Hypergeometric summation representations of the Stieltjes constants

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(Received 2011)

March 9, 2011

Abstract

The Stieltjes constants γ_k appear in the regular part of the Laurent expansion of the Riemman and Hurwitz zeta functions. We demonstrate that these coefficients may be written as certain summations over mathematical constants and specialized hypergeometric functions ${}_pF_{p+1}$. This family of results generalizes a representation of the Euler constant in terms of a summation over values of the trigonometric integrals Si or Ci. The series representations are suitable for acceleration. As byproducts, we evaluate certain sine-logarithm integrals and present the leading asymptotic form of the particular ${}_pF_{p+1}$ functions.

Key words and phrases

Riemann zeta function, Stieltjes constants, generalized hypergeometric function, Gamma function, digamma function, Euler constant, series representation, integral representation, Hurwitz zeta function, cosine integral, sine integral

2010 AMS codes

11M06, 11M35, 11Y60, 33C20

Introduction and statement of results

Recently we developed series representations of the Euler constant γ and values of the Riemann zeta function at integer argument, together with other mathematical constants, in terms of summations over the trigonometric integrals Si and Ci [8] (Propositions 2 and 3, Corollaries 7 and 9). The present work is in a sense a generalization of those results. We present series representations of the Stieltjes (generalized Euler) constants that involve sums over certain generalized hypergeometric functions ${}_pF_{p+1}$. There are underlying connections to Si and Ci and certain logarithmic integrals of those functions, and our presentation includes some developments of special function theory.

We let $\zeta(s) = \zeta(s,1)$ be the Riemann zeta function [11, 15, 21, 24], Γ the Gamma function, $\psi = \Gamma'/\Gamma$ be the digamma function (e.g., [1]) with $\gamma = -\psi(1)$ the Euler constant, $\psi^{(k)}$ be the polygamma functions [1], and ${}_pF_q$ be the generalized hypergeometric function [2]. Although we concentrate on the Stieltjes constants $\gamma_k = \gamma_k(1)$ corresponding to the Riemann zeta function, we briefly describe how the approach carries over to those for the Hurwitz zeta function. (See the discussion section.) The Stieltjes constants $\gamma_k(a)$ [5, 6, 4, 7, 17, 18, 23, 25] arise in the regular part of the Laurent expansion of the Hurwitz zeta function $\zeta(s,a)$:

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n(a)(s-1)^n, \tag{1.1}$$

where $\gamma_0(a) = -\psi(a)$.

The Stieltjes constants may be expressed through the limit relation [3]

$$\gamma_n(a) = \frac{(-1)^n}{n!} \lim_{N\to\infty} \left[\sum_{k=0}^{N} \frac{\ln^n(k+a)}{k+a} - \frac{\ln^{n+1}(N+a)}{n+1}\right], \qquad n \geq 0. \tag{1.2}$$

Here, $a \notin \{0, -1, -2, \ldots\}$. The sequence $\{\gamma_k(a)\}_{k\geq 0}$ has rapid growth in magnitude with k for k large and changes in sign due to both k and a. Subsequences of the same sign of arbitrarily long length occur. For an asymptotic expression for these constants, even valid for moderate values of k, [16] (Section 2) may be consulted. The Stieltjes constants appear in applications including asymptotic analysis whether in computer science or high energy physics.

Proposition 1. (a)

$$\gamma_1 = 2\sum_{n=1}^{\infty}\left[1 + \frac{1}{2}(\gamma-2)\gamma - \frac{\pi^2}{24} - \frac{\pi^2 n^2}{6}\ {}_3F_4\left(1,1,1;2,2,2,\frac{5}{2};-\pi^2 n^2\right)\right.$$
$$\left. + \ln(2\pi n)\left(\gamma - 1 + \frac{1}{2}\ln(2\pi n)\right)\right] + \frac{1}{2} - \gamma, \tag{1.3}$$

and (b) the slightly accelerated form

$$\gamma_1 = 2\sum_{n=1}^{\infty}\left[1 + \frac{1}{2}(\gamma-2)\gamma - \frac{\pi^2}{24} - \frac{5}{32\pi^4}\frac{1}{n^4} - \frac{\pi^2 n^2}{6}\ {}_3F_4\left(1,1,1;2,2,2,\frac{5}{2};-\pi^2 n^2\right)\right.$$
$$\left. + \ln(2\pi n)\left(\gamma - 1 + \frac{1}{2}\ln(2\pi n)\right)\right] + \frac{73}{144} - \gamma, \tag{1.4}$$

and (c)

$$\gamma_2 = 1 - 2(\gamma+\gamma_1) + 2\sum_{n=1}^{\infty}\left\{2(1-\gamma) + \gamma^2 + \frac{\gamma^3}{3} - \frac{\pi^2}{12} + \gamma\frac{\pi^2}{12} - \frac{2}{3}\zeta(3)\right.$$
$$+\frac{\pi^2 n^2}{6}\ {}_4F_5\left(1,1,1,1;2,2,2,2,\frac{5}{2};-\pi^2 n^2\right)$$

$$-2\ln(2\pi n)\left[1-\gamma+\frac{\gamma^2}{2}-\frac{\pi^2}{24}+\frac{1}{2}\ln(2\pi n)(\gamma-1)+\frac{1}{6}\ln^2(2\pi n)\right]\Bigg\}. \tag{1.5}$$

Remark. The summand in (1.5) is $O(n^{-3})$ as $n\to\infty$. All parts of the Proposition may be further accelerated in their rate of convergence.

Proposition 2. The Stieltjes constant γ_j may be expressed as a summation over n of mathematical constants, terms $\ln^k(2\pi n)$, with $k=1,\ldots,j$, and the term

$$\pi^2\sum_{n=1}^\infty n^2\ {}_{j+2}F_{j+3}\left(1,1,\ldots,1;2,2,\ldots,2,\frac{5}{2};-\pi^2n^2\right). \tag{1.6}$$

The following section of the paper contains the proof of the Propositions. Section 3 contains various supporting and reference Lemmas. Some of these Lemmas present results of special function theory and may indeed be of occasional interest in themselves. Certain logarithmic integrals of the Si and Ci functions are considered in the Appendix.

Proof of Propositions

We let $P_1(x)=B_1(x-[x])=x-[x]-1/2$ be the first periodic Bernoulli polynomial, with $\{x\}=x-[x]$ the fractional part of x. Being periodic, P_1 has the Fourier series [1] (p. 805)

$$P_1(x)=-\sum_{j=1}^\infty\frac{\sin 2\pi jx}{\pi j}. \tag{2.1}$$

Proposition 1. (a) We take $a=1$ in the representation

$$\zeta(s,a)=\frac{a^{-s}}{2}+\frac{a^{1-s}}{s-1}-s\int_0^\infty\frac{P_1(x)}{(x+a)^{s+1}}dx, \qquad \sigma\equiv\text{Re } s>0, \tag{2.2}$$

so that

$$\zeta(s)=\frac{1}{2}+\frac{1}{s-1}-s\int_1^\infty\frac{P_1(x)}{x^{s+1}}dx, \qquad \sigma>-1, \tag{2.3}$$

and

$$\zeta'(s) + \frac{1}{(s-1)^2} = -\int_1^\infty \frac{P_1(x)}{x^{s+1}}dx + s\int_1^\infty \frac{P_1(x)}{x^{s+1}}\ln x\; dx. \tag{2.4}$$

Taking $s \to 1$ in (2.4) and using (1.1) gives

$$\gamma_1 = -\int_1^\infty \frac{P_1(x)}{x^2}\ln x\; dx + \frac{1}{2} - \gamma. \tag{2.5}$$

Here, we have used the well known integral that also results from (2.3)

$$\int_1^\infty \frac{P_1(x)}{x^2}dx = \sum_{j=1}^\infty \int_j^{j+1} \frac{(x-j-1/2)}{x^2}dx$$

$$= \sum_{j=1}^\infty \left[\ln\left(\frac{j+1}{j}\right) - \frac{1}{2}\left(\frac{1}{j+1} + \frac{1}{j}\right)\right] = \frac{1}{2} - \gamma. \tag{2.6}$$

From Lemmas 2, 3, and 6 we have

$$\frac{1}{\kappa}\int_1^\infty \frac{\sin \kappa x}{x^2}\ln x\; dx = 1 + \frac{1}{2}(\gamma-2)\gamma - \frac{\pi^2}{24} - \frac{\kappa^2}{24}\, {}_3F_4\left(1,1,1;2,2,2,\frac{5}{2};-\frac{\kappa^2}{4}\right)$$

$$+\frac{\ln \kappa}{2}(2\gamma - 2 + \ln \kappa). \tag{2.7}$$

Taking $\kappa = 2\pi n$ and using (2.1) and (2.5) so that

$$\gamma_1 = \frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_1^\infty \frac{\sin(2\pi n x)}{x^2}\ln x\; dx + \frac{1}{2} - \gamma, \tag{2.8}$$

gives the first part of the Proposition from (2.7).

(b) The summand of (1.3) is $O(n^{-4})$ as $n \to \infty$. By adding and subtracting $\frac{5}{32}\sum_{n=1}^\infty \frac{1}{n^4} = \frac{5}{32}\frac{\zeta(4)}{\pi^4} = \frac{1}{288}$ we transform the summand to be $O(n^{-6})$ as $n \to \infty$.

(c) From (2.4) we obtain

$$\zeta''(s) - \frac{2}{(s-1)^3} = 2\int_1^\infty \frac{P_1(x)}{x^{s+1}}\ln x\; dx - s\int_1^\infty \frac{P_1(x)}{x^{s+1}}\ln^2 x\; dx. \tag{2.9}$$

Then from (1.1)

$$\gamma_2 = \int_1^\infty \frac{P_1(x)}{x^2}(2 - \ln x)\ln x \; dx, \tag{2.10}$$

where by (2.5) the term

$$2\int_1^\infty \frac{P_1(x)}{x^2}\ln x \; dx = 1 - 2(\gamma + \gamma_1). \tag{2.11}$$

For the term

$$\int_1^\infty \frac{P_1(x)}{x^2}\ln^2 x \; dx = -\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_1^\infty \frac{\sin(2\pi n x)}{x^2}\ln^2 x \; dx, \tag{2.12}$$

we apply Lemmas 3, 4, and 6. We omit further details.

Proposition 2. As may be proved from (2.2) by induction, for integers $j \geq 1$ we have

$$\zeta^{(j)}(s,a) = (-1)^j a^{1-s}\sum_{k=0}^j \binom{j}{k}(j-k)!\frac{\ln^k a}{(s-1)^{j-k+1}} + \frac{(-1)^j}{2}a^{-s}\ln^j a$$
$$+(-1)^j\int_0^\infty \frac{P_1(x)}{(x+a)^{s+1}}\ln^{j-1}(x+a)\left[j - s\ln(x+a)\right] \; dx. \tag{2.13}$$

At $a = 1$ we have

$$\zeta^{(j)}(s) = \frac{(-1)^j j!}{(s-1)^{j+1}} + (-1)^j\int_1^\infty \frac{P_1(x)}{x^{s+1}}\ln^{j-1} x\,(j - s\ln x) \; dx. \tag{2.14}$$

Then by (1.1) we have

$$\gamma_j = \int_1^\infty \frac{P_1(x)}{x^2}\ln^{j-1} x(j - \ln x)dx$$
$$= -\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_1^\infty \frac{\sin(2\pi n x)}{x^2}\ln^{j-1} x(j - \ln x)dx. \tag{2.15}$$

Herein on the right side, j multiplies a contribution from γ_{j-1}. We then appeal to Lemmas 2, 3, and 4 for logarithmic-sine integrals, and the result follows.

Lemmas

Herein the cosine integral is defined by

$$\mathrm{Ci}(z) \equiv -\int_z^\infty \frac{\cos t}{t} dt = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} dt. \tag{3.1}$$

As usual, $(w)_n = \Gamma(w+n)/\Gamma(w)$ denotes the Pochhammer symbol.

Lemma 1. (Hypergeometric form of the cosine integral)

$$\mathrm{Ci}(z) = \gamma + \ln z - \frac{z^2}{4} \, {}_2F_3\left(1,1;2,2,\frac{3}{2};-\frac{z^2}{4}\right). \tag{3.2}$$

Proof. This easily follows from the expression

$$\mathrm{Ci}(z) = \gamma + \ln z + \sum_{\ell=1}^\infty \frac{(-1)^\ell z^{2\ell}}{2\ell(2\ell)!}. \tag{3.3}$$

Lemma 2. (a) For $a \neq 0$,

$$\int_x^y \frac{\mathrm{Ci}(az)}{z} dz = \gamma \ln(y/x) + \frac{1}{2}[\ln^2(ay) - \ln^2(ax)]$$

$$-\frac{a^2}{8}\left[y^2 \, {}_3F_4\left(1,1,1;2,2,2,\frac{3}{2};-\frac{a^2y^2}{4}\right) - x^2 \, {}_3F_4\left(1,1,1;2,2,2,\frac{3}{2};-\frac{a^2x^2}{4}\right)\right], \tag{3.4}$$

and (b) for $b > a > 0$

$$\int_a^b \frac{\sin \kappa x}{x^2} \ln x \, dx = \kappa \left\{\mathrm{Ci}(\kappa b)(1+\ln b) - \mathrm{Ci}(\kappa a)(1 + \ln a) - \gamma \ln(b/a)\right.$$

$$\left. + \frac{1}{2}[\ln^2(\kappa a) - \ln^2(\kappa b)] + \frac{\kappa^2}{8}\left[b^2 \, {}_3F_4\left(1,1,1;2,2,2,\frac{3}{2};-\frac{\kappa^2 b^2}{4}\right) - a^2 \, {}_3F_4\left(1,1,1;2,2,2,\frac{3}{2};-\frac{\kappa^2 a^2}{4}\right)\right]\right\}$$

$$+\frac{\sin \kappa a}{a}(1+\ln a) - \frac{\sin \kappa b}{b}(1 + \ln b). \tag{3.5}$$

Proof. (a) We first note that $1/(\ell+1) = (1)_\ell/(2)_\ell$ and $(2\ell+2)! = \Gamma(2\ell+3) = 2\cdot 4^\ell (3/2)_\ell(\ell+1)\ell!$. The latter relation follows from the duplication formula $\Gamma(2j+1) = 4^j\Gamma(j+1/2)j!/\Gamma(1/2)$. Then by using (3.3) we have

$$\int_x^y \frac{\text{Ci}(az)}{z}dz = \int_x^y [\gamma+\ln(az)]\frac{dz}{z} + \sum_{\ell=1}^\infty \frac{(-1)^\ell a^{2\ell}}{2\ell(2\ell)!}\int_x^y z^{2\ell-1}dz$$

$$= \gamma\ln(y/x) + \frac{1}{2}[\ln^2(ay)-\ln^2(ax)] + \sum_{\ell=1}^\infty \frac{(-1)^\ell a^{2\ell}}{(2\ell)^2(2\ell)!}(y^{2\ell}-x^{2\ell}). \tag{3.6}$$

The sums are then rewritten according to

$$\sum_{\ell=1}^\infty \frac{(-1)^\ell a^{2\ell}}{(2\ell)^2(2\ell)!}y^{2\ell} = -\frac{1}{4}\sum_{\ell=0}^\infty \frac{(-1)^\ell a^{2\ell+2}}{(\ell+1)^2(2\ell+2)!}y^{2\ell+2}$$

$$= -\frac{a^2}{8}\sum_{\ell=0}^\infty (-1)^\ell \frac{(1)_\ell^2}{(2)_\ell^2}\frac{a^{2\ell}}{4^\ell}\frac{1}{(3/2)_\ell}\frac{(1)_\ell}{(2)_\ell}\frac{y^{2\ell+2}}{\ell!}$$

$$= -\frac{a^2}{8}y^2\ {}_3F_4\left(1,1,1;2,2,2,\frac{3}{2};-\frac{a^2y^2}{4}\right). \tag{3.7}$$

(b) We first have, by integrating by parts,

$$\int_a^b \frac{\sin\kappa x}{x^2}\ln x\ dx = \int_a^b\left[\frac{\sin\kappa x}{x}+\kappa\cos\kappa x\ln x\right]\frac{dx}{x} - \frac{\sin\kappa x}{x}\ln x\Big|_a^b$$

$$= \kappa\int_a^b \frac{\cos\kappa x}{x}\ln x\ dx + \kappa[\text{Ci}(\kappa b)-\text{Ci}(\kappa a)] + \frac{\sin\kappa a}{a}(1+\ln a) - \frac{\sin\kappa b}{b}(1+\ln b). \tag{3.8}$$

Next, also by integration by parts,

$$\int_a^b \frac{\cos\kappa x}{x}\ln x\ dx = -\int_a^b \frac{\text{Ci}(\kappa x)}{x}dx + \text{Ci}(\kappa b)\ln b - \text{Ci}(\kappa a)\ln a. \tag{3.9}$$

We then apply part (a) to (3.9) and combine with (3.8).

Lemma 3. (a) For $j\ge 1$ an integer,

$$\int_1^\infty \frac{\sin\kappa x}{x^2}\ln^j x\ dx = j\int_1^\infty \frac{\ln^{j-1}t}{t}\left[-\kappa\text{Ci}(\kappa t)+\frac{\sin\kappa t}{t}\right]dt. \tag{3.10}$$

(b) With $j=1$,

$$\int_1^\infty \frac{\sin \kappa x}{x^2} \ln x \; dx = -\kappa \left[\int_1^\infty \frac{\mathrm{Ci}(\kappa t)}{t} dt + \mathrm{Ci}(\kappa) \right] + \sin \kappa. \tag{3.11}$$

Proof. We interchange a double integral on the basis of Tonelli's theorem,

$$\begin{aligned} \int_1^\infty \frac{\sin \kappa x}{x^2} \ln^j x \; dx &= j \int_1^\infty \frac{\sin \kappa x}{x^2} \int_1^x \frac{\ln^{j-1} t}{t} dt \\ &= j \int_1^\infty \frac{\ln^{j-1} t}{t} dt \int_t^\infty \frac{\sin \kappa x}{x^2} dx \\ &= j \int_1^\infty \frac{\ln^{j-1} t}{t} \left[-\kappa \mathrm{Ci}(\kappa t) + \frac{\sin \kappa t}{t} \right] dt, \end{aligned} \tag{3.12}$$

with the aid of an integration by parts. (b) follows easily.

Part (d) of the next result gives a prescription for successively obtaining certain needed log-sine integrals, while part (b) complements Lemma 3.

Lemma 4. (a)

$$\int_1^\infty \frac{\cos(bx)}{x} \ln x \; dx = -\int_1^\infty \frac{\mathrm{Ci}(bx)}{x} dx, \tag{3.13}$$

(b)

$$\int_1^\infty \frac{\sin(bx)}{x^2} \ln x \; dx = \int_1^\infty \left[-b\mathrm{Ci}(bx) + \frac{\sin(bx)}{x} \right] dx, \tag{3.14}$$

(c)

$$\int_1^\infty \frac{\cos(bx)}{x} \ln^j x \; dx = -j \int_1^\infty \frac{\mathrm{Ci}(bx)}{x} \ln^{j-1} x \; dx, \tag{3.15}$$

(d) Let

$$f_j(b) = \int_1^\infty \frac{\mathrm{Ci}(bx)}{x} \ln^j x \; dx = \int_b^\infty \frac{\mathrm{Ci}(y)}{y} \ln^j \left(\frac{y}{b}\right) dy, \tag{3.16}$$

with $f_j(\infty) = 0$, such that

$$-j \int_0^b f_{j-1}(b) db = \int_1^\infty \frac{\sin(bx)}{x^2} \ln^j x \; dx \equiv g_j(b), \tag{3.17}$$

with $g_j(0) = g_j(\infty) = 0$ and $g_0(b) = \sin b - b\mathrm{Ci}(b)$. Then for $j \geq 1$

$$f_j(b) = -j \int \frac{f_{j-1}(b)}{b} db, \tag{3.18}$$

and

$$g_j(b) = -jb \int \frac{g_{j-1}(b)}{b^2} db + c_j b, \tag{3.19}$$

where c_j is a constant.

(e) Let for $j \geq 1$

$$h_j(b) = \int_1^\infty f(bx) \ln^j x \, dx = \frac{1}{b} \int_b^\infty f(y) \ln^j \left(\frac{y}{b}\right) dy, \tag{3.20}$$

where $f(\infty) = 0$ and $f \to 0$ sufficiently fast at infinity for the integral to converge. Then

$$h_j(b) = -\frac{j}{b} \int h_{j-1}(b) db. \tag{3.21}$$

Proof. For (a) and (c) we integrate by parts, using that $\mathrm{Ci}(x)$ is $\mathrm{O}(1/x)$ as $x \to \infty$:

$$\begin{aligned}\int_1^\infty \frac{\cos(bx)}{x} \ln^j x \, dx &= -j \int_1^\infty \frac{\mathrm{Ci}(bx)}{x} \ln^{j-1} x \, dx + \mathrm{Ci}(bx) \ln^j x \Big|_1^\infty \\ &= -j \int_1^\infty \frac{\mathrm{Ci}(bx)}{x} \ln^{j-1} x \, dx. \end{aligned} \tag{3.22}$$

For (b) we use (a) and

$$\int_0^b \mathrm{Ci}(bx) db = b\mathrm{Ci}(bx) - \frac{\sin(bx)}{x}, \tag{3.23}$$

so that

$$\begin{aligned}\int_1^\infty \frac{\sin(bx)}{x^2} \ln x \, dx &= \int_0^b \int_1^\infty \frac{\cos(bx)}{x} \ln x \, dx db \\ &= -\int_0^b \int_1^\infty \frac{\mathrm{Ci}(bx)}{x} dx db. \end{aligned} \tag{3.24}$$

For (d) we use the property

$$\frac{\partial}{\partial b} \ln^j \left(\frac{y}{b}\right) = -\frac{j}{b} \ln^{j-1} \left(\frac{y}{b}\right). \tag{3.25}$$

Then for $j \geq 1$

$$\begin{aligned} f_{j-1}(b) &= \int_b^\infty \frac{\text{Ci}(y)}{y} \ln^{j-1} \left(\frac{y}{b}\right) dy \\ &= -\frac{b}{j} \int_b^\infty \frac{\text{Ci}(y)}{y} \frac{\partial}{\partial b} \ln^j \left(\frac{y}{b}\right) dy \\ &= -\frac{b}{j} \frac{\partial}{\partial b} \int_b^\infty \frac{\text{Ci}(y)}{y} \ln^j \left(\frac{y}{b}\right) dy \\ &= -\frac{b}{j} \frac{\partial}{\partial b} f_j(b), \end{aligned} \tag{3.26}$$

from which (3.15) follows. For (3.16), we have

$$g_j(b) = b \int_b^\infty \frac{\sin y}{y^2} \ln^j \left(\frac{y}{b}\right) dy, \tag{3.27}$$

from which follows

$$\frac{\partial}{\partial b} g_j(b) = \frac{1}{b} g_j(b) - \frac{j}{b} g_{j-1}(b). \tag{3.28}$$

The solution of this linear differential equation gives (3.19). That $g_j(\infty) = 0$ follows from the Riemann-Lebesgue Lemma applied to (3.17). For (e) we have $h_{j-1}(b) = -(b/j)\partial_b h_j(b)$ from which (3.21) follows.

Corollary 1. By iteration of (3.19), we see that $g_j(b)$ contains a term with $\frac{(-1)^{j+1}}{j+1} b \ln^{j+1} b$.

Remark. Imposing $g_1(\infty) = 0$ gives $c_1 = \gamma^2/2 - \pi^2/24$.

We note that Lemma 5 just below and the like allows for the straightforward integration of the ${}_pF_{p+1}$ functions arising from the logarithmic-trigonometric integrals involved.

Lemma 5. For $p \neq 0$, $q \neq -1$, and $(q+1)/p \neq -1$,

$$\int \kappa^q \ {}_3F_4(a_1, a_2, a_3; b_1, b_2, b_3, c; -\kappa^p/4) d\kappa$$
$$= \frac{\kappa^{q+1}}{q+1} \ {}_4F_5 \left(a_1, a_2, a_3, \frac{q+1}{p}; b_1, b_2, b_3, c, \frac{q+1}{p} + 1; -\frac{\kappa^p}{4} \right). \tag{3.29}$$

Proof. We have

$$\int \kappa^q \ {}_3F_4(a_1, a_2, a_3; b_1, b_2, b_3, c; -\kappa^p/4) d\kappa = \sum_{\ell=0}^{\infty} \frac{(a_1)_\ell (a_2)_\ell (a_3)_\ell}{(b_1)_\ell (b_2)_\ell (b_3)_\ell} \frac{1}{(c)_\ell} \left(-\frac{1}{4} \right)^\ell \frac{1}{\ell!} \int \kappa^{p\ell+q} d\kappa$$
$$= \sum_{\ell=0}^{\infty} \frac{(a_1)_\ell (a_2)_\ell (a_3)_\ell}{(b_1)_\ell (b_2)_\ell (b_3)_\ell} \frac{1}{(c)_\ell} \left(-\frac{1}{4} \right)^\ell \frac{1}{\ell!} \frac{\kappa^{p\ell+q+1}}{(p\ell+q+1)}. \tag{3.30}$$

We then use

$$\frac{q+1}{p\ell+q+1} = \frac{\left(\frac{q+1}{p}\right)_\ell}{\left(\frac{q+1}{p}+1\right)_\ell}, \tag{3.31}$$

to complete the Lemma.

We then call out a special case at $c = (q+1)/p$.

Corollary 2. For $p \neq 0$,

$$\int \kappa^q \ {}_3F_4 \left(a, a, a; b, b, b, \frac{q+1}{p}; -\frac{\kappa^p}{4} \right) d\kappa$$
$$= \frac{\kappa^{q+1}}{q+1} \ {}_3F_4 \left(a, a, a; b, b, b, \frac{q+1}{p} + 1; -\frac{\kappa^p}{4} \right). \tag{3.32}$$

In particular, this result applies with $p = q = 2$ when integrating the function of (3.6) with respect to a.

Lemma 6. (Leading asymptotic form of special ${}_pF_{p+1}$ functions) For $z \to \infty$ we have (a)

$${}_2F_3 \left(1, 1; 2, 2, \frac{3}{2}; -z \right) \sim \frac{1}{z} \left(\gamma + \ln 2 + \frac{1}{2} \ln z \right), \tag{3.33}$$

(b)

$$
{}_3F_4\left(1,1,1;2,2,2,\frac{5}{2};-z\right) \sim \frac{1}{z}\left[6(1-\gamma)+3\gamma^2-\frac{\pi^2}{4}-6\ln 2+6\gamma\ln 2+3\ln^2 2\right]
$$

$$
+\frac{3}{z}(\gamma-1+\ln 2)\ln z+\frac{3}{4}\frac{\ln^2 z}{z}, \tag{3.34}
$$

and (c)

$$
{}_4F_5\left(1,1,1,1;2,2,2,2,\frac{5}{2};-z\right) \sim \frac{1}{z}\left[12(\gamma-1)-6\gamma^2+2\gamma^3+\frac{\pi^2}{2}-\frac{\gamma}{2}\pi^2-6\ln^2 2+6\gamma\ln^2\right.
$$

$$
+2\ln^3 2+6\ln 2(\gamma-1)\ln z+3\ln^2 2\ln z-\frac{3}{2}\ln^2 z+\frac{3}{2}(\gamma+\ln 2)\ln^2 z+\ln^3 z
$$

$$
\left.+(6(1-\gamma)+3\gamma^2-\frac{\pi^2}{4})\ln(4z)+4\zeta(3)\right]. \tag{3.35}
$$

Proof. (a) This follows immediately from the definition of Ci in (3.1) and the relation (3.2). However, we use a more general procedure based upon the Barnes integral representation of ${}_pF_q$ ([20], Section 2.3). We have

$$
{}_2F_3\left(1,1;2,2,\frac{3}{2};-z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(-s)\Gamma(s+1)g(s)z^s ds, \tag{3.36}
$$

where the path of integration is a Barnes contour, indented to the left of the origin but staying to the right of -1, and

$$
\Gamma(s+1)g(s) = \Gamma\left(\frac{3}{2}\right)\frac{\Gamma^2(s+1)}{\Gamma^2(s+2)}\frac{1}{\Gamma(s+3/2)} = \Gamma\left(\frac{3}{2}\right)\frac{1}{(s+1)^2\Gamma(s+3/2)}. \tag{3.37}
$$

The contour can be thought of as closed in the right half plane, over a semicircle of infinite radius. We then move the contour to the left of $s=-1$, picking up the residue there. We have

$$
\frac{d}{ds}\frac{\Gamma(-s)}{\Gamma(s+3/2)} = -\frac{\Gamma(-s)}{\Gamma(s+3/2)}[\psi(-s)+\psi(s+3/2)], \tag{3.38}
$$

and $\psi(1/2) = -\gamma - 2\ln 2$. Therefore, we have for s near -1

$$\Gamma\left(\frac{3}{2}\right)\frac{\Gamma(-s)}{\Gamma(s+3/2)} = \frac{1}{2} + (\gamma + \ln 2)(s+1) + O[(s+1)^2], \tag{3.39}$$

with $z^s = 1/z + \ln z(s+1)/z + O[(s+1)^2]$ and find

$$\Gamma\left(\frac{3}{2}\right)\mathrm{Res}_{s=-1}\ \frac{\Gamma(-s)z^s}{(s+1)^2\Gamma(s+3/2)} = \frac{1}{z}\left(\gamma + \ln 2 + \frac{1}{2}\ln z\right), \tag{3.40}$$

giving (3.25).

(b) We proceed similarly, with

$${}_3F_4\left(1,1,1;2,2,2,\frac{5}{2};-z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(-s)\Gamma(s+1)g(s)z^s ds, \tag{3.41}$$

and

$$\Gamma(s+1)g(s) = \Gamma\left(\frac{5}{2}\right)\frac{\Gamma^3(s+1)}{\Gamma^3(s+2)}\frac{1}{\Gamma(s+5/2)}. \tag{3.42}$$

Again using an expansion like (3.39) and evaluating the residue at $s = -1$ gives (3.34).

(c) Similarly, we use

$${}_4F_5\left(1,1,1,1;2,2,2,2,\frac{5}{2};-z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(-s)\Gamma(s+1)g(s)z^s ds, \tag{3.43}$$

with

$$\Gamma(s+1)g(s) = \Gamma\left(\frac{5}{2}\right)\frac{\Gamma^4(s+1)}{\Gamma^4(s+2)}\frac{1}{\Gamma(s+5/2)} = \Gamma\left(\frac{5}{2}\right)\frac{1}{(s+1)^4\Gamma(s+5/2)}, \tag{3.44}$$

and compute

$$\Gamma\left(\frac{5}{2}\right)\mathrm{Res}_{s=-1}\ \frac{\Gamma(-s)z^s}{(s+1)^4\Gamma(s+3/2)}.$$

Remarks. We note that results like part (b) of this Lemma allows us to find the limit as $y \to \infty$ for the integral of Lemma 2.

We have presented the algebraic part of the asymptotic form of the ${}_pF_{p+1}$ functions, that is the leading portion. The higher order terms come from the exponential expansion of these functions, and they are infinite in number. According to ([20], Section 2.3, p. 57) this expansion has the form

$$E(-z) = (2iz^{1/2})^{\theta} e^{2i\sqrt{z}} \sum_{k=0}^{\infty} \frac{A_k}{(2iz^{1/2})^k}, \tag{3.45}$$

where $\theta = 1 - p - \alpha$ and $A_0 = (2\pi)^{-1/2} 2^{-1/2-\theta}$. Here, α is either $3/2$ (for the $p = 2$ case) or otherwise $5/2$. The other coefficients A_k come from an inverse factorial expansion of the function $g(s)$,

$$g(s) = 2 \cdot 4^s \left[\sum_{j=0}^{\infty} \frac{A_j}{\Gamma(2s + 1 - \theta + j)} + \frac{O(1)}{\Gamma(2s + 1 - \theta + M)} \right]. \tag{3.46}$$

The full asymptotic expansion of the ${}_pF_{p+1}$ functions includes the algebraic portion $H(-z)$ given in the Lemma, together with (3.42), as

$${}_pF_{p+1}\left(1, 1, \ldots, 1; 2, 2, \ldots, 2, \frac{5}{2}; -z\right) \sim E(-z) + E(z) + H(-z). \tag{3.47}$$

Lemma 7. (Alternative integral representation)

$${}_3F_4\left(1, 1, 1; 2, 2, 2, \frac{5}{2}; -\frac{\kappa^2}{4}\right) = \frac{3}{2}\frac{1}{\kappa^3} \int_0^{\infty} x^2 [-\kappa \cos(e^{-x/2}\kappa) + e^{x/2} \sin(e^{-x/2}\kappa)] dx. \tag{3.48}$$

Proof. This follows by using the special case Laplace transform

$$\int_0^{\infty} x^2 e^{-rx} dx = \frac{\Gamma(3)}{r^3}. \tag{3.49}$$

Then, with the integral being absolutely convergent, we may interchange summation and integration in

$$
{}_3F_4\left(1,1,1;2,2,2,\frac{5}{2};-\frac{\kappa^2}{4}\right)=\frac{1}{2}\sum_{j=0}^{\infty}\int_0^\infty x^2e^{-(j+1)x}\frac{1}{(5/2)_j}\left(-\frac{\kappa^2}{4}\right)^j\frac{1}{j!}dx, \qquad (3.50)
$$

and the result follows.

We mention the connection between integrals expressible as certain differences of the confluent hypergeometric function ${}_1F_1$, differences of the incomplete gamma function $\gamma(x,y)$, and a ${}_1F_2$ function. We have

Lemma 8. We have for Re $\mu > -1$

$$
\begin{aligned}
\int_0^1 x^{\mu-1}\sin(ax)dx &= -\frac{i}{2\mu}[\ {}_1F_1(\mu;\mu+1;ia)-\ {}_1F_1(\mu;\mu+1;-ia)] \\
&= -\frac{i}{2}(ia)^{-\mu}[\gamma(\mu,-ia)-(-1)^\mu\gamma(\mu,ia)] \\
&= \frac{a}{\mu+1}\ {}_1F_2\left(\frac{1+\mu}{2};\frac{3}{2},\frac{3+\mu}{2};-\frac{a^2}{4}\right). \qquad (3.51)
\end{aligned}
$$

We note that by logarithmic differentiation with respect to μ this result generates a family of logarithmic sine integrals. I.e., for Re $\mu > -1$,

$$
\int_0^1 x^{\mu-1}\sin(ax)\ln^k x\ dx=\left(\frac{\partial}{\partial\mu}\right)^k\int_0^1 x^{\mu-1}\sin(ax)dx. \qquad (3.52)
$$

Proof. The first line of (3.51) is from ([12], p. 420, 3.761.1). The second line follows from the relation ([12], p. 1063)

$$
\gamma(\alpha,x)=\frac{1}{\alpha}x^\alpha\ {}_1F_1(\alpha;\alpha+1;-x). \qquad (3.53)
$$

For the third line we write

$$
{}_1F_1(\mu;\mu+1;ia)-\ {}_1F_1(\mu;\mu+1;-ia)=\sum_{j=0}^{\infty}\frac{(\mu)_j}{(\mu+1)_j}\frac{(ia)^j}{j!}[1-(-1)^j]
$$

$$= 2i \sum_{m=0}^{\infty} \frac{(\mu)_{2m+1}}{(\mu+1)_{2m+1}} \frac{a^{2m+1}}{(2m+1)!}. \tag{3.54}$$

The Lemma is now completed with the use of the duplication formula, as at the beginning of the proof of Lemma 2, and the relation

$$\frac{(\mu)_{2m+1}}{(\mu+1)_{2m+1}} = \frac{\mu}{2} \frac{1}{(\mu/2+m+1/2)} = \frac{\mu}{2(\mu+1)} \frac{\left(\frac{1+\mu}{2}\right)_m}{\left(\frac{3+\mu}{2}\right)_m}. \tag{3.55}$$

Discussion

We mention how hypergeometric summatory representations may be obtained for the constants $\gamma_k(a)$. When $k=0$ we must recover from (2.2) the known representations (e.g., [11], p. 107)

$$\ln\Gamma(a) = \left(a - \frac{1}{2}\right)\ln a - a + \frac{1}{2}\ln(2\pi) - \int_0^\infty \frac{P_1(t)}{t+a} dt, \tag{4.1}$$

and

$$\psi(a) = -\gamma_0(a) = \ln a - \frac{1}{2a} + \int_0^\infty \frac{P_1(t)}{(t+a)^2} dt, \tag{4.2}$$

leading to, for Re $a > 0$,

$$\psi(a) = -\gamma_0(a) = \ln a - \frac{1}{2a} + \sum_{j=1}^{\infty} \left[2\cos(2\pi ja)\mathrm{Ci}(2\pi ja) - \sin(2\pi ja)[\pi - 2\mathrm{Si}(2\pi ja)]\right]. \tag{4.3}$$

Matters are more interesting, and challenging, for $k \geq 1$.

From (2.2) we have

$$\zeta'(s,a) = -\frac{a^{-s}\ln a}{2} - \frac{a^{1-s}\ln a}{s-1} - \frac{a^{1-s}}{(s-1)^2}$$

$$+ \int_0^\infty \frac{P_1(x)}{(x+a)^{s+1}}[s\ln(x+a) - 1]dx, \tag{4.4}$$

so that

$$-\gamma_1(a) = -\frac{\ln a}{2a} + \frac{\ln^2 a}{2} + \int_0^\infty \frac{P_1(x)}{(x+a)^2}[\ln(x+a) - 1]dx. \tag{4.5}$$

Again the Fourier series (2.1), applies so that the term

$$\int_0^\infty \frac{P_1(x}{(x+a)^2}[\ln(x+a) - 1]dx = -\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty \frac{\sin(2\pi nx)}{(x+a)^2}[1 - \ln(x+a)]dx$$

$$= -\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_a^\infty \frac{\sin[2\pi n(x-a)]}{x^2}(1 - \ln x)dx$$

$$= -\frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_a^\infty \frac{[\sin(2\pi nx)\cos 2\pi na - \cos(2\pi nx)\sin 2\pi na]}{x^2}(1 - \ln x)dx. \tag{4.6}$$

A special case occurs at $a = 1/2$, when $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$, implying that $\gamma_1(1/2) = \gamma_1 - 2\gamma\ln 2 - \ln^2 2$. We have from (4.4) and (4.5)

$$-\gamma_1(1/2) = \ln 2 + \frac{1}{2}\ln^2 2 - \frac{1}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n}\int_{1/2}^\infty \frac{\sin(2\pi nx)}{x^2}(1 - \ln x)dx. \tag{4.7}$$

Here

$$\int_{1/2}^\infty \frac{\sin(2\pi nx)}{x^2}dx = -\kappa \text{Ci}(\kappa/2) + 2\sin(\kappa/2), \tag{4.8}$$

with the last term vanishing when $\kappa = 2\pi n$. The logarithmic integral in (4.6) can be decomposed as $\int_{1/2}^\infty = \int_{1/2}^1 + \int_1^\infty$ and Lemma 2 and (2.7) can be applied respectively. This leads to a hypergeometric summatory expression for $\gamma_1(1/2)$.

Lastly, we mention that our methods for treating logarithmic integrals may be applied to other functions, including the Bessel functions of the first kind J_n. These integrals may be of the form

$$\int_1^\infty \frac{J_n(ax)}{x^j}\ln^k(bx)dx,$$

and in particular for $n = 1$ we expect analogies with our previous trigonometric logarithmic integrals. However, we do not pursue that topic here.

Summary

We have presented summations over particular ${}_pF_{p+1}$ generalized hypergeometric functions and mathematical constants giving the Stieltjes constants γ_k. We have described the asymptotic form of the ${}_pF_{p+1}$ functions with repeated parameters, and indeed the extended asymptotics may be used to accelerate the convergence of the hypergeometric-based summations. We have considered certain logarithmic-sine integrals. We showed that these are essentially certain iterated integrals with the cosine integral Ci at their base. Our results generalize significantly simpler summatory expressions for the Euler constant γ in terms of evaluations of Ci or the sine integral Si.

Appendix: Logarithmic integrals

The following two integrals are given in the Table [12] (p. 647):

$$\int_0^\infty \mathrm{Ci}(x)\ln x \ dx = \frac{\pi}{2}, \tag{A.1}$$

and

$$\int_0^\infty \mathrm{si}(x)\ln x \ dx = \gamma + 1. \tag{A.2}$$

Here,

$$\mathrm{si}(x) \equiv -\int_x^\infty \frac{\sin t}{t} dt. \tag{A.3}$$

We show that these two integrals follow as special cases of the following.

Proposition A1. For $a > 0$,

$$\int_0^\infty e^{-at}\cos t(1-\ln t)dt = \frac{1}{2(1+a^2)}\left\{2\cot^{-1}a + a\left[\ln\left(1+\frac{1}{a^2}\right) + 2(1+\gamma+\ln a)\right]\right\}, \tag{A.4}$$

and

$$\int_0^\infty e^{-at}\sin t(1-\ln t)dt = \frac{1}{2(1+a^2)}\left[-2a\cot^{-1}a + \ln\left(1+\frac{1}{a^2}\right) + 2(1+\gamma+\ln a)\right]. \tag{A.5}$$

We first observe the special cases under formal interchange of integrations

$$\begin{aligned}\int_0^\infty \mathrm{Ci}(x)\ln x \ dx &= -\int_0^\infty \ln x \int_x^\infty \frac{\cos t}{t} dtdx \\ \text{``}=\text{''} &-\int_0^\infty \frac{\cos t}{t}\int_0^t \ln x \ dxdt \\ &= \int_0^\infty \cos t(1-\ln t)dt, \end{aligned} \tag{A.6}$$

and similarly

$$\int_0^\infty \mathrm{si}(x)\ln x \; dx = -\int_0^\infty \ln x \int_x^\infty \frac{\sin t}{t} dt dx$$

$$\text{``}=\text{''} - \int_0^\infty \frac{\sin t}{t} \int_0^t \ln x \; dx dt$$

$$= \int_0^\infty \sin t(1-\ln t) dt. \tag{A.7}$$

Indeed the integrals (A.6) and (A.7) are not convergent. However, the $a \to 0$ limit in the Proposition gives the integrals (A.1) and (A.2).

Proof. We just prove (A.4), as (A.5) goes similarly. From the Gamma function integral

$$\int_0^\infty e^{-at} t^{z-1} dt = a^{-z}\Gamma(z) \tag{A.8}$$

we have

$$\int_0^\infty e^{-at} t^{z-1} \ln t \; dt = \frac{\partial}{\partial z} a^{-z}\Gamma(z)$$

$$= a^{-z}\Gamma(z)[\psi(z) - \ln a]. \tag{A.9}$$

We recall the relation $\psi(n+1) = H_n - \gamma$, where H_n is the nth harmonic number, and then for integers $j \geq 0$

$$\int_0^\infty e^{-at} t^{2j} \ln t \; dt = a^{-2j+1}\Gamma(2j+1)[\psi(2j+1) - \ln a]$$

$$= a^{-(2j+1)}(2j)![H_{2j} - \gamma - \ln a]. \tag{A.10}$$

Then integrating term-by-term we have

$$\int_0^\infty e^{-at} \cos t \ln t \; dt = \sum_{j=0}^\infty \frac{(-1)^j}{(2j)!} \int_0^\infty e^{-at} t^{2j} \ln t \; dt$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} a^{-(2j+1)} (2j)! [H_{2j} - \gamma - \ln a]$$

$$= -\frac{(\gamma + \ln a)}{1 + a^2} + \sum_{j=0}^{\infty} (-1)^j H_{2j} a^{-(2j+1)}. \qquad (A.11)$$

The latter sum may be performed with the substitution of an integral representation for the digamma function, or by manipulating the generating function

$$\sum_{n=0}^{\infty} (-1)^n H_n z^n = -\frac{\ln(1+z)}{1+z}, \qquad (A.12)$$

and recalling that $\cot^{-1} z = (1/(2i)) \ln[(1+iz)/(1-iz)]$. We have

$$\sum_{n=0}^{\infty} [(-1)^n + 1] H_n z^n = 2 \sum_{m=0}^{\infty} H_{2m} z^{2m} = -\frac{\ln(1+z)}{1+z} + \frac{\ln(1-z)}{z-1}. \qquad (A.13)$$

We then put $z = ia$ to obtain the sum of (A.11). Using the elementary integral $\int_0^\infty e^{-at} \cos t dt = a/(1+a^2)$ completes the Proposition.

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