Non-Hermitian oscillator Hamiltonians and multiple Charlier polynomials

Hiroshi Miki

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Sakyo-Ku, Kyoto 606 8501, Japan

Luc Vinet

Centre de recherches mathématiques, Université de Montréal, P. O. Box 6128, Centre-ville Station, Montréal (Québec), H3C 3J7, Canada

Alexei Zhedanov

Donetsk Institute for Physics and Technology, Donetsk 83 114, Ukraine

Abstract

A set of r non-Hermitian oscillator Hamiltonians in r dimensions is shown to be simultaneously diagonalizable. Their spectra is real and the common eigenstates are expressed in terms of multiple Charlier polynomials. An algebraic interpretation of these polynomials is thus achieved and the model is used to derive some of their properties.

Keywords: multiple Charlier polynomials, non-Hermitian Hamiltonians, algebraic model.

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Non-Hermitian Hamiltonians with real spectra are currently being actively investigated with respect to their mathematical underpinnings and their physical applications. (See the most recent special issues dedicated to this topic [1, 2, 3].) We present here a set of r non-Hermitian oscillator Hamiltonians in r dimensions that all have real eigenvalues. This system is also seen to provide an algebraic model for the multiple Charlier polynomials and is exploited to derive some properties of these special functions. The approach is related to the interpretation that was given of the ordinary Charlier polynomials in [4, 5] and of the d-orthogonal Charlier polynomials in [6].

The monic r-multiple Charlier polynomials [7, 8, 9] $C_{\vec{n}}^{\vec{\sigma}}(k)$ are indexed by a multi-index $\vec{n} = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$ with length $|\vec{n}| = n_1 + \cdots + n_r$. They are orthogonal with respect to r Poisson measures with different positive parameters $\sigma_1, \cdots, \sigma_r$ (collectively denoted by $\vec{\sigma}$):

$$\sum_{k=0}^{\infty} C_{\vec{n}}^{\vec{\sigma}}(k) k^l \frac{\sigma_j^l}{k!} = 0, \quad l = 0, 1, \cdots, n_j - 1 \tag{1}$$

for $1 \le j \le r$. (In the following we shall omit the suffix $\vec{\sigma}$.) They have been shown [7, 10] to obey the r nearest-neighbor recurrence relations:

$$kC_{\vec{n}}(k) = C_{\vec{n}+\vec{e_1}}(k) + (\sigma_1 + |\vec{n}|)C_{\vec{n}}(k) + \sum_{j=1}^{r} n_j \sigma_j C_{\vec{n}-\vec{e_j}}(k),$$

$$\vdots \tag{2}$$

$$kC_{\vec{n}}(k) = C_{\vec{n}+\vec{e_r}}(k) + (\sigma_r + |\vec{n}|)C_{\vec{n}}(k) + \sum_{j=1}^{r} n_j \sigma_j C_{\vec{n}-\vec{e_j}}(k),$$

where $\vec{e_j} = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the j-th standard unit vector with 1 on the j-th entry. Subtracting the above relations pair-wise, one finds that the polynomials $C_{\vec{n}}(k)$ satisfy as a consequence

$$C_{\vec{n}+\vec{e_i}}(k) - C_{\vec{n}+\vec{n_j}}(k) + (\sigma_i - \sigma_j)C_{\vec{n}}(k) = 0 \tag{3}$$

for all $i, j \in \{1, \cdots, r\}$.

Let us introduce the Heisenberg-Weyl algebra $W(r)$ associated to harmonic oscillators in r-dimensions. H is generated by the annihilation and

creation operators a_i and a_i^+ (resp.), $i = 1, \cdots, r$, that satisfy the commutation relations

$$[a_i, a_j] = [a_i^+, a_j^+] = 0, \quad [a_i, a_j^+] = \delta_{ij}, \quad i, j = 1, \cdots, r. \tag{4}$$

Denote by $|n_1, \cdots, n_r\rangle = |n_1\rangle \cdots |n_r\rangle$ the normalized simultaneous eigenstates of the r number operators $N_i = a_i^+ a_i$:

$$a_i^+ a_i |n_1, \cdots, n_i, \cdots, n_r\rangle = n_i |n_1, \cdots, n_i, \cdots, n_r\rangle, \tag{5}$$
$$n_i \in \mathbb{N}, \quad i = 1, \cdots, r,$$
$$\langle m_1, \cdots, m_r | n_1, \cdots, n_r\rangle = \delta_{m_1, n_1} \cdots \delta_{m_r, n_r}. \tag{6}$$

Remember that

$$[a_i^+ a_i, a_j] = -\delta_{ij} a_j, \quad [a_i^+ a_i, a_j^+] = \delta_{ij} a_j^+. \tag{7}$$

The algebra $W(r)$ is represented in this number state basis in the standard way:

$$a_i |n_1, \cdots, n_i, \cdots, n_r\rangle = \sqrt{n_i} |n_1, \cdots, n_i - 1, \cdots, n_r\rangle,$$
$$a_i^+ |n_1, \cdots, n_i, \cdots, n_r\rangle = \sqrt{n_i + 1} |n_1, \cdots, n_i + 1, \cdots, n_r\rangle. \tag{8}$$

Consider now the set of r Hamiltonians $H_i, i = 1, \cdots, r$, defined as follows:

$$H_i = \sum_{j=1}^{r} a_j^+ a_j + \sum_{j=1}^{r} \sigma_j a_j^+ + a_i + \sigma_i, \quad i = 1, \cdots, r. \tag{9}$$

It is straightforward to see that the multiple Charlier polynomials simultaneously diagonalize the r non-Hermitian oscillator Hamiltonians. To that end, form the states

$$|k\rangle\rangle = N_k^{(r)} \sum_{\vec{n}=0}^{\infty} \frac{C_{\vec{n}}(k)}{\sqrt{n_1! \cdots n_r!}} |n_1, \cdots, n_r\rangle, \quad k \in \mathbb{N}. \tag{10}$$

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Let us act on $|k\rangle\rangle$ with H_i:

$$
\begin{aligned}
H_i \left|k\right\rangle\rangle =& N_k^{(r)} \sum_{\vec{n}=0}^{\infty} \frac{C_{\vec{n}}(k)}{\sqrt{n_1! \cdots n_r!}} \\
& \cdot \left\{ (|\vec{n}| + \sigma_i) \left|n_1, \cdots, n_i, \cdots, n_r\right\rangle \right. \\
& + \sqrt{n_i} \left|n_1, \cdots, n_i - 1, \cdots, n_r\right\rangle \\
& + \left. \sum_{j=1}^{r} \sigma_j \sqrt{n_j + 1} \left|n_1, \cdots, n_i + 1, \cdots, n_r\right\rangle \right\} \\
=& N_k^{(r)} \sum_{\vec{n}=0}^{\infty} \frac{1}{\sqrt{n_1! \cdots n_r!}} \\
& \cdot \left\{ C_{\vec{n}+\vec{e}_i}(k) + (\sigma_i + |\vec{n}|) C_{\vec{n}}(k) + \sum_{j=1}^{r} n_j \sigma_j C_{\vec{n}-\vec{e}_j}(k) \right\},
\end{aligned}
\tag{11}
$$

Invoking the recurrence relations (2), we thus have indeed

$$
H_i \left|k\right\rangle\rangle = k H_i \left|k\right\rangle\rangle \tag{12}
$$

for all $i = 1, \cdots, r$. So although non-Hermitian, the operators H_i have a real spectrum given by the non-negative integers, the states $|k\rangle\rangle$ are uniquely defined (up to a constant factor) as the joint eigenstates of the operators $H_i, i = 1, \cdots, r$ with eigenvalues equal to k. Interestingly the Hamiltonians H_i do not commute pairwise. Indeed it is readily found that

$$
[H_i, H_j] = a_i - a_j + (\sigma_i - \sigma_j) \tag{13}
$$

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for all $i, j = 1. \cdots , r$. Remark however that

$$
\begin{aligned}
(a_i - a_j) \, |k\rangle\rangle = & N_k^{(r)} \sum_{\vec{n}=0}^{\infty} \frac{C_{\vec{n}}(k)}{\sqrt{n_1! \cdots n_r!}} \\
& \cdot \left\{ \sqrt{n_i} \, |n_1, \cdots , n_i - 1, \cdots , n_r\rangle \right. \\
& \left. - \sqrt{n_j} \, |n_1, \cdots , n_i - 1, \cdots , n_r\rangle \right\} \\
= & N_k^{(r)} \sum_{\vec{n}=0}^{\infty} \frac{1}{\sqrt{n_1! \cdots n_r!}} \\
& \cdot \left(C_{\vec{n}+\vec{e}_i}(k) - C_{\vec{n}+\vec{e}_j}(k) \right) |n_1, \cdots , n_r\rangle
\end{aligned}
\tag{14}
$$

Hence, by property (3),

$$
[H_i, H_j] \, |k\rangle\rangle = 0.
\tag{15}
$$

The operators H_i, thus on commute "on shell", thereby reconciling the fact that they do not commute and yet have common eigenvectors. As such the set of Hamiltonians H_i form a "weakly" integrable system. It is also useful to observe that the operators H_i can be obtained from the standard harmonic oscillator Hamiltonian in r dimensions

$$
H_0 = \sum_{j=1}^{r} a_j^+ a_j
\tag{16}
$$

by similarity transformations. Using the Baker-Campbell-Hausdorff formula

$$
e^A Y e^{-B} = A + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A[A, B]]] + \cdots ,
\tag{17}
$$

it is easily shown that

$$
H_i = S_i H_0 S_i^{-1}
\tag{18}
$$

with

$$
S_i = e^{a_i} \prod_{j=1}^{r} e^{-\sigma_j a_j^+}, \quad i = 1, \cdots , r.
\tag{19}
$$

This explains obviously why the spectrum of the H_i are real.

We shall now show how this framework can be used to derive properties of the multiple Charlier polynomials. We shall focus on the step relations

they obey and their explicit expression. Suppose we have operators X such that

$$[H_i, X] = -X \tag{20}$$

for all $i = 1, \cdots, r$. Then,

$$H_i \left(X \left| k \right\rangle\rangle \right) = (k-1) X \left| k \right\rangle\rangle, \tag{21}$$

that is, $X \left| k \right\rangle\rangle$ is a simultaneous eigenvectors of the H_i with eigenvalue $k - 1$ and is hence proportional to $\left| k - 1 \right\rangle\rangle$. Similarly, if there is an operator Y such that

$$[H_i, Y] = Y \tag{22}$$

for all $i = 1, \cdots, r$, the vector $Y \left| k \right\rangle\rangle$ will be proportional to $\left| k + 1 \right\rangle\rangle$. It is immediate to find these operators X and Y from the known lowering and raising operators of H_0, by exploiting the fact that the H_i's are related to H_0 by the similarity transformations (18)-(19). Let $X_0 = \sum_{i=1}^{r} \alpha_i a_i$, where α_i are some constants. It is clear that $[H_0, X_0] = -X_0$. To obtain Xs that will obey (20), one has to determine the constants α_i so that

$$S_1 X_0 S_1^{-1} = S_2 X_0 S_2^{-1} = \cdots = S_r X_0 S_r^{-1}. \tag{23}$$

In this case, it is found that (23) imposes no constraints on the α_is and hence the r operators

$$X_j = S_i a_j S_i^{-1} = a_j + \sigma_j \tag{24}$$

enjoy the property (20). They will be found to imply r step relations for the multiple Charlier polynomials. Similarly, using $Y_0 = \sum_{i=1}^{r} \beta_i a_i^+$ as the generic raising operator of H_0, the condition analogous to (24) is found to require that all the coefficients β_i be equal. Up to a trivial constant factor there is thus only one raising operator Y verifying (22) and it is given by

$$Y = S_i \left(\sum_{j=1}^{r} a_j^+ \right) S_i = \left(\sum_{j=1}^{r} a_j^+ \right) + 1. \tag{25}$$

In order to obtain step relations for the multiple Charlier polynomials from the action of $X_j, j = 1, \cdots, r$ and Y on both sides of (10), we need to know the precise action of these operators on $\left| k \right\rangle\rangle$. The key is to relate $\left| k \right\rangle\rangle$ to an eigenstate of H_0.

Let $\left| k \right\rangle^*$ be a state such that

$$H_0 \left| k \right\rangle^* = k \left| k \right\rangle^*. \tag{26}$$

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Take $S_1 |k\rangle^*$. Obviously,

$$H_1 S_1 |k\rangle^* = S_1 H_0 |k\rangle^* = k S_1 |k\rangle^* . \tag{27}$$

That is, $S_1 |k\rangle^*$ is an eigenstate of H_1 with eigenvalue k. If now,

$$S_i |k\rangle^* = \gamma_i S_1 |k\rangle^* , \quad i = 2, \cdots, r \tag{28}$$

for all $i \neq 1$, with γ_i some constants, because of (18), $S_1 |k\rangle^*$ will necessarily be a common eigenstate of all $H_i, i = 1, \cdots, r$, with eigenvalue k. Since this last property defines $|k\rangle\rangle$ (up to a constant factor), we could then posit

$$|k\rangle\rangle = S_1 |k\rangle^* . \tag{29}$$

Let us then identify the state $|k\rangle\rangle$ for which (29) will hold. It is determined by the conditions (26) and (28). The most general $|k\rangle^*$ satisfying (26) will be of the form

$$|k\rangle^* = M_k^{(r)} \sum_{l_2,\cdots,l_r} d_{l_2,\cdots,l_r} \left| k - |\vec{l}|, l_2, \cdots, l_r \right\rangle , \tag{30}$$

where $d_{l_2,\cdots,l_r}$ are some coefficients to be specified from (25) and $M_k^{(r)}$ is the normalization factor. The sum in (30) is performed over all $l_i \in \{0, \cdots, k\}$ such that $|\vec{l}| = (l_2 + \cdots + l_r) \leq k$.

Using the property $e^A e^B = e^{[A,B]} e^B e^A$ which is valid when the operators A and B commute with their commutator, one readily finds from (19) that

$$S_i = e^{\sigma_1 - \sigma_i} S_i e^{-a_1} e^{a_i} , \quad i = 2, \cdots, r. \tag{31}$$

If we choose the coefficients $\{d_{l_2,\cdots,l_r}\}$ so that

$$e^{-a_1} e^{-a_i} |k\rangle^* = |k\rangle^* \tag{32}$$

for all $i = 2, \cdots, r$, condition (28) will be satisfied with $\gamma_i = e^{\sigma_1 - \sigma_i}$. Now (32) is tantamount to demanding that

$$(a_1 - a_i) |k\rangle^* = 0, \quad i = 2, \cdots, r. \tag{33}$$

The $r - 1$ constraints (33) are straightforwardly found to imply the following $r - 1$ recurrence relations for the coefficients $\{d_{l_2,\cdots,l_r}\}$:

$$\sqrt{l_i} d_{l_2,\cdots l_i+1,\cdots,l_r} = \sqrt{k - |\vec{l}|} d_{l_2,\cdots l_i,\cdots,l_r} \tag{34}$$

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which are solved by

$$d_{l_2,\cdots,l_r} = \sqrt{\frac{k!}{(k-|\vec{l}|)!l_2!\cdots l_r!}} d_{0,\cdots,0}. \tag{35}$$

Hence

$$|k\rangle^* = \tilde{M}_k^{(r)} \sum_{l_2,\cdots,l_r} \sqrt{\frac{k!}{(k-|\vec{l}|)!l_2!\cdots l_r!}} \left|k-|\vec{l}|, l_2, \cdots, l_r\right\rangle, \tag{36}$$

where $\tilde{M}_k^{(r)} = M_k^{(r)} d_{0,\cdots,0}$. This normalization factor is readily determined from the condition that $|k\rangle^*$ be normalized: $^*\langle k|k\rangle^* = 1$. This yields

$$\tilde{M}_k^{(r)} = \frac{1}{r^{\frac{k}{2}}} \tag{37}$$

with the help of the multinomial formula. The action of the lowering and raising operators of $|k\rangle^*$ can now be directly computed. One finds that

$$a_i |k\rangle^* = \sqrt{\frac{k}{r}} |k-1\rangle^* \tag{38}$$

and also that

$$(a_1^+ + a_2^+ + \cdots + a_r^+) |k\rangle^* = \sqrt{r(k+1)} |k+1\rangle^*. \tag{39}$$

Although, we have privileged the use of S_1 in the preceding considerations, this choice must not matter and indeed the vector $|k\rangle^*$ that has been found is seen to symmetrically satisfy the constraints

$$(a_i - a_j) |k\rangle^* = 0 \tag{40}$$

for all pair $i, j = 1, \cdots, r$.

We are now ready to complete the derivations of the step relations. Using (24), (29) and (38), we can now write that

$$(a_j + \sigma_j) |k\rangle\rangle = |k-1\rangle\rangle. \tag{41}$$

Using (10), this yields

$$\sqrt{\frac{k}{r}} N_{k-1}^{(r)} C_{\vec{n}}(k-1) = N_k^{(r)}(C_{\vec{n}+\vec{e_j}}(k) + \sigma_j C_{\vec{n}}(k)). \tag{42}$$

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To proceed further we need to find $N_k^{(r)}$. In order to do that, set $\vec{n} = 0$ in (42) and use the recurrence relations (2) to observe that

$$C_{\vec{e_j}(k)} = k - \sigma_j. \tag{43}$$

We thus have

$$\frac{N_{k-1}^{(r)}}{N_k^{(r)}} = \sqrt{rk} \tag{44}$$

which implies

$$N_k^{(r)} = \frac{1}{\sqrt{k!}r^{\frac{k}{2}}} N_0 \tag{45}$$

with

$$N_0 = \langle 0, \cdots, 0 | S_1 | 0, \cdots, 0 \rangle = e^{-\sigma_1}. \tag{46}$$

Returning to (42), we finally have using (44), the following set of r "backward" step relations:

$$kC_{\vec{n}}(k-1) = C_{\vec{n}+\vec{e_j}}(k) + \sigma_j C_{\vec{n}}(k), \quad j = 1, \cdots, r. \tag{47}$$

Similarly, for the raising operator, we see from (25), (29) and (39) that we have

$$(a_1^+ + \cdots + a_r^+ + 1) |k\rangle\rangle = \sqrt{r(k+1)} |k+1\rangle\rangle. \tag{48}$$

Using again the coherent sum (10), with $N_k^{(r)}$ now given by (45), steps analogous to those that led to (47) bring one to find the "forward" relation that (48) entail:

$$C_{\vec{n}}(k+1) = C_{\vec{n}}(k) + \sum_{j=1}^{n} n_j C_{\vec{n}-\vec{e_j}}(k). \tag{49}$$

The relation (47) and (49) can obviously be combined to obtain a difference equation for the multiple polynomials $C_{\vec{n}}^k$.

The Hamiltonian H_0 is known to admit a $U(r)$ invariance algebra, generated by the bilinears $a_i^+ a_j, i, j = 1, \cdots, r$ that all commute with H_0. One might be curious to know what is their inference on the discrete integrable system on the lattice $\mathbb{N}^r$ for which the multiple Charlier polynomials provide solutions. As for the joint ladder operators X and Y, the common symmetries R of the H_i ($[H_i, R] = 0, i = 1, \cdots, r$.) are obtained from the symmetries R_0 of H_0 that have all their conjugates $S_i R_0 S_i^{-1}, i = 1, \cdots, r$ differ at most by a

constant (that does not matter). Resolving those conditions, it is found that the common symmetries must be of the form

$$
\begin{aligned}
R_{ij} = -R_{ji} &= S_1[(a_1^+ + \cdots + a_r^+)(a_i - a_j)]S_1^{-1} \\
&= (a_1^+ + \cdots + a_r^+ + 1)(a_i + \sigma_i - a_j - \sigma_j)
\end{aligned}
\tag{50}
$$

From (40), it is seen that they all annihilate $|k\rangle\rangle$ owing to the factor $(a_i - a_j)$ in their expression in the first line. Note that $S_1(a_1 - a_j)S_1^{-1} = (a_i + \sigma_i - a_j - \sigma_j)$.

Observe also that these joint constants of motion are in involution:

$$
[R_{ij}, R_{kl}] = 0.
\tag{51}
$$

By making explicit, the implication of

$$
R_{ij}\,|k\rangle\rangle = 0
\tag{52}
$$

when R_{ij} acts on the r.h.s. of (10), one obtains the relations

$$
\begin{aligned}
&\sum_{s\neq i} n_i C_{\vec{n}+\vec{e}_s-\vec{e}_i}(k) - \sum_{s\neq j} n_j C_{\vec{n}+\vec{e}_s-\vec{e}_j}(k) \\
&+ (n_i - n_j + \sigma_i - \sigma_j)C_{\vec{n}}(k) + C_{\vec{n}+\vec{e}_i}(k) - C_{\vec{n}+\vec{e}_j}(k) \\
&+ (\sigma_i - \sigma_j)\sum_{s=1}^{r} n_s C_{\vec{n}-\vec{e}_s}(k) = 0
\end{aligned}
\tag{53}
$$

which are in fact consequences of (3) when combined with the recurrence relations (2).

Our final considerations will bear on the explicit expression of the multiple Charlier polynomials and on the generating function. Given (29), that is $|k\rangle\rangle = S_1 |k\rangle^*$, and the fact that S_1 and $|k\rangle^*$ are known explicitly, it is quite clear that an expression for the multiple Charlier polynomials should follow

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from working out the action of S_1 on $|k\rangle^*$. Let us then proceed.

$$S_1 = e^{-\sigma_1} e^{-\sigma_1 a_1^+} e^{a_1} \cdot e^{-\sigma_2 a_2^+} \cdots e^{-\sigma_r a_r^+}, \tag{54}$$

$$S_1 \left| k - |\vec{l}|, l_2, \cdots, l_r \right\rangle$$
$$= e^{-\sigma_1} e^{-\sigma_1 a_1^+} e^{a_1} \left| k - |\vec{l}| \right\rangle \cdot \prod_{j=2}^{r} e^{-\sigma_j a_j^+} |l_j\rangle, \tag{55}$$

$$e^{-\sigma_j a_j^+} |l_j\rangle = \sum_{n_j=0}^{\infty} |n_j\rangle \langle n_j| e^{-\sigma_j a_j^+} |l_j\rangle$$
$$= \sum_{n_j=0}^{\infty} \frac{(-\sigma)^{n_j - l_j}}{(n_j - l_j)!} \sqrt{\frac{n_j!}{l_j!}} |n_j\rangle, \tag{56}$$

$$e^{-\sigma_1 a_1^+} e^{a_1} \left| k - |\vec{l}| \right\rangle = \sum_{n_1=0}^{\infty} |n_1\rangle \langle n_1| e^{-\sigma_1 a_1^+} e^{a_1} \left| k - |\vec{l}| \right\rangle. \tag{57}$$

The matrix elements occurring in (57) can be obtained following techniques used in [4] and [5]. If one sets

$$\psi_{n,k} = \langle k| e^{-\sigma_1 a_1^+} e^{a_1} |n\rangle, \tag{58}$$

from

$$\langle k| a_1^+ a_1 e^{-\sigma_1 a_1^+} e^{a_1} |n\rangle = \langle k| e^{-\sigma_1 a_1} e^{a_1} (a_1^+ a_1 - a_1 - \sigma_1 a_1^+ + \sigma_1) |n\rangle, \tag{59}$$

one readily finds that $\psi_{n,k}$ obey the recurrence relation

$$k\psi_{n,k} = (n + \sigma_1)\psi_{n,k} - \sqrt{n}\psi_{n-1,k} - \sigma_1\sqrt{n+1}\psi_{n+1,k}. \tag{60}$$

This implies that

$$\psi_{n,k} = \frac{1}{\sqrt{n!}}(-\sigma_1)^n p_n(k)\psi_{0,k}, \tag{61}$$

where $p_n(k)$ are polynomials of order n in k which are identified with the monic Charlier polynomials. The "initial" value $\psi_{0,k}$ is directly computed to be

$$\psi_{0,k} = \frac{(-\sigma_1)^k}{\sqrt{k!}}. \tag{62}$$

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This therefore implies that

$$e^{-\sigma_1 a_1^+} e^{a_1} \left| k - |\vec{l}| \right\rangle = \sum_{n_1=0}^{\infty} \frac{(-\sigma_1)^{n_1+|\vec{l}|-k}}{\sqrt{n_1!(k-|\vec{l}|)!}} p_{k-|\vec{l}|}(n_1) \left| n_1 \right\rangle. \tag{63}$$

One then uses (55), (56) and (63) to write down the action of S_1 on $|k\rangle^*$ as given by (36) and (37). First, one calls upon the following expression for the standard Charlier polynomials [11]

$$p_n(k) = (-\sigma_1)^n \, {}_2F_0\left(\begin{matrix} -n, -k \\ - \end{matrix} \middle| -\frac{1}{\sigma_1} \right)$$

$$= (-\sigma_1)^n \sum_{s=0}^{\infty} \frac{(-n)_s(-k)_s}{s!} \left(-\frac{1}{\sigma_1} \right)^s, \tag{64}$$

where we are using the Pochhammer symbol defined by

$$(a)_r = a(a+1)\cdots(a+r-1). \tag{65}$$

One then identifies the coefficients of $|n_1, \cdots, n_r\rangle$ on both sides of $|k\rangle\rangle = S_1 |k\rangle^*$ to find that

$$C_{\vec{n}}(k) = \sum_{s=0}^{n_1} \sum_{\{l_i\}} \frac{k!(|\vec{l}|-k)_r}{(k-|\vec{l}|)!} \cdot (-n_1)_r \cdot \frac{n_2!\cdots n_r!}{(n_2-l_2)!\cdots(n_r-l_r)!}$$

$$\cdot \frac{(-\sigma_1)^{n_1-s}}{s!} \cdot \frac{(-\sigma_2)^{l_2-s}}{l_2!} \cdots \frac{(-\sigma_r)^{l_r-s}}{l_r!}. \tag{66}$$

After some simplification, one finally arrives at the following explicit expression:

$$C_{\vec{n}}(k) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_r=0}^{n_r} (-n_1)_{l_1}(-n_2)_{l_2} \cdots (-n_r)_{l_r}(-k)_{l_1+\cdots+l_r}$$

$$\cdot \frac{(-\sigma_1)^{n_1-l_1}}{l_1!} \cdot \frac{(-\sigma_2)^{n_2-l_2}}{l_2!} \cdots \frac{(-\sigma_r)^{n_r-l_r}}{l_r!}. \tag{67}$$

This formula has been given in [7] for the case $r = 2$.

Last, a generating function can be obtained from (29), by introducing the Bargmann representation where the states $|n_1, n_2, \ldots, n_t\rangle$ and the oscillator operators $a_i, a_i^+, \; i = 1, 2, \ldots, r$ are realized by

$$|n_1, n_2, \ldots, n_r\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots n_r!}} z_1^{n_1} z_2^{n_2} \ldots z_r^{n_r}, \quad a_i = \partial_{z_i}, \; a_i^+ = z_i \tag{68}$$

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Relations (8) are obviously enforced in this model.

Recall from (28) and (29) that

$$e^{\sigma_i - \sigma_1} S_i |k\rangle^* = |k\rangle\rangle \tag{69}$$

for any $i = 1, 2, \ldots, r$. In the Bargmann representation

$$S_i = e^{-\sigma_1} e^{-(\sigma_1 z_1 + \cdots + \sigma_r z_r)} e^{\partial_{z_i}} \tag{70}$$

and

$$|k\rangle^* = \sqrt{\frac{k!}{r^k}} \sum_{l_1 + l_2 + \cdots + l_r = k} \frac{z^{l_1}}{l_1!} \cdots \frac{z^{l_r}}{l_r!} \tag{71}$$

Since

$$e^{\partial_{z_i}} f(z_1, z_2, \ldots, z_i, \ldots, z_r) = f(z_1, z_2, \ldots, z_i + 1, \ldots, z_r),$$

we have, combining (69) and (71)

$$\frac{e^{-\sigma_1} \sqrt{k!}}{r^{k/2}} e^{-(\sigma_1 z_1 + \cdots + \sigma_r z_r)} \sum_{l_1, \ldots l_r} \frac{z_1^{l_1} \cdots (z_i + 1)^{l_i} z_r^{l_r}}{l_1! \ldots l_r!} =$$
$$\frac{e^{-\sigma_1}}{\sqrt{k! r^k}} \sum_{n_1, \ldots n_r} \frac{C_{\vec{n}}(k)}{n_1! \ldots n_r!} z_1^{n_1} \ldots z_r^{n_r}. \tag{72}$$

This readily gives, the following generating function with the help of the multimonomial formula

$$e^{-(\sigma_1 z_1 + \cdots + \sigma_r z_r)} (z_1 + z_2 + \cdots + z_r + 1)^k =$$
$$\sum_{\vec{n}} C_{\vec{n}}(k) \frac{z_1^{n_1}}{n_1!} \cdots \frac{z_r^{n_r}}{n_r!} \tag{73}$$

Notice that, as should be, any operator S_i, $i = 1, \ldots, r$ can be used to derive the identity (73).

In summary, let us recall some of our findings. We have presented a remarkable set of non-Hermitian harmonic oscillator Hamiltonians in r-dimensions with real spectra. Their common eigenfunctions have been seen to be given in terms of multiple Charlier polynomials. Had we passed to a coordinate representation, we could have obtained relations involving products of Hermite polynomials and their translates [4, 5]. This physical settings has provided an algebraic model for the multiple Charlier polynomials that

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has been used to offer alternate demonstrations of some of their structural relations. It would be of interest in our opinion to pursue the algebraic interpretation of multiple polynomials. The multiple Meixner polynomials in particular, should lend themselves also to an oscillator modelization. We plan to return to this in a future publication.

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