

Periodically driven dynamics of a particle moving in the field of Coulomb plus confining potential

D.U.Matrasulov, P.K.Khabibullaev, D.M.Otajanov

Heat Physics Department of the Uzbek Academy of Sciences,
28 Katartal St.,700135 Tashkent, Uzbekistan

and

F.C.Khanna

Physics Department University of Alberta

Edmonton Alberta, T6G 2J1 Canada

Abstract

Periodically driven dynamics of a particle moving in the field Coulomb plus confining potential is treated for one and three dimensional cases. Critical value of the external field strength at which chaotization will occur is evaluated analytically based on the resonance overlap criterion. The analysis of the phase-space dynamics is presented.

1 Introduction

Periodically driven dynamics has been subject of extensive research for past few decades both in classical and quantum contexts [1]-[6]. An interesting feature of the periodically perturbed dynamical system is the chaotization of its motion under certain conditions. This implies divergence of the phase space trajectories and unlimited energy growth(in the classical case) leading to acceleration. Qualitative and quantitative analysis of the model systems such as

kicked rotor shows that there is linear time dependence in the average energy of the periodically driven system leads to a diffusion in the phase space [6]. Corresponding quantum studies have shown that such a diffusion is suppressed in the quantum case [6]. Therefore periodically driven systems are convenient testing ground for the study of dynamical chaos in time-dependent case. Great variety of periodically driven systems have been treated from the viewpoint of dynamical chaos theory from model systems such as kicked rotor [6], pendula [2] or billiards [4] to realistic systems like atoms and molecules [5, 7, 8, 9, 3, 10] and surface-state electrons in liquid helium [11] in the time-periodic fields. Comprehensive theoretical and experimental studies of the hydrogen atom in a monochromatic field showed that diffusive excitation of the atom leading to ionization during relatively long time can occur. This fact makes hydrogen atom in a monochromatic field a convenient system for the theoretical and experimental study of chaos in the time-dependent dynamical systems [3, 7, 8, 9]. A theoretical analysis, of the behavior of a classical hydrogen atom interacting with monochromatic field, based on the resonance overlap criterion [9]-[10], shows that for some critical value of the external field strength ϵ_{cr} , the electron enters into chaotic regime of motion, marked by unlimited diffusion along orbits, leading to ionization. Experimentally, this phenomenon was first observed by Bayfield and Koch [12]. Such an ionization was called chaotic [3, 7] or diffusive [9] ionization. During the last three decades chaotic ionization of nonrelativistic atom was investigated by many authors theoretically [9, 3, 11] as well as experimentally [7, 8, 12].

In this paper we study a model which may be considered as a QCD counterpart of the periodically driven hydrogen atom. Namely, we address the problem of regular and chaotic motion of a particle bounded in the field of Coulomb plus confining potential in the presence of a time-periodic external perturbation. Such system may be used for modelling periodically driven quark-antiquark state, so-called quarkonium.

Using resonance analysis based on the Chirikov criterion for stochasticity we estimate critical values of the external field strength at which quarkonium motion enters into chaotic regime.

Quarkonium in a monochromatic field can be considered as an analog of the hydrogen atom in a monochromatic field, in which Coulomb potential is replaced by Coulomb plus confining potential. Quarkonia have been the subject of extensive experimental as well as theoretical

studies for the last two decades [13]-[14]. In the framework of potential model the description of quark motion in hadrons is reduced to solving classical or quantum mechanical equations of motion with Coulomb plus confining potential [14].

Study of the nonlinear dynamics of hadrons is of importance due to the recent advances made in creation of hadronic and quark-gluon matters in the collisions experiments of ultrarelativistic heavy ions where creation of hadronic or quark-gluon matter is possible [15]. Quarkonia in quark-gluon matter can be considered as a system perturbed by time-dependent force, that may lead to chaotization of the quarkonium motion. Indeed, the recent studies on quarkonium dynamics showed that regular motion can be expected at a small values of color screening mass but the chaotic motion is expected at a large one [16]. Periodically driven quarkonium can be also realized in the interaction of mesons with laser fields. This paper is organized as follows. In next section we will treat a simple model, a one-dimensional quarkonium under a periodic perturbation. In section 3 we extend our treatment to the three-dimensional case. Some concluding remarks are presented in section 4.

2 One-dimensional model

For simplicity we consider first a one-dimensional model described by a potential

$$V(x) = \begin{cases} -\frac{Z}{x} + \lambda x & \text{for } x > 0 \\ \infty & \text{for } x \leq 0 \end{cases},$$

where $Z = \frac{4}{3}\alpha_s$, α_s being the effective strong coupling constant and λ gives strength of the confining potential. As is well known, in the case of the hydrogen atom interacting with a monochromatic field, one-dimensional model provides an excellent description of the experimental chaotization thresholds for real three-dimensional hydrogen atom [7, 11, 3]. The same success is to be expected in the case of quarkonium.

The unperturbed Hamiltonian for the above potential is

$$H_0 = \frac{p^2}{2} - \frac{Z}{x} + \lambda x. \quad (1)$$

We will treat the interaction of the system given by Hamiltonian (1) with the periodic external potential of the form

$$U(x, t) = \epsilon x \cos \omega t. \quad (2)$$

with ϵ and ω being the field strength and frequency, respectively. Thus the total Hamiltonian of the periodically driven quarkonium is

$$H = H_0 + U(x, t) \quad (3)$$

Formally, the Hamiltonian (1) is equivalent to that of the hydrogen atom in constant homogenous electric field. Chaotic dynamics of hydrogen atom in constant electric field under the influence of time-periodic field was treated earlier [17, 18]. To explore periodically driven dynamics of our system we introduce so-called action-angle variables and rewrite its Hamiltonian in terms of these variables.

The action variable is defined as [5]:

$$n = \frac{1}{2\pi} \int_c^a \sqrt{2(H_0 - V(x))} dx = \frac{\sqrt{2\lambda}}{2\pi} \int_c^a \sqrt{\frac{(a-x)(x-c)}{x}} dx, \quad (4)$$

where the constants a and c are the turning points of particle and are given by

$$a = \frac{H_0 + \sqrt{H_0^2 + 4Z\lambda}}{2\lambda}, \quad c = \frac{H_0 - \sqrt{H_0^2 + 4Z\lambda}}{2\lambda}$$

Since $c < 0$ for the action we have

$$\begin{aligned} n &= \frac{1}{2\pi} \int_0^a \sqrt{2(H_0 - V(x))} dx = \\ &= B \sqrt{a + \frac{1}{a}} \cdot \left[\left(a - \frac{1}{a}\right) E(k) + \frac{1}{a} K(k) \right]. \end{aligned} \quad (5)$$

where

$$B = \frac{2\sqrt{2}}{3\pi\lambda^{\frac{1}{4}}},$$

here $E(k)$ and $K(k)$ are the elliptic integrals [19] and

$$k^2 = \frac{a^2}{a^2 + 1}. \quad (6)$$

The angle variable is defined as [5]

$$\theta = \frac{\partial S}{\partial n}$$

where

$$S = \int^x p(x, H(n)) dx$$

Then in terms of n and x variables the angle can be written as

$$\theta(x, n) = B \sqrt{x + \frac{1}{x}} \left[\left(x + \frac{1}{x}\right) \frac{E(k) - K(k)}{k} \frac{dk}{dn} + \frac{1}{x} K(k) \frac{E(k) - k\sqrt{1-k^2}K(k)}{k\sqrt{1-k^2}} \frac{dk}{dn} \right], \quad (7)$$

where $k = k(n)$ is defined by the eq. (6).

Furthermore, we consider the following two cases: $a \gg 1$ and $a \ll 1$. For both cases we obtain approximate expression for unperturbed Hamiltonian as a function of angle variable. To do this we solve the eq.(5) with respect to H_0 and use asymptotic estimates for the elliptic integrals, $E(k)$ and $K(k)$ [19] for $a \ll 1$ and $a \gg 1$ that correspond to $k \ll 1$ and $k \approx 1$, respectively.

For $a \gg 1$ we have:

$$H_0 = Z^2 A n^{2/3} \cdot \left[1 - \frac{\lambda \ln(4B^{-2/3} n^{2/3})}{A^2 n^{4/3}} \right], \quad (8)$$

with

$$A = \frac{3\pi\lambda^{2/3}}{2\sqrt{2}}.$$

Corresponding proper frequency is

$$\omega_0 = \frac{2}{3} Z^2 \left[\frac{A}{n^{1/3}} + \frac{\lambda}{A n^{5/3}} [\ln(4A\sqrt{\lambda} n^{2/3})] - 1 \right] \quad (9)$$

For second case, $a \ll 1$ we get

$$H_0 = 0.5 Z^2 (9.7 \lambda n^2 - n^{-2}); \quad (10)$$

The proper frequency for this Hamiltonian is

$$\omega_0 = Z^2 (n^{-3} + 9.7 n \lambda). \quad (11)$$

The Hamiltonian of the perturbed system can be written as

$$H = H_0 + \epsilon \sum x_k \cos(k\theta - \omega t), \quad (12)$$

with

$$x_k(n) = - \int_0^{2\pi} x(n, \theta) e^{ik\theta} d\theta \quad (13)$$

being Fourier amplitude of the perturbation. For $a \ll 1$ we have

$$x_k(n) \approx - \frac{4E(n)}{\lambda} \frac{1}{k} \sin^2 \frac{\pi k \sqrt{\lambda}}{2}. \quad (14)$$

For $a \gg 1$ we obtain

$$x_k(n) = - \frac{2An^{2/3}}{\pi^2 \lambda k^2}. \quad (15)$$

As is well known [1, 2, 5], dynamics of a periodically driven system may become chaotic if the resonance condition is fulfilled and external field strength exceeds some critical value. To estimate this critical field strength, ϵ_{cr} for our system we use Chirikov's resonance overlap criterion [1, 5, 9], which can be written as:

$$\frac{\Delta\nu_k + \Delta\nu_{k+1}}{\omega_0(k+1) - \omega_0(k)} > 2.5, \quad (16)$$

with

$$\Delta\nu_k = \left(\frac{\epsilon x_k}{\omega'_0} \right)$$

being the width of the k -th resonance [1, 5] and

$$\omega'_0 = d\omega_0/dn.$$

From the resonance condition we get

$$\omega_0(k) - \omega_0(k+1) = \frac{\omega}{k} - \frac{\omega}{k+1} = \frac{\omega}{k(k+1)}.$$

Applying this criterion to our system give by (12) we have for $a \gg 1$

$$\epsilon_{cr} = \frac{0.07Z^2\omega\pi^2\lambda}{n^2} \cdot \frac{k(k+1)}{(k+1)^2 + k^2} \times$$

$$\left\{ 1 + \frac{\lambda}{A^2 n^{\frac{4}{3}}} \left[5 \ln \left(4A\lambda^{-\frac{1}{2}} n^{\frac{2}{3}} \right) - 7 \right] \right\} \quad (17)$$

and for $a \ll 1$:

$$\begin{aligned} \epsilon_{cr} &= \frac{0.3\omega\lambda}{k(k+1)n^2} \cdot \frac{29\lambda n^4 - 9}{29\lambda n^4 - 3} \times \\ &\times \left[\frac{1}{k} \sin^2 \left(k\sqrt{\lambda} \frac{\pi}{2} \right) + \frac{1}{k+1} \sin^2 \left((k+1)\sqrt{\lambda} \frac{\pi}{2} \right) \right]^{-1} \end{aligned} \quad (18)$$

Table 1 presents the values of the critical field for $a \ll 1$ $u\bar{u}$, $d\bar{d}$, $s\bar{s}$, $c\bar{c}$ and $b\bar{b}$ quarkonia at the following values of parameters: $\omega = 10^9 Hz$, $\alpha_s = 0.112$ [20], $\lambda = 0.2 GeV^2$, $n = 5; 7; 10$. For light (u, d, s) quarkonia we use formula (17) and formula (18) for $b\bar{b}$ - and $c\bar{c}$ - quarkonia. In the Table 2 the critical values of the field strength are displayed for the same values of parameters but at $a \gg 1$. The huge difference between the corresponding values in these two tables can be explained by the fact that these two limit cases make our system (to some extent) equivalent to hydrogenlike atom ($a \ll 1$) and highly excited confined (by linear potential only) system. Therefore the difference between these two cases is caused by the big difference between the binding energies.

In Fig.1 the critical field strength is plotted as a function of the action n . Again, three cases are considered: periodically driven motion in the field Coulomb potential ($\lambda = 0$), periodically driven motion in the field of Coulomb plus linear potential for the cases $a \ll 1$ and for $a \gg 1$.

The parameters Z and λ are chosen as: $Z = 0.15$; $\lambda = 0.4$. As is seen from this figure, unlike to periodically driven hydrogen-like atom, where field strength is a decreasing function of n , for periodically driven quarkonium field strength increases first up to certain value of n , then decreasing occurs. The curve for quarkonium at $a \ll 1$ is closer to that for hydrogen-like atom, while for $a \gg 1$ the difference between the critical fields for quarkonium and hydrogen-like atom becomes significant. This can be explained by the fact that for $a \ll 1$ the energy of the unperturbed quarkonium becomes closer to that for hydrogen-like atom.

It is well known that the phase-space trajectories of the regular motion lie on tori (so-called KAM tori). According to Kolmogorov-Arnold-Moser theorem for sufficiently small fields most of the trajectories remain regular. If the value of the external perturbation exceeds some value,

which is called the critical field strength, KAM tori start to break down and chaotization of the motion will occur [5].

In Figs 2-4 the phase-space portraits are plotted and compared for periodically driven pure Coulomb potential (hydrogenlike atom), quarkonium system at $a \ll 1$ and quarkonium system at $a \gg 1$ for various values of the ratio ϵ_{cr}/ϵ .

Fig. 2 presents phase-space portraits the case $\epsilon_{cr}/\epsilon = 0.1$; the panels a, b, c are the phase space portraits at hydrogenlike atom, quarkonium for $a \ll 1$ and quarkonium for $a \gg 1$, respectively. Figs. 3 and 4 correspond to the cases when $\epsilon_{cr}/\epsilon = 0.5$ and $\epsilon_{cr}/\epsilon = 0.9$, respectively.

The following values of the parameters Z, λ were chosen for the plots: $Z = 0.15$; $\lambda = 0.4$.

As is seen from these plots periodically driven motion in the field of Coulomb potential is more chaotic than that in the field of Coulomb plus linear potential. In other words confining potential makes periodically driven dynamics more regular. Among two limit cases, $a \ll 1$ and $a \gg 1$, dynamics for $a \ll 1$ is more chaotic than that for $a \gg 1$, which also confirms that confining force suppresses chaotization of motion.

3 Three-dimensional model

The Hamiltonian for the three-dimensional model is

$$H_0 = \frac{p_r^2}{2} - \frac{Z}{r} + \lambda r + \frac{L^2}{r^2}.$$

where L is the orbital angular momentum and p_r is the radial momentum.

The action can be expressed in terms of elliptic integrals [21]:

$$\begin{aligned} n = \int_c^a p dr = \int_c^a \sqrt{2(E - \frac{L^2}{r^2} + \frac{Z}{r} - \lambda r)} dr = \\ \left[(2Z/3 - L^2/c + Ec/3)K(k) + E(a - c)/3E(k) + \right. \\ \left. + L^2(c^{-1} - b^{-1}) \Pi(\beta^2, k) \right] g/\sqrt{\lambda} \end{aligned} \quad (19)$$

with a and c being the turning points, K, E, Π are complete elliptic integrals of the first, second and third kind, respectively [19], and the constants are given as

$$k^2 = (a - b)/(a - c),$$

$$\beta = ck^2/b,$$

$$g = 2/\sqrt{a-c}.$$

From eq.(19) the unperturbed Hamiltonian as a function of n can be found approximately for $E/\lambda \gg 1$, (which corresponds to $n \gg 1$ or large quarkonium masses):

$$H_0 = \left(\frac{3}{2}\lambda n\right)^{2/3} \left[1 + \frac{\pi L}{3n}\right].$$

The proper frequency is

$$\omega_0 = \frac{\partial H_0}{\partial n} = \left(\frac{2\lambda^2}{3}\right)^{1/3} \cdot \left[n^{-1/3} - \frac{\pi L}{6}n^{-4/3}\right]$$

Then the Hamiltonian of the three-dimensional quarkonium in a monochromatic field can be written as

$$H = H_0 + \epsilon a \cos(\omega t) \times \left\{-\frac{3}{2}e \sin \varphi + 2 \sum [x_k \sin \psi \cos k\phi + y_k \cos \psi \sin k\phi]\right\}, \quad (20)$$

where

$$x_k = \frac{2i}{\omega_0 k T} \int e^{i\omega_0 k t} \dot{x} dt, \quad y_k = \frac{2i}{\omega_0 k T} \int e^{i\omega_0 k t} \dot{y} dt,$$

and ψ and ϕ are the Euler angles. Again, using the resonance overlap criterion (16) in which the resonance width is defined by

$$\Delta\nu_k = \left(\frac{\epsilon r_k}{\omega'_0}\right),$$

where

$$r_k = \sqrt{x_k^2 + y_k^2},$$

we obtain an estimate for the critical field:

$$\epsilon_{cr} = \frac{0.07\lambda\omega}{k(k+1)\pi n^2} \left(1 - \frac{\pi L}{n}\right) \left\{ \sqrt{\frac{16\pi^2}{9} + \frac{1}{k^2}} + \sqrt{\frac{16\pi^2}{9} + \frac{1}{(k+1)^2}} \right\}^{-1} \times \left[1 - \frac{L^2}{4\pi^4 n^2}\right]. \quad (21)$$

This estimate for the critical field assumes that $n \gg 1$. If the external field strength has the value exceeding ϵ_{cr} , breaking of KAM surfaces in the phase space will occur and the quarkonium diffuses in action and the motion becomes chaotic.

4 Conclusions

In this work we explored periodically driven dynamics of a particle bounded in the field of Coulomb plus linear potential. Using resonance overlap criterion the estimates of for the critical field strength at which dynamics enters into chaotic regime of motion are obtained. The analysis of phase-space dynamics by plotting phase space portraits shows that the periodically driven dynamics of quarkonium is more regular compared to that for hydrogen-like atom. Also, there appears a pick in n -dependence of the critical field strength, that can be explained by the role of confining potential. The obtained results are applied for the estimation of critical field strength needed for chaotization of various quarkonia. The model considered in this work can be used for modelling nonlinear dynamics of quarkonia perturbed by the field of a quark-gluon plasma or laser radiation.

5 Acknowledgements

This work is supported by the NATO reintegration grant (Ref. No. 981787), by INTAS Young Scientist fellowship (Ref. No. 06-1000023-6008) and by a grant of the Uzbek Academy of Sciences (FA-F2-084). The work of FCK is supported by NSERCC.

References

- [1] B.V.Chirikov *Phys.Rep.*, **52**, 159, (1979)
- [2] D.F. Escande, *Phys. Rep.* **121**, 167, (1985)
- [3] G.Casati, B.V.Chirikov and D.L.Shepelyansky, *Phys.Rep.*, **154**, 77, (1987)

- [4] B. Eckhardt, *Phys. Rep.* **163**, 207, (1988)
- [5] R.Z.Sagdeev, D.A.Usikov, G.M.Zaslavsky, *Nonlinear Physics: from pendulum to turbulence and chaos* (Academic Publisher, NY 1988)
- [6] G.M.Izrailev, *Phys.Rep.*, **196** 299 (1990)
- [7] R.V.Jensen , S.M.Susskind and M.M.Sanders *Phys.Rep.* **201**, 1 (1991)
- [8] P.M.Koch, K.A.H. van Leeuwen *Phys.Rep.* **255** (1998) 289
- [9] N.B.Delone, V.P.Krainov and D.L.Shepelyansky, *Usp. Fiz. Nauk.* **140**, 335, (1983)
- [10] D.U.Matrasulov, *Phys.Rev. A* **50** 700 (1999)
- [11] R.V.Jensen, *Phys.Rev. A*, **30**, 386, (1984)
- [12] J.E.Bayfield and P.M.Koch, *Phys.Rev.Lett.*, **33**, 258, (1974)
- [13] ALEPH Collaboration, R.Barate, *et al.*, *Phys.Lett. B*, **425** 215 (1998).
- [14] S.N.Mukherjee *et.al.* *Phys.Rep.* **231** (1993) 203
- [15] NA50 collaboration, *Phys. Lett. B* **477** 28 (2000)
- [16] Jian-zhong Gu *et al*, *Phys.Rev.C*, **60** 035211 (1999).
- [17] G. P. Berman, G. M. Zaslavskii, and A. R. Kolovskii, *Sov. Phys. JETP*, **61** 925 (1985)
- [18] Mark J. Stevens and Bala Sundaram, *Phys.Rev. A*, **36**, 417, (1987)
- [19] M.A. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Nat. Bur. Stand. Washington D.C.,1964;
- [20] Particle Data Group, R.M.Barnett *et al* *Phys.Rev.D*, **54** 1 (1996)
- [21] M.Seetharaman, R.Raghavan and S.S.Vasan *J.Phys.A* **16** 455(1983)

TABLE 1. The values of the critical field strength for heavy quarkonia.

No	Quarkonium	Quark mass (in MeV)	Critical field (V/fm)		
			$n = 5$	$n = 7$	$n = 10$
1	$c\bar{c}$	300	1.215	0.6163	0.3008
2	$b\bar{b}$	1560	$5.761 \cdot 10^2$	$2.901 \cdot 10^2$	$1.407 \cdot 10^2$

TABLE 2. The values of the critical field strength for light quarkonia.

No	Quarkonium	Quark mass (in MeV)	Critical field (V/fm)		
			$n = 5$	$n = 7$	$n = 10$
1	$u\bar{u}$	1	$1.018 \cdot 10^{19}$	$5.192 \cdot 10^{18}$	$2.544 \cdot 10^{18}$
2	$d\bar{d}$	2	$8.141 \cdot 10^{19}$	$4.153 \cdot 10^{19}$	$2.035 \cdot 10^{19}$
3	$s\bar{s}$	30	$9.158 \cdot 10^{22}$	$4.673 \cdot 10^{22}$	$2.29 \cdot 10^{22}$

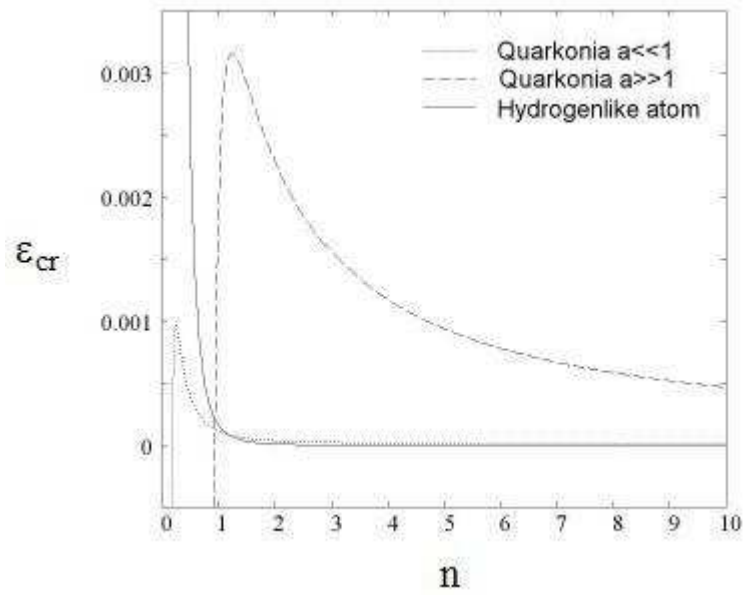


Figure 1: Critical field strength (in the system of units where quarkonium mass is equal to 1) as a function of n , for periodically driven hydrogenlike atom, quarkonium in $a \gg 1$ case and quarkonium in $a \ll 1$ cases.

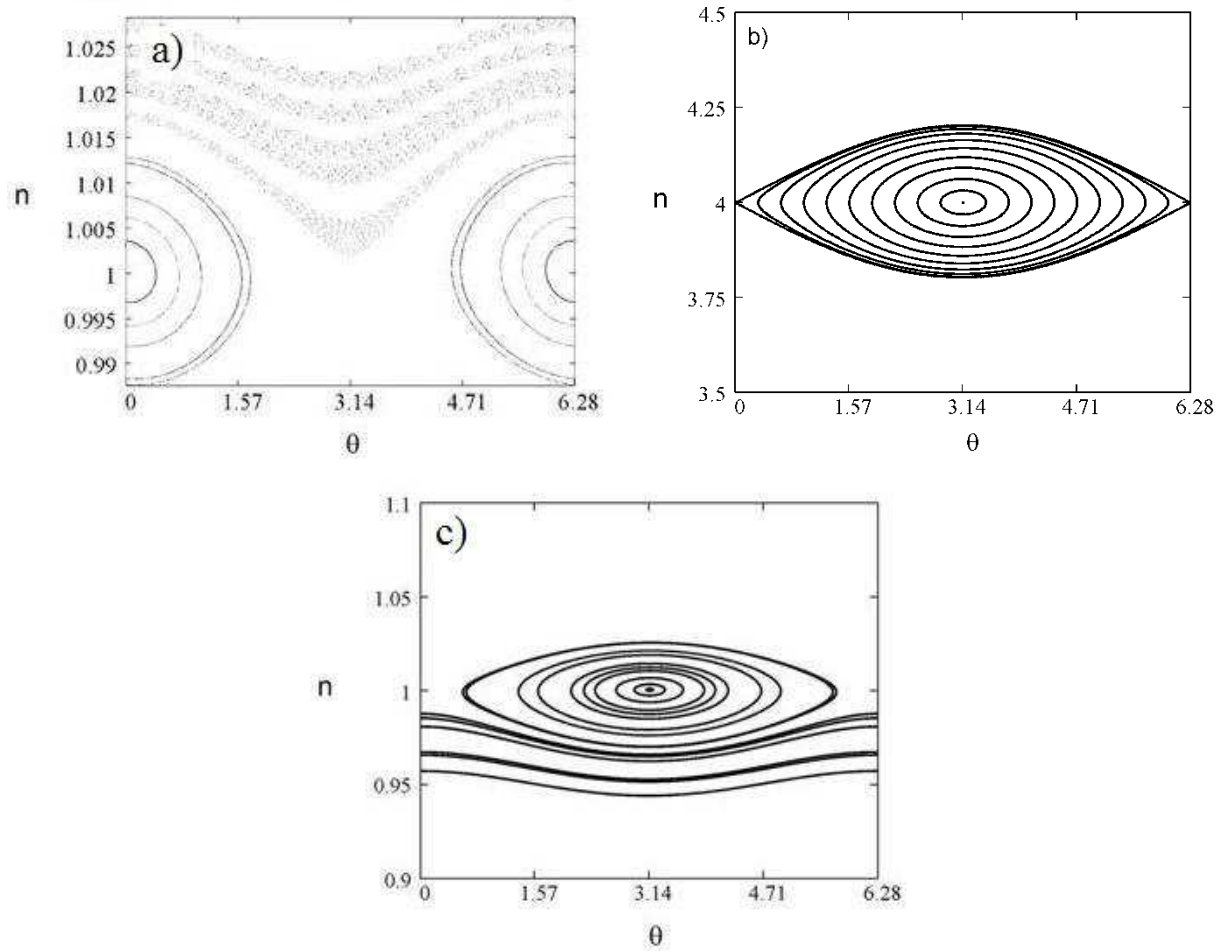


Figure 2: Phase space portraits of the periodically driven: a) hydrogenlike atom; b) quarkonium ($a \gg 1$ case); c) quarkonium ($a \ll 1$ case), for $\epsilon_{cr}/\epsilon = 0.1$.

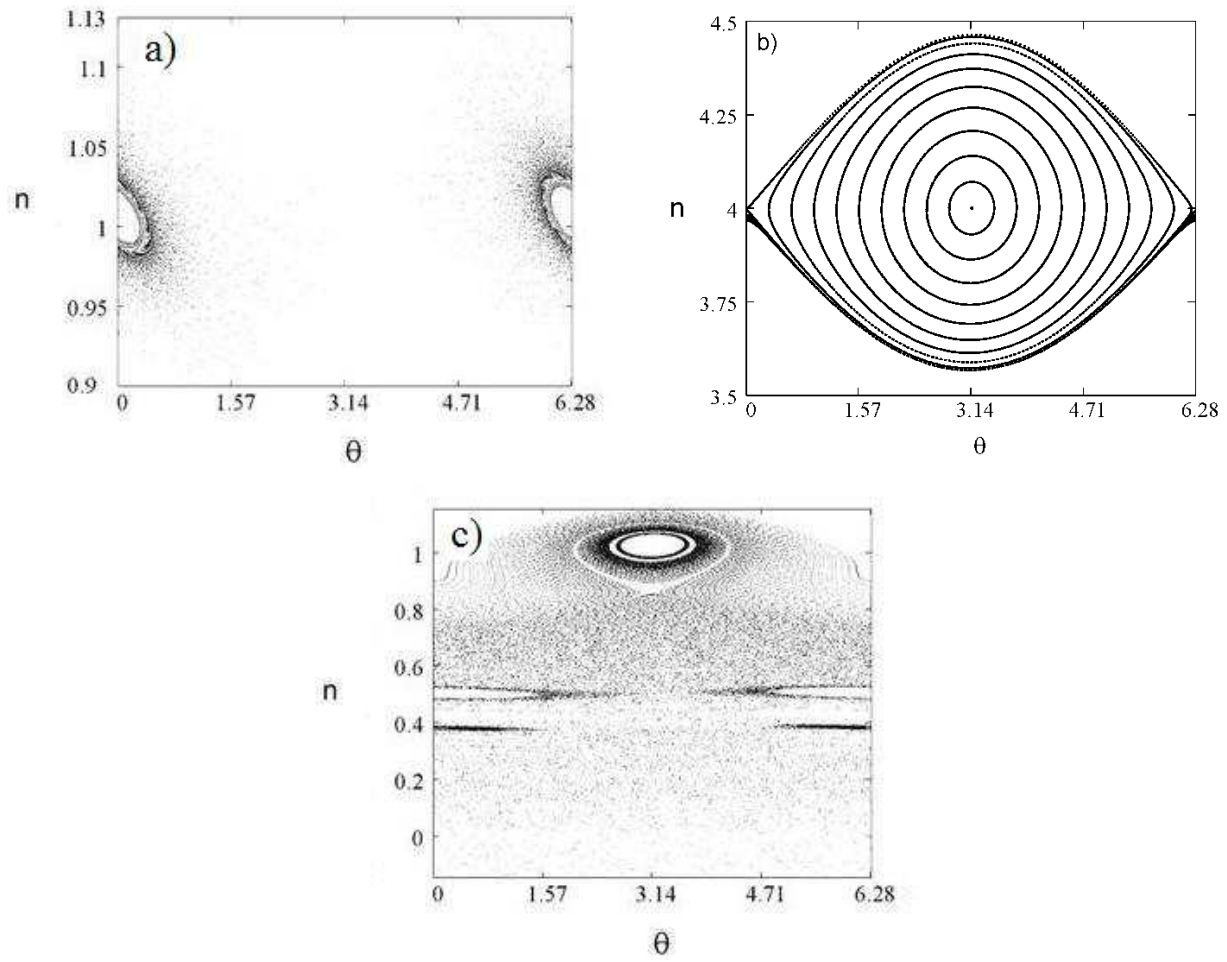


Figure 3: Phase space portraits of the periodically driven: a) hydrogenlike atom ; b) quarkonium ($a \gg 1$ case); c) quarkonium ($a \ll 1$ case), for $\epsilon_{cr}/\epsilon = 0.5$.

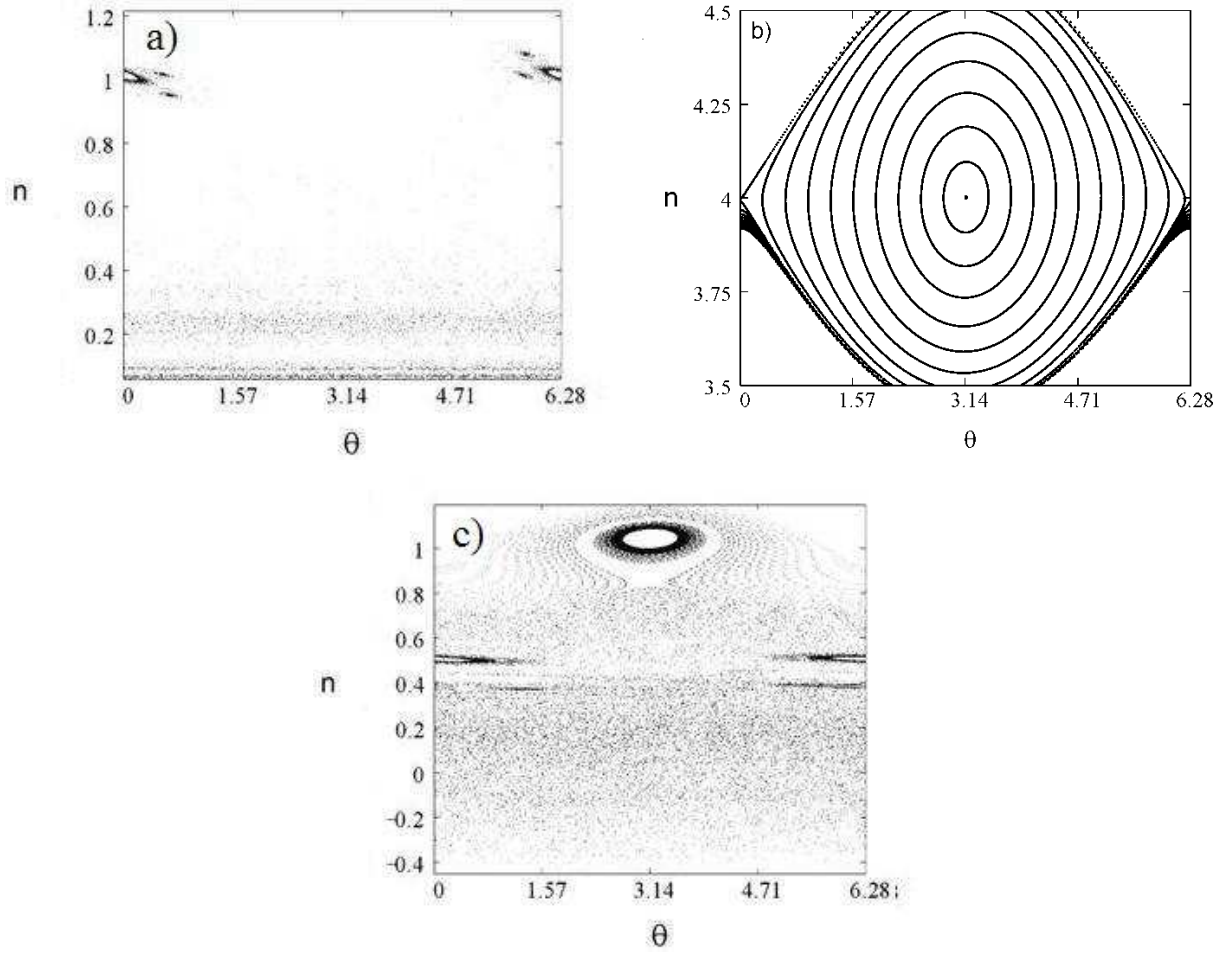


Figure 4: Phase space portraits of the periodically driven: a) hydrogenlike atom ; b) quarkonium ($a \gg 1$ case); c) quarkonium ($a \ll 1$ case), for $\epsilon_{cr}/\epsilon = 0.9$.