SOMMERFELD RADIATION CONDITION AT THRESHOLD

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Dedicated to Hiroshi Isozaki on the occasion of his Sixtieth Birthday.

Abstract. We prove Besov space bounds of the resolvent at low energies in any dimension for a class of potentials that are negative and obey a virial condition with these conditions imposed at infinity only. We do not require spherical symmetry. The class of potentials includes in dimension ≥ 3 the attractive Coulomb potential. There are two boundary values of the resolvent at zero energy which we characterize by radiation conditions. These radiation conditions are zero energy versions of the well-known Sommerfeld radiation condition.

Contents

1. Introduction

In this paper we study low-energy spectral theory for the Schrödinger operator $H = -\Delta + V$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 1$, where the potential V obeys the following condition. We use the notation $\langle x \rangle = \sqrt{x^2+1}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $\mu \in (0,2)$ the notation $s_0 = 1/2 + \mu/4$.

Condition 1.1. Let $V = V_1 + V_2$ be a real-valued function defined on $\mathbb{R}^d$; $d \geq 1$. There exists $\mu \in (0,2)$ such that the following conditions (1)–(5) hold.

(1) There exists $\epsilon_1 > 0$ such that $V_1(x) \leq -\epsilon_1 \langle x \rangle^{-\mu}$.
(2) $V_1 \in C^\infty(\mathbb{R}^d)$. For all $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that
$$\langle x \rangle^{\mu+|\alpha|} |\partial^\alpha V_1(x)| \leq C_\alpha.$$
(3) There exists $\tilde{\epsilon}_1 > 0$ such that $-|x|^{-2}\left(x \cdot \nabla(|x|^2 V_1)\right) \geq -\tilde{\epsilon}_1 V_1$.
(4) There exists $\delta, C, R > 0$ such that
$$|V_2(x)| \leq C|x|^{-2s_0-\delta},$$
for $|x| > R$.
(5) $V_2 \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ for $d = 1,2,3$, $V_2 \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p > 2$ if $d = 4$ while $V_2 \in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d)$ for $d \geq 5$.

Due to (4) and (5) the operator $V_2(-\Delta + i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$, see for example [RS, Theorems X.20 and X.21] for the case $d \geq 4$. Whence H is self-adjoint. The Schrödinger operator with an attractive Coulomb potential in dimension $d \geq 3$ is a particular example.

While low-energy spectral asymptotics for Schrödinger operators is a well studied subject for classes of potentials of fast decay the literature is more sparse for classes of potentials with decay $O(r^{-2})$ or slower. We refer to [Ya, Na, FS, SW] and references therein. We remark that Condition 1.1 is closely related to the conditions used in [FS], in fact in the present paper we aim at proving more precise resolvent bounds than done in [FS] (Besov space bounds), and characterize boundary values $R(0 \pm \mathrm{i}0)$ of the resolvent at zero energy. The latter is achived in terms of certain microlocal estimates traditionally referred to as Sommerfeld radiation conditions. For positive energies the limiting absorption principle (LAP) and Sommerfeld radiation conditions are fundamental for stationary scattering theory and they are used at many places in the literature, see for example [Sa, AH, GY] and [Hö1, Section 30.2] (two-body problems), [Is, Va] (many-body problems) and [Mø] (abstract framework). Moreover radiation conditions are an integral part of for one of the oldest method of proving LAP, cf. for example [Sa]. We consider them as interesting of their own right, in particular including the case of zero energy cf. [DS1].

Neither [FS] nor the present paper deal with scattering theory. On the other hand there are indeed applications to scattering theory at low energies as demonstrated in the recent works [DS1, DS2]. However this is for a smaller class of potentials (essentially radially symmetric potentials). We plan in a future publication [Sk] to study scattering theory at low energyies for a larger class than the ones of [DS1, DS2] however within the one defined by Condition 1.1. For this study Besov space bounds and uniqueness induced by versions of the Sommerfeld radiation condition will be useful. We remark that these results are also present in some form in [DS1] although, as indicated above, this is under stronger conditions on the potentials. Besides the Besov space bounds are not shown in the strongest form as done here and they are obtained somewhat indirectly (in fact only the imaginary part of the boundary value of the resolvent is estimated). We present (most of) our results in Subsection 1.1. They are all under Condition 1.1.

1.1. **Results.**

1.1.1. *Resolvent bounds.* Let us recall a main result from [FS]. Let $\theta \in (0, \pi)$, $\lambda_0 > 0$ and define

$$\Gamma_\theta = \{z \in \mathbb{C} \setminus \{0\} \,\big|\, \arg z \in (0, \theta),\ |z| \le \lambda_0\}. \tag{1.1}$$

In [FS] λ_0 is exclusively taken equal to one although this is only for convenience of presentation. In this paper we fix any $\lambda_0 > 0$ at this point and suppress henceforth any dependence of this constant (as done in the notation (1.1)). At this point we also fix $\theta \in (0, \pi)$, but keep (somewhat inconsistently) the dependence of θ of the set (1.1).

For $\mu \in (0, 2)$, $K > 0$ and $\lambda \ge 0$ let

$$f = f_\lambda(x) = (\lambda + K\langle x\rangle^{-\mu})^{1/2};\ x \in \mathbb{R}^d. \tag{1.2}$$

Here λ will be taken as $|z|$ for z in the closure of Γ_θ and K can for parts of our presentation and analysis be taken arbitrary. More precisely the latter is true for Theorems 1.2 and 1.3 presented below. Consequently we take, for convenience, $K = 1$ in these theorems as well as in Subsection 2.2 (where Theorem 1.3 is proved). As for Section 3 we choose a different value of K, see (1.6).

For a Hilbert space $\mathcal{H}$ (which in our case will be $L^2(\mathbb{R}^d)$) we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ (a similar notation will be used for Banach spaces). A $\mathcal{B}(\mathcal{H})$-valued function $T(\cdot)$ on Γ_θ is said to be uniformly Hölder continuous in Γ_θ if there exist $C, \gamma > 0$ such that

$$\|T(z_1) - T(z_2)\| \le C|z_1 - z_2|^\gamma \text{ for all } z_1, z_2 \in \Gamma_\theta.$$

We consider the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ under Condition 1.1. The resolvent is denoted by $R(z) = (H - z)^{-1}$. In the statement below of (a version of) [FS, Theorem 1.1] some conditions on the potential V are slightly changed. Comments at this point are given after the statement.

Theorem 1.2 (LAP)**.** *Suppose Condition 1.1. For all $s > s_0$ the family of operators $T(z) = \langle x\rangle^{-s} R(z) \langle x\rangle^{-s}$ is uniformly Hölder continuous in Γ_θ. In particular the limits*

$$T(0 + \mathrm{i}0) = \langle x\rangle^{-s} R(0 + \mathrm{i}0)\langle x\rangle^{-s} = \lim_{z\to 0, z\in\Gamma_\theta} T(z),$$
$$T(0 - \mathrm{i}0) = \langle x\rangle^{-s} R(0 - \mathrm{i}0)\ \langle x\rangle^{-s} = \lim_{z\to 0, z\in\Gamma_\theta} T(\bar z)$$

exist in $\mathcal{B}(L^2(\mathbb{R}^d))$.

For all $s > 1/2$ there exists $C > 0$ such that for all $z \in \Gamma_\theta$

$$\|\langle x\rangle^{-s} f_{|z|}^{1/2} R(z) f_{|z|}^{1/2} \langle x\rangle^{-s}\| \le C. \tag{1.3}$$

We have already noted that due to (4) and (5) the operator $V_2(-\Delta+i)^{-1}$ is a compact, which occurs as a separate condition in [FS, Theorem 1.1] (the condition (5) is a new condition compared to [FS, Theorem 1.1]). Another condition from [FS, Theorem 1.1] that we omitted above is a version of unique continuation at infinity. This version, [FS, Assumption 2.1], is automatically satisfied, given (5), due to results of [FH] (for $d \leq 3$) and [JK] (for $d \geq 3$). Applying it with $V \to V - \lambda$ for any $\lambda \geq 0$ in conjunction with [FS, Theorem 2.4] we have $\sigma_{\mathrm{pp}}(H) \cap [0, \infty) = \emptyset$ (for $d = 1, 2, 3$ absence of strictly positive eigenvalues follows alternatively from [FH, Corollary 1.4]). The absence of non-negative eigenvalues is of course a consequence of (1.3), however this property is part of the proof of the latter bound.

We note that imposing the conditions (1) and (3) only near infinity may seem, with the other conditions of Condition 1.1, to weaken the assumptions. However this is not the case cf. a discussion in [FS, Section 3]. On the other hand it suffices to have the bounds (2) for $|\alpha| \leq 2$. More precisely this is the case for Theorems 1.2 and 1.3. For the microlocal estimates of Section 3 though we need V_1 to be a "symbol" and all bounds of (2) are then needed.

Notice also that (1.3) is stronger than boundedness of the family $T(\cdot)$ (which is a consequence of the uniform Hölder continuity).

A main result of this paper (recall that we impose Condition 1.1 throughout the paper) is the following improvement of (1.3) in terms of Besov spaces as defined in the beginning of Subsection 2.1 with the operator A there being given as multiplication by $|x|$ on the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d_x)$.

Theorem 1.3 (Besov space bound)**.** *There exists $C > 0$ such that for all $z \in \Gamma_\theta$*

$$\|f_{|z|}^{1/2} R(z) f_{|z|}^{1/2}\|_{\mathcal{B}(B(|x|),B(|x|)^*)} \leq C. \tag{1.4}$$

1.1.2. *Sommerfeld radiation condition.* We shall give an outline of the results of Section 3. These results are on microlocal estimates of solutions to the equation $Hu = v$. In particular we estimate and characterize the particular solution provided by Theorems 1.2 and 1.3. This particular solution is constructed as follows in terms of Besov spaces. First note that the relevant Besov space at zero energy is $B^\mu := \langle x \rangle^{-\mu/4} B(|x|)$, cf. Theorem 1.3. We have the following characterization of the corresponding dual space (recall $s_0 := 1/2 + \mu/4$)

$$u \in (B^\mu)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-s_0} \|F(|x| < R)u\| < \infty.$$

A slightly smaller space is given by

$$u \in (B^\mu)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-s_0} \|F(|x| < R)u\| = 0.$$

Now suppose $v \in B^\mu$. Then due to Theorems 1.2 and 1.3 there exists the weak-star limit

$$u = R(0 + \mathrm{i}0)v = \operatorname*{w^\star\text{–}lim}_{z\to 0, z\in\Gamma_\theta} R(z)v \in (B^\mu)^*. \tag{1.5}$$

Note that indeed this u is a (distributional) solution to the equation $Hu = v$.

Let us state a microlocal property of this solution. We shall use (1.2) with

$$K = \epsilon_1 \tilde{\epsilon}_1/(2 - \mu), \tag{1.6}$$

where the ϵ's come from Condition 1.1. In terms of f_0 we then introduce symbols

$$a_0 = \frac{\xi^2}{f_0(x)^2}, \quad b_0 = \frac{\xi}{f_0(x)} \cdot \frac{x}{\langle x \rangle},$$

and we prove that

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in (B^\mu)^*_0 \text{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \text{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, 1)). \tag{1.7}$$

Here we use Weyl quantization (although this is not the only choice). For (1.5), (1.7), another version of (1.7) as well as another microlocal property (high energy estimates) we refer the reader to Proposition 3.4.

The support property of $\tilde{\chi}_-$ in (1.7) mirrors that the particular solution studied is "outgoing", and we refer to (1.7) as a *Sommerfeld radiation condition.* This condition (in fact a weaker version) suffices for a characterization as expresssed in the following result. We refer the reader to Theorem 3.6 for a slightly stronger result as well as another version.

Theorem 1.4 (Uniqueness of outgoing solution)**.** *Suppose $v \in B^\mu$. Suppose u is a distributional solution to the equation $Hu = v$ belonging to the space $\langle x \rangle^{-s} L^2(\mathbb{R}^d)$ for some $s \in \mathbb{R}$, and suppose that there there exists $\sigma \in (0, 1]$ such that*

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in (B^\mu)^*_0 \textit{ for all } \chi_- \in C_c^\infty(\mathbb{R}) \textit{ and } \tilde{\chi}_- \in C_c^\infty((-\infty, \sigma)). \tag{1.8}$$

Then $u = R(0 + \mathrm{i}0)v$. In particular (1.7) *holds.*

The proof of Theorem 3.6 (yielding in particular Theorem 1.4) relies partly on a "propagation of singularities" result. This result is stated as Proposition 3.5. We note that the "incoming" solution $u = R(0 - \mathrm{i}0)v$ can be characterized similarly. Our results generalize [DS1, Proposition 4.10] at zero energy. For similar results for positive energies and for larger classes of potentials see [Hö1, Theorem 30.2.10] and [GY].

2. Improved resolvent bounds

In Subsection 2.2 we prove Theorem 1.3. The proof will be based on various results for abstract Besov spaces to be given in Subsection 2.1.

2.1. Abstract Besov spaces.

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $R_0 = 0$ and $R_j = 2^{j-1}$ for $j \in \mathbb{N}$. We define correspondingly characteristic functions $F_j = F(R_{j-1} \leq |\cdot| < R_j)$ and the space

$$B = B(A) = \big\{u \in \mathcal{H} \big| \sum_{j \in \mathbb{N}} R_j^{1/2} \|F_j(A)u\| =: \|u\|_B < \infty\big\}. \tag{2.1}$$

We can identify (using the embeddings $\langle A \rangle^{-1}\mathcal{H} \subseteq B \subseteq \mathcal{H} \subseteq B^*$, $\langle A \rangle := \sqrt{A^2 + 1}$) the dual space B^* as

$$B^* = B(A)^* = \big\{u \in \langle A \rangle \mathcal{H} \big| \sup_{j \geq 1} R_j^{-1/2} \|F_j(A)u\| =: \|u\|_{B^*} < \infty\big\}. \tag{2.2}$$

Alternatively, the elements u of B^* are those *sequences* $u = (u_j) \subseteq \mathcal{H}$ with $u_j \in \mathrm{Ran}(F_j(A))$ and $\sup_{j \in \mathbb{N}} R_j^{-1/2} \|u_j\| < \infty$. This abstract space was also considered in [JP] (note however that $B(A)^*$ is identified incorrectly in [JP] as the completion of $\mathcal{H}$

in the norm $\|\cdot\|_{B^*}$). For other previous related works we refer to [AH, GY, Wa, Ro] and [Hö1, Subsections 14.1 and 30.2]. We note the bounds, cf. [Hö1, Subsections 14.1],

$$\|u\|_{B^*} \leq \sup_{R>1} R^{-1/2}\|F(|A|<R)u\| \leq 2\|u\|_{B^*}. \tag{2.3}$$

Introducing *abstract weighted spaces* $L^2_s = L^2_s(A) = \langle A\rangle^{-s}\mathcal{H}$ and $B^*_0 = B^*_0(A)$, the completion of $\mathcal{H}$ in the space B^*, we have the embeddings

$$L^2_s \subseteq B \subseteq L^2_{1/2} \subseteq \mathcal{H} \subseteq L^2_{-1/2} \subseteq B^*_0 \subseteq B^* \subseteq L^2_{-s}, \text{ for all } s>1/2. \tag{2.4}$$

All embeddings are continuous and corresponding bounding constants can be chosen as absolute constants, i.e. independently of A and $\mathcal{H}$. In particular

$$\|u\|_{\mathcal{H}} \leq \|u\|_B \text{ for all } u \in B. \tag{2.5}$$

Note also that

$$u \in B^*_0 \text{ if and only if } u \in B^* \text{ and } \lim_{R\to\infty} R^{-1/2}\|F(|A|<R)u\| = 0. \tag{2.6}$$

We refer to the spaces B, B^* and B^*_0 as *abstract Besov spaces*. Recall the following interpolation type result, here stated abstractly. The proof is the same as that of the concrete versions [AH, Theorem 2.5], [Hö1, Theorem 14.1.4], [JP, Proposition 2.3] and [Ro, Subsection 4.3].

Lemma 2.1. *Let A_1 and A_2 be self-adjoint operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $s>1/2$. Suppose $T \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)\cap\mathcal{B}(L^2_s(A_1),L^2_s(A_2))$. Then $T \in \mathcal{B}(B(A_1),B(A_2))$, and there is a constant $C=C(s)>0$ (independent of T) such that*

$$\|T\|_{\mathcal{B}(B(A_1),B(A_2))} \leq C\big(\|T\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} + \|T\|_{\mathcal{B}(L^2_s(A_1),L^2_s(A_2))}\big). \tag{2.7}$$

Corollary 2.2. *Let A_1 and A_2 be self-adjoint operators on a Hilbert spaces $\mathcal{H}$. Suppose that $\langle A_2\rangle^s\langle A_1\rangle^{-s} \in \mathcal{B}(\mathcal{H})$ for some $s>1/2$. Then $B(A_1)\subseteq B(A_2)$ and*

$$\|u\|_{B(A_2)} \leq C\|u\|_{B(A_1)} \text{ for all } u \in B(A_1). \tag{2.8}$$

The norm on $B(A)$ in (2.1) is not the only possible choice: Define for any $p>1$

$$B_p = B(A)_p = \big\{u \in \mathcal{H}\big|\ \sum_{j\in\mathbb{N}} \tilde{R}_j^{1/2}\|\tilde{F}_j(A)u\| =: \|u\|_{B_p} < \infty\big\}, \tag{2.9}$$

where $\tilde{F}_j = F(\tilde{R}_{j-1} \leq |\cdot| < \tilde{R}_j)$, $\tilde{R}_0 = 0$ and $\tilde{R}_j = p^{j-1}$ for $j \geq 1$. For $p=2$ this agrees with (2.1).

Lemma 2.3. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. For all $p>1$ the space $B(A)_p = B(A)$ and there exists $C=C(p)>0$ (i.e. independent of A) such that for all $u \in B(A)$*

$$C^{-1}\|u\|_{B(A)_p} \leq \|u\|_{B(A)} \leq C\|u\|_{B(A)_p}. \tag{2.10}$$

Proof. The first term in the expression $\sum_{j\in\mathbb{N}} \tilde{R}_j^{1/2}\|\tilde{F}_j(A)u\|$ is bounded by $\|u\|_{B(A)}$, cf. (2.5). So let us look at a term with $j\geq 2$. We estimate

$$\tilde{R}_j^{1/2}\|\tilde{F}_j(A)u\| \leq \sqrt{p}\|\tilde{F}_j(A)|A|^{1/2}u\| \leq \sqrt{p}\sum_{k=2}^{\infty}\|\tilde{F}_j(A)F_k(A)|A|^{1/2}u\|.$$

Now a small consideration shows that for all $k \geq 2$ there are at most $2 + [\ln 2/\ln p]$ number of j's with $j \geq 2$ for which $\tilde{F}_j F_k \neq 0$. Whence

$$\sum_{j\geq 2} \tilde{R}_j^{1/2} \|\tilde{F}_j(A)u\|$$
$$\leq \sqrt{p}\big(2 + \ln 2/\ln p\big) \sum_{k=2}^{\infty} \|F_k(A)|A|^{1/2}u\| \leq \sqrt{p}\big(2 + \ln 2/\ln p\big)\|u\|_{B(A)},$$

yielding the first inequality in (2.10) for any $C \geq 1 + \sqrt{p}\big(2 + \ln 2/\ln p\big)$.

By the same method one shows the second inequality in (2.10) for any $C \geq 1 + \sqrt{2}\big(2 + \ln p/\ln 2\big)$. □

Lemma 2.4. *Let A be operator on a Hilbert space $\mathcal{H}$, such that $A \geq I$, and $s > -1$ be given. Then $A^{-s/2} : B(A) \to B(A^{1+s})$ is a homeomorhic isomorphism, and there is a constant $C = C(s) > 0$ (i.e. independent of A) such that*

$$\|A^{-s/2}\|_{\mathcal{B}(B(A),B(A^{1+s}))} \leq C \text{ and } \|A^{s/2}\|_{\mathcal{B}(B(A^{1+s}),B(A))} \leq C. \tag{2.11}$$

Proof. We let $p = 2^{1/(1+s)}$, $C = 2^{\max(s/2,0)/(1+s)}$ and estimate for all $u \in B(A)$

$$\begin{aligned}
\|A^{-s/2}u\|_{B(A^{1+s})} &= \sum_{j\geq 2} R_j^{1/2} \|F_j(A^{1+s})A^{-s/2}u\| \\
&\leq C \sum_{j\geq 2} 2^{\left(1-\frac{s}{1+s}\right)(j-1)/2} \left\| F\left(2^{\frac{j-2}{1+s}} \leq A < 2^{\frac{j-1}{1+s}}\right)u \right\| \\
&\leq C \sum_{j\geq 2} p^{(j-1)/2} \big\| F(p^{j-2} \leq A < p^{j-1})u \big\| \\
&= C\|u\|_{B(A)_p}.
\end{aligned}$$

Now we obtain the first estimate in (2.11) by invoking Lemma 2.3.

The second estimate in (2.11) follows from the first with $A \to A^{1+s}$ and $s \to -s/(1+s)$.

□

Lemma 2.5. *Suppose A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, $c \in \mathbb{R}$, $u \in B(A)$ and either $|c| > 1$ or $u = F(|cA| \geq 1)u$, then $u \in B(cA)$ with*

$$\|u\|_{B(cA)} \leq 8|c|^{1/2}\|u\|_{B(A)}. \tag{2.12}$$

Proof. Suppose first that $|c| > 1$. Pick $i \geq 2$ such that $R_{i-1} < |c| \leq R_i$. Then for all $j \geq i + 1$

$$F_j(ct) \leq F(R_{j-1}/R_i \leq |t| < R_j/R_{i-1}) \leq F_{j-i+1}(t) + F_{j-i+2}(t).$$

Whence for any $u \in B(A)$ we can estimate

$$\begin{aligned}\|u\|_{B(cA)} &\leq \Big(\sup_{j\geq i+1}\big(R_j/R_{j-i+1}\big)^{1/2}+\sup_{j\geq i+1}\big(R_j/R_{j-i+2}\big)^{1/2}\Big)\|u\|_{B(A)}+\sum_{j=1}^{i}R_j^{1/2}\|u\|_{\mathcal{H}}\\ &\leq \big(2^{(i-1)/2}+2^{(i-2)/2}+2^{i/2}(\sqrt{2}+1)\big)\|u\|_{B(A)}\\ &\leq \big(\sqrt{2}+1+2(\sqrt{2}+1)\big)|c|^{1/2}\|u\|_{B(A)}\\ &\leq 8|c|^{1/2}\|u\|_{B(A)}.\end{aligned}$$

Suppose now that $|c| \leq 1$ and that $u = F(|cA| \geq 1)u$. Clearly we can assume that $c \neq 0$. Then we can pick $i \geq 2$ such that $R_{i-1} \leq 1/|c| < R_i$, and we note that $F_1(cA)u = 0$. For $j \geq 2$ we have

$$F_j(ct) \leq F(R_{j-1}R_{i-1} \leq |t| < R_jR_i) \leq F_{j+i-2}(t) + F_{j+i-1}(t).$$

Whence

$$\begin{aligned}\|u\|_{B(cA)} &\leq \Big(\sup_{j\geq 2}\big(R_j/R_{j+i-2}\big)^{1/2}+\sup_{j\geq 2}\big(R_j/R_{j+i-1}\big)^{1/2}\Big)\|u\|_{B(A)}\\ &\leq \big(2^{(2-i)/2}+2^{(1-i)/2}\big)\|u\|_{B(A)}\\ &\leq 3|c|^{1/2}\|u\|_{B(A)}.\end{aligned}$$

□

We note the following abstract version of a result from [JP, Mo2] (proven by using suitable decompositions of unity and the Cauchy-Schwarz inequality, see also [Wa, Subsection 2.2]).

Lemma 2.6. *Let A_1 and A_2 be self-adjoint operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Suppose that uniformly in $z \in \Gamma_\theta$ and $m, n \in \mathbb{Z}$,*

$$\|F(m \leq A_2 < m+1)TF(n \leq A_1 < n+1)\| \leq C. \tag{2.13}$$

Then $T \in \mathcal{B}(B(A_1), B(A_2)^*)$, and with the constant C from (2.13) *we have*

$$\|T\|_{\mathcal{B}(B(A_1),B(A_2)^*)} \leq 2C. \tag{2.14}$$

We note the following (partial) abstract criterion for (2.13), cf. [Mo2, (I.10)] (see also [Wa]). Recall that a bounded operator T on a Hilbert space is called *accretive* if $T + T^* \geq 0$, cf. for example [RS, Chapter X].

Lemma 2.7. *Let A be self-adjoint operator on Hilbert spaces $\mathcal{H}$, and suppose $T \in \mathcal{B}(\mathcal{H})$ is accretive. Suppose the following bounds uniformly in $n \in \mathbb{Z}$,*

$$\begin{aligned}\|F(n \leq A < n+1)TF(n \leq A < n+1)\| &\leq C_1,\\ \|F(A < n)TF(n \leq A < n+1)\| &\leq C_2,\\ \|F(n \leq A < n+1)TF(A \geq n)\| &\leq C_3.\end{aligned}$$

Then (2.13) *holds with $A_1 = A_2 = A$, the accretive T and with $C = 2C_1 + C_2 + C_3$.*

2.2. **Besov space bound of resolvent.** In this subsection we shall prove Theorem 1.3. We shall prepare for the proof of (1.4) in terms of three lemmas.

Due to (1.3), (2.4) and resolvent identities, cf. [FS, (5.12)], it suffices for (1.4) to prove the bound with V_2 in Condition 1.1 taken to be zero (i.e. only Condition 1.1 (1)–(3) are imposed and $V = V_1$). Let in this subsection

$$A = (x \cdot p + p \cdot x)/2;\; p = -\mathrm{i}\nabla_x. \tag{2.16}$$

Notice that the commutator

$$\mathrm{i}[H, A] = 2H + W;\; W(x) = -2V(x) - x \cdot \nabla V(x) \geq \epsilon_1 \tilde{\epsilon}_1 \langle x\rangle^{-\mu}. \tag{2.17}$$

Lemma 2.8. *Suppose $V_2 = 0$ in Theorem 1.2. Then with A given by* (2.16)

$$\sup_{z\in\Gamma_\theta} \big\| f_{|z|} R(z) f_{|z|} \big\|_{\mathcal{B}(B(A),B(A)^*)} \leq C = C(\theta). \tag{2.18}$$

Proof. Following [FS, Section 3] (a modification of the method of [Mo1]) we introduce

$$R_z(\epsilon) = (H - \mathrm{i}\epsilon \mathrm{i}[H, A] - z)^{-1};\; \epsilon \operatorname{Im} z > 0. \tag{2.19}$$

We recall the quadratic estimate [FS, Lemma 3.1],

$$\|f_{|z|} R_z(\epsilon) T\|^2 \leq C|\epsilon|^{-1} \|T^* R_z(\epsilon) T\|. \tag{2.20}$$

valid for $z \in \Gamma_\theta$ or $\bar{z} \in \Gamma_\theta$, $\epsilon \operatorname{Im} z > 0$ and $0 < |\epsilon| \leq \epsilon(\theta)$ sufficiently small and for all bounded operators T.

We recall, cf. [FS, Lemma 3.3],

$$\frac{d}{d\epsilon} R_z(\epsilon) = (1 - 2\mathrm{i}\epsilon)^{-1} \left\{ R_z(\epsilon) A - A R_z(\epsilon) + \mathrm{i}\epsilon R_z(\epsilon)(x \cdot \nabla W) R_z(\epsilon) \right\}. \tag{2.21}$$

Also we recall the following bound valid for any $s \in \mathbb{R}$, cf. [FS, (4.9)],

$$|\partial_x^\alpha f_\lambda^s| \leq C_\alpha f_\lambda^s \langle x\rangle^{-|\alpha|}, \tag{2.22}$$

with C_α independent of $\lambda \geq 0$.

Now we shall prove three bounds which are uniform in $z \in \Gamma_\theta$, $\epsilon > 0$ and $0 < \epsilon \leq \epsilon(\theta)$,

$$\|F_z(\epsilon)\| \leq C \text{ for } F_z(\epsilon) := \langle A\rangle^{-1} f_{|z|} R_z(\epsilon) f_{|z|} \langle A\rangle^{-1}, \tag{2.23a}$$

$$\|F_z^-(\epsilon)\| \leq C \text{ for } F_z^-(\epsilon) := \mathrm{e}^{\epsilon A} F(A < 0) f_{|z|} R_z(\epsilon) f_{|z|} \langle A\rangle^{-2}, \tag{2.23b}$$

$$\|F_z^+(\epsilon)\| \leq C \text{ for } F_z^+(\epsilon) := \langle A\rangle^{-2} f_{|z|} R_z(\epsilon) f_{|z|} F(A \geq 0) \mathrm{e}^{-\epsilon A}. \tag{2.23c}$$

Re (2.23a)**.** Due to (2.20)

$$\|F_z(\epsilon)\| \leq \epsilon^{-1} C \text{ for } 0 < \epsilon \leq \epsilon(\theta). \tag{2.24}$$

Next we note the bounds, due to (2.22),

$$\|f_{|z|}^{-1} A f_{|z|} \langle A\rangle^{-1}\| \leq C \text{ and } \|\langle A\rangle^{-1} f_{|z|} A f_{|z|}^{-1}\| \leq C. \tag{2.25}$$

Using (2.20), (2.21) and (2.25) we obtain

$$\Big\|\frac{d}{d\epsilon} F_z(\epsilon)\Big\| \leq C\big(\epsilon^{-1/2} \|F_z(\epsilon)\|^{1/2} + \|F_z(\epsilon)\|\big). \tag{2.26}$$

Clearly (2.23a) follows from (2.24) and (2.26) by two integrations.

Re (2.23b)**.** Due to (2.20) and (2.23a)

$$\|F_z^-(\epsilon(\theta))\| \leq \epsilon(\theta)^{-1/2}C. \tag{2.27}$$

Using (2.21) we compute

$$\begin{aligned} &\frac{d}{d\epsilon}F_z^-(\epsilon) = T_1 + \cdots + T_4; \\ &\qquad T_1 = \big(1-(1-2\mathrm{i}\epsilon)^{-1}\big)\mathrm{e}^{\epsilon A}F(A<0)Af_{|z|}R_z(\epsilon)f_{|z|}\langle A\rangle^{-2}, \\ &\qquad T_2 = (1-2\mathrm{i}\epsilon)^{-1}\mathrm{e}^{\epsilon A}F(A<0)[A,f_{|z|}]R_z(\epsilon)f_{|z|}\langle A\rangle^{-2}, \\ &\qquad T_3 = (1-2\mathrm{i}\epsilon)^{-1}\mathrm{e}^{\epsilon A}F(A<0)f_{|z|}R_z(\epsilon)Af_{|z|}\langle A\rangle^{-2}, \\ &\qquad T_4 = \mathrm{i}\epsilon(1-2\mathrm{i}\epsilon)^{-1}\mathrm{e}^{\epsilon A}F(A<0)f_{|z|}R_z(\epsilon)(x\cdot\nabla W)R_z(\epsilon)f_{|z|}\langle A\rangle^{-2}. \end{aligned} \tag{2.28}$$

Using (2.20), (2.22) and (2.23a) we estimate

$$\|T_j\| \leq \epsilon^{-1/2}C \text{ for } 0<\epsilon\leq\epsilon(\theta) \text{ and } j=1,\dots,4. \tag{2.29}$$

Notice that for all of the terms T_1–T_4 we apply (2.20) with $T = f_{|z|}\langle A\rangle^{-1}$ and in addition for T_4 we apply (2.20) with $T = f_{|z|}$. Clearly (2.23b) follows from (2.27)–(2.29) by one integration.

Re (2.23c)**.** We mimic the proof of (2.23b).

Next we note that the above arguments apply to $A \to A-n$ for any $n\in\mathbb{Z}$ yielding bounds being independent of n. Taking $\epsilon\to 0$ we thus obtain the following bounds for the accretive operator $T(z) = -\mathrm{i}f_{|z|}R(z)f_{|z|}$, all being uniform in $z\in\Gamma_\theta$ and $n\in\mathbb{Z}$,

$$\begin{aligned} \|\langle A-n\rangle^{-1}T(z)\langle A-n\rangle^{-1}\| &\leq \tilde{C}, \\ \|F(A<n)T(z)\langle A-n\rangle^{-2}\| &\leq \tilde{C}, \\ \|\langle A-n\rangle^{-2}T(z)F(A\geq n)\| &\leq \tilde{C}. \end{aligned}$$

Due to these bounds and Lemmas 1.3 and 2.10 we conclude (2.18) with $C = 16\tilde{C}$. □

We shall use Weyl quantization on $\mathbb{R}^d$ denoted by $\mathrm{Op}^{\mathrm{w}}(c)$, cf. [Hö1, FS]. The following (λ-dependent) symbols will play a prominant role (cf. [DS1, FS]):

$$a = a_\lambda = \frac{\xi^2}{f_\lambda(x)^2}, \quad b = b_\lambda = \frac{\xi}{f_\lambda(x)}\cdot\frac{x}{\langle x\rangle}. \tag{2.31}$$

It is convenient to introduce the following metric

$$g = g_\lambda = \langle x\rangle^{-2}\mathrm{d}x^2 + f_\lambda^{-2}\mathrm{d}\xi^2,$$

and the corresponding symbol classes $S(m,g)$. Here $m = m_\lambda = m_\lambda(x,\xi)$ will be a *uniform weight function*, see [FS, Subsection 4.2] for this terminology and an account of basic pseudodifferential operator results. Here we have $\lambda\in[0,\lambda_0]$ (for the fixed $\lambda_0>0$) and the function m obeying bounds uniform in this parameter, see [FS, Lemma 4.3 (ii)] for details (note however that the discussion there concerns uniformity in a parameter $E\in(0,1]$ rather than in $\lambda\in[0,\lambda_0]$ which can be considered as a trivial modification).

In the present paper we shall use the notation $S_{\mathrm{unif}}(m_{|z|}, g_{|z|})$, given a uniform weight function m, to signify the symbol class of smooth symbols $c = c_z \in S(m_{|z|}, g_{|z|})$ with z in the closure of the set $\Gamma_\theta \subset \mathbb{C}$ satisfying

$$|\partial_x^\gamma \partial_\xi^\beta c_z(x,\xi)| \leq C_{\gamma,\beta} m_{|z|}(x,\xi) \langle x\rangle^{-|\gamma|} f_{|z|}^{-|\beta|}. \tag{2.32}$$

Note that these bounds are uniform in z belonging to the closure of the set Γ_θ. We also notice that the "Planck constant" for this class is $\langle x\rangle^{-1} f_{|z|}^{-1}$, in particular $\langle x\rangle^{\mu/2-1}$ is a "uniform Planck constant". The corresponding class of Weyl quantized operators is denoted by $\Psi_{\mathrm{unif}}(m_{|z|}, g_{|z|})$. Whence for example $f_{|z|}^s \in \Psi_{\mathrm{unif}}(f_{|z|}^s, g_{|z|})$, cf. (2.22). Here is a list of other examples (referring to (2.31) for definition)

$$\begin{aligned}
a_{|z|} &\in S_{\mathrm{unif}}(a_{|z|}+1, g_{|z|}),\\
F(a_{|z|}), I - F(a_{|z|}) &\in S_{\mathrm{unif}}(1, g_{|z|}) \text{ for all } F \in C_c^\infty(\mathbb{R}),\\
F(a_{|z|})G(b_{|z|}) &\in S_{\mathrm{unif}}(1, g_{|z|}) \text{ for all } F, G \in C_c^\infty(\mathbb{R}),\\
h, h - z &\in S_{\mathrm{unif}}(f_{|z|}^2(a_{|z|}+1), g_{|z|}) \text{ for } h := \xi^2 + V(x) = \xi^2 + V_1(x).
\end{aligned}$$

Now, with reference to the constant C_0 in Condition 1.1 (2) (i.e. the constant with $\alpha = 0$) we can bound

$$f_{|z|}^{-2}(x)\big|V(x) - z\big| \leq C_0' := \max(C_0, 1). \tag{2.34}$$

Let a real-valued $\chi_- \in C_c^\infty(\mathbb{R})$ be given such that $\chi_-(t) = 1$ for $t \in [0, C_0'+1]$, and let $\chi_+ = 1 - \chi_-$. Then the symbol

$$r = r_z := (h-z)^{-1}\chi_+(a_{|z|}) \in S_{\mathrm{unif}}(f_{|z|}^{-2}(a_{|z|}+1)^{-1}, g_{|z|}).$$

In particular, using the calculus (see [FS, Subsection 4.2]), we can write

$$\mathrm{Op}^{\mathrm{w}}(\chi_+(a_{|z|}))f_{|z|} = S_z^{\mathrm{left}} f_{|z|}^{-1}(H-z) + T_z^{\mathrm{left}}\langle x\rangle^{-1} f_{|z|}, \tag{2.35a}$$

$$f_{|z|}\mathrm{Op}^{\mathrm{w}}(\chi_+(a_{|z|})) = (H-z) f_{|z|}^{-1} S_z^{\mathrm{right}} + f_{|z|}\langle x\rangle^{-1} T_z^{\mathrm{right}}, \tag{2.35b}$$

where the operators $S_z^{\mathrm{left}}, T_z^{\mathrm{left}}, S_z^{\mathrm{right}}, T_z^{\mathrm{right}}$ have symbols

$$s_z^{\mathrm{left}}, t_z^{\mathrm{left}}, s_z^{\mathrm{right}}, t_z^{\mathrm{right}} \in S_{\mathrm{unif}}(1, g_{|z|}),$$

respectively. Notice for example for (2.35a) that

$$\mathrm{Op}^{\mathrm{w}}(\chi_+(a_{|z|})) - \mathrm{Op}^{\mathrm{w}}(r_{|z|}) f_{|z|}(H-z) f_{|z|}^{-1} \in \Psi_{\mathrm{unif}}((\langle x\rangle f_{|z|})^{-1}, g_{|z|}),$$

and that this argument can be repeated, say k times, improving the remainder to be in $S_{\mathrm{unif}}((\langle x\rangle f_{|z|})^{-k}, g_{|z|})$. Whence if $k(1-\mu/2) \geq 1$ indeed (2.35a) follows.

Lemma 2.9. *Under the condition of Lemma 2.8 there exists $C = C(\theta) > 0$ such that*

$$\sup_{z\in\Gamma_\theta} \big\| f_{|z|} R(z) f_{|z|} \big\|_{\mathcal{B}(B(f_{|z|}\langle x\rangle), B(f_{|z|}\langle x\rangle)^*)} \leq C. \tag{2.36}$$

Proof. Let for convenience $P = f_{|z|}R(z)f_{|z|}$, i.e. the quantity we want to bound, and $T_- = \mathrm{Op}^{\mathrm{w}}(\chi_-(a_{|z|}))$ (as defined above). By inserting repeatedly $I = \mathrm{Op}^{\mathrm{w}}(\chi_+(a_{|z|})) + \mathrm{Op}^{\mathrm{w}}(\chi_-(a_{|z|}))$ either to the left or right of terms proportional to P and using (2.35a) and (2.35b) we obtain an expansion in various terms. Only the following four terms are not obviously in $\Psi_{\mathrm{unif}}(1, g_{|z|}) \subseteq \mathcal{B}(\mathcal{H})$ and hence bounded by a constant that is independent of $z \in \Gamma_\theta$ (such a constant is henceforth referred to as a "uniform bounding constant"):

$$\begin{aligned} P_1 &= T_- P\, T_-, \\ P_2 &= T_- P\, \langle x\rangle^{-1} T_z^{\mathrm{right}}, \\ P_3 &= T_z^{\mathrm{left}} \langle x\rangle^{-1} P\, T_-, \\ P_4 &= T_z^{\mathrm{left}} \langle x\rangle^{-1} P\, \langle x\rangle^{-1} T_z^{\mathrm{right}}. \end{aligned}$$

Due to (1.3) also $P_4 \in \mathcal{B}(\mathcal{H})$ with a uniform bounding constant.

We calculate $[A, T_-] \in \Psi_{\mathrm{unif}}(1, g_{|z|})$ and then in turn

$$\langle A\rangle T_- \langle f_{|z|}\langle x\rangle\rangle^{-1} \in \mathcal{B}(\mathcal{H}),$$

with a uniform bounding constant. Whence due to Lemma 2.1 applied with $s = 1$ (using for (2.37b) that $T_- = T_-^*$)

$$T_- \in \mathcal{B}(B(f_{|z|}\langle x\rangle), B(A)), \tag{2.37a}$$

$$T_- \in \mathcal{B}(B(A)^*, B(f_{|z|}\langle x\rangle)^*), \tag{2.37b}$$

with uniform bounding constants.

The contribution from the term P_1 is treated by (2.37a), (2.37b) and Lemma 2.8.

It remains to treat the contributions from the terms P_2 and P_3. For that we use the resolvent identities

$$R(z) = R(\mathrm{i}) + (z - \mathrm{i})R(z)R(\mathrm{i}) = R(\mathrm{i}) + (z - \mathrm{i})R(\mathrm{i})R(z). \tag{2.38}$$

We shall treat the term P_2 by using the first identity in (2.38), omitting the (similar) arguments for P_3 (which is based on the second identity). By Lemma 2.1 (used again with $s = 1$)

$$f_{|z|}^{-1} R(\mathrm{i}) f_{|z|} \langle x\rangle^{-1} T_z^{\mathrm{right}} \in \mathcal{B}(B(f_{|z|}\langle x\rangle), B(A)), \tag{2.39}$$

with a uniform bounding constant. Now using (2.38) there are two terms to bound. The contribution from the first term is trivially treated (it is in $\mathcal{B}(\mathcal{H})$), while the contribution from the second term is treated using (2.37b), (2.39) and Lemma 2.8. We obtain a uniform bound $\|P_2\|_{\mathcal{B}(B(f_{|z|}\langle x\rangle), B(f_{|z|}\langle x\rangle)^*)} \le C$. □

Lemma 2.10. *There exists $C = C(\mu) > 0$ such that*

$$\|f_{|z|}^{-1/2} u\|_{B(f_{|z|}\langle x\rangle)} \le C\|u\|_{B(|x|)} \text{ for all } z \in \Gamma_\theta \text{ and } u \in B(|x|). \tag{2.40}$$

Proof. Due to Corollary 2.2 the bound (2.40) is equivalent to the bound (possibly changing the constant)

$$\|f_{|z|}^{-1/2} u\|_{B(f_{|z|}\langle x\rangle)} \le C\|u\|_{B(\langle x\rangle)} \text{ for all } z \in \Gamma_\theta \text{ and } u \in B(|x|) = B(\langle x\rangle). \tag{2.41}$$

Now to show (2.41) we decompose

$$I = F_- + F_+,\ F_- := F(|z| < \langle x\rangle^{-\mu}),\ F_+ := F(|z| \ge \langle x\rangle^{-\mu}),$$

and estimate for all $u \in B(|x|)$

$$\begin{aligned}\|f_{|z|}^{-1/2}F_-u\|_{B(f_{|z|}\langle x\rangle)} &\le \sum_{j=2}^{\infty} R_j^{1/2}\|F(R_{j-2}\le \langle x\rangle^{1-\mu/2} < R_j)\langle x\rangle^{\mu/4}F_-u\| \\ &\le \sup_{i\ge 2}\Big(R_i^{1/2}/R_{i-1}^{1/2}\Big)\sum_{j=2}^{\infty} R_{j-1}^{1/2}\|F(R_{j-2}\le \langle x\rangle^{1-\mu/2} < R_{j-1})\langle x\rangle^{\mu/4}u\| \\ &\quad + \sum_{j=2}^{\infty} R_j^{1/2}\|F(R_{j-1}\le \langle x\rangle^{1-\mu/2} < R_j)\langle x\rangle^{\mu/4}u\|.\end{aligned}$$

By using Lemma 2.4 with $A = \langle x\rangle$ and $s = -\mu/2$ to both terms on the right-hand side we obtain then that

$$\|f_{|z|}^{-1/2}F_-u\|_{B(f_{|z|}\langle x\rangle)} \le (\sqrt{2}+1)C(s)\|u\|_{B(\langle x\rangle)}. \tag{2.42}$$

Similarly we estimate

$$\begin{aligned}\|f_{|z|}^{-1/2}F_+u\|_{B(f_{|z|}\langle x\rangle)} &\le |z|^{-1/4}\sum_{j=2}^{\infty} R_j^{1/2}\|F(R_{j-2}\le \sqrt{|z|}\langle x\rangle < R_j)F_+u\| \\ &\le (\sqrt{2}+1)|z|^{-1/4}\|F_+u\|_{B(\sqrt{|z|}\langle x\rangle)}.\end{aligned}$$

By using Lemma 2.5 with $A = \langle x\rangle$, $c = \sqrt{|z|}$ and $u \to F_+u$ (note that indeed $F_+u = F(|cA|\ge 1)F_+u$) we obtain in turn that

$$\|F_+u\|_{B(\sqrt{|z|}\langle x\rangle)} \le 8|z|^{1/4}\|F_+u\|_{B(\langle x\rangle)} \le 8|z|^{1/4}\|u\|_{B(\langle x\rangle)}.$$

Consequently

$$\|f_{|z|}^{-1/2}F_+u\|_{B(f_{|z|}\langle x\rangle)} \le 8(\sqrt{2}+1)\|u\|_{B(\langle x\rangle)}. \tag{2.43}$$

We obtain (2.41) by combining (2.42) and (2.43) (notice that the sum of associated bounding constants only depends on μ).

□

Proof of Theorem 1.3. We can assume that $V_2 = 0$. Due to Lemma 2.10

$$f_{|z|}^{-1/2} \in \mathcal{B}(B(|x|), B(f_{|z|}\langle x\rangle)), \tag{2.44a}$$

$$f_{|z|}^{-1/2} \in \mathcal{B}(B(f_{|z|}\langle x\rangle)^*, B(|x|)^*), \tag{2.44b}$$

with uniform bounding constants. We decompose

$$f_{|z|}^{1/2}R(z)f_{|z|}^{1/2} = f_{|z|}^{-1/2}\big(f_{|z|}R(z)f_{|z|}\big)f_{|z|}^{-1/2},$$

and apply (2.44a), Lemma 2.9 and (2.44b) to the three factors on the right-hand side ordered from the right to the left, respectively. □

3. Sommerfeld radiation condition at zero energy

We shall give a characterization of the boundary values $R(0 \pm \mathrm{i}0)$ of Theorem 1.2 in terms of "radiation conditions" in a Besov space. This may be seen as a zero-energy analogue of the (strictly) positive-energy result [Hö1, Theorem 30.2.10] which is valid for a larger class of potentials. Under less general conditions than Condition 1.1 such a result was proved in [DS1], see [DS1, Proposition 4.10].

We recall some bounds of [FS]. First it is convenient to change the definition of the symbols f, a and b used in Subsection 2.2. Specifically we define here and henceforth f by (1.2) with K given by (1.6). We define the symbols a and b by (2.31) with this value of K and modify similarly the metric g. (The symbol class $S_{\mathrm{unif}}(m_{|z|}, g_{|z|})$ is independent of the constant $K > 0$ however.) Let us note the following modification of (2.34)

$$f_{|z|}^{-2}(x)\big|V_1(x) - z\big| \leq C_0' := \max(C_0/K, 1). \tag{3.1}$$

Consider real-valued $\chi_- \in C_c^\infty(\mathbb{R})$ such that $\chi_-(t) = 1$ in a neighbourhood of $[0, C_0']$ and such that $\chi_-'(t) \leq 0$ for $t > 0$. Let $\chi_+ = 1 - \chi_-$. Then we obtain obvious modifications of (2.35a) and (2.35b), in fact we could replace the power $\langle x\rangle^{-1}$ occuring to the right in (2.35a) and (2.35b) by any negative power of $\langle x\rangle$ by the same arguments used for (2.35a) and (2.35b) and we could distribute powers of f differently. Whence in particular the pseudodifferential calculus leads to the following result.

Lemma 3.1. *Let χ_+ be given as above. Then for all $n \in \mathbb{N}$ there exist*

$$s_z^{\mathrm{left}}, t_z^{\mathrm{left}} \in S_{\mathrm{unif}}((a_{|z|} + 1)^{-1}, g_{|z|}) \subseteq S_{\mathrm{unif}}(1, g_{|z|})$$

such that with $S_z^{\mathrm{left}} = \mathrm{Op^w}(s_z^{\mathrm{left}})$ and $T_z^{\mathrm{left}} = \mathrm{Op^w}(t_z^{\mathrm{left}})$

$$\mathrm{Op^w}(\chi_+(a_{|z|}) = S_z^{\mathrm{left}} f_{|z|}^{-2}(H - V_2 - z) + f_{|z|}^{-1/2}\langle x\rangle^{-n} T_z^{\mathrm{left}} \langle x\rangle^{-n} f_{|z|}^{1/2}. \tag{3.2}$$

We combine (1.3), (2.38), (3.2) and the calculus to conclude the following generalization of [FS, (4.5b)]. (The method of proof in [FS] is at this point more complicated.) The continuity assertion below is shown as in [DS1, Subsection 4.1].

Lemma 3.2. *Let χ_- and χ_+ be given as above. Let $\delta > 0$ be given as in Condition 1.1 (4). Then for all $s > 1/2$, $t \in [0, \delta + 1 - \mu/2]$ there exists $C > 0$ such that for $z \in \Gamma_\theta$*

$$\|T_+(z)| \leq C;\ T_+(z) := \langle x\rangle^{t-s} f_{|z|}^{1/2} \mathrm{Op^w}(\chi_+(a_{|z|}) R(z) f_{|z|}^{1/2} \langle x\rangle^{-t-s}. \tag{3.3}$$

Moreover there exists $T_+(0 + \mathrm{i}0) = \lim_{z \to 0, z \in \Gamma_\theta} T_+(z)$ in $\mathcal{B}(L^2(\mathbb{R}^d))$.

We also have a generalization of [FS, (4.5d)]. For that we consider an arbitrarily given real-valued $\tilde{\chi}_- \in C_c^\infty((-\infty, 1))$. Again we use the δ from Condition 1.1.

Lemma 3.3. *Let χ_-, χ_+ and $\tilde{\chi}_-$ be given as above. Then for all $s > 1/2$, $t \in [0, \delta)$ there exists $C > 0$ such that for $z \in \Gamma_\theta$*

$$\|T_-(z)| \leq C;\ T_-(z) := \langle x\rangle^{t-s} f_{|z|}^{1/2} \mathrm{Op^w}(\chi_-(a_{|z|}) \tilde{\chi}_-(b_{|z|})) R(z) f_{|z|}^{1/2} \langle x\rangle^{-t-s}. \tag{3.4}$$

Moreover there exists $T_-(0 + \mathrm{i}0) = \lim_{z \to 0, z \in \Gamma_\theta} T_-(z)$ in $\mathcal{B}(L^2(\mathbb{R}^d))$.

Note that the symbol $\chi_-(a_{|z|})\tilde{\chi}_-(b_{|z|}) \in S_{\mathrm{unif}}(1, g_{|z|})$ while this is not the case for the symbol $\tilde{\chi}_-(b_{|z|})$ itself, and that the important condition on the support of $\tilde{\chi}_-$ (yielding the localization $b_{|z|} \leq 1-\sigma < 1$ for some $\sigma > 0$) is equivalent to the support condition used for the symbol b in [FS, (4.5d)] (this is a consequence of the normalization (1.6)). Taking this remark into account the bound (3.4) is essentially contained in [FS]. The necessary modification of the proof of the weaker result [FS, (4.5d)] is explained in the beginning of [DS1, Subsection 4.1]. This explanation will not be repeated here. Note that a version of (3.4) is in fact stated in [DS1] as [DS1, (4.3d)]. The continuity assertion of Lemma 3.3 is shown as in [DS1, Subsection 4.1]. Also we remark that there are versions of Lemmas 3.2 and 3.3 with the localization factors put to the right of the resolvent, cf. [FS, Theorem 4.1], however these statements will not be used in this paper.

We shall focus on the energy zero for which $f = f_0 = \sqrt{K}\langle x\rangle^{-\mu/2}$ and introduce the Besov space $B^\mu := B(\langle x\rangle^{2s_0}) = B(|x|^{2s_0})$ adapted to this energy. Recall here and henceforth the notation of Condition 1.1, $s_0 = 1/2 + \mu/4$. Note that $B^\mu = f_0^{1/2}B(\langle x\rangle)$, cf. Lemma 2.4. The corresponding spaces $(B^\mu)^*$ and $(B^\mu)^*_0$ are characterized as follows, cf. (2.3) and (2.6).

$$u \in (B^\mu)^* \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \sup_{R>1} R^{-s_0}\|F(|x|<R)u\| < \infty, \tag{3.5a}$$

$$u \in (B^\mu)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-s_0}\|F(|x|<R)u\| = 0. \tag{3.5b}$$

In fact the expression to the right in (3.5a) defines a norm on $B(\langle x\rangle^{2s_0})^*$ which is equivalent to the canonical norm of (2.2). Yet another equivalent norm on $(B^\mu)^*$ is given in terms of an arbitrary $\epsilon \in (0,1)$ by the expression $\|F(|x|\leq 1)u\| + \sup_{R>1} R^{-s_0}\|F(\epsilon R \leq |x| < R)u\|$, and similarly

$$u \in (B^\mu)^*_0 \Leftrightarrow u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } \lim_{R\to\infty} R^{-s_0}\|F(\epsilon R \leq |x| < R)u\| = 0. \tag{3.5c}$$

Henceforth we abbreviate $L^2_s(|x|) = L^2_s$ for any $s \in \mathbb{R}$. The corresponding norm is denoted by $\|\cdot\|_s$, and we abbreviate $L^2_0(|x|) = L^2$ and $\|\cdot\|_0 = \|\cdot\|$. Let $L^2_{-\infty} = \cup_{s\in\mathbb{R}} L^2_s$.

Due to Theorem 1.2, Theorem 1.3, Lemma 3.2 and Lemma 3.3 we have

Proposition 3.4. *Let χ_-, χ_+ and $\tilde{\chi}_-$ be given as in Lemma 3.3. For all $v \in B^\mu$ there exists the weak-star limit*

$$u = R(0 + \mathrm{i}0)v = \mathrm{w}^\star\text{–}\lim_{z\to 0, z\in\Gamma_\theta} R(z)v \in (B^\mu)^*, \tag{3.6a}$$

and

$$\mathrm{Op^w}(\chi_+(a_0))u,\ \mathrm{Op^w}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in (B^\mu)^*_0. \tag{3.6b}$$

If $v \in L^2_s$ for some $s > s_0$ we have the following stronger conclusion compared to (3.6b),

$$\mathrm{Op^w}(\chi_+(a_0))u,\ \mathrm{Op^w}(\chi_-(a_0)\tilde{\chi}_-(b_0))u \in L^2_t \text{ for all } t < \min(s - s_0, \delta) - s_0. \tag{3.7}$$

In particular for any such v we can take $t = -s_0$ in (3.7).

Proof. Note that indeed (3.6a) follows from Theorems 1.2 and 1.3 (by a density argument). Similarly it suffices for (3.6b) to show the statements for $v \in L^2_1$ (using here that $\mathrm{Op^w}(\chi_+(a_0))$ and $\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))$ are bounded operators on $(B^\mu)^*$, cf. Lemma 2.1). However for such v it follows from Lemmas 3.2 and 3.3 that

$$\mathrm{Op^w}(\chi_+(a_0))u,\ \mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_{-s_0} \subseteq (B^\mu)^*_0.$$

Clearly (3.7) is also a consequence of Lemmas 3.2 and 3.3. □

Note that s_0 defines the critical weighted L^2 space for the function u defined in (3.6a), more precisely $u \in L^2_s$ for all $s < -s_0$ while no stronger assertion of this type is given. Whence this $u \in L^2_{-s_0-\delta}$ and it constitutes a particular (distributional) solution to the equation $Hu = v$. We study this situation somewhat generally in the following version of the "propagation of singularities" result [DS1, Proposition 4.5], see also [Hö2, Me, Va].

Proposition 3.5. *Let χ_- be given as in Lemma 3.3.*

i) Suppose $v \in B^\mu$, that $u \in L^2_{-s_0-\delta}$ obeys $Hu = v$, and the following localization for some $\sigma > 0$

$$\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_{-s_0} \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,\sigma-1)). \tag{3.8a}$$

Then

$$\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_{-s_0} \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,1)). \tag{3.8b}$$

ii) Suppose $v \in L^2_{s+2s_0}$ for some $s \in \mathbb{R}$, that $u \in L^2_{s-\delta}$ obeys $Hu = v$, and the following localization for some $\sigma > 0$

$$\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_{s} \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,\sigma-1)). \tag{3.9a}$$

Then

$$\mathrm{Op^w}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_{s} \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,1)). \tag{3.9b}$$

Proof. Obviously i) follows from ii) with $s = -s_0$. We shall show ii) by mimicking the scheme of the proof of [DS1, Proposition 4.5]. Since the observable b used in [DS1] is somewhat different to our b_0 and the class of potentials considered is smaller than here, there are indeed various differences to tackle. We intend to give a self-contained presentation.

We introduce the notation $s_1 = 2s_0 - \mu + \delta$, and explain the role of this parameter: Suppose $u \in L^2_{s-s_1}$ obeys $Hu = v \in L^2_{s+2s_0}$. Note that since $s_1 > \delta$ this is a more general situation than in ii). Then due to Lemma 3.1 we can find $S \in \mathcal{B}(L^2)$ such that $X^sSX^{-s} \in \mathcal{B}(L^2)$ and

$$\mathrm{Op^w}(\chi_+(a))u - SX^{-s_1}u \in L^2_s,$$

and whence we conclude that

$$\mathrm{Op^w}(\chi_+(a))u \in L^2_s. \tag{3.10}$$

A consequence of this remark is that the statements (3.9a) and (3.9b) do not depend on which $\chi_- = 1 - \chi_+$ that enter (in particular the factors of χ_- in these statements may differ from one another).

At various points below we need to bound possible local singularities of the potential V_2. Since these by assumption are located in a bounded region they are easily treated by the general bounds (in turn easily proven using the relative compactness of V_2)

$$\|V_2 F(|x| \leq R)u\| \leq C_t(\|Hu\|_t + \|u\|_t) \text{ for } t \in \mathbb{R}. \tag{3.11}$$

The reason why we need the a priory information $u \in L^2_{s-\delta}$ in ii) rather than just $u \in L^2_{-\infty}$ (as in [DS1]) lies in contributions from $V_2 F(|x| > R)u$. We will use that the function $|x|^{2s_0+\delta} V_2 F(|x| > R)$ is bounded (for R as in Condition 1.1 (4)), but since we are not imposing further regularity of this function our method of proof does not allow us to weaken the condition $u \in L^2_{s-\delta}$.

Henceforth we abbreviate the symbols $a_0 = a$ and $b_0 = b$. An important ingredient of the proof is the following estimation of a Poisson bracket, cf. [DS1, (4.30)]. Here we have $h_1 = \xi^2 + V_1(x)$.

$$\begin{aligned} \{h_1, b\} &= \langle x\rangle^{\mu/2-1}\big(W - (2-\mu)K\langle x\rangle^{-\mu} b^2 + 2h_1\big)/\sqrt{K} \\ &\geq (2-\mu)\sqrt{K}\langle x\rangle^{-2s_0}\big(1 - b^2\big) + \langle x\rangle^{\mu/2-1} 2h_1/\sqrt{K}. \end{aligned} \tag{3.12}$$

We introduce for $\kappa \in (0,1]$ the notation $X_\kappa = (\kappa|x|^2 + 1)^{1/2}$ and abbreviate $X_1 = X = \langle x\rangle$. These observables have Poisson brackets

$$\{h_1, X_\kappa\} = 2\kappa\xi \cdot x/X_\kappa = 2f_0 b \tfrac{\kappa X}{X_\kappa} \text{ and } \{h_1, X\} = 2\xi \cdot x/X = 2f_0 b. \tag{3.13}$$

Step I. Let

$$\epsilon_2 = \min(1 - \mu/2, \delta). \tag{3.14}$$

The quantity $X^{-\epsilon_2}$ will play the role of a "Planck constant". (The number $2\epsilon_2$ will play the role of the parameter ϵ_2 used in [DS1].) First we prove (3.9b) with the replacement $s \to \tilde{s} = s + \epsilon_2 - \delta$ for any $u \in L^2_{\tilde{s}-\epsilon_2} = L^2_{s-\delta}$ obeying $Hu = v \in L^2_{\tilde{s}+2s_0} \supseteq L^2_{s+2s_0}$. Note that indeed $\tilde{s} \leq s$. In particular if (3.9a) is valid for s then (3.9a) is also valid for $s \to \tilde{s}$ (to be used in Step II below). So we assume in addition (3.9a) with $s \to \tilde{s}$ for any such u, and we aim at proving (3.9b) with $s \to \tilde{s}$. Clearly we can (and will) assume that $\sigma < 1$ in (3.9a).

We shall use that

$$\mathrm{Op^w}(\chi_+(a))u \in L^2_{\tilde{s}}, \tag{3.15}$$

cf. (3.10).

Now let $\tilde{\chi}_-$ be given as in (3.9b). In particular this means that $\operatorname{supp}\tilde{\chi}_- \subseteq (-\infty, k]$ for some $k \in (0,1)$. Pick a non-positive function $f \in C_c^\infty((\sigma/3 - 1, 1))$ with $f'(t) \geq 0$ on $[\sigma/2 - 1, \infty)$ and $f(t) < 0$ on $[\sigma/2 - 1, (k+1)/2]$. Pick real-valued functions $f_1, f_2 \in C_c^\infty((-1,1))$ with $f_1^2(t) + f_2^2(t) = 1$ on $\operatorname{supp} f$ and $\operatorname{supp} f_1 \subset (\sigma/3 - 1, \sigma - 1)$ while $\operatorname{supp} f_2 \subset (\sigma/2 - 1, 1)$. We introduce for any $K_0 > 0$ and $\kappa \in (0,1]$ the observables

$$b_\kappa = X^{s_0} a_\kappa,\ a_\kappa = X^{\tilde{s}} X_\kappa^{-\epsilon_2} \exp(-K_0 b) f(b) \chi_-(a); \tag{3.16}$$

here X_κ is defined above. Notice that we removed the subscripts for the old observables a and b and used these in (3.16) to introduce new observables with subscripts (not to be mixed up with the old notation). We will prove bounds involving quantizations of a_κ and b_κ that will be uniform in κ. The proof will then be completed by taking $\kappa \to 0$.

We look at the right hand side of (3.12). The first term has the following positive lower bound on $\operatorname{supp} b_\kappa$:

$$\cdots \geq cX^{-2s_0};\ c = (2-\mu)\sqrt{K}\big(1 - \sup\{t^2 | t \in \operatorname{supp} f\}\big).$$

First we fix K_0: A part of the Poisson bracket with b_κ^2 is

$$\{h_1, X^{2\tilde{s}+2s_0} X_\kappa^{-2\epsilon_2}\} = Y_\kappa b X^{2\tilde{s}} X_\kappa^{-2\epsilon_2}, \tag{3.17}$$

where $Y_\kappa = Y_\kappa(|x|)$ is uniformly bounded in κ, cf. (3.13). We pick $K_0 > 0$ such that for all $\kappa \in (0,1]$

$$2K_0 c \geq |Y_\kappa| + 2 \text{ on } \operatorname{supp} b_\kappa.$$

From (3.12), (3.17) and the properties of K_0 and f, we conclude the following bound at $\{f'(b) \geq 0\}$:

$$\{h_1, b_\kappa^2\} \leq -2a_\kappa^2 + O\big(X^{\mu+\tilde{s}}\big) h_1 a_\kappa + O\big(X^{2\tilde{s}}\big)(\chi_-^2)'(a).$$

Next we multiply both sides by $f_2^2(b)\,(= 1 - f_1^2(b))$ and obtain after a rearrangement

$$\begin{aligned}\{h_1, b_\kappa^2\} \leq\ & -2a_\kappa^2 + h_1 X^\mu d_\kappa a_\kappa \\ & + K_1 f_1^2(b)\chi_-^2(a) X^{2\tilde{s}} - K_2(\chi_-^2)'(a) X^{2\tilde{s}},\ d_\kappa \in S(X^{\tilde{s}}, g_0);\end{aligned} \tag{3.18}$$

here $K_1, K_2 > 0$ are independent of κ, and the family of symbols $\{d_\kappa : \kappa \in (0,1]\}$ is bounded in the specified symbol class.

We introduce $A_\kappa = \operatorname{Op^w}(a_\kappa)$, $B_\kappa = \operatorname{Op^w}(b_\kappa)$ and the regularization

$$u_R = \chi(X/R < 1)u$$

in terms of a parameter $R > 1$. Here and henceforth we use the notation $\chi(t > \epsilon)$ for any $\epsilon > 0$ to denote a smooth increasing function $= 1$ for $t > \epsilon$ and $= 0$ for $t < \frac{1}{2}\epsilon$ and we define $\chi(\cdot < \epsilon) = 1 - \chi(\cdot > \epsilon)$. Let $H_1 = H - V_2$. The following arguments rely heavily on the calculus, cf. [Hö1, Theorems 18.5.4, 18.6.3, 18.6.8].

First we compute the expectation

$$\langle \mathrm{i}[H_1, B_\kappa^2]\rangle_u = \lim_{R\to\infty} \langle \mathrm{i}[H_1, B_\kappa^2]\rangle_{u_R} = -2\operatorname{Im}\langle v, B_\kappa^2 u\rangle + 2\operatorname{Im}\langle V_2 u, B_\kappa^2 u\rangle. \tag{3.19}$$

Next we estimate

$$|-2\operatorname{Im}\langle v, B_\kappa^2 u\rangle| \leq C_1 \|v\|_{\tilde{s}+2s_0}\big(\|A_\kappa u\| + \|u\|_{\tilde{s}-\epsilon_2}\big) \leq \tfrac{1}{4}\|A_\kappa u\|^2 + C_2, \tag{3.20}$$

and (using the estimate (3.11) with any $t \leq \tilde{s} - \delta$)

$$\begin{aligned}|2\operatorname{Im}\langle V_2 u, B_\kappa^2 u\rangle| &\leq C_3\big(\|u\|_{\tilde{s}-\delta} + \|v\|_{\tilde{s}+2s_0}\big)\big(\|A_\kappa u\| + \|u\|_{\tilde{s}-\epsilon_2}\big) \\ &\leq \tfrac{1}{4}\|A_\kappa u\|^2 + C_4.\end{aligned} \tag{3.21}$$

From (3.19)–(3.21) we conclude that

$$|\langle \mathrm{i}[H_1, B_\kappa^2]\rangle_u| \leq \tfrac{1}{2}\|A_\kappa u\|^2 + C_2 + C_4. \tag{3.22}$$

On the other hand, using (3.9a) (with $s \to \tilde{s}$), (3.15) and (3.18), we infer that

$$\begin{aligned}\langle \mathrm{i}[H_1, B_\kappa^2]\rangle_u &= \lim_{R\to\infty} \langle \mathrm{i}[H_1, B_\kappa^2]\rangle_{u_R} \\ &\leq -2\|A_\kappa u\|^2 + C_5\|H_1 u\|_{\tilde{s}+\mu}\|A_\kappa u\| + C_6.\end{aligned}$$

Here the second term arises as a bound of $\operatorname{Re}\langle \mathrm{Op^w}(d_\kappa)X^\mu H_1 u, A_\kappa u\rangle$. Using the bound $\|(H-V_2)u\|_{\tilde s+\mu} \le C\big(\|v\|_{\tilde s+\mu} + \|u\|_{\tilde s-\delta}\big)$ it follows that

$$\langle \mathrm{i}[H_1, B_\kappa^2]\rangle_u \le -\tfrac{3}{2}\|A_\kappa u\|^2 + C_7. \tag{3.23}$$

Combining (3.22) and (3.23) yields

$$\|A_\kappa u\|^2 \le C_2 + C_4 + C_7,$$

which in combination with the property, $f(t) < 0$ on $[\sigma/2-1, (k+1)/2]$, in turn gives a uniform bound

$$\|X_\kappa^{-\epsilon_2}\mathrm{Op^w}(\chi_-(a)\tilde\chi_-(b))u\|^2_{\tilde s} \le C. \tag{3.24}$$

We let $\kappa \to 0$ in (3.24) and infer (3.9b) with $s \to \tilde s$.

Step II. Define for $m \in \mathbb{N}$ the number $\tilde s_m = \min(s, s + m\epsilon_2 - \delta)$, where ϵ_2 is given by (3.14). We learn from Step I that (3.9b) is valid with $s \to \tilde s_1$. We proceed inductively to prove that (3.9b) is valid with $s \to \tilde s = \tilde s_m$ given that we know the statement for $\tilde s_{m-1}$ for some $m = 2, 3, \dots$ This is done by mimicking the corresponding step in [DS1] although in the present context it is doable in a somewhat simpler way. The idea is to use the procedure of Step I for a localized version of u, say $u_\epsilon = I_\epsilon u$. The factor I_ϵ is a pseudodifferential operator with symbol $= 1$ in a slightly bigger region than the support of the symbol a_κ of Step I (now used with $\tilde s = \tilde s_m$). Explicitly I_ϵ can be constructed as follows: With f and χ_- given as in (3.16) we pick $f_\epsilon \in C_c^\infty((-\infty, 1))$ with $f_\epsilon(t) = 1$ in a neighbourhood of the support of f and we pick χ_ϵ of the same type of function as χ_- but with $\chi_\epsilon(t) = 1$ in a neighbourhood of the support of χ_-, and then we define $I_\epsilon = \mathrm{Op^w}(\chi_\epsilon(a)f_\epsilon(b))$. By the induction hypothesis $u_\epsilon \in L^2_{\tilde s_{m-1}}$. We need to consider contributions from the commutator $[H_1, I_\epsilon]$ when mimicking the procedure of Step I using the same constructions (3.16). But these are "nice to any order" due to the support properties of the involved symbols and the calculus. To see this more concretely consider the analogue of (3.20): We consider now

$$\langle I_\epsilon v + [H_1, I_\epsilon]u, B_\kappa^2 u_\epsilon\rangle = \langle I_\epsilon v, B_\kappa^2 u_\epsilon\rangle + \langle B_\kappa^2[H_1, I_\epsilon]u, u_\epsilon\rangle.$$

The contribution from the first term is treated as before (using now the induction hypothesis), and the contribution from the second term is indeed harmless (since the symbol of $B_\kappa^2[H_1, I_\epsilon]$ is in $S(X^t, g_0)$ for all $t \in \mathbb{R}$). Similarly the expression $\langle V_2 u, B_\kappa^2 u\rangle$ of (3.21) needs to be replaced by

$$\langle I_\epsilon V_2 u, B_\kappa^2 u_\epsilon\rangle,$$

which indeed can be estimated as before (using the induction hypothesis). Whence we obtain (3.22) with $u \to u_\epsilon$.

Similarly we can show (3.23), and hence in turn conclude (3.24), for $u \to u_\epsilon$. Again we complete the proof by letting $\kappa \to 0$. □

We have two versions of uniqueness of the outgoing solution at zero energy stated in one theorem as follows.

Theorem 3.6. *Let χ_- be given as in Lemma 3.3. Let δ and s_0 be given as in Condition 1.1.*

i) Let $v \in B^\mu$. Then the equation $Hu = v$ (in the distributional sense) has a unique solution $u \in L^2_{-\infty}$ obeying

$$\exists\, \sigma \in (0,1):\ \mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde\chi_-(b_0))u \in (B^\mu)^*_0 \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,\sigma)). \tag{3.25a}$$

this solution is given by $u = R(0+\mathrm{i}0)v$ as defined in (3.6a). *In particular*

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde\chi_-(b_0))u \in (B^\mu)^*_0 \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,1)). \tag{3.25b}$$

ii) Let $v \in L^2_s$ for some $s > s_0$. Then the equation $Hu = v$ has a unique solution $u \in L^2_{-s_0-\delta}$ obeying

$$\exists\, \sigma > 0:\ \mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_{-s_0} \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,\sigma-1)). \tag{3.26a}$$

This solution is given by $u = R(0+\mathrm{i}0)v$ as defined in (3.6a). *Whence for all $t < \min(s-s_0,\delta) - s_0$*

$$\mathrm{Op}^{\mathrm{w}}(\chi_-(a_0)\tilde\chi_-(b_0))u \in L^2_t \text{ for all } \tilde\chi_- \in C_c^\infty((-\infty,1)). \tag{3.26b}$$

In particular we can take $t = -s_0$ in (3.26b).

Proof. By Proposition 3.4 the function $u = R(0+\mathrm{i}0)v$ defined as in (3.6a) for any v given as in i) or ii) obeys (3.25b) or (3.26b) (for any such t), respectively. By Proposition 3.5 i) the condition (3.26a) is stronger than (3.25b), which in turn obviously is stronger than (3.25a). Whence it only remains to show that (3.25a) is sufficient for uniqueness. In fact we can take $v = 0$, and it suffices to show that (3.25a) for some $u \in L^2_{-\infty}$ obeying $Hu = 0$ implies that $u = 0$. For that we essentially mimic the proof of [DS1, Proposition 4.10].

Step I. We shall show that any such $u \in L^2_s$ for all $s < -s_0$. Suppose $u \in L^2_t$ for some $t < -s_0$ (which is an a priory information). Introducing

$$\epsilon = -\min\big((t+s_0)/2,\ t+s_0+\epsilon_2\big) \text{ and } t_1 = -s_0 - \epsilon$$

(the parameter ϵ_2 is given in (3.14)), we check that

$$t_1 > t \geq t_1 - \epsilon_2. \tag{3.27}$$

Let $u_R = \chi(X/R < 1)u$, $R > 1$, be a given regularization of u as in the proof of Proposition 3.5. We abbreviate $\chi_R = \chi(X/R) = \chi(X/R < 1)$. Here and henceforth we also use the notation of (3.13) (in particular we again abbreviate $a_0 = a$ and $b_0 = b$). Obviously, undoing the commutator, we have

$$\langle \mathrm{i}[H, X^{-2\epsilon}\chi_R]\rangle_u = 0. \tag{3.28}$$

On the other hand, cf. (3.13),

$$\mathrm{i}[H, X^{-2\epsilon}\chi_R] = 2\mathrm{Re}\,\big(f_0 h_{\epsilon,R}\mathrm{Op}^{\mathrm{w}}(b)\big);$$
$$h_{\epsilon,R}(x) = -2\epsilon X^{-1-2\epsilon}\chi(X/R) + X^{-2\epsilon}R^{-1}\chi'(X/R).$$

By using Lemma 3.1, (3.11), (3.27) and the calculus (cf. [Hö1, Theorems 18.5.4 and 18.6.3]) we obtain that

$$2\mathrm{Re}\,\langle f_0 h_{\epsilon,R}\mathrm{Op}^{\mathrm{w}}\big(b\chi_+(a)\big)\rangle_u = 2\mathrm{Re}\,\langle f_0 h_{\epsilon,R}\mathrm{Op}^{\mathrm{w}}\big(b\chi_-(a)\chi_+(a)\big)\rangle_u + O(R^0) = O(R^0),$$

and whence that

$$\langle \mathrm{i}[H, X^{-2\epsilon}\chi_R]\rangle_u = 2\mathrm{Re}\,\langle f_0 h_{\epsilon,R}\mathrm{Op^w}\big(b\chi_-^2(a)\big)\rangle_u + O(R^0).$$

We invoke then (3.25a) and deduce

$$\langle \mathrm{i}[H, X^{-2\epsilon}\chi_R]\rangle_u = 2\mathrm{Re}\,\langle f_0 h_{\epsilon,R}\mathrm{Op^w}\big(b\chi(b>\sigma/2)\chi_-^2(a)\big)\rangle_u + O(R^0),$$

which in turn yields (by applying [Hö1, Theorem 18.6.8] and "reversing" the arguments above) that

$$\langle \mathrm{i}[H, X^{-2\epsilon}\chi_R]\rangle_u \le -\epsilon\sigma\langle f_0 X^{-1-2\epsilon}\chi_R\rangle_u + O(R^0). \tag{3.29}$$

By combining (3.28) and (3.29) we obtain

$$\langle f_0 X^{-1-2\epsilon}\chi_R\rangle_u \le C, \tag{3.30}$$

for some constant C which is independent of $R>1$. Whence, letting $R\to\infty$ we see that $u\in L^2_{t_1}$.

More generally, we define for $m\in\mathbb{N}$

$$t_m = -s_0 + \min\big((t_{m-1}+s_0)/2,\ t_{m-1}+s_0+\epsilon_2\big),\ t_0 := t,$$

and iterate the above procedure. We conclude that $u\in L^2_{t_m}$, and hence that indeed $u\in L^2_s$ for all $s<-s_0$.

Step II. We shall show that $u\in(B^\mu)^*_0$. We apply the same scheme as in Step I, now with $\epsilon=0$ and using the same factor of $\chi(b>\sigma/2)$. This leads to

$$-R^{-1}\langle f_0\chi'\big(X/R\big)\rangle_u = o(R^0),$$

and hence indeed $u\in(B^\mu)^*_0$, cf. (3.5c).

Step III. We shall show that $u=0$. First we introduce $\epsilon_3=\epsilon_2/2$ and let $s\in(-s_0,\epsilon_3-s_0)$ be given arbitrarily. Our goal is to show that $u\in L^2_s$. An iteration procedure will then give that $u\in L^2$, and hence that $u=0$.

We consider for $\kappa\in(0,1/2]$ (and in terms of notation of (3.13))

$$b_\kappa = X^{s_0}a_\kappa;\ a_\kappa = \Big(\tfrac{X}{X_\kappa}\Big)^s X_\kappa^{-s_0}\chi(-b>1/2)\chi_-(a). \tag{3.31}$$

Using (3.13) we calculate the Poisson bracket

$$\Big\{h_1, \Big(\tfrac{X}{X_\kappa}\Big)^{2s_0+2s}\Big\} = 4(1-\kappa)(s_0+s)X_\kappa^{-3}\Big(\tfrac{X}{X_\kappa}\Big)^{2s_0+2s-1} f_0 b.$$

This is negative on the support of b_κ with the (negative) upper bound

$$\begin{aligned}\cdots &\le -2^{-1}(s_0+s)\Big(X^{2s_0-1}f_0\Big)X_\kappa^{-2}\Big(\big(\tfrac{X}{X_\kappa}\big)^s X_\kappa^{-s_0}\Big)^2\\ &= -cX_\kappa^{-2}\Big(\big(\tfrac{X}{X_\kappa}\big)^s X_\kappa^{-s_0}\Big)^2;\ c=2^{-1}(s_0+s)/\sqrt{K}.\end{aligned} \tag{3.32}$$

Similarly, by (3.12),

$$\begin{aligned}&\big\{h_1, \chi(-b>1/2)\big\}\\ &\quad\le -(2-\mu)\sqrt{K}\chi'(-b>1/2)X^{-2s_0}(1-b^2) - 2\chi'(-b>1/2)X^{\mu/2-1}h_1/\sqrt{K}.\end{aligned}$$

Note that the first term is non-positive.

We introduce the quantizations $A_\kappa = \mathrm{Op}^{\mathrm{w}}(a_\kappa)$ and $B_\kappa = \mathrm{Op}^{\mathrm{w}}(b_\kappa)$, and the states $u_R = \chi_R u$, $R > 1$. Since $u \in (B^\mu)^*_0$ due to Step II

$$\lim_{R\to\infty} \langle \mathrm{i}[H, B_\kappa^2]\rangle_{u_R} = 0. \tag{3.33}$$

Since $\delta > 2\epsilon_3$ and $s - \epsilon_3 < -s_0$ we also that

$$|\langle \mathrm{i}[V_2, B_\kappa^2]\rangle_{u_R}| \leq C,$$

where C is a positive constant independent of $R > 1$ and $\kappa \in (0, 1/2]$.

On the other hand due to the above considerations the expectation of $\mathrm{i}[H_1, B_\kappa^2]$ in u_R tends to be negative. Using Lemma 3.1, (3.11), (3.33) and the above estimations of symbols we obtain by letting $R \to \infty$

$$c\|X_\kappa^{-1} A_\kappa u\|^2 \ \big(= \lim_{R\to\infty} c\|X_\kappa^{-1} A_\kappa u_R\|^2 \big) \leq C,$$

where c is given by (3.32) and C is independent of κ.

Whence, letting $\kappa \to 0$, we conclude that

$$\mathrm{Op}^{\mathrm{w}}\Big(\chi(-b > 1/2)\chi_-(a)\Big)u \in L^2_s. \tag{3.34a}$$

Upon replacing the factor $\chi(-b > 1/2)$ in (3.31) by $\chi(b > 1/2)$, we can argue similarly and obtain

$$\mathrm{Op}^{\mathrm{w}}\Big(\chi(b > 1/2)\chi_-(a)\Big)u \in L^2_s. \tag{3.34b}$$

In combination with Lemma 3.1, (3.11) and Proposition 3.5 the bounds (3.34a) and (3.34b) yield that $u \in L^2_s$.

Next the above procedure is iterated: Assuming that $u \in L^2_s$ for all $s < t_m := m\epsilon_3 - s_0$ (for some $m \in \mathbb{N}$), it leads to $u \in L^2_s$ for all $s < t_{m+1}$. Consequently, $u \in L^2_s$ for all $s \in \mathbb{R}$. In particular $u \in L^2$, and therefore $u = 0$.

□

Corollary 3.7. *Suppose $u \in (B^\mu)^*_0$ solves the equation $Hu = 0$. Then $u = 0$.*

Remark 3.8. There is a similar characterization of the operator $R(0-\mathrm{i}0) = \lim_{z\to 0, z\in\Gamma(\theta)} R(\bar z)$ as the one for $R(0 + \mathrm{i}0)$ in Theorem 3.6. We are not stating the result here.

References

[AH] S. Agmon, L. Hörmander: *Asymptotic properties of solutions of differential equations with simple characteristics*, J. d'Analyse Math., *30* (1976), 1–38.

[DS1] J. Dereziński, E. Skibsted, *Quantum scattering at low energies*, J. Funct. Anal., **257** (2009), 1828–1920.

[DS2] J. Dereziński, E. Skibsted, *Scattering at zero energy for attractive homogeneous potentials*, Ann. H. Poincaré **10** (2009), 549–571.

[FH] R. Froese and I. Herbst, *Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators*, Comm. Math. Phys. **87** no. 3 (1982/83), 429–447.

[FS] S. Fournais, E. Skibsted, *Zero energy asymptotics of the resolvent for a class of slowly decaying potentials*, Math. Z. **248** (2004), 593–633.

[GY] Y. Gatel, D. Yafaev, *On the solutions of the Schrödinger equation with radiation conditions at infinity: the long-range case*, Ann. Inst. Fourier, Grenoble **49**, no. 5 (1999), 1581–1602.

[Hö1] L. Hörmander, *The analysis of linear partial differential operators. II-IV*, Berlin, Springer 1983–85.

[Hö2] L. Hörmander, *On the existence and the regularity of solutions of linear pseudo-differential equations*, Enseignement Math. **17** no. 2 (1971), 99–163.

[Is] H. Isozaki, *A generalization of the radiation condition of Sommerfeld for N-body Schrödinger operators*, Duke Math. J. **74**, no. 2 (1994), 557–584.

[JP] A. Jensen, P. Perry, *Commutator methods and Besov space estimates for Schrödinger operators*, J. Operator Theory **14** (1985), 181–188.

[JK] D. Jerison and C.E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. (2) **121**, no. 3 (1985),463–494, With an appendix by E. M. Stein.

[Me] R. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Spectral and scattering theory (Sanda, 1992) (M. Ikawa, ed.), Marcel Dekker (1994), 85–130.

[Mo1] É. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Comm. Math. Phys. **78** no. 3 (1980/81), 391–408.

[Mo2] É. Mourre, *Operateurs conjugués et propriétés de propagation*, Comm. Math. Phys. **91** (1983), 297–300.

[Mø] J.S. Møller, *An abstract radiation condition and applications to N-body systems*, Rev. Math. Phys. **12**, no. 5 (2000), 767–803.

[Na] S. Nakamura, *Low energy asymptotics for Schrödinger operators with slowly decreasing potentials*, Comm. Math. Phys. **161**, no. 1 (1994), 63–76.

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics I-IV*, New York, Academic Press 1972-78.

[Ro] J. Royer, *Limiting absorption principle for the dissipative Helmholtz equation*, Comm. PDE. **35** (2010), 1458–1489.

[Sa] Y. Saitō, *Spectral representations for Schrödinger operators with a long-range potentials*, Lecture Notes in Matematics **727**, Berlin, Springer 1979.

[Sk] E. Skibsted: Renormalized two-body low-energy scattering, in preparation.

[SW] E. Skibsted, X. P. Wang: *Two-body threshold spectral analysis, the critical case*, J. Funct. Anal. **260** (2011), 1766-1794.

[Va] A. Vasy, *Propagation of singularities in three-body scattering*, Astérique, **262** (2000).

[Wa] X. P. Wang, *Microlocal estimates of the stationary Schrödinger equation in semi-classical limit*, Partial differential equations and applications, Séminaires et Congrès SMF **15** (2007), 265–308.

[Ya] D. Yafaev, *The low energy scattering for slowly decreasing potentials*, Comm. Math. Phys. **85**, no. 2 (1982), 177–196.

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