Inverse problems for Jacobi operators II: Mass perturbations of semi-infinite mass-spring systems

Rafael del Rio
Departamento de Métodos Matemáticos y Numéricos
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
C.P. 04510, México D.F.
delrio@leibniz.iimas.unam.mx

Mikhail Kudryavtsev
Department of Mathematics
Institute for Low Temperature Physics and Engineering
Lenin Av. 47, 61103
Kharkov, Ukraine
kudryavtsev@onet.com.ua

Luis O. Silva*
Departamento de Métodos Matemáticos y Numéricos
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
C.P. 04510, México D.F.
silva@leibniz.iimas.unam.mx

Abstract

We consider an inverse spectral problem for infinite linear mass-spring systems with different configurations obtained by changing the first mass. We give results on the reconstruction of the system from the spectra of two configurations. Necessary and sufficient conditions for two real sequences to be the spectra of two modified systems are provided.

Mathematics Subject Classification(2010): 34K29, 47A75, 47B36, 70F17,
Keywords: Infinite mass-spring system; Jacobi matrices; Two-spectra inverse problem.
*Partially supported by CONACYT (México) through grant CB-2008-01-99100

1. Introduction

The inverse problem of reconstructing a Jacobi operator from its spectrum and the spectrum of a rank-one perturbation has been amply studied in both the finite (see [5, 6, 9, 11, 14]) and semi-infinite cases (see [8, 12, 13, 19, 23]). In the semi-infinite case, a Jacobi operator, denoted by J, is the operator in the Hilbert space $l_2(\mathbb{N})$ whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is a Jacobi matrix of the form

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $q_n \in \mathbb{R}$ and $b_n > 0$ for any $n \in \mathbb{N}$. J has always self-adjoint extensions denoted by $J^{(g)}$, where $g \in \mathbb{R} \cup \{\infty\}$. This includes the case $J = J^*$, where $J^{(g)} = J$ for all $g \in \mathbb{R} \cup \{\infty\}$ (see in Section 2 the precise definition of the operators). Thus, the two spectra inverse problem takes as input data the spectra of two rank-one perturbations of $J^{(g)}$ while the solution of it is the finding of the matrix (1.1) and the "boundary condition at infinity" g if necessary. Rank-one perturbations can be seen as a change of the "boundary condition at the origin" for the corresponding difference equation (see [19, Appendix]).

In this work we treat the two spectra inverse problem for two Jacobi operators which are not obtained from each other by a rank-one perturbation. The particular kind of perturbation studied here has a physical motivation; it is the extension to the semi-infinite case of an inverse problem for finite mass-spring systems studied in [18] and [7].

It is known that the dynamics of a finite mass-spring system is characterized by the spectral properties of a finite Jacobi matrix [11]. Accordingly, in solving the inverse problem for mass-spring systems mentioned above, [18] provides necessary and sufficient conditions for two point sets to be the spectra of two finite Jacobi matrices corresponding to two mass-spring systems, one of which has a mass and a spring modified. The results of [18] are related to the study of microcantilevers [21, 22], which are modeled by a spring-mass system whose masses and springs constants correspond to the mechanical parameters of the system. The inverse problem treated in [18] could be used as a theoretical framework for the problem of measuring micromasses with a help of microcantilevers [21, 22].

Let us consider a semi-infinite spring-mass system with masses $\{m_j\}_{j=1}^{\infty}$ and spring constants $\{k_j\}_{j=1}^{\infty}$ as in Figure 1. By a standard reasoning (see [11, 17]) one verifies that the infinite system of Fig. 1 is modeled by the spectral properties of the

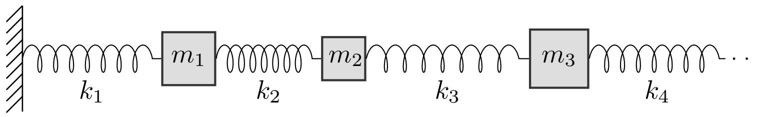

Figure 1: Semi-infinite mass-spring system

Jacobi operator J with

$$q_j = -\frac{1}{m_j}\left(\frac{k_{j+1}}{l_{j+1}} + \frac{k_j}{l_j}\right), \qquad b_j = \frac{k_{j+1}}{l_{j+1}\sqrt{m_i m_{j+1}}}, \qquad j \in \mathbb{N},$$

where l_j is the length of the j-th spring at equilibrium. We remark that in [11, 17] the obtained matrix corresponds to $-J$. An alternative physical interpretation is provided by a one dimensional harmonic crystal [24, Sec. 1.5].

In this work we consider the spectrum of $J^{(g)}$ to be discrete (if $J \neq J^*$ this is always the case). Then, the movement of our mechanical system is a superposition of harmonic oscillations whose frequencies are the square roots of the modules of the eigenvalues.

Along with the self-adjoint operator $J^{(g)}$ we consider the family of operators $J^{(g)}(\theta)$ $(\theta > 0)$ being self-adjoint extensions of the Jacobi operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is

$$\begin{pmatrix} \theta^2 q_1 & \theta b_1 & 0 & 0 & \cdots \\ \theta b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}. \tag{1.2}$$

Going from $J^{(g)}$ to $J^{(g)}(\theta)$ corresponds to changing the first mass by $\Delta m = m_1(\theta^{-2} - 1)$. In other words, θ^2 is the ratio of the original mass m_1 to the new mass $m_1 + \Delta m$. This is illustrated in Fig. 2. It is worth mentioning that we also consider here the cases when $\Delta m < 0$, equivalently, $\theta > 1$, although physical applications correspond to $\theta < 1$ [21, 22].

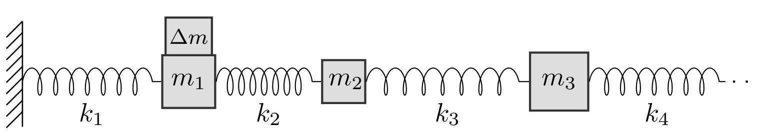

Figure 2: Perturbed semi-infinite mass-spring system

The problem of reconstructing the initial and the perturbed matrices by their spectra can be then interpreted from the physical point of view as the problem of finding the mechanical parameters of the spring-mass system from the frequencies of its oscillations before and after the modification.

This work is organized as follows. In Section 2 we lay down the notation, introduce the Jacobi operators and its perturbations, and present some preparatory facts related with the inverse spectral problems of such operators. Section 3 gives an account of the spectral properties of the family of perturbed Jacobi operators. The problem of reconstruction is treated in Section 4. Finally, Section 5 gives necessary and sufficient conditions for two sequences of real numbers to be the spectra of $J^{(g)}$ and its perturbation.

2. Preliminaries

Let Υ be a second order symmetric difference expression such that for any sequence $f = \{f_k\}_{k\in\mathbb{N}}$

$$(\Upsilon f)_1 := q_1 f_1 + b_1 f_2\,, \tag{2.1}$$

$$(\Upsilon f)_k := b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1}\,, \quad k \in \mathbb{N}\setminus\{1\}, \tag{2.2}$$

where, for $n \in \mathbb{N}$, b_n is positive and q_n is real. Let $l_{\text{fin}}(\mathbb{N})$ be the linear space of complex sequences with a finite number of non-zero elements. In the Hilbert space $l_2(\mathbb{N})$, let us consider the operator whose domain is $l_{\text{fin}}(\mathbb{N})$ and acts as the expression Υ. This operator is symmetric since it is densely defined and Hermitian, and thus it is closable. Now, let J be the closure of this operator.

We have defined the operator J so that the semi-infinite Jacobi matrix (1.1) is its matrix representation with respect to the canonical basis $\{\delta_n\}_{n=1}^{\infty}$ in $l_2(\mathbb{N})$ (see [2, Sec. 47] for the definition of the matrix representation of an unbounded symmetric operator). Indeed, J is the minimal closed symmetric operator satisfying

$$\begin{array}{ll}\langle \delta_n, J\delta_n\rangle = q_n\,, \quad \langle \delta_{n+1}, J\delta_n\rangle = \langle \delta_n, J\delta_{n+1}\rangle = b_n\,, & \\ \langle J\delta_n, \delta_{n+k}\rangle = \langle \delta_n, J\delta_{n+k}\rangle = 0\,, & \end{array} \quad n\in\mathbb{N}\,,\ k\in\mathbb{N}\setminus\{1\}\,.$$

We shall refer to J as the *Jacobi operator* and to (1.1) as its associated matrix.

The operator J^* turns out to be given by

$$\operatorname{dom}(J^*) = \{f \in l_2(\mathbb{N}) : \Upsilon f \in l_2(\mathbb{N})\} \qquad J^* f = \Upsilon f\,,$$

which follows directly from the definition of J [1, Chap. 4 Sec. 1.1], [20, Thm. 2.7].

If one gives the complex number f_1, the solution of the difference equation,

$$(\Upsilon f) = \zeta f\,, \qquad \zeta\in\mathbb{C}\,,$$

is uniquely determined from (2.1) and (2.2) by recurrence. For the elements of this solution when $f_1 = 1$, the following notation is standard [1, Chap. 1, Sec. 2.1]

$$P_{n-1}(\zeta) := f_n\,, \qquad n \in \mathbb{N}\,,$$

where the polynomial $P_k(\zeta)$ (of degree k) is referred to as the k-th orthogonal polynomial of the first kind associated with the matrix (1.1). The polynomials of the second kind $Q_k(\zeta)$ are defined as the solutions of

$$(\Upsilon f)_k = \zeta f_k \qquad k \in \mathbb{N} \setminus \{1\}$$

under the assumption that $f_1 = 0$ and $f_2 = b_1^{-1}$.

The sequence $P(\zeta) := \{P_{k-1}(\zeta)\}_{k=1}^{\infty}$ is not in $l_{\text{fin}}(\mathbb{N})$, but it may happen that

$$\sum_{k=0}^{\infty} |P_k(\zeta)|^2 < \infty\,, \tag{2.3}$$

in which case $P(\zeta) \in \ker(J^* - \zeta I)$. Since J is symmetric, if the series in (2.3) is convergent for one ζ in the upper half plane $\mathbb{C}_+$ (the lower half plane $\mathbb{C}_-$), then it is convergent in all $\mathbb{C}_+$ ($\mathbb{C}_-$). Actually, because of the reality of $P_{k-1}(\zeta)$ for all $k \in \mathbb{N}$, the series in (2.3) will then be convergent in all $\mathbb{C} \setminus \mathbb{R}$ and J has deficiency indices $(1, 1)$. When the series in (2.3) is divergent for one ζ in $\mathbb{C}\setminus\mathbb{R}$, J has deficiency indices $(0, 0)$ and the operator is self-adjoint since J is closed. There are known conditions on the matrix (1.1) which guarantee that J is self-adjoint [1, Chap. 1, Addenda and Problems], [3, Chap. 7, Thms. 1.2–1.4].

Let us now introduce the operators that will be at the center of our considerations in this work.

Consider first the case $J \neq J^*$. Define the sequence $v(g) = \{v_k(g)\}_{k=1}^{\infty}$ such that $\forall k \in \mathbb{N}$

$$v_k(g) := P_{k-1}(0) + gQ_{k-1}(0)\,, \quad g \in \mathbb{R}$$

and

$$v_k(\infty) := Q_{k-1}(0)\,.$$

Let $J^{(g)}$ be the restriction of J^* to the set

$$\left\{ f = \{f_k\}_{k\in\mathbb{N}} \in \operatorname{dom}(J^*) : \lim_{k\to\infty} b_k(v_k(g)f_{k+1} - f_k v_{k+1}(g)) = 0 \right\}\,.$$

When $g \in \mathbb{R}\cup\{\infty\}$, $J^{(g)}$ runs over all self-adjoint extensions of J. Moreover, different values of g imply different self-adjoint extensions [24, Lemma 2.20].

In the case $J = J^*$, let us define $J^{(g)} := J$ for all $g \in \mathbb{R} \cup \{\infty\}$.

Alongside the operator $J^{(g)}$ we consider the operators $J_n^{(g)}$ ($n \in \mathbb{N}$) defined as the restriction of $J^{(g)}$ to $l_2(\mathbb{N}) \ominus \operatorname{span}\{\delta_1, \dots, \delta_n\}$. Thus, $J_n^{(g)}$ is a self-adjoint extension of the Jacobi operator whose associated matrix is (1.1) with the first n columns and n rows removed.

Finally we introduce the perturbed operators $J^{(g)}(\theta)$. They are defined as follows.

Consider $J^{(g)}$ with fixed $g \in \mathbb{R} \cup \{\infty\}$ and take any $\theta > 0$. Then

$$J^{(g)}(\theta) := J^{(g)} + q_1(\theta^2 - 1)\,\langle \delta_1, \cdot \rangle\, \delta_1 + b_1(\theta - 1)(\langle \delta_1, \cdot \rangle\, \delta_2 + \langle \delta_2, \cdot \rangle\, \delta_1)\,,$$

where we take the inner product to be antilinear in its first argument. By this definition $J^{(g)}(\theta)$ is a self-adjoint extension of the Jacobi operator whose associated matrix is (1.2). Note that $J^{(g)}(\theta)$ is a finite-rank perturbation of $J^{(g)}$ and thus $\mathrm{dom}(J^{(g)}) = \mathrm{dom}(J^{(g)}(\theta))$.

Fix $g \in \mathbb{R} \cup \{\infty\}$ and take the resolution of the identity $E^{(g)}(t)$ of $J^{(g)}$, so

$$J^{(g)} = \int_{\mathbb{R}} t dE^{(g)}(t)\,.$$

Since $J^{(g)}$ is simple [1, Sec. 2.2 Chap. 4], it is particularly useful to consider the function

$$\rho^{(g)}(t) := \left\langle \delta_1, E^{(g)}(t)\delta_1 \right\rangle\,, \quad t \in \mathbb{R}\,. \tag{2.4}$$

It turns out that all the moments of the measure generated by $\rho^{(g)}$ are finite [1, Thm. 4.1.3], that is,

$$s_k = \int_{\mathbb{R}} t^k d\rho^{(g)}(t) < \infty \qquad \forall k \in \mathbb{N} \cup \{0\}\,, \tag{2.5}$$

and the polynomials are dense in $L_2(\mathbb{R}, d\rho^{(g)})$ [1, Thms. 2.3.2, 4.1.4], [20, Prop. 4.15].

In this work we also make use of the so-called Weyl m-function

$$m^{(g)}(\zeta) := \left\langle \delta_1, (J^{(g)} - \zeta I)^{-1}\delta_1 \right\rangle\,, \quad \zeta \notin \sigma(J^{(g)})\,. \tag{2.6}$$

The functions (2.4) and (2.6) are related by the Borel transform, viz.,

$$m^{(g)}(\zeta) = \int_{\mathbb{R}} \frac{d\rho^{(g)}(t)}{t - \zeta}\,,$$

so $m^{(g)}$ is a Herglotz function, i. e.,

$$\frac{\operatorname{Im} m^{(g)}(\zeta)}{\operatorname{Im} \zeta} > 0\,, \qquad \operatorname{Im} \zeta > 0\,.$$

Using the von Neumann expansion for the resolvent (cf.[24, Chap. 6, Sec. 6.1])

$$(J^{(g)} - \zeta I)^{-1} = -\sum_{k=0}^{N-1} \frac{(J^{(g)})^k}{\zeta^{k+1}} + \frac{(J^{(g)})^N}{\zeta^N}(J^{(g)} - \zeta I)^{-1}\,,$$

where $\zeta \in \mathbb{C} \setminus \sigma(J^{(g)})$, one can easily obtain the following asymptotic formula

$$m^{(g)}(\zeta) = -\frac{1}{\zeta} - \frac{q_1}{\zeta^2} - \frac{b_1^2 + q_1^2}{\zeta^3} + O(\zeta^{-4}) \,, \tag{2.7}$$

as $\zeta \to \infty$ ($\operatorname{Im} \zeta \geq \epsilon$, $\epsilon > 0$).

The inverse Stieltjes transform allows to recover the spectral function (2.4) from its corresponding Weyl m-function (2.6). So they are in one-to-one correspondence. Furthermore, either (2.4) or (2.6) uniquely determines the Jacobi operator $J^{(g)}$, i. e., the matrix (1.1) and the parameter g in the non-self-adjoint case. Indeed, there are two general methods for recovering the matrix (1.1) that work without any assumption on the spectrum. One method, developed in [9] (see also [23]), makes use of the asymptotic behavior of the Weyl m-function and the Riccati equation [9, Eq. 2.15], [23, Eq. 2.23],

$$b_n^2 m_n^{(g)}(\zeta) = q_n - \zeta - \frac{1}{m_{n-1}^{(g)}(\zeta)} \,, \quad n \in \mathbb{N} \,, \tag{2.8}$$

where $m_n^{(g)}(\zeta)$ is the Weyl m-function of the Jacobi operator $J_n^{(g)}$ ($m_0 = m$).

The other method of reconstruction (see [3, Chap. 7, Sec. 1.5, particularly, Thm. 1.11]) has its starting point in the sequence $\{t^k\}_{k=0}^\infty$, $t \in \mathbb{R}$. From (2.5) all the elements of the sequence $\{t^k\}_{k=0}^\infty$ are in $L_2(\mathbb{R}, d\rho^{(g)})$ and one can apply, in this Hilbert space, the Gram-Schmidt procedure of orthonormalization to the sequence $\{t^k\}_{k=0}^\infty$. One, thus, obtains a sequence of polynomials $\{P_k(t)\}_{k=0}^\infty$ normalized and orthogonal in $L_2(\mathbb{R}, d\rho^{(g)})$. These polynomials satisfy a three term recurrence equation [3, Chap. 7, Sec. 1.5], [20, Sec. 1]

$$tP_{k-1}(t) = b_{k-1}P_{k-2}(t) + q_k P_{k-1}(t) + b_k P_k(t) \,, \quad k \in \mathbb{N} \setminus \{1\} \,, \tag{2.9}$$

$$tP_0(t) = q_1 P_0(t) + b_1 P_1(t) \,, \tag{2.10}$$

where all the coefficients b_k ($k \in \mathbb{N}$) turn out to be positive and q_k ($k \in \mathbb{N}$) are real numbers. The system (2.9) and (2.10) defines a Jacobi matrix which is either the matrix representation of either $J^{(g)}$ or a restriction of $J^{(g)}$ depending on whether $J \neq J^*$.

The function (2.6), equivalently (2.4), determines the parameter g which defines the self-adjoint extension when the reconstructed matrix turns out to be the matrix representation of a non-self-adjoint operator. Indeed, consider a pole γ of $m^{(g)}$ (there is always one when $J \neq J^*$) and evaluate $P_k(\gamma)$, $k \in \mathbb{N}$. Then either

$$\lim_{k\to\infty} b_k (Q_{k-1}(0) P_k(\gamma) - P_{k-1}(\gamma) Q_{k-1}(0)) = 0$$

which means that $g = \infty$, or

$$g = \frac{\lim_{k\to\infty} b_k(P_{k-1}(0)P_k(\gamma) - P_{k-1}(\gamma)P_{k-1}(0))}{\lim_{k\to\infty} b_k(Q_{k-1}(0)P_k(\gamma) - P_{k-1}(\gamma)Q_{k-1}(0))}\,.$$

The details of this recipe are explained for instance in [19, Sec. 2].

Since any simple self-adjoint operator in a infinite dimensional Hilbert space is unitarily equivalent to some operator $J = J^*$ [1, Thm. 4.2.3],[2, Sec. 69], in the case $J = J^*$, $\sigma(J^{(g)})$ may be any non-empty closed infinite set in $\mathbb{R}$. In particular $J^{(g)}$ may have discrete spectrum, that is, $\sigma_{ess}(J^{(g)}) = \emptyset$. When $J \neq J^*$, this is always the case, that is all self-adjoint extensions $J^{(g)}$ of the non-self-adjoint operator J have discrete spectrum [24, Lem. 2.19].

Assume that J has discrete spectrum (this always happen if $J \neq J^*$), so the spectrum is a sequence of real numbers, $\{\lambda_k\}_k$, without finite points of accumulation. The simplicity of $J^{(g)}$ implies that all eigenvalues are of multiplicity one. In this case the function $\rho^{(g)}(t)$, defined by (2.4), can be written as follows

$$\rho^{(g)}(t) = \sum_{\lambda_k < t} \frac{1}{\alpha_k}\,, \tag{2.11}$$

where the coefficients $\{\alpha_k\}_k$ are called the normalizing constants and according to [3, Chap. 7, Thm. 1.17] are given by

$$\alpha_n = \sum_{k=0}^{\infty} |P_k(\lambda_n)|^2\ . \tag{2.12}$$

Thus, from (2.11) and (2.6) one has that

$$m^{(g)}(\zeta) = \sum_k \frac{1}{\alpha_k(\lambda_k - \zeta)}\,. \tag{2.13}$$

Remark 1. In the case of discrete spectrum, the set of poles of the meromorphic Weyl m-function coincides with $\sigma(J^{(g)})$. By (2.8), the set of zeros coincides with $\sigma(J_1^{(g)})$. The zeros and poles of the Weyl m-function are simple and interlace as occurred to any meromorphic Herglotz function. Interlacing means that between two contiguous poles there is exactly one zero.

Remark 2. By elementary perturbation theory (Weyl theorem), $J^{(g)}$ has discrete spectrum if and only if $J^{(g)}(\theta)$ has discrete spectrum. Note that $J^{(g)}(\theta)$ has simple spectrum since it is a self-adjoint extension of a Jacobi operator.

3. Direct spectral analysis of $J^{(g)}$ and $J^{(g)}(\theta)$

We begin this section by noting that

$$J_1^{(g)} = J_1^{(g)}(\theta)\,, \qquad \forall \theta > 0\,.$$

Fix $g \in \mathbb{R} \cup \{\infty\}$ and consider the Weyl m-functions $m^{(g)}$, $\widehat{m}^{(g)}$ of the operators $J^{(g)}$ and $J^{(g)}(\theta)$. Therefore, taking into account that $m_1^{(g)}$ and $\widehat{m}_1^{(g)}$ coincide, (2.8) implies that

$$\theta^2 \left(\zeta + \frac{1}{m^{(g)}(\zeta)} \right) = \zeta + \frac{1}{\widehat{m}^{(g)}(\zeta)}\,, \tag{3.1}$$

Let us now consider the function

$$\mathfrak{m}(\zeta) := \frac{m^{(g)}(\zeta)}{\widehat{m}^{(g)}(\zeta)}\,. \tag{3.2}$$

Remark 3. In view of Remark 2, if $J^{(g)}$ has discrete spectrum, the function $\mathfrak{m}$ is meromorphic by (3.2). Since the zeros of $m^{(g)}$ and $\widehat{m}^{(g)}$ are the same (see Remark 1), it follows that for all $\theta > 0$ the set of poles of $\mathfrak{m}$ is a subset of $\sigma(J^{(g)})$, while $\sigma(J^{(g)}(\theta))$ contains all the zeros of $\mathfrak{m}$. Observe also that, from (3.1), $0 \in \sigma(J^{(g)})$ if and only if $0 \in \sigma(J^{(g)}(\theta))$. Moreover, whenever $\theta \neq 1$, (3.1) implies that the sets $\sigma(J^{(g)})$ and $\sigma(J^{(g)}(\theta))$ can intersect only at 0.

Remark 4. By [15, Chap. 7, Sec. 3.5, Thm. 3.9] the zeros of $\mathfrak{m}$ are analytic functions of the parameter θ. The same is true for the eigenvectors of $J^{(g)}(\theta)$.

Proposition 3.1. *Let $J^{(g)}$ have discrete spectrum and let $\{\lambda_k(\theta)\}_k$ be the set of eigenvalues of $J^{(g)}(\theta)$ ($\theta > 0$). For a fixed k the following holds*

$$\frac{d}{d\theta}\lambda_k(\theta) = \frac{2\lambda_k(\theta)}{\theta \alpha_k(\theta)}\,,$$

where $\alpha_k(\theta)$ is the normalizing constant corresponding to $\lambda_k(\theta)$.

Proof. Let us denote by $f(\theta)$ the eigenvector of $J^{(g)}(\theta)$ corresponding to $\lambda_k(\theta)$. We assume that $f(\theta)$ is normalized in such a way that

$$\langle \delta_1, f(\theta) \rangle = 1\,. \tag{3.3}$$

Pick any small real τ (it suffices that $|\tau| < \theta$). Then, taking into account that $\mathrm{dom}(J^{(g)}) = \mathrm{dom}(J^{(g)}(\theta))$ and the self-adjointness of $J^{(g)}(\theta)$ for any $\theta > 0$, we have that

$$
\begin{aligned}
(\lambda_k(\theta+\tau)-\lambda_k(\theta))\left\langle f(\theta), f(\theta+\tau)\right\rangle &= \left\langle f(\theta), J^{(g)}(\theta+\tau) f(\theta+\tau)\right\rangle \\
&\quad - \left\langle J^{(g)}(\theta) f(\theta), f(\theta+\tau)\right\rangle \\
&= \left\langle f(\theta), (J^{(g)}(\theta+\tau) - J^{(g)}(\theta) + J^{(g)}(\theta)) f(\theta+\tau)\right\rangle \\
&\quad - \left\langle J^{(g)}(\theta) f(\theta), f(\theta+\tau)\right\rangle \\
&= \left\langle f(\theta), (J^{(g)}(\theta+\tau) - J^{(g)}(\theta)) f(\theta+\tau)\right\rangle .
\end{aligned}
$$

From (3.3) it follows that the entries $f(\theta+\tau)$ and $f(\theta)$ are the polynomials of the first kind associated to the matrix of $J^{(g)}(\theta+\tau)$ and $J^{(g)}(\theta)$, so

$$
f_2(\theta+\tau) = \frac{\lambda_k(\theta+\tau) - (\theta+\tau)^2 q_1}{(\theta+\tau) b_1}\,, \qquad f_2(\theta) = \frac{\lambda_k(\theta) - \theta^2 q_1}{\theta b_1}\,,
$$

Now, taking into account these last equalities and (3.3), together with

$$
J^{(g)}(\theta+\tau) - J^{(g)}(\theta) = \begin{pmatrix} (2\theta\tau+\tau^2) q_1 & \tau b_1 & 0 & 0 & \cdots \\ \tau b_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} ,
$$

one obtains that

$$
(\lambda_k(\theta+\tau)-\lambda_k(\theta))\left\langle f(\theta), f(\theta+\tau)\right\rangle = \tau\left(\frac{\lambda_k(\theta+\tau)}{\theta+\tau} + \frac{\lambda_k(\theta)}{\theta}\right)
$$

Therefore, on the basis of Remark 4, one has

$$
\lim_{\tau\to 0}\frac{\lambda_k(\theta+\tau)-\lambda_k(\theta)}{\tau} = \lim_{\tau\to 0}\frac{1}{\left\langle f(\theta), f(\theta+\tau)\right\rangle}\left(\frac{\lambda_k(\theta+\tau)}{\theta+\tau} + \frac{\lambda_k(\theta)}{\theta}\right) = \frac{2\lambda_k(\theta)}{\theta\alpha_k(\theta)}\,.
$$

□

The proposition below can be proven by means of Remark 3, 4, and Proposition 3.1. However, we present an alternative proof based on the following expression

$$
\mathfrak{m}(\zeta) = \zeta(\theta^2-1) m^{(g)}(\zeta) + \theta^2\,, \tag{3.4}
$$

which follows from (3.1) and (3.2).

Proposition 3.2. *Fix $g \in \mathbb{R}\cup\{\infty\}$ and let $J^{(g)}$ have discrete spectrum. The spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ interlace in $\mathbb{R}_+$ and $\mathbb{R}_-$. Moreover, $\sigma(J^{(g)}(\theta))$ in $\mathbb{R}_+$ ($\mathbb{R}_-$) is shifted with respect to $\sigma(J^{(g)})$ to the left (right) if $\theta < 1$, and to the right (left) if $\theta > 1$.*

Proof. Take two positive and contiguous eigenvalues of $\sigma(J^{(g)})$, $\lambda < \widetilde{\lambda}$. Due to (2.13), one has

$$\lim_{\substack{t\to\widetilde{\lambda}^- \\ t\in\mathbb{R}}} m^{(g)}(t) = +\infty\,, \qquad \lim_{\substack{t\to\lambda^+ \\ t\in\mathbb{R}}} m^{(g)}(t) = -\infty\,. \tag{3.5}$$

Now, in (3.4) assume that $\theta > 1$. Thus, because of the positivity of $\lambda, \widetilde{\lambda}$, (3.4) and (3.5) imply that

$$\lim_{\substack{t\to\widetilde{\lambda}^- \\ t\in\mathbb{R}}} \mathfrak{m}(t) = +\infty\,, \qquad \lim_{\substack{t\to\lambda^+ \\ t\in\mathbb{R}}} \mathfrak{m}(t) = -\infty\,.$$

Since $\mathfrak{m}$ is analytic on the interval $(\lambda, \widetilde{\lambda})$, it should cross the 0-axis an odd number of times. If it crosses this axis three or more times as in Figure 3 (a), then, by Remarks 1 and 3, there are at least two elements of $\sigma(J_1^{(g)})$ in $(\lambda, \widetilde{\lambda})$. But, because of Remark 1, this would contradict the fact that $\lambda, \widetilde{\lambda}$ are contiguous. Observe that

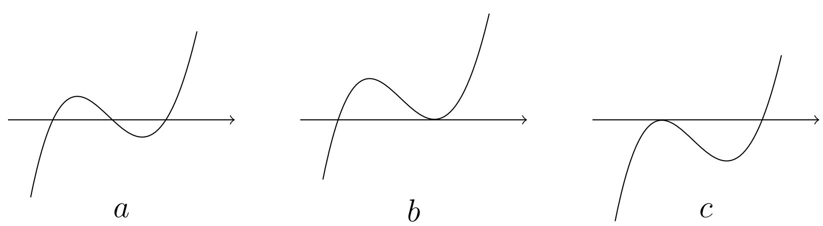

Figure 3: Impossible crossings of the 0-axis by $\mathfrak{m}$

one should discard the possibility of one crossing of the 0-axis and a tangential touch of it as in b and c. But again the impossibility of this follows from the fact that the poles of m_θ are simple (see Remark 1). By the same token the spectra interlace in $\mathbb{R}_-$. The case $\theta < 1$ is treated in a similar way. The second assertion follows directly from Proposition 3.1. □

4. Reconstruction of $J^{(g)}$ and $J^{(g)}(\theta)$

In this section we find some necessary conditions for the spectra of $J^{(g)}(\theta)$ ($\theta > 0$). Also we provide a reconstruction algorithm of the Jacobi matrix and establish uniqueness of the reconstruction. A central part of our approach is the Weyl m-function and its properties. We begin our discussion by setting out a convention for enumerating the elements of the spectra.

Convention. For a given countable set of real numbers S without finite points of accumulation, let M be an infinite subset of consecutive integers such that there is a one-to-one increasing function $h : M \to S$ with the property that, $h^{-1}(0) = 0$. Thus, M is semi-bounded from above (below) if and only if the same holds for S. We write $S = \{\lambda_k\}_{k\in M}$, where $\lambda_k = h(k)$. Note that in the sequence $\{\lambda_k\}_{k\in M}$ only λ_0 is allowed to be zero. Thus, if $-1, 1 \in M$, then

$$\lambda_{-1} < 0 < \lambda_1 \,.$$

In the sequel, the spectra of all operators will be enumerated according to this convention.

When $\{\lambda_k\}_{k\in M}$ is considered together with a sequence interlacing with it, we use the same set M for enumerating both sequences. For instance, if $\{\lambda_k\}_{k\in M}$ and $\{\mu_k\}_{k\in M}$ are interlacing and not semi-bounded, then one can assume that

$$\lambda_k < \mu_k < \lambda_{k+1} \,, \quad \forall k \in M.$$

The following auxiliary result can be found in [19, Sec. 4]. We sketch the proof here for the reader's convenience.

Lemma 4.1. *Let $J^{(g)}$ have discrete spectrum and assume that $\sigma(J^{(g)}) = \{\lambda_k\}_{k\in M}$, and $\sigma(J_1^{(g)}) = \{\eta_k\}_{k\in M}$. Then, the following formula holds for the Weyl m-function of $J^{(g)}$*

$$m^{(g)}(\zeta) = C\frac{\zeta - \eta_0}{\zeta - \lambda_0}\prod_{\substack{k\in M\\ k\neq 0}}\left(1 - \frac{\zeta}{\eta_k}\right)\left(1 - \frac{\zeta}{\lambda_k}\right)^{-1} , \tag{4.1}$$

Moreover, $C < 0$ and

$$\eta_k < \lambda_k < \eta_{k+1} \quad \forall k \in M \,, \tag{4.2}$$

if $\sigma(J^{(g)})$ is semi-bounded from above, while, $C > 0$ and

$$\lambda_k < \eta_k < \lambda_{k+1} \quad \forall k \in M \tag{4.3}$$

otherwise.

Proof. Assume first that $\sigma(J^{(g)})$ is semi-bounded from below. Since the greatest lower bound of J does not exceed the greatest lower bound of $J_1^{(g)}$, the smallest element of $\{\lambda_k\}_{k\in M}$ is less than the smallest of $\{\eta_k\}_{k\in M}$ (see [4, Chap. 6 Sec. 1.3]). Thus one can enumerate the sequences $\{\lambda_k\}_{k\in M}$ and $\{\eta_k\}_{k\in M}$ so that they obey our convention and (4.3). According to [16, Chap. 7, Sec.1, Thm. 1], (4.1) holds with $C > 0$.

Clearly, when $\sigma(J^{(g)})$ is not semi-bounded, the sequences can be arranged to obey (4.3), and then (4.1) holds with $C > 0$.

Now suppose that $\sigma(J^{(g)})$ is semi-bounded from above. Then $\sigma(-J^{(g)})$ is semi-bounded from below and, consequently, the greatest of $\{\eta_k\}_{k\in M}$ is less than the greatest of $\{\lambda_k\}_{k\in M}$. Thus $\{\lambda_k\}_{k\in M}$, and $\{\eta_k\}_{k\in M}$ cannot be arranged according to (4.3). However, we are still able to use (4.3) for arranging the zeros and poles of the meromorphic Herglotz function $-\frac{1}{m^{(g)}}$, that is, we use (4.2). Therefore [16, Chap. 7, Sec.1, Thm. 1] gives

$$-\frac{1}{m^{(g)}(\zeta)} = \widetilde{C}\frac{\zeta-\lambda_0}{\zeta-\eta_0}\prod_{\substack{k\in M\\ k\neq 0}}\left(1-\frac{\zeta}{\lambda_k}\right)\left(1-\frac{\zeta}{\eta_k}\right)^{-1}, \qquad \widetilde{C}>0\,,$$

For completing the proof it only remains to note that the last equation can be rewritten as asserted in the lemma. The infinite product in (4.1) is convergent because of (4.2) (see the proof of [16, Chap. 7, Sec.1, Thm. 1]). ☐

Another auxiliary simple result to be use later is the following lemma.

Lemma 4.2. *Let $J^{(g)}$ have discrete spectrum and $\{\lambda_k(\theta)\}_k$ be the set of eigenvalues of $J^{(g)}(\theta)$. Then, the series*

$$\sum_{k\in M}\frac{\lambda_k(\theta)}{\alpha_k(\theta)} \tag{4.4}$$

converges uniformly in $[\theta_1,\theta_2]\subset\mathbb{R}_+$ to $s_1(\theta)$ (see (2.5)).

Proof. From (2.5) and (2.11), it follows that the series converges pointwise to $s_1(\theta)$. The series

$$\sum_{k\in M}\frac{\lambda_k^2(\theta)}{\alpha_k(\theta)} \tag{4.5}$$

converges also pointwise to the function $s_2(\theta)$. Since this function is continuous in $[\theta_1,\theta_2]$, then (4.5) is uniformly convergent in that interval (see [25, Sec. 1.31]). Now, for any $\theta\in[\theta_1,\theta_2]$ and $|\lambda_k|>1$, one has

$$|\lambda_k|<\lambda_k^2\,,$$

so (4.4) is uniformly convergent in $[\theta_1,\theta_2]$. ☐

Remark 5. Proposition 3.2 tells that the interlacing of the sequences $\sigma(J^{(g)}) = \{\lambda_k\}_k$ and $\sigma(J(\theta)) = \{\mu_k\}_k$ is different in $\mathbb{R}_+$ and $\mathbb{R}_-$. So let us agree to enumerate the sequences according to our convention (the subscripts of the sequences run over M and only the eigenvalues with subscript equal zero are allowed to be zero) and obeying

$$\lambda_k<\mu_k<\lambda_{k+1}\quad\text{in }\mathbb{R}_+\,,\qquad \mu_k<\lambda_k<\mu_{k+1}\quad\text{in }\mathbb{R}_-\,,$$

when $\theta>1$, and

$$\mu_k<\lambda_k<\mu_{k+1}\quad\text{in }\mathbb{R}_+\,,\qquad \lambda_k<\mu_k<\lambda_{k+1}\quad\text{in }\mathbb{R}_-\,,$$

if $\theta<1$.

Proposition 4.1. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $0 < \theta_1 < \theta_2$. Let $J^{(g)}$ have discrete spectrum and assume that $\sigma(J^{(g)}(\theta_1)) = \{\lambda_k\}_{k\in M}$ and $\sigma(J^{(g)}(\theta_2)) = \{\mu_k\}_{k\in M}$, where the sequences have been arranged according to Remark 5. Then,*

$$\sum_{k\in M}(\mu_k - \lambda_k) = q_1(\theta_2^2 - \theta_1^2)\,.$$

Proof. Observe that from Proposition 3.1 it follows that

$$\mu_k - \lambda_k = 2\int_{\theta_1}^{\theta_2} \frac{\lambda_k(\theta)d\theta}{\theta\alpha_k(\theta)}\,.$$

Consider a sequence $\{M_n\}_{n=1}^{\infty}$ of subsets of M, such that $M_n \subset M_{n+1}$ and $\cup_n M_n = M$. Thus

$$\sum_{k\in M}(\mu_k - \lambda_k) = 2\lim_{n\to\infty}\int_{\theta_1}^{\theta_2}\left(\sum_{k\in M_n}\frac{\lambda_k(\theta)}{\alpha_k(\theta)}\right)\frac{d\theta}{\theta}$$

By Lemma 4.2 and the fact that

$$s_1(\theta) = \left\langle \delta_1, J^{(g)}\delta_1\right\rangle = q_1\theta^2\,,$$

one obtains

$$\sum_{k\in M}(\mu_k - \lambda_k) = 2q_1\int_{\theta_1}^{\theta_2}\theta d\theta = q_1(\theta_2^2 - \theta_1^2)$$

□

Proposition 4.2. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $\theta > 0$. Let $J^{(g)}$ have discrete spectrum and assume that $\sigma(J^{(g)}) = \{\lambda_k\}_{k\in M}$ and $\sigma(J^{(g)}(\theta)) = \{\mu_k\}_{k\in M}$, where the sequences have been arranged according to Remark 5. Then,*

$$\mathfrak{m}(\zeta) = \prod_{k\in M}\frac{\zeta - \mu_k}{\zeta - \lambda_k}\,.$$

Proof. Consider a sequence $\{M_n\}_{n=1}^{\infty}$ of subsets of M, such that $M_n \subset M_{n+1}$ and

$\cup_n M_n = M$. From (4.1) and (3.2) it follows that

$$\begin{aligned}\mathfrak{m}(\zeta) &= C\frac{\zeta-\mu_0}{\zeta-\lambda_0}\lim_{n\to\infty}\frac{\displaystyle\prod_{\substack{k\in M_n\\k\neq 0}}\left(1-\frac{\zeta}{\eta_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}}{\displaystyle\prod_{\substack{k\in M_n\\k\neq 0}}\left(1-\frac{\zeta}{\eta_k}\right)\left(1-\frac{\zeta}{\mu_k}\right)^{-1}}\\ &= C\frac{\zeta-\mu_0}{\zeta-\lambda_0}\prod_{\substack{k\in M\\k\neq 0}}\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}. \end{aligned} \tag{4.6}$$

On the other hand, by Proposition 4.1, it holds true that

$$\prod_{\substack{k\in M\\k\neq 0}}\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1} = \prod_{\substack{k\in M\\k\neq 0}}\frac{\lambda_k}{\mu_k}\prod_{\substack{k\in M\\k\neq 0}}\frac{\zeta-\mu_k}{\zeta-\lambda_k} \tag{4.7}$$

From (2.7) and (3.4) it follows that

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geq\epsilon>0}}\mathfrak{m}(\zeta) = 1\,. \tag{4.8}$$

Also, on the basis that the second product on the r. h. s of (4.7) converges uniformly, one has

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geq\epsilon}}\prod_{k\in M}\frac{\zeta-\mu_k}{\zeta-\lambda_k} = \lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geq\epsilon}}\prod_{k\in M}\left(1+\frac{\mu_k-\lambda_k}{\lambda_k-\zeta}\right) = 1\,. \tag{4.9}$$

Thus, (4.6), (4.7), (4.8), and (4.9) imply that

$$C = \prod_{\substack{k\in M\\k\neq 0}}\frac{\mu_k}{\lambda_k}$$

and the proposition is proven. □

Corollary 4.1. *Fix $g\in\mathbb{R}\cup\{\infty\}$ and $\theta>0$. Let $J^{(g)}$ have discrete spectrum and assume that $\sigma(J^{(g)})=\{\lambda_k\}_k$ and $\sigma(J^{(g)}(\theta))=\{\mu_k\}_k$, where the sequences have been arranged according to Remark 5. Then,*

$$\theta^2 = \prod_{k\in M}\frac{\eta-\mu_k}{\eta-\lambda_k}\,.$$

where η is any element of $\sigma(J_1^{(g)})$. Moreover, when $0 \notin \sigma(J^{(g)})$,

$$\theta^2 = \prod_{k \in M} \frac{\mu_k}{\lambda_k} \,. \tag{4.10}$$

and, if $0 \in \sigma(J^{(g)})$,

$$\theta^2 = \frac{1}{\alpha_0 - 1} \left\{ \alpha_0 \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} - 1 \right\} , \tag{4.11}$$

where α_0 is given in (2.12).

Proof. The first two identities for θ^2 are a straightforward consequence of Proposition 4.2 and (3.4). As regards to (4.11), note that, from (2.13), one has

$$\alpha_k^{-1} = - \operatorname*{Res}_{\zeta = \lambda_k} m(\zeta) \,. \tag{4.12}$$

Thus, according to (3.4),

$$\theta^2 - \alpha_0^{-1}(\theta^2 - 1) = \mathfrak{m}(0) = \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} \,. \tag{4.13}$$

□

Remark 6. Due to (4.13) and the properties of the normalizing constants, when $0 \in \sigma(J^{(g)})$, one the following inequalities hold depending on the value of $\theta \neq 1$:

$$\theta^2 < \mathfrak{m}(0) = \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} < 1 \qquad 1 < \mathfrak{m}(0) = \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} < \theta^2 \,.$$

Theorem 4.1. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $\theta > 0$. Let $J^{(g)}$ have discrete spectrum and assume that $0 \notin \sigma(J)$. The spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ ($\theta \neq 1$) uniquely determine the Jacobi matrix (1.1), that is the operator J, the parameter θ defining the perturbation, and the parameter g specifying the self-adjoint extension when $J \neq J^*$.*

Proof. Given the sequences $\sigma(J^{(g)})$ and $\sigma(J^{(g)}(\theta))$, one finds the parameter θ from (4.10). Proposition 4.2 yields the function $\mathfrak{m}$ and equation (3.4) the Weyl function $m^{(g)}$. According to the Preliminaries this function allows to recover the matrix associated to the Jacobi operator and the parameter g which determines the self-adjoint extension when $J \neq J^*$. □

Theorem 4.2. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $\theta > 0$. Let $J^{(g)}$ have discrete spectrum and assume that $0 \in \sigma(J)$. The spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ ($\theta \neq 1$), together with either q_1 or α_0, uniquely determine the matrix associated to J, the parameter θ, and the parameter g when $J \neq J^*$. Alternatively, the spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$*

and the parameter $\theta \neq 1$ uniquely determine the matrix corresponding to J and the parameter g if necessary.

Proof. This follows immediately from the proof of the previous theorem, taking into account (4.11). Note that θ can be determined either by Proposition 4.1 or by the asymptotic formula

$$\mathfrak{m}(\zeta) = 1 + \frac{q_1(1-\theta^2)}{\zeta} + O(\zeta^{-2}),$$

as $\zeta \to \infty$ ($\operatorname{Im} \zeta \geq \epsilon$, $\epsilon > 0$), obtained by combining (2.7) and (3.4). □

5. Necessary and sufficient conditions for the spectra of $J^{(g)}$ and $J^{(g)}(\theta)$

Theorem 5.1. *Given two infinite real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ without finite points of accumulation, such that none of them contains the zero, there is a unique positive θ, a unique operator J, and a unique $g \in \mathbb{R} \cup \{\infty\}$ if $J \neq J^*$, such that $\{\mu_k\}_k$ is the spectrum of $J^{(g)}(\theta)$ and $\{\lambda_k\}_k$ is the spectrum of $J^{(g)}$ if and only if the following conditions are satisfied.*

a) $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ interlace in $\mathbb{R}_+$, $\mathbb{R}_-$ with one sequence shifted to the right (left) in $\mathbb{R}_+$, ($\mathbb{R}_-$) with respect to the other one. Thus, the sequences can be ordered according to Remark 5.

b) The following series converges

$$\sum_{k\in M} (\mu_k - \lambda_k)$$

By condition b) the products $\prod_{\substack{k\in M\\k\neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}$, $\prod_{k\in M} \frac{\mu_k}{\lambda_k}$ are convergent, so define

$$\tau_n := \frac{(\mu_n - \lambda_n) \prod_{\substack{k\in M\\k\neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}}{\lambda_n \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1 \right)}, \quad \forall n \in M. \tag{5.1}$$

c) The sequence $\{\tau_n\}_{n\in M}$ is such that, for $m = 0, 1, 2, \ldots$, the series

$$\sum_{k\in M} \lambda_k^{2m} \tau_k \quad \textit{converges.}$$

d) If a sequence of complex numbers $\{\beta_k\}_{k\in M}$ *is such that the series*

$$\sum_{k\in M} |\beta_k|^2 \tau_k \quad \textit{converges}$$

and, for $m = 0, 1, 2, \dots,$

$$\sum_{k\in M} \beta_k \lambda_k^m \tau_k = 0\,,$$

then $\beta_k = 0$ *for all* $k \in M$.

Proof. In view of Propositions 3.2 and 4.1, for proving the necessity of the conditions, it only remains to show that for all $n \in M$, $\tau_n = \alpha_n^{-1}$. Indeed *c*) and *d*) will follow from the fact that all moments of the spectral measure (2.11) exist and that the polynomials are dense in $L_2(\mathbb{R}, \rho)$.

From (4.12), (3.4), and Proposition 4.2 , it follows that

$$\begin{aligned}\alpha_n^{-1} &= \frac{1}{\theta^2 - 1} \lim_{\zeta\to\lambda_n} \frac{\lambda_n - \zeta}{\zeta} \mathfrak{m}(\zeta) \\ &= \frac{\mu_n - \lambda_n}{\lambda_n(\theta^2 - 1)} \prod_{\substack{k\in M\\ k\neq n}} \frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k}\,.\end{aligned}$$

Hence, by Corollary 4.1, one verifies that $\tau_n = \alpha_n^{-1}$.

We now prove that conditions *a*), *b*), *c*), and *d*) are sufficient.

The condition *a*) implies that

$$\frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k} > 0 \qquad \forall k \in M\,,\ \ k \neq n$$

On the other hand, by *b*) one can define the number

$$\kappa = \prod_{k\in M} \frac{\mu_k}{\lambda_k} \tag{5.2}$$

which is clearly positive and also $\kappa > 1$ if $\mu_k > \lambda_k$ for all $k \in M$ and $\kappa < 1$ if $\mu_k < \lambda_k$ for all $k \in M$. Thus,

$$\frac{\mu_n - \lambda_n}{\lambda_n(\kappa - 1)} > 0 \qquad \forall n \in M$$

Hence, for all $n \in M$, $\tau_n > 0$, so define the function

$$\rho(t) := \sum_{\lambda_k < t} \tau_k\,. \tag{5.3}$$

It follows from *c*) that the moments of the measure corresponding to ρ are finite.

Now, on the basis of $a)$ and $b)$, define the functions

$$\widetilde{\mathfrak{m}}(\zeta) := \prod_{k\in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k}$$

and

$$\widetilde{m}(\zeta) := \frac{\widetilde{\mathfrak{m}}(\zeta) - \prod_{k\in M} \frac{\mu_k}{\lambda_k}}{\zeta \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1 \right)}, \qquad \zeta \neq 0 . \tag{5.4}$$

Thus, taking into account (5.1), one has

$$\operatorname*{Res}_{\zeta=\lambda_n} \widetilde{m}(\zeta) = \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1 \right)^{-1} \lim_{\zeta\to\lambda_n} \frac{\lambda_n - \zeta}{\zeta} \widetilde{\mathfrak{m}}(\zeta) = -\tau_n . \tag{5.5}$$

In view of what was done earlier,

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\geq\epsilon>0}} \widetilde{\mathfrak{m}}(\zeta) = 1 . \tag{5.6}$$

Therefore,

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\geq\epsilon>0}} \widetilde{m}(\zeta) = \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1 \right)^{-1} \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\geq\epsilon>0}} \frac{\widetilde{\mathfrak{m}}(\zeta)}{\zeta} = 0 \tag{5.7}$$

By (5.5) and (5.7), [16, Chap. VII, Sec.1 Theorem 2] implies that

$$\widetilde{m}(\zeta) = \sum_{k\in M} \frac{\tau_k}{\lambda_k - \zeta} . \tag{5.8}$$

On the other hand, using (5.6), one obtains

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\geq\epsilon>0}} \zeta\widetilde{m}(\zeta) = \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1 \right)^{-1} \lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\geq\epsilon>0}} \left(\widetilde{\mathfrak{m}}(\zeta) - \prod_{k\in M} \frac{\mu_k}{\lambda_k} \right) = -1 .$$

But

$$\lim_{\substack{\zeta\to\infty \\ \operatorname{Im}\zeta\geq\epsilon>0}} \zeta\widetilde{m}(\zeta) = -\sum_{k\in M} \tau_k ,$$

so it has been proven that, for the function given in (5.3),

$$\int_{\mathbb{R}} d\rho(t) = 1 .$$

Thus the measure corresponding to ρ is appropriately normalized and all the moment exist, so in $L_2(\mathbb{R},\rho)$ apply the Gram-Schmidt procedure of orthonormalization to the sequence $\{t_k\}_{k=0}^{\infty}$ to obtain a Jacobi matrix as was explained in the Preliminaries. Denote by J the operator whose matrix representation is the obtained matrix (cf. [2, Sec. 47]). Now, depending on the sequence of moments, J is self-adjoint or not. If $J = J^*$, the function ρ is the resolution of the identity of J, while if $J \neq J^*$, ρ corresponds to the resolution of the identity of a self-adjoint extension of J. This is a consequence of condition *d*) since it means that the polynomials are dense in $L_2(\mathbb{R},\rho)$ [20, Prop. 4.15].

Finally, denote by $J^{(g)}$ the self-adjoint extension of J corresponding to ρ and consider the operator $J^{(g)}(\theta)$ obtained from $J^{(g)}$ as indicated in the Preliminaries with θ given by (4.10). By construction the sequence $\{\lambda_k\}_{k\in M}$ is the spectrum of $J^{(g)}$. For the proof to be complete it only remains to show that $\{\mu_k\}_{k\in M}$ is the spectrum of $J^{(g)}(\theta)$. For the function given in (3.2), taking into account (3.4) and (2.13), one has

$$\mathfrak{m}(\zeta) = \theta^2 + \zeta\left(\theta^2 - 1\right)\sum_{k\in M}\frac{1}{\alpha_k(\lambda_k - \zeta)}\,.$$

On the other hand, from (5.4) and (5.8), it follows that

$$\widetilde{\mathfrak{m}}(\zeta) = \theta^2 + \zeta\left(\theta^2 - 1\right)\sum_{k\in M}\frac{\tau_k}{\lambda_k - \zeta}\,.$$

But we have already proven that $\alpha_k^{-1} = \tau_k$ for $k \in M$. Thus $\mathfrak{m} = \widetilde{\mathfrak{m}}$, meaning that the zeros of $\mathfrak{m}$ are given by the sequence $\{\mu_k\}_{k\in M}$. □

Theorem 5.2. *Let $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ be two infinite real sequences without finite points of accumulation, such that each of them contains exactly one element equal zero, and consider any positive real number $\theta \neq 1$. There exists a unique operator J, and a unique $g \in \mathbb{R}\cup\{\infty\}$ if $J \neq J^*$, such that $\{\mu_k\}_k$ is the spectrum of $J^{(g)}(\theta)$ and $\{\lambda_k\}_k$ is the spectrum of $J^{(g)}$ if and only if the conditions a), b), c), and d) hold with*

$$\tau_n := \frac{\mu_n - \lambda_n}{\lambda_n\left(\theta^2 - 1\right)}\prod_{\substack{k\in M\\k\neq n}}\frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}\,,\quad n \in M\,, n \neq 0\,,$$

$$\tau_0 := \left(\theta^2 - 1\right)^{-1}\left(\theta^2 - \prod_{\substack{k\in M\\k\neq 0}}\frac{\mu_k}{\lambda_k}\right)\,,$$

where

$$\theta^2 \begin{cases} < \displaystyle\prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k} & \textit{if } \{\mu_k\}_k \textit{ is shifted to the left in } \mathbb{R}_+ \textit{ w.r.t. } \{\lambda_k\}_k\,, \\ > \displaystyle\prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k} & \textit{otherwise.} \end{cases} \tag{5.9}$$

Proof. The proof is analogous to the proof of Theorem 5.1. Recall that by our convention for enumerating the sequences $\lambda_0 = \mu_0 = 0$. Thus, for proving the necessity of the conditions *a)*–*d)*, one only should verify that $\tau_0 = a_0^{-1}$ and (5.9) holds. This is immediate in view of (4.13) and Remark 6. The sufficiency of the conditions is established as in the proof of Theorem 5.1. Here, one substitutes (5.2) by

$$\kappa = \prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k}$$

and (5.4) by

$$\widetilde{m}(\zeta) := \frac{\widetilde{\mathfrak{m}}(\zeta) - \theta^2}{\zeta\,(\theta^2 - 1)}\,, \qquad \zeta \neq 0\,.$$

Then, one verifies that $\operatorname{Res}_{\zeta=\lambda_n} \widetilde{m}(\zeta) = -\tau_n$ for all $n \in M$ and $\sum_{k\in M} \tau_k = 1$. Note that (5.9) guarantees that $\tau_n > 0$ for all $n \in M$. The rest of the proof repeats that of Theorem 5.1 taking into account that now the zeros of $\mathfrak{m}$ are given by $\{\mu_k\}_{k\in M} \setminus \{0\}$. ☐

Theorem 5.3. *Given two infinite real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ without finite points of accumulation, such that none of them contains the zero, there is a unique positive θ and a unique operator $J = J^*$ such that $\{\mu_k\}_k$ is the spectrum of $J^{(g)}(\theta)$ and $\{\lambda_k\}_k$ is the spectrum of J if and only if conditions a), b), c), together with*

d')
$$\lim_{n\to\infty} \frac{\det \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{pmatrix}}{\det \begin{pmatrix} s_4 & s_5 & \cdots & s_{n+2} \\ s_5 & s_6 & \cdots & s_{n+3} \\ \cdots & \cdots & \cdots & \cdots \\ s_{n+2} & s_{n+3} & \cdots & s_{2n} \end{pmatrix}} = 0\,,$$

where $s_n := \sum_{k\in M} \lambda_k^n \tau_k$ for n in $\mathbb{N} \cup \{0\}$ are fulfilled. Note that by our convention on the notation $J^{(g)}(\theta)$ is a non-singular finite-rank perturbation of J which does not depend on g.

Proof. We again repeat the reasoning of the proof of Theorem 5.1. Clearly, s_n ($n \in$

$\mathbb{N} \cup \{0\}$) are the numbers given in (2.5). Thus, on the basis of Hamburger criterion (see [1, Chap. 2 Addenda and Problems, Example 9]), *d')* holds when $J = J^*$. For the sufficiency, note that, due to [1, Chap. 2 Addenda and Problems, Example 9], *d')* implies that the measure corresponding to the function given in (5.3) is the unique solution of the moment problem, so $J = J^*$ and *d)* is not needed. □

Remark 7. Admittedly, *d')* is not easy to check, however it allows to give necessary and sufficient conditions in the self-adjoint case. Note that one can also give the analogous self-adjoint version of Theorem 5.2 by substituting condition *d)* for *d')*.

References

[1] Akhiezer, N. I.: *The classical moment problem and some related questions in analysis*. Hafner Publishing Co., New York, 1965.

[2] Akhiezer, N. I. and Glazman, I. M.: *Theory of linear operators in Hilbert space*. Dover Publications Inc., New York, 1993.

[3] Berezans′kiĭ, J. M.: *Expansions in eigenfunctions of selfadjoint operators*. Translations of Mathematical Monographs **17**. American Mathematical Society, Providence, R.I., 1968.

[4] Birman, M. Sh. and Solomjak, M. Z.: *Spectral theory of selfadjoint operators in Hilbert space*. Mathematics and its Applications (Soviet Series). D. Reidel, Dordrecht, 1987.

[5] Chu, M.T., and Golub, G.H.: *Inverse eigenvalue problems: Theory algorithms and applications*. Numerical Mathematics and Scientific Computation Oxford University Press, New York, 2005.

[6] de Boor, C. and Golub, G.H., The numerically stable reconstruction of a Jacobi matrix from spectral data. Linear Alg. Appl. **21** (3), 1978, pp. 245–260.

[7] del Rio, R. and Kudryavtsev, M.: Inverse problems for Jacobi operators I: Interior mass-spring perturbations in finite systems, `arXiv:1106.1691`.

[8] Fu, L. and Hochstadt, H.: Inverse theorems for Jacobi matrices. *J. Math. Anal. Appl.* **47** (1974) 162–168.

[9] Gesztesy, F. and Simon, B.: m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices. *J. Anal. Math.* **73** (1997), 267–297.

[10] Gesztesy, F. and Simon, B.: On local Borg-Marchenko uniqueness results. *Comm. Math. Phys.* **211** (2000) 273–287.

[11] Gladwell, G. M. L.: *Inverse problems in vibration*. Second edition. Solid Mechanics and its Applications, **119**. Kluwer Academic, Dordrecht, 2004.

[12] Guseĭnov, G.: The determination of the infinite Jacobi matrix from two spectra. *Mat. Zametki* **23** (1978) 709–720.

[13] Halilova, R. Z.: An inverse problem. *Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Tehn. Mat. Nauk* **1967**(3-4) (1967), 169–175. In Russian.

[14] Hochstadt, H.: On the construction of a Jacobi matrix from spectral data. *Linear Algebra and Appl.* **8** (1974), 435–446.

[15] Kato, T.: *Perturbation theory for linear operators.* Second ed. Grundlehren der Mathematischen Wissenschaften, **132**. Springer, Berlin-New York, 1976.

[16] Levin, B. Ja.: *Distribution of zeros of entire functions.* Translations of Mathematical Monographs **5**. American Mathematical Society, Providence, R.I., 1980.

[17] Marchenko, V. A. and Misyura, T. V.: *Señalamientos metodológicos y didácticos al tema: Problemas inversos de la teoría espectral de operadores de dimensión finita.* Monografías IIMAS-UNAM **12** No. 28 México 2004.

[18] Ram, Y.M.: Inverse Eigenvalue Problem for a Modified Vibrating System, SIAM Appl. Math **53** (1993), 1763–1775.

[19] Silva, L. O. and Weder, R.: On the two spectra inverse problem for semi-infinite Jacobi matrices.*Math. Phys. Anal. Geom.* **3**(9) (2006), 263–290.

[20] Simon, B.: The classical moment problem as a self-adjoint finite difference operator. *Adv. Math.* **137**(1) (1998), 82–203.

[21] Spletzer, M., Raman, A., Sumali, H. and Sullivan, J. P.: Highly sensitive mass detection and identification using vibration localization in coupled microcantilever arrays. *Applied Physics Letters* **92** (2008) 114102.

[22] Spletzer, M., Raman, A., Wu, A. Q. and Xu, X.: Ultrasensitive mass sensing using mode localization in coupled microcantilevers. *Applied Physics Letters* **88** (2006) 254102.

[23] Teschl, G.: Trace formulas and inverse spectral theory for Jacobi operators. *Comm. Math. Phys.* **196**(1) (1998), 175–202.

[24] Teschl, G.: *Jacobi operators and completely integrable nonlinear lattices.* Mathematical Surveys and Monographs **72**. American Mathematical Society, Providence, RI, 2000.

[25] Titchmarsh, E. C.: *The theory of functions.* Oxford University Press, London, 1952.