Series representations of the Riemann and Hurwitz zeta functions and series and integral representations of the first Stieltjes constant

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Abstract

We develop series representations for the Hurwitz and Riemann zeta functions in terms of generalized Bernoulli numbers (Nörlund polynomials), that give the analytic continuation of these functions to the entire complex plane. Special cases yield series representations of a wide variety of special functions and numbers, including log Gamma, the digamma, and polygamma functions. A further byproduct is that $\zeta(n)$ values emerge as nonlinear Euler sums in terms of generalized harmonic numbers. We additionally obtain series and integral representations of the first Stieltjes constant $\gamma_1(a)$. The presentation unifies some earlier results.

Key words and phrases

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Introduction and statement of results

Let $\zeta(s,a)$ be the Hurwitz zeta function and $\zeta(s) = \zeta(s,1)$ the Riemann zeta function [10, 13, 18]. We develop series representations for these functions in terms of coefficients with the generalized Bernoulli numbers $B_n^{(\alpha)}$, and these have several consequences. We illustrate that our results also extend to Dirichlet L functions. As Corollaries, we obtain series representations of the log Gamma function, the polygamma functions, and special numbers including the ordinary and generalized harmonic numbers. Specific application of our series is to the Stieltjes constants $\gamma_k(a)$, leading to new integral representations. We are able to unify some earlier results, specifically including the very recent ones of Rubinstein [14]. The Discussion section gives details as to how our framework subsumes that presentation. While an experimental mathematics approach lead to a representation and demonstration of certain derivative values $\alpha_k'(1)$, this is a natural result within our approach. We then provide some other observations concerning the Nörlund numbers $B_n^{(n)}$, implications of our series representation of $\zeta(s,a)$, and finish with selected concluding remarks.

The generalized Bernoulli numbers are explicitly given by [16]

$$B_n^{(\alpha)} = \sum_{k=0}^{n} \binom{\alpha + n}{n - k}\binom{\alpha + k - 1}{k}\frac{n!}{(n+k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{n+k}, \qquad (1.1)$$

wherein α may be complex. On the right side, a factor with the Stirling number of the second kind S is evident,

$$k!S(n,k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n. \qquad (1.2)$$

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$B_n^{(\alpha)}$ is a rational polynomial of degree n, with highest coefficient $(-1/2)^n$. The first few such are $B_0^{(\alpha)} = 1$, $B_1^{(\alpha)} = -\alpha/2$, $B_2^{(\alpha)} = \alpha(3\alpha-1)/12$, and $B_3^{(\alpha)} = -\alpha^2(\alpha-1)/8$. For these Nörlund polynomials, $\alpha = 1$ is a simple root if $n > 1$ is odd, and $\alpha = 0$ is a simple root if $n > 0$ is even, and a double root if $n > 1$ is odd.

The generalized Bernoulli polynomials (e.g., [12, 16]) are given by

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)} x^{n-k}, \tag{1.3}$$

$B_n^{(\alpha)}(x) = (-1)^n B_n^{(\alpha)}(\alpha - x)$, and they enter the generating function

$$\left(\frac{z}{e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!}, \qquad |z| < 2\pi. \tag{1.4}$$

As usual, $B_n^{(1)}(x) = B_n(x)$ and $B_n^{(1)}(0) = B_n(0) = B_n$ denote the Bernoulli polynomials and numbers, respectively.

Throughout we write $\sigma = \text{Re } s$. In the following Γ is the Gamma function, $(a)_j = \Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol, $\psi = \Gamma'/\Gamma$ is the digamma function, $\gamma = -\psi(1)$ is the Euler constant, $\psi^{(j)}$ is the polygamma function, and $_pF_q$ is the generalized hypergeometric function (e.g., [1, 2, 9, 19]).

The Hurwitz zeta function, defined by $\zeta(s, a) = \sum_{n=0}^{\infty}(n + a)^{-s}$ for $\sigma > 1$ and Re $a > 0$ extends to a meromorphic function in the entire complex s-plane. In the Laurent expansion

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a)(s - 1)^n, \tag{1.5}$$

$\gamma_n(a)$ are the Stieltjes constants [3, 4, 5, 17, 20], and by convention one takes $\gamma_k = \gamma_k(1)$.

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We have

Proposition 1. Let $s \neq 1$ and Re $a > 0$. Then we have

$$\zeta(s,a) = \frac{\Gamma(a)}{\Gamma(s-1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} \frac{\Gamma(s+k-1)}{\Gamma(s+k+a-1)}. \qquad (1.6)$$

In particular,

$$\zeta(s) = \frac{1}{\Gamma(s)} \frac{1}{(s-1)} + \frac{1}{\Gamma(s-1)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)^2}. \qquad (1.7)$$

Equivalently to (1.6) we have for Re $a > 0$,

$$\Gamma(s)[s\zeta(s+1,a) - a^{-s}] = \Gamma(a) \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k!} \left[\frac{(s-1)B_k^{(s+k-1)}}{s+k-1} - \frac{sB_k^{(s+k)}}{s+k} \right] \frac{\Gamma(s+k)}{\Gamma(s+k+a)}. \qquad (1.8)$$

Corollary 1. For integers $n \geq 1$ we have

$$\psi^{(n)}(x) = (-1)^{n+1} n \Gamma(x) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(n+k)}}{(n+k)} \frac{(n+k-1)!}{\Gamma(n+k+x)}. \qquad (1.9)$$

This representation holds due to the relation $\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x)$. In particular, for the trigamma function we recover

$$\psi'(x) = \Gamma(x) \sum_{k=0}^{\infty} \frac{k!}{(k+1)} \frac{1}{\Gamma(x+k+1)} = \sum_{k=0}^{\infty} \frac{k!}{(k+1)} \frac{1}{(x+k)} \frac{1}{(x)_k}, \qquad (1.10)$$

and the functional equation $\psi'(x+1) = \psi'(x) - 1/x^2$:

$$\psi'(x+1) = \psi'(x) - \sum_{k=0}^{\infty} \frac{k!\Gamma(x)}{\Gamma(x+k+2)} = \psi'(x) - \frac{\Gamma^2(x)}{\Gamma^2(x+1)} = \psi'(x) - \frac{1}{x^2}. \qquad (1.11)$$

In the case of the tetragamma function, the functional equation follows from

$$\psi''(x+1) = \psi''(x) - 2\Gamma(x) \sum_{k=1}^{\infty} \frac{k!H_k}{\Gamma(x+k+2)}, \qquad (1.12)$$

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where H_k is the kth harmonic number, and the latter sum is given by

$$\sum_{k=1}^{\infty} \frac{k!}{\Gamma(x+k+2)} \sum_{\ell=1}^{k} \frac{1}{\ell} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{k=\ell}^{\infty} \frac{k!}{\Gamma(x+k+2)}$$

$$= \frac{1}{x} \sum_{\ell=1}^{\infty} \frac{\Gamma(\ell)}{\Gamma(x+\ell+1)} = \frac{1}{x^3 \Gamma(x)}. \tag{1.13}$$

In connection with (1.10)-(1.13), some elementary relations have been relegated to Appendix A.

We recover a well known value in the next result.

Corollary 2. We have

$$\zeta(0, a) = \frac{1}{2} - a = -B_1(a), \tag{1.14}$$

where B_1 is the first Bernoulli polynomial.

We also have series representations for the digamma and log Gamma functions, and for the first Stieltjes constant.

Proposition 2. Let Re $a > 0$. Then (a)

$$\psi(a) - \ln a = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{(a)_n} \left[B_n^{(n)} + n B_{n-1}^{(n-1)} \right], \tag{1.15}$$

(b) for Re $x > -1$,

$$\ln \Gamma(x+1) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} B_k^{(k)}(1) \left[\frac{1}{k-1} - \frac{x}{k} - \frac{\Gamma(k-1)\Gamma(x+1)}{\Gamma(k+x)} \right], \tag{1.16}$$

and (c)

$$\frac{1}{2} \ln^2 a + \gamma_1(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} [B_n^{(n)} + n B_{n-1}^{(n-1)}] \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\ln(k+a)}{(k+a)}. \tag{1.17}$$

We emphasize that the sum on the right side of (1.16) is well defined. For k near 1 we have $\Gamma(k-1) = 1/(k-1) - \gamma + O(k-1)$ and for k near 0 we have $\Gamma(k-1) = -1/k +$

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$\gamma - 1 + O(k)$. Then the quantity on the right side in brackets is $-x[\psi(x) + \gamma] + x - 1$ for $k = 0$ and $\psi(x + 1) - x + \gamma$ for $k = 1$.

Proposition 3. (a) For Re $a > 0$ we have

$$\gamma_1(a) = \frac{1}{2}[\psi'(a) - \psi^2(a)] - \frac{1}{a} \int_0^1 v \, {}_4F_3(1, 1, 1, v + 1; 2, 2, a + 1; 1) dv, \tag{1.18}$$

(b) we have

$$\gamma_1 = \frac{\pi^2}{6} + \int_0^1 \left(\gamma\psi(x) + \frac{1}{2}[\psi^2(x) - \psi'(x)] \right) dx, \tag{1.19}$$

and

$$\gamma_1 = \frac{\pi^2}{6} - \frac{1}{2} \int_0^1 \left[2\frac{\gamma}{x} + \psi'(x) - \psi^2(x) \right] dx, \tag{1.20}$$

and (c)

$$\gamma_2(a) = \frac{1}{3} \left[-\psi^3(a) + 3\psi(a)\psi'(a) - \psi''(a) \right] + 2\Gamma(a) \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \frac{1}{\Gamma(k + a)} \left[\frac{dB_k^{(s+k-1)}}{ds} \right]_{s=1}$$

$$+ 2\Gamma(a) \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \frac{B_k^{(k)}}{\Gamma(k + a)} [-1 + \gamma k + k\psi(k) - k\psi(k + a)]. \tag{1.21}$$

Corollary 3. We have

$$2[\zeta(2) - \gamma_1] = \gamma^2 + 1 + 2 \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k - 1} - \sum_{m=4}^{\infty} \frac{(-1)^m}{m - 1} \sum_{k=2}^{m-2} \zeta(k)\zeta(m - k). \tag{1.22}$$

A sum of (1.22) has many alternative forms, a few of which are collected in the following. We let $\Gamma(x, y)$ be the incomplete Gamma function.

Corollary 4.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \zeta(k + 1) = \int_0^{\infty} [\gamma + \Gamma(0, t) + \ln t] \frac{dt}{e^t - 1}$$

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$$= \sum_{j=1}^{\infty} \frac{1}{j} \ln \left(1 + \frac{1}{j} \right)$$

$$= \int_0^{\infty} [\gamma + \psi \left(1 + e^{-t} \right)] dt = \int_0^{\infty} [\gamma + \psi \left(e^{-t} \right) + e^t] dt. \qquad (1.23)$$

This sum, with approximate numerical value 1.25774688694, has been encountered before in analytic number theory [6] in the form $\sum_{k=1}^{\infty} \ln(k+1)/[k(k+1)]$. The latter reference gives several other representations of this sum.

A recursion for the derivatives of the Nörlund polynomials is given in the following.

Proposition 4. For $n \geq 0$,

$$\partial_\alpha B_n^{(\alpha)} = -\frac{n}{2} B_{n-1}^{(\alpha)} - \sum_{j=0}^{n-2} \binom{n}{j} \frac{B_{n-j}}{(n-j)} B_j^{(\alpha)}. \qquad (1.24)$$

With a (null) convention for the sum, this includes the initial cases $\partial_\alpha B_0^{(\alpha)} = 0$ and $\partial_\alpha B_1^{(\alpha)} = -1/2$.

Proposition 1 can be applied to Dirichlet L-functions. As a first quick example, the Dirichlet L-function defined by

$$L(s) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \qquad \text{Re } s > 1, \qquad (1.25)$$

corresponding to quadratic characters modulo 4, can be expressed as

$$L(s) = 4^{-s}[\zeta(s, 1/4) - \zeta(s, 3/4)] = 1 + 4^{-s}[\zeta(s, 5/4) - \zeta(s, 3/4)]. \qquad (1.26)$$

In particular, we have for nonnegative integers m the special values

$$L(2m+1) = -\frac{(2\pi)^{2m+1}}{2(2m+1)!} B_{2m+1}(1/4). \qquad (1.27)$$

Generally the values of L at odd or even integer argument may be expressed in terms of Euler or Bernoulli polynomials at rational argument and these in turn expressed in

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terms of the Hurwitz zeta function. Therefore we may in this way obtain many other computable series representations for $L(2m)$ and $L(2m+1)$. These include the special cases of $L(1) = \pi/4$, $L(2) = G \simeq 0.91596559$, Catalan's constant, and $L(3) = \pi^3/32$.

In general, Dirichlet L functions may be written as a combination of Hurwitz zeta functions. For instance, for χ a principal (nonprincipal) character modulo m and $\sigma > 1$ ($\sigma > 0$) we have

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} = \frac{1}{m^s} \sum_{k=1}^{m} \chi(k) \zeta\left(s, \frac{k}{m}\right). \tag{1.28}$$

Proof of Propositions

Proposition 1. We will use the generating function [12] (p. 147)

$$\left(\frac{\ln(x+1)}{x}\right)^z = z \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{B_k^{(z+k)}}{(z+k)}, \qquad |x| < 1, \tag{2.1}$$

valid for complex z. We have for $\sigma > 1$

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(a-1)t}}{e^t - 1} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - e^{-t}} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^1 \frac{[-\ln(1-u)]^{s-1}}{u} (1-u)^{a-1} du$$

$$= \frac{(s-1)}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} \int_0^1 u^{s+k-2} (1-u)^{a-1} du$$

$$= \frac{1}{\Gamma(s-1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} B(s+k-1, a), \tag{2.2}$$

where the Beta function factor

$$B(s+k-1, a) = \frac{\Gamma(a)\Gamma(s+k-1)}{\Gamma(s+k+a-1)}. \tag{2.3}$$

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With $B_k^{(s+k-1)}$ being polynomials in s, (2.2) gives the analytic continuation of $\zeta(s, a)$ to the whole complex plane, completing the Proposition.

(1.8) is based upon the use of (2.1) and the integral

$$\int_0^1 \left[\frac{1}{\ln(1-y)} + \frac{1}{y} \right] (1-y)^{a-1} \ln^s \left(\frac{1}{1-y} \right) dy = \Gamma(s)[s\zeta(s+1, a) - a^{-s}]. \qquad (2.4)$$

Based upon (2.11) below for $a^{-s}\Gamma(s)$, we see the equivalence of (1.7) and (1.5). For (1.7) we may note the particular value $[B_k^{(k-1)}/(k-1)]_{k=1} = -\int_0^1 t\,dt = -1/2$.

Corollary 2. We recall that $B_0^{(\alpha)} = 1$ and $B_1^{(s)}/s = -1/2$. When $s \to 0$ in (1.6), only the $k = 0$ and $k = 1$ terms contribute, as $1/\Gamma(s-1) = -s + (1-\gamma)s^2 + O(s^3)$. We obtain $\zeta(0, a) = 1 - a - 1/2 = 1/2 - a$.

Remarks. With $B_n^{(\alpha)}(x)$ the generalized Bernoulli polynomial for $\nu \le n$,

$$B_\nu^{(n+1)}(x) = \frac{\nu!}{n!} \frac{d^{n-\nu}}{dx^{n-\nu}} (x-1)(x-2) \cdots (x-n) = (-1)^n \frac{\nu!}{n!} \frac{d^{n-\nu}}{dx^{n-\nu}} (1-x)_n. \qquad (2.5)$$

We have $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$. It follows as a very special case of (2.5) that $B_k^{(k+1)} = (-1)^k k!$. This latter relation allows the recovery of the series $\zeta(2) = \sum_{k=1}^\infty 1/k^2$ from (1.7).

As also follows from (2.5), $B_k^{(k+2)} = (-1)^k k! H_{k+1}$, where $H_n = \sum_{k=1}^n 1/k$ is the nth harmonic number. With these values, we have from (1.7) an Euler series (e.g., [8], Appendix B) for $\zeta(3)$:

$$\zeta(3) = \frac{1}{4} + \sum_{k=1}^\infty \frac{H_{k+1}}{(k+2)^2} = \sum_{k=1}^\infty \frac{H_k}{(k+1)^2}. \qquad (2.6)$$

Integral arguments of the digamma and polygamma functions are directly related to the harmonic H_n and generalized harmonic $H_n^{(r)}$ numbers: $H_n = \psi(n+1) - \psi(1) =$

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$\psi(n+1)+\gamma,$

$$H_n^{(r)} = \frac{(-1)^{r-1}}{(r-1)!} \left[\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1) \right], \qquad (2.7)$$

Therefore Corollary 1 gives another form of these special numbers. The generalized harmonic numbers are given by

$$H_n^{(r)} \equiv \sum_{j=1}^{n} \frac{1}{j^r}, \qquad H_n \equiv H_n^{(1)}. \qquad (2.8)$$

As also follows from (2.5), we have the relation

$$B_n^{(n+3)} = (-1)^n n! [H_{n+2}^2 - H_{n+2}^{(2)}]. \qquad (2.9)$$

Proceeding as for the Proposition, we have for Re $a > 0$,

$$a^{-s}\Gamma(s) = \int_0^\infty x^{s-1} e^{-ax} dx = \int_0^1 [-\ln(1-t)]^{s-1} (1-t)^{a-1} dt, \qquad (2.10)$$

so that using (2.1) we have

$$a^{-s}\Gamma(s) = (s-1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} \frac{\Gamma(a)\Gamma(s+k)}{\Gamma(s+k+a)}. \qquad (2.11)$$

We have found that (1.9) and (1.15) are given in Nörlund's book [12] in a section dealing with numerical differentiation and integration (pp. 243 and 244, respectively).

Proposition 2. (a) We introduce the constants ([5], Proposition 11)

$$p_{n+1} = -\frac{1}{n!} \int_0^1 (-x)_n dx = \frac{(-1)^{n+1}}{n!} \sum_{k=1}^{n} \frac{s(k,n)}{k+1}, \qquad (2.12)$$

where $s(k, \ell)$ is the Stirling number of the first kind. These constants enter the generating function

$$\sum_{n=1}^{\infty} p_{n+1} z^{n-1} = \frac{1}{z} + \frac{1}{\ln(1-z)}, \qquad |z| < 1. \qquad (2.13)$$

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Lemma 1. We have

$$k! p_{k+1} = (-1)^{k-1}[B_k^{(k)} + kB_{k-1}^{(k-1)}]. \tag{2.14}$$

We have

$$\int_0^1 (x)_k dx = \int_0^1 (1-y)_k dy = (-1)^k B_k^{(k)} = \int_0^1 (k-y)(1-y)_{k-1} dy$$

$$= -\int_0^1 y(1-y)_{k-1} dy + k\int_0^1 (1-y)_{k-1} dy, \tag{2.15}$$

giving $(-1)^k B_k^{(k)} = -k! p_{k+1} + k(-1)^{k-1} B_{k-1}^{(k-1)}$, from which the Lemma follows.

Part (a) of the Proposition then immediately follows from Proposition 5(a) of [7].

Here we give direct verifications:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{(a)_n} \left[B_n^{(n)} + nB_{n-1}^{(n-1)} \right] = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(a)_n} \int_0^1 [(x)_n - n(x)_{n-1}] dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(a)_n} \int_0^1 (x-1)_n dx = \int_0^1 [\psi(a) - \psi(a-x+1)] dx = \psi(a) - \ln a. \tag{2.16}$$

Otherwise, we may employ

$$\frac{1}{(a)_n} = \frac{\Gamma(a)}{\Gamma(a+n)} = \frac{1}{\Gamma(n)} \int_0^{\infty} e^{-at}(1-e^{-t})^{n-1} dt, \qquad \text{Re } a > 0, \tag{2.17}$$

to write

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{(a)_n} \left[B_n^{(n)} + nB_{n-1}^{(n-1)} \right] = \int_0^1 \int_0^{\infty} e^{-at} \frac{(e^t - e^{xt})}{(1-e^t)} dt dx$$

$$= \int_0^{\infty} e^{-at} \left[\frac{1}{e^{-t} - 1} + \frac{1}{t} \right] dt = \psi(a) - \ln a. \tag{2.18}$$

The latter integral representation is a standard one (e.g., [9], p. 943).

(b) From (2.1) and $B_k^{(n+1)}(1) = \left(1 - \frac{k}{n}\right) B_k^{(n)}$ it follows that

$$\left(\frac{x}{\ln(x+1)}\right)^z = \sum_{k=0}^{\infty} \frac{x^k}{k!} B_k^{(k-z+1)}(1). \tag{2.19}$$

We use the integral representation [11] (p. 343)

$$\ln \Gamma(x+1) = \int_0^1 \frac{[1 - xt - (1-t)^x]}{t \ln(1-t)} dt = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} B_k^{(k)}(1) \int_0^1 [1 - xt - (1-t)^x] t^{k-2} dt,$$

$$\tag{2.20}$$

with the last integral given by the Beta function $B(k-1, x+1)$, and part (b) follows.

For (c) we may use Lemma 1 and Proposition 5(b) of [7], or else apply the integral representation (2.4).

Remarks. It is easy to show that

$$B_n^{(n)} = \sum_{k=0}^{n} (-1)^k \frac{s(n,k)}{k+1}, \tag{2.21}$$

where $s(n,n) = 1$ and $s(n,0) = \delta_{n0}$ in terms of the Kronecker symbol δ_{jk}.

From [12] (p. 147), $(k-1)B_k^{(k)}(1) = -B_k^{(k-1)} = (k-1)(-1)^{k-1} \int_0^1 t(1-t)_{k-1} dt$, so that for the coefficients of part (b) we have $B_k^{(k)}(1) = (-1)^{k-1} \int_0^1 t(1-t)_{k-1} dt = (-1)^{k-1} k! p_{k+1}$.

In fact we have [12] (p. 148)

$$(-1)^n (1-x)_n = \sum_{r=0}^{n} \binom{n}{r} x^r B_{n-r}^{(n+1)}. \tag{2.22}$$

Therefore, performing the integrations, we obtain

$$\int_0^1 (x)_n dx = (-1)^n \sum_{r=0}^{n} \binom{n}{r} \frac{B_{n-r}^{(n+1)}}{(r+1)}, \tag{2.23}$$

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for $n > 0$,

$$(-1)^n \int_0^1 x(1-x)_n dx = -\frac{1}{n} B_{n+1}^{(n)} = (-1)^n \sum_{r=0}^n \binom{n}{r} \frac{B_{n-r}^{(n+1)}}{(r+2)}, \qquad (2.24)$$

and

$$(-1)^n \int_0^1 (-x)_n dx = (-1)^n \sum_{r=0}^n \binom{n}{r} (2^{r+1}-1) \frac{B_{n-r}^{(n+1)}}{(r+1)}. \qquad (2.25)$$

We record the latter result in the following form for the constants of (2.13).

Lemma 2.

$$p_{n+1} = \frac{(-1)^{n+1}}{n!} \sum_{r=0}^n \binom{n}{r} (2^{r+1}-1) \frac{B_{n-r}^{(n+1)}}{(r+1)}. \qquad (2.26)$$

From differentiating the generating function (1.4) there is the relation $B_k^{(\alpha+1)} = \frac{\alpha-k}{\alpha} B_k^{(\alpha)} - k B_{k-1}^{(\alpha)}$. At $\alpha = k-1$ this yields $k! p_{k+1} = (-1)^{k-1} B_k^{(k-1)}/(k-1)$. According to the relations above (2.22), this expression is equivalent to (2.12).

Comparing (2.1) with the generating function

$$[\ln(x+1)]^n = n! \sum_{k=n}^\infty s(k,n) \frac{x^k}{k!}, \qquad (2.27)$$

we have

$$s(m,n) = \frac{(m-1)!}{(n-1)!} \frac{B_{m-n}^{(m)}}{(m-n)!}. \qquad (2.28)$$

A generalization of Stirling numbers of the first kind in the first argument is presented in Appendix B.

Proposition 3. With $B_0^{(\alpha)} = 1$, Proposition 1 gives

$$\zeta(s,a) = \frac{\Gamma(a)}{\Gamma(s+a-1)} \frac{1}{(s-1)} + \frac{\Gamma(a)}{\Gamma(s-1)} \sum_{k=1}^\infty \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} \frac{\Gamma(s+k-1)}{\Gamma(s+k+a-1)}. \qquad (2.29)$$

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We then expand both sides about $s = 1$, using for instance

$$\frac{1}{\Gamma(s-1)} = (s-1) + \gamma(s-1)^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right)(s-1)^3 + O[(s-1)^4], \qquad (2.30a)$$

and

$$\frac{\Gamma(a)}{\Gamma(s+a-1)} = 1 - \psi(a)(s-1) + \frac{1}{2}[\psi^2(a) - \psi'(a)](s-1)^2$$

$$+ \frac{1}{6}\left[-\psi^3(a) + 3\psi(a)\psi'(a) - \psi''(a)\right](s-1)^3 + O[(s-1)^4]. \qquad (2.30b)$$

From the $O(s-1)$ terms we recover the known relation $\gamma_0(a) = -\psi(a)$. In addition

we have

$$\left[\zeta'(s,a) + \frac{1}{(s-1)^2}\right]_{s\to 1^+} = -\gamma_1(a) = \frac{1}{2}[\psi^2(a) - \psi'(a)] + \Gamma(a)\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\frac{B_k^{(k)}}{\Gamma(k+a)}. \qquad (2.31)$$

Now [12] (p. 147)

$$B_k^{(k)} = (-1)^k \int_0^1 (1-t)_k dt = (-1)^k \int_0^1 (t)_k dt. \qquad (2.32)$$

We easily have

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{(t)_k}{\Gamma(k+a)} = \frac{1}{\Gamma(a+1)}t\ _4F_3(1,1,1,t+1;2,2,a+1;1), \qquad (2.33)$$

and part (a) readily follows.

Part (b) then follows at $a = 1$. For (1.20), in light of $\psi(x) = -1/x - \gamma + (\pi^2/6)x + O(x^2)$ as $x \to 0$, we write the term $\gamma\psi(x) = \gamma[\psi(x) + 1/x - 1/x]$. We then employ the integral

$$\int_0^1 \left[\psi(x) + \frac{1}{x}\right]dx = \int_0^1 \psi(x+1)dx = \ln\Gamma(x+1)|_0^1 = 0. \qquad (2.34)$$

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Part (c) proceeds similarly to (a), also using

$$B_k^{(s+k-1)} = B_k^{(k)} + \left[\frac{dB_k^{(s+k-1)}}{ds}\right]_{s=1}(s-1) + \frac{1}{2}\left[\frac{d^2 B_k^{(s+k-1)}}{ds^2}\right]_{s=1}(s-1)^2 + O[(s-1)^3].$$

(2.35)

Expression (1.21) follows from

$$\left[\zeta''(s,a) - \frac{2}{(s-1)^3}\right]_{s\to 1+} = \gamma_2(a).$$

(2.36)

Remark. We have the Laurent expansion about $s = 1$ of the logarithmic derivative of the zeta function,

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} - \sum_{p=0}^{\infty}\eta_p(s-1)^p, \qquad |s-1| < 3,$$

(2.37)

where $\eta_0 = -\gamma$. From Proposition 2 or Corollary 3 we then obtain corollary expressions for $\eta_1 = \gamma^2 + 2\gamma_1$.

Corollary 3. We use Proposition 3(b) along with the expansions (e.g., [9], p. 944 or [1], p. 259)

$$\psi(x+1) = \psi(x) + \frac{1}{x} = -\gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1}, \qquad |x| < 1,$$

(2.38)

$$\psi'(x) = \frac{1}{x^2} + \sum_{k=1}^{\infty}(-1)^{k+1}k\zeta(k+1)x^{k-1},$$

(2.39)

and

$$\psi^2(x) = \frac{1}{x^2} + 2\frac{\gamma}{x} + \gamma^2 - 2\sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-2} - 2\gamma\sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1}$$

$$+ \sum_{m=4}^{\infty}\sum_{k=2}^{m-2}(-1)^m\zeta(k)\zeta(m-k)x^{m-2}.$$

(2.40)

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We insert these into

$$2[\zeta(2) - \gamma_1] = \int_0^1 \left[2\frac{\gamma}{x} + \psi'(x) - \psi^2(x) \right] dx, \qquad (2.41)$$

integrate term by term, Abel sum the series

$$\lim_{x \to 1^-} \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1) x^k = 1, \qquad (2.42)$$

and use the easily proved sum $\sum_{k=2}^{\infty} (-1)^k \zeta(k)/k = \gamma$, giving the Corollary.

Corollary 4. The first expression follows by using a standard integral represen-

tation for the zeta function. The next follows by use of a geometric series, and

the integral representation with the digamma function follows by using the Laplace

transform representation of $1/k$.

Proposition 4. From the generating function (1.4) we have

$$\left(\frac{z}{e^z - 1} \right)^\alpha e^{xz} \ln \left(\frac{z}{e^z - 1} \right) = \sum_{n=1}^{\infty} \frac{\partial B_n^{(\alpha)}(x)}{\partial \alpha} \frac{z^n}{n!}, \qquad |z| < 2\pi. \qquad (2.43)$$

In order to expand the log factor, we first note that

$$\frac{d}{dz} \ln \left(\frac{z}{e^z - 1} \right) = \frac{1}{z} - 1 - \frac{1}{e^z - 1} = \frac{1}{z} - 1 - \frac{1}{z} \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad (2.44)$$

yielding

$$\frac{d}{dz} \ln \left(\frac{z}{e^z - 1} \right) = -1 - \sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} z^n. \qquad (2.45)$$

Upon integrating,

$$\ln \left(\frac{z}{e^z - 1} \right) = -z - \sum_{n=0}^{\infty} \frac{B_{n+1} z^{n+1}}{(n+1)(n+1)!} = -\frac{z}{2} - \sum_{n=1}^{\infty} \frac{B_{n+1} z^{n+1}}{(n+1)(n+1)!}. \qquad (2.46)$$

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Then at $x = 0$ in (2.43) we have

$$\sum_{n=1}^{\infty} \frac{\partial B_n^{(\alpha)}}{\partial \alpha} \frac{z^n}{n!} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \left(-\frac{z}{2} - \sum_{m=1}^{\infty} \frac{B_{m+1} z^{m+1}}{(m+1)(m+1)!} \right)$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{z^{n+1}}{n!} - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{B_n^{(\alpha)} B_{m+1} z^{n+m+1}}{n!(m+1)!(m+1)}. \qquad (2.47)$$

Reordering the double sum,

$$\sum_{n=1}^{\infty} \frac{\partial B_n^{(\alpha)}}{\partial \alpha} \frac{z^n}{n!} = -\frac{1}{2} \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{z^{n+1}}{n!} - \sum_{\ell=1}^{\infty} \sum_{n=0}^{\ell-1} \frac{B_n^{(\alpha)} B_{\ell-n+1}}{n!(\ell-n+1)!} \frac{z^{\ell+1}}{(\ell-n+1)}, \qquad (2.48)$$

from which the Proposition follows.

Remarks. It is seen that the recursion (1.24) also applies for $x \neq 0$. Since $B_{2n-1} = 0$ for $n > 1$, the sum of (2.44) may be written as $\frac{1}{2} \sum_{m=1}^{\infty} \frac{B_{2m} z^{2m}}{m(2m)!}$. From (2.43) we have similarly for higher order derivatives

$$\left(\frac{z}{e^z - 1} \right)^{\alpha} e^{xz} \ln^j \left(\frac{z}{e^z - 1} \right) = \sum_{n=j}^{\infty} \frac{\partial^j B_n^{(\alpha)}(x)}{\partial \alpha^j} \frac{z^n}{n!}, \qquad |z| < 2\pi, \qquad (2.49)$$

implying

$$\sum_{n=j}^{\infty} \left(\frac{\partial^j B_n^{(\alpha)}(x)}{\partial \alpha^j} \right)_{\alpha=0} \frac{z^n}{n!} = e^{xz} \ln^j \left(\frac{z}{e^z - 1} \right). \qquad (2.50)$$

Discussion

Rubinstein [14] developed several expansions for the Riemann zeta function using certain polynomials $\alpha_k(s)$. That work employed the generating function

$$\left(-\frac{\ln(1-t)}{t} \right)^{s-1} = \sum_{k=0}^{\infty} \alpha_k(s) t^k, \qquad |t| < 1. \qquad (3.1)$$

Comparing with (2.1) we identify

$$\alpha_k(s) = (s-1) \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)}. \qquad (3.2)$$

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We have

$$\alpha'_k(s) = \frac{(-1)^k}{(k-1)!} \frac{B_k^{(s+k-1)}}{(s+k-1)^2} + \frac{(-1)^k}{k!} \frac{(s-1)}{(s+k-1)} \frac{d}{ds} B_k^{(s+k-1)}, \qquad (3.3)$$

where, by (1.1), and using the functional equation of the digamma function,

$$\partial_\alpha B_n^{(\alpha)} = [\psi(n+\alpha+1) - \psi(\alpha)] B_n^{(\alpha)}$$

$$-\sum_{k=0}^{n} \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \frac{1}{(k+\alpha)} \frac{n!}{(n+k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{n+k}, \qquad (3.4)$$

and $\psi(n+\alpha+1) - \psi(\alpha) = \sum_{r=0}^{n} 1/(\alpha+r)$. We have precisely by (2.32)

$$\alpha'_k(1) = \frac{(-1)^k}{k!} \frac{B_k^{(k)}}{k} = \frac{1}{kk!} \int_0^1 (t)_k dt, \qquad (3.5)$$

that was found by indirect means in [14].

We also have for $k \geq 0$ and $m \geq 0$,

$$\alpha_k(-k) = \frac{(-1)^k}{k!}, \qquad (3.6a)$$

$$\alpha_{2m+2}(-2m-1) = \frac{B_{2m+2}}{(2m+2)!}, \qquad (3.6b)$$

and

$$\alpha_{2m+2}(-2m) = -(2m+1) \frac{B_{2m+2}}{(2m+2)!}. \qquad (3.6c)$$

By using the generating function (1.5), one easily sees that $B_n^{(-1)} = 1/(n+1)$, so that from (3.2), $\alpha_k(-k) = (k+1)(-1)^k B_k^{(-1)}/k!$, giving (3.6a).

On the Nörlund numbers $B_n^{(n)}$

These numbers have the known asymptotic form $B_n^{(n)} \sim (-1)^n n!/\ln n$ as $n \to \infty$.

From (2.32) it is easy to see how a standard generating function for them arises:

$$\sum_{n=0}^{\infty} \frac{B_n^{(n)}}{n!} z^n = \int_0^1 (1+z)^{-t} dt = \frac{z}{(1+z)\ln(1+z)}. \qquad (4.1)$$

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From (2.32) and the relation above (2.22) for $B_r^{(r-1)}$ it is also clear how to develop expressions given in [12] (p. 244):

$$\ln\left(\frac{x+1}{x}\right) = \sum_{r=0}^{\infty} \frac{(-1)^r B_r^{(r)}}{(x+1)_r}, \tag{4.2}$$

and

$$\ln\left(\frac{x+1}{x}\right) = \frac{1}{x} - \frac{1}{2x(x+1)} - \frac{1}{x} \sum_{r=2}^{\infty} \frac{(-1)^r B_r^{(r-1)}}{(r-1)} \frac{1}{(x+1)_r}. \tag{4.3}$$

We note that differentiation with respect to x of these formulas gives further sum identities for generalized Bernoulli numbers.

Given the generating function (2.19) and that for harmonic numbers,

$$\sum_{k=0}^{\infty} H_{k+1} z^k = -\frac{\ln(1-z)}{z(1-z)}, \tag{4.4}$$

one suspects various relations between generalized Bernoulli numbers and harmonic numbers. As an example, we present the following. Although this relation could be proven with generating functions, we give a proof employing special function theory.

Proposition 5. For integers $n \geq 0$,

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}}{(n-r)!} B_{n-r}^{(n-r)} H_{r+1} = n+1. \tag{4.5}$$

Proof. We begin by reordering sums and applying Chu-Vandermonde summation,

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}}{(n-r)!} B_{n-r}^{(n-r)} H_{r+1} = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{(n-r)!} B_{n-r}^{(n-r)} \sum_{\ell=1}^{r+1} \frac{1}{\ell} = \sum_{\ell=1}^{n+1} \frac{1}{\ell} \sum_{r=\ell-1}^{n} \frac{(-1)^{n-r}}{(n-r)!} B_{n-r}^{(n-r)}$$

$$= \sum_{\ell=1}^{n+1} \frac{1}{\ell} \sum_{r=\ell-1}^{n} \frac{1}{(n-r)!} \int_0^1 (t)_{n-r} dt$$

$$= \sum_{\ell=1}^{n+1} \frac{1}{\ell} \int_0^1 \frac{\Gamma(t+n-\ell+2)}{\Gamma(t+1)\Gamma(n-\ell+2)} dt$$

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$$= \int_0^1 \frac{\Gamma(t+n+2)}{\Gamma(n+2)\Gamma(t+1)} [\psi(t+n+2) - \psi(t+1)] dt$$

$$= \frac{1}{n+1} \int_0^1 \left(\frac{d}{dt} \frac{1}{B(n+1,t+1)} \right) dt = n+1. \tag{4.6}$$

Above, we manipulated a terminating $_3F_2$ function at unit argument, since

$$\sum_{\ell=1}^{n+1} \frac{1}{\ell} \frac{\Gamma(t+n-\ell+2)}{\Gamma(n-\ell+2)} z^\ell = \sum_{\ell=0}^{n} \frac{(1)_\ell^2}{(2)_\ell} \frac{\Gamma(t+n-\ell+1)}{\Gamma(n-\ell+1)} \frac{z^{\ell+1}}{\ell!}$$

$$= \sum_{\ell=0}^{n} \frac{(1)_\ell^2}{(2)_\ell} \frac{\Gamma(t+n+1)}{\Gamma(n+1)} \frac{(-n)_\ell}{(-t-n)_\ell} \frac{z^{\ell+1}}{\ell!} = z \frac{\Gamma(t+n+1)}{\Gamma(n+1)} \, _3F_2(1,1,-n;2,-t-n;z). \tag{4.7}$$

Because

$$_3F_2(1,1,a;2,b;1) = \left(\frac{b-1}{a-1} \right) [\psi(b-1) - \psi(b-a)], \tag{4.8}$$

we have

$$\frac{\Gamma(t+n+1)}{\Gamma(n+1)} \, _3F_2(1,1,-n;2,-t-n;1) = \frac{\Gamma(t+n+1)}{\Gamma(n+1)} \frac{(t+n+1)}{(n+1)} [\psi(-t-n-1) - \psi(-t)]$$

$$= \frac{\Gamma(t+n+2)}{\Gamma(n+2)} [\psi(t+n+2) - \psi(t+1)], \tag{4.9}$$

by recalling that $\psi(1-x) - \psi(x) = \pi \cot(\pi x)$.

Other relations from (1.6)

From the representation (1.6) we have

$$\partial_a \zeta(s,a) = -s\zeta(s+1,a) = \psi(a)\zeta(s,a)$$

$$-\frac{\Gamma(a)}{\Gamma(s-1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} \frac{\Gamma(s+k-1)}{\Gamma(s+k+a-1)} \psi(s+k+a-1). \tag{5.1}$$

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We also have the special case

$$\zeta\left(s, \frac{1}{2}\right) = \frac{\Gamma(1/2)}{\Gamma(s-1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B_k^{(s+k-1)}}{(s+k-1)} \frac{\Gamma(s+k-1)}{\Gamma(s+k-1/2)} = (2^s - 1)\zeta(s). \quad (5.2)$$

It would be desirable to otherwise have proofs of the right-most equalities in (4.1) and (5.2). The duplication formula for the Gamma function may be useful in this regard for (5.2).

The Bernoulli polynomials satisfy the multiplication formula

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right), \quad (5.3)$$

that is easily verified with the generating function (1.4) with $\alpha = 1$. One may ask whether there is a generalization to a formula such as

$$B_n^{(\alpha)}(mx) = m^{n-\alpha} f_1(n, \alpha) \sum_{k=0}^{m-1} \left[B_n^{(\alpha)}\left(x + \frac{\alpha k}{m}\right) + f_2(x, n, \alpha) \right], \quad (5.4)$$

where f_1 and f_2 are such that $f_1 = 1$ and $f_2 = 0$ for $\alpha = 1$.

Summary remarks

The series (1.6) and (1.7) converge everywhere in the complex plane and include as special cases the relations $\zeta(1-m, a) = -B_m(a)/m$ and the trivial zeros $\zeta(-2n) = 0$. The treatment has been sufficiently general to subsume expansion of the zeta function at integer argument in terms of Stirling numbers. We may emphasize, among other features, that nonlinear Euler sums for $\zeta(n)$ naturally emerge as special cases. As an example, we have from (2.9)

$$\zeta(4) = \frac{1}{2} \sum_{k=2}^{\infty} \frac{[H_k^2 - H_k^{(2)}]}{(k+1)^2}, \quad (6.1)$$

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in terms of generalized harmonic numbers $H_n^{(r)}$.

Proposition 3 may be the only known integral representation for the first Stieltjes constant γ_1 in terms of the digamma and trigamma functions.

It seems that the second line of (3.4) for the derivatives $\partial_\alpha B_n^{(\alpha)}$ could have an alternative, more compact form, possibly involving $B_j^{(\alpha)}$ and B_n values. Such an expression would be very convenient in further developments for the Stieltjes and other constants. For instance, it could be immediately applied to Proposition 3(c).

As mentioned in connection with (5.4), further investigation of the Nörlund polynomials appears to be in order.

Appendix A: Beta function-based relations

We have for Re $x > 0$ and $n \geq 0$ an integer, the special case of the Beta function

$$B(x, n+1) = \int_0^1 t^{x-1}(1-t)^n dt = \frac{n!}{x(x+1)\cdots(x+n)} = \frac{n!}{(x)_{n+1}}, \qquad (A.1)$$

that may be verified by induction. So as relates to (1.10)-(1.12), we have

$$\sum_{k=0}^\infty \frac{k!}{(k+1)} \frac{1}{(x)_{k+1}} = \sum_{k=0}^\infty \frac{1}{(k+1)} \int_0^1 t^{x-1}(1-t)^k dt$$

$$= \int_0^1 \frac{t^{x-1}}{t-1} \ln t \, dt = \psi'(x). \qquad (A.2)$$

Here, we have recalled (e.g. by [9], p. 943)

$$\psi^{(j)}(z) = \int_0^1 \frac{t^{z-1}}{t-1} \ln^j t \, dt, \qquad \text{Re } z > 0. \qquad (A.3)$$

In regard to (1.12), we recall the generating function with harmonic numbers

$$\sum_{k=1}^\infty H_k z^k = \frac{\ln(1-z)}{z-1} = \frac{\text{Li}_1(z)}{1-z}, \qquad (A.4)$$

where Li_j is the polylogarithm function, and $H_0 = 0$, and its integrated form,

$$\sum_{k=1}^{\infty} \frac{H_k}{k+1} z^{k+1} = \frac{1}{2} \ln^2(1-z). \tag{A.5}$$

Then from (A.1) we have

$$\sum_{k=1}^{\infty} \frac{k! H_k}{(x)_{k+2}} = \sum_{k=1}^{\infty} \frac{H_k}{k+1} \int_0^1 t^{x-1}(1-t)^{k+1} dt$$

$$= \frac{1}{2} \int_0^1 t^{x-1} \ln^2 t \; dt = \frac{1}{2} \int_0^{\infty} v^2 e^{-xv} dv = \frac{1}{x^3}. \tag{A.6}$$

By binomially expanding the integrand of (A.1) and/or by differentiating with respect to x we obtain other relations. For instance, we have

$$\partial_x B(x, n+1) = \int_0^1 t^{x-1}(1-t)^n \ln t \; dt = \frac{n!}{(x)_{n+1}}[\psi(x) - \psi(x+n+1)] = -\sum_{\ell=0}^{n} \frac{(-1)^\ell}{(x+\ell)^2} \binom{n}{\ell}. \tag{A.7}$$

Alternative points of view of $B(x, n+1) = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(x+\ell)} \binom{n}{\ell}$ are in terms of partial fractions or as divided differences of $1/x$.

Appendix B: Generalized Stirling numbers of the first kind

We let $s(j, k)$ denote the Stirling numbers of the first kind with integer arguments. We proceed to generalize these with the first argument complex. In the following ψ again denotes the digamma function, $\psi^{(j)}$ the polygamma functions, and $(z)_k = \Gamma(z+k)/\Gamma(z)$ the Pochhammer symbol.

It is shown in [5] (Lemma 1) that

$$\left(\frac{d}{ds}\right)^\ell (s)_j \Bigg|_{s=1} = (-1)^{j+\ell} \ell! s(j+1, \ell+1). \tag{B.1}$$

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From this we make the extension to $\lambda \in C$

$$s(\lambda, k) = \frac{(-1)^{\lambda+k}}{(k-1)!} \left(\frac{d}{ds}\right)^{k-1} (s)_{\lambda-1}\big|_{s=1} . \qquad (B.2)$$

We then obviously have agreement with the usual Stirling numbers of the first kind when λ is a nonnegative integer. When λ is a positive integer, $(s)_{\lambda-1}$ is a polynomial of degree $\lambda-1$ in s. Hence by the definition (B.2), $s(\lambda, k) = 0$ when $k > \lambda$. Moreover, we show that these extended Stirling numbers satisfy the same recursion relation as their classical counterparts.

Lemma B1. We have

$$s(\lambda, k) = s(\lambda - 1, k - 1) - (\lambda - 1)s(\lambda - 1, k). \qquad (B.3)$$

Proof. We first note $(s)_{\lambda-1} = (s + \lambda - 2)(s)_{\lambda-2}$ so that from (B.2)

$$s(\lambda, k) = \frac{(-1)^{\lambda+k}}{(k-1)!} \left(\frac{d}{ds}\right)^{k-2} \left[(s)_{\lambda-2} + (s + \lambda - 2)\frac{d}{ds}(s)_{\lambda-2}\right]_{s=1}$$

$$= \frac{(-1)^{\lambda+k}}{(k-1)!} \left[(-1)^{\lambda+k}(k-2)!s(\lambda - 1, k - 1) + \left(\frac{d}{ds}\right)^{k-2}(s + \lambda - 2)\frac{d}{ds}(s)_{\lambda-2}\right]_{s=1} . \qquad (B.4)$$

By using the product rule we find for the last term

$$\left(\frac{d}{ds}\right)^{k-2}(s + \lambda - 2)\frac{d}{ds}(s)_{\lambda-2}\bigg|_{s=1} = D_s^{k-2}(s)_{\lambda-2}|_{s=1} + (\lambda - 1)D_s^{k-1}(s)_{\lambda-2}|_{s=1}$$

$$= (-1)^{\lambda+k}(k-2)!\binom{k-2}{k-3}s(\lambda - 1, k - 1) + (-1)^{\lambda+k-1}(\lambda - 1)(k-1)!s(\lambda - 1, k). \quad (B.5)$$

Then (B.4) becomes

$$s(\lambda, k) = \frac{(-1)^{\lambda+k}}{(k-1)!}[(-1)^{\lambda+k}(k-1)(k-2)!s(\lambda-1, k-1) + (-1)^{\lambda+k-1}(\lambda-1)(k-1)!s(\lambda-1, k)],$$

$$(B.6)$$

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and the Lemma follows.

Remarks. We have

$$\frac{d}{ds}(s)_\lambda = (s)_\lambda[\psi(s+\lambda) - \psi(s)], \qquad (B.7)$$

so that the 'higher' $s(\lambda, k)$ values may be obtained via Bell polynomials. We recall the harmonic H_n and generalized harmonic $H_n^{(r)}$ numbers: $H_n = \psi(n+1) - \psi(1) = \psi(n+1) + \gamma$,

$$H_n^{(r)} = \frac{(-1)^{r-1}}{(r-1)!}\left[\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1)\right]. \qquad (B.8)$$

Our definition of $s(\lambda, k)$ then extends as desired. For instance, we have

$$s(\lambda, 2) = (-1)^\lambda(\lambda-1)!H_{\lambda-1}, \qquad s(\lambda, 3) = (-1)^{\lambda+1}\frac{(\lambda-1)!}{2}[H_{\lambda-1}^2 - H_{\lambda-1}^{(2)}], \quad (B.9a)$$

and

$$s(\lambda, 3) = (-1)^\lambda\frac{(\lambda-1)!}{6}[H_{\lambda-1}^3 - 3H_{\lambda-1}H_{\lambda-1}^{(2)} + 2H_{\lambda-1}^{(3)}]. \qquad (B.9b)$$

Here, it is understood that $(\lambda-1)! = \Gamma(\lambda)$.

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