controls the dissipative (thus irreversible) part of the dynamics.

It is worth noting that the thermodynamical picture outlined above is also consistent with the results on the canonical quantization of dissipative systems in quantum field theory [25].

## V. NONCOMMUTATIVE GEOMETRY AND THE DISSIPATIVE INTERFERENCE PHASE

We have seen that doubling of the algebra amounts to consider the system, its environment and their reciprocal interaction, which in turn determines the dissipative character of the dynamics ruling the time evolution. The strict relation which exists in Alain Connes’ construction between the doubling of the algebra and the noncommutative geometry, finds a realization in the strict and immediate relation between dissipation and noncommutative geometry in the plane of the doubled coordinates  $(x_1, x_2)$ . The reason is that dissipation implies the appearance of a ‘‘dissipative interference phase’’, a notion which we will clarify in the present section.

Although in the following we consider the example of the damped harmonic oscillator and of its time-reversed image, our conclusions also apply to more general cases.



Since we will consider paths in the doubled coordinate plane, it is convenient to work with the  $(x_+, x_-)$  coordinates, introduced in Section III (which slightly differ in their definition from the  $(x_1, x_2)$  coordinates).

We remark that  $H$  given by Eq. (29) does not change its form when  $x_1, x_2, p_1, p_2$  are replaced by  $x_+, x_-, p_+, p_-$ , respectively. The components in the  $(x_+, x_-)$  plane of forward and backward in time velocity  $v_{\pm} = \dot{x}_{\pm}$  are then obtained as

$$v_{\pm} = \frac{\partial H}{\partial p_{\pm}} = \pm \frac{1}{m} \left( p_{\pm} \mp \frac{\gamma}{2} x_{\mp} \right), \quad (69)$$

and they do not commute

$$[v_+, v_-] = i\hbar \frac{\gamma}{m^2}. \quad (70)$$

It is thus impossible to fix these velocities  $v_+$  and  $v_-$  as being identical [37]. By putting  $mv_{\pm} = \hbar K_{\pm}$ , Eq. (70) gives

$$[K_+, K_-] = \frac{i\gamma}{\hbar} \equiv \frac{i}{L^2}, \quad (71)$$

and a canonical set of conjugate position coordinates  $(\xi_+, \xi_-)$  may be defined by  $\xi_{\pm} = \mp L^2 K_{\mp}$  so that

$$[\xi_+, \xi_-] = iL^2. \quad (72)$$

The commutation relation Eq. (72) characterizes the noncommutative geometry in the plane  $(x_+, x_-)$ .

We now show that an Aharonov–Bohm-type phase interference can always be associated with the noncommutative  $(X, Y)$  plane where

$$[X, Y] = iL^2; \quad (73)$$

$L$  denotes the geometric length scale in the plane [37].

Consider a particle moving in the plane along two paths,  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , starting and finishing at the same point, in a forward and in a backward direction, respectively. Let  $\mathcal{A}$  denote the resulting area enclosed by the paths. We will show that the phase interference  $\vartheta$  may be written as

$$\vartheta = \frac{\mathcal{A}}{L^2}. \quad (74)$$

A phase space action integral

$$\mathcal{S}(\mathcal{P}) = \int_{\mathcal{P}} p_i dq^i, \quad (75)$$

may be associated with each path  $\mathcal{P}$  (in phase space) for motion at fixed energy. The phase interference  $\vartheta$  between the two paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  is given by the difference

$$\vartheta = \frac{1}{\hbar} \int_{\mathcal{P}_1} p_i dq^i - \frac{1}{\hbar} \int_{\mathcal{P}_2} p_i dq^i = \frac{1}{\hbar} \oint_{\mathcal{P}=\partial\Omega} p_i dq^i, \quad (76)$$

with  $\mathcal{P}$  the closed path going from the initial point to the final point via path  $\mathcal{P}_1$  and returning back to the

initial point via  $\mathcal{P}_2$ . It constitutes the boundary of a two-dimensional surface  $\Omega$ :  $\mathcal{P} = \partial\Omega$ . Then, due to Stokes theorem, i.e.,

$$\vartheta = \frac{1}{\hbar} \oint_{\mathcal{P}=\partial\Omega} p_i dq^i = \frac{1}{\hbar} \int_{\Omega} (dp_i \wedge dq^i), \quad (77)$$

the phase interference  $\vartheta$  between two alternative paths turns out to be proportional to the “area”  $\mathcal{A}$  of the surface  $\Omega$  in phase space  $(p_1, \dots, p_f; q^1, \dots, q^f)$ .

Equation (73) in the noncommutative plane can be written as

$$[X, P_X] = i\hbar \quad \text{where} \quad P_X = \left( \frac{\hbar Y}{L^2} \right), \quad (78)$$

and Eq. (77) then reads

$$\vartheta = \frac{1}{\hbar} \int_{\Omega} (dP_X \wedge dX) = \frac{1}{L^2} \int_{\Omega} (dY \wedge dX), \quad (79)$$

which proves Eq. (74), i.e., the quantum phase interference between two alternative paths in the plane is determined by the noncommutative length scale  $L$  and the enclosed area  $\mathcal{A}$ .

Notice that the existence of a phase interference is connected to the zero point fluctuations in the coordinates; indeed Eq. (73) implies a zero point uncertainty relation  $(\Delta X)(\Delta Y) \geq L^2/2$ .

For Eq. (71) in the dissipative case, i.e.,

$$L^2 = \frac{\hbar}{\gamma}, \quad (80)$$

we then conclude that, provided  $x_+ \neq x_-$ , the quantum dissipative phase interference  $\vartheta = \mathcal{A}/L^2 = \mathcal{A}\gamma/\hbar$  is associated with the two paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  in the noncommutative plane.

## VI. CONCLUSIONS

We have considered the implications of the central ingredient in Alain Connes’ noncommutative spectral geometry construction — which provides a purely geometric explanation of the Standard Model — namely the doubling of the algebra  $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$  acting on the space  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ . Firstly, we have shown that the doubling of the algebra is strictly related to dissipation and the gauge field structure. As a result, the two-sheeted geometry must not be considered as just a simple almost commutative space, which is the simplest generalization beyond commutative geometries, but instead, the construction which can lead to gauge fields, required to explain the Standard Model. To be more precise, the two-sheeted space is not an assumption but a requirement so that gauge fields can be naturally accommodated. Secondly, by exploiting ’t Hooft’s conjecture, according which loss of information within the framework of completely deterministic dynamics, might lead to a quantum evolution,

we have argued that dissipation, implied by the algebra doubling, may lead to quantum features. We have thus suggested that Alain Connes' classical construction carries *implicit* in the doubling of the algebra the seeds of quantization.

We have shown that in Alain Connes' two-sheeted construction, the doubled degree of freedom is associated with "unlikely processes" in the classical limit. Thus, in the perturbative approach one may drop higher order terms in the expansion, since they correspond to unlikely processes at the classical level. However, since the higher order terms in the expansion are the ones responsible for dissipation (quantum corrections), the second sheet cannot be neglected at the classical level, if one does not want to preclude quantization effects. Put it differently, the second sheet — representing gauge fields — cannot be neglected once the universe entered the radiation dominated era. However, at the Grand Unified Theories scale, when inflation took place, the effect of gauge fields, in other words the discrete space of two points, is fairly shielded.

At the end of Section III we have mentioned that in the formalism of the algebra doubling the deformed Hopf algebra plays a relevant rôle. Let us close with a final comment on this point. One central ingredient of Hopf algebras is the operator doubling implied by the coalgebra. The map  $\mathcal{A} \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$  in Eq. (1) is just the Hopf coproduct map  $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A} \equiv \mathcal{A}_1 \otimes \mathcal{A}_2$  which duplicates the algebra. We have also recalled in Section III that the doubled space of states  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  is constructed out of the zero energy eigenstate  $|0(\theta)\rangle$  of the doubled Hamiltonian system by repeated application of the creation operators of the fields  $\psi(\theta; x)$  and  $\tilde{\psi}(\theta; x)$  obtained from  $\psi(x)$  and  $\tilde{\psi}(x)$ , respectively, by means of the Bogoliubov transformation of "angle"  $\theta$ . We now remark that Bogoliubov transformations are known to be obtained by convenient combinations of the *deformed* coproduct operation of the

form  $\Delta a_q^\dagger = a_q^\dagger \otimes q^{1/2} + q^{-1/2} \otimes a_q^\dagger$ , where  $q \equiv q(\theta)$  is the deformation parameters and  $a_q^\dagger$  are the creation operators in the  $q$ -deformed Hopf algebra [33]. These deformed coproduct maps are noncommutative and the deformation parameter is related to the system life-time in its dissipative evolution or to the condensate content of  $|0(\theta)\rangle$  (constrained by the  $\theta$ -state condition Eq. (42)). In this connection it is interesting to observe that the  $q$ -derivative is a finite difference derivative, which has to be compared with the fact that in Alain Connes' construction the derivative in the discrete direction is a finite difference quotient, as mentioned in Section II.

A relevant point is that the deformation parameter *labels* the  $\theta$ -representations  $\{|0(\theta)\rangle\}$  and, for  $\theta \neq \theta'$ ,  $\{|0(\theta)\rangle\}$  and  $\{|0(\theta')\rangle\}$  are unitarily inequivalent representations of the canonical (anti-)commutation rules. This is a characteristic feature of quantum field theory [26, 32]. Its physical meaning is that an order parameter exists, which assumes different  $\theta$ -dependent values in each of the representations. In other words, the deformed Hopf algebra structure induces the *foliation* of the whole Hilbert space into physically inequivalent subspaces where symmetry breakdown occurs, the broken symmetry vacuum being characterized by the specific value of the order parameter. Variations in the order parameter (derivatives in the deformation parameter, or, in the language of Section IV, translations in the  $u$  parameter clasifying 't Hooft families of states) thus describe phase transitions in the system evolution. We thus see how the basic ingredient of the doubling of the algebra in Alain Connes' construction has built in, not only the seed of quantization, but also the noncommutative deformation of Hopf algebra, and has the far reaching physical consequence of the spontaneous breakdown of symmetry, as in fact observed and crucial in the Standard Model.

- 
- [1] A. Connes, *Noncommutative Geometry*, Academic Press, New York (1994).
- [2] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Hindustan Book Agency, India (2008).
- [3] A. H. Chamseddine, A. Connes and M. Marcolli, *Adv. Theor. Math. Phys.* **11**, 991 (2007) [arXiv:hep-th/0610241].
- [4] T. v. d. Broek and W. D. van Suijlekom, [arXiv:1003.3788 [hep-th]].
- [5] W. Nelson and M. Sakellariadou, *Phys. Rev. D* **81**, 085038 (2010) [arXiv:0812.1657 [hep-th]].
- [6] W. Nelson and M. Sakellariadou, *Phys. Lett. B* **680**, 263 (2009) [arXiv:0903.1520 [hep-th]].
- [7] M. Marcolli and E. Pierpaoli, arXiv:0908.3683 [hep-th].
- [8] M. Buck, M. Fairbairn and M. Sakellariadou, *Phys. Rev. D* **82** 043509 (2010) [arXiv:1005.1188 [hep-th]].
- [9] W. Nelson, J. Ochoa and M. Sakellariadou, *Phys. Rev. Lett.* **105** (2010) 101602 [arXiv:1005.4279 [hep-th]].
- [10] W. Nelson, J. Ochoa and M. Sakellariadou, *Phys. Rev. D* **82** (2010) 085021 [arXiv:1005.4276 [hep-th]].
- [11] M. Sakellariadou, *Int. J. Mod. Phys. D* **20** 785 (2011) [arXiv:1008.5348 [hep-th]].
- [12] M. Sakellariadou, *PoS CNCFG* **2010** 028 (2010) [arXiv:1101.2174 [hep-th]].
- [13] G. 't Hooft, *Class. Quant. Grav.* **16** 3263 (1999) [arXiv:gr-qc/9903084].
- [14] G. 't Hooft, in "*Basics and Highlights of Fundamental Physics*", Erice, (1999) [arXiv:hep-th/0003005].
- [15] G. 't Hooft, *J. Phys.: Conf. Ser.* **67** 012015 (2007) [arXiv:quant-ph/0604008].
- [16] A. H. Chamseddine and A. Connes, *Phys. Rev. Lett.* **99**, 191601 (2007) [arXiv:0706.3690 [hep-th]].
- [17] A. H. Chamseddine and A. Connes, *J. Math. Phys.* **47**, 063504 (2006).
- [18] A. H. Chamseddine and A. Connes, *Phys. Rev. Lett.* **77**, 4868 (1996).
- [19] A. H. Chamseddine and A. Connes, *Comm. Math. Phys.*

- 186**, 731 (1977).
- [20] A. H. Chamseddine and A. Connes, *Comm. Math. Phys.* **293**, 867 (2010) [arXiv:0812.0165 [hep-th]].
- [21] R.P.Feynman, *Statistical Mechanics: A set of Lectures*, W.A. Benjamin, Reading, MA (1972).
- [22] M. Blasone, Y. N. Srivastava, G. Vitiello and A. Widom, *Annals Phys.* **267**, 61 (1998).
- [23] H. Bateman, *Phys. Rev.* **38**, 815 (1931).
- [24] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill Book Co. (1953).
- [25] E. Celeghini, M. Rasetti and G. Vitiello, *Annals Phys.* **215** 156 (1992).
- [26] M. Blasone, P. Jizba and G. Vitiello, *Quantum Field Theory and its macroscopic manifestations*, Imperial College Press, London, (2011)
- [27] Y. N. Srivastava, G. Vitiello and A. Widom, *Annals Phys.* **238**, 200 (1995) [arXiv:hep-th/9502044].
- [28] Y. Tsue, A. Kuriyama and M. Yamamura, *Prog. Theor. Phys.* **91**, 469 (1994).
- [29] M. Blasone, E. Graziano, O. K. Pashaev and G. Vitiello, *Annals Phys.* **252** 115 (1996) [arXiv:hep-th/9603092].
- [30] E. Celeghini, E. Graziano, K. Nakamura and G. Vitiello, *Phys. Lett.* **B285**, 98 (1992).
- [31] E. Celeghini, E. Graziano, K. Nakamura and G. Vitiello, *Phys. Lett.* **B304**, 121 (1993).
- [32] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and condensed states*, North-Holland, Amsterdam, The Netherlands (1982).
- [33] E. Celeghini, S. De Martino, S. De Siena, A. Iorio, M. Rasetti and G. Vitiello, *Phys. Lett.* **A244**, 455 (1998).
- [34] M. Blasone, P. Jizba and G. Vitiello, *Phys. Lett. A* **287** 205 (2001) [arXiv:hep-th/0007138].
- [35] M. Blasone, E. Celeghini, P. Jizba and G. Vitiello, *Phys. Lett. A* **310**, 393 (2003) [arXiv:quant-ph/0208012]
- [36] M. V. Berry, *Proc. Roy. Soc. Lond.* **A392**, 45 (1984).
- [37] S. Sivasubramanian, Y. Srivastava, G. Vitiello and A. Widom, *Phys. Lett.* **A311**, 97 (2003) [arXiv:quant-ph/0301005].