THE QUILLEN METRIC, ANALYTIC TORSION AND TUNNELING FOR HIGH POWERS OF A HOLOMORPHIC LINE BUNDLE

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ABSTRACT. Let L be a line bundle over a compact complex manifold X (possibly non-Kähler) and denote by h_L and h_X fixed Hermitian metrics on L and TX, respectively. We generalize the asymptotics for the induced Quillen metric on the determinant line associated to a higher tensor power of L to the non-Kähler setting. In the case when L is ample we also obtain the leading asymptotics for the Ray-Singer analytic torsion of a (possbly non-positively curved) metric on L, without assuming h_X is Kähler. The key point of the proofs is to relate the asymptotics of the torsions above to "tunneling", i.e. to the distribution of the exponentially small eigenvalues of the correponding Dolbeault-Kodaira Laplacians. The proof thus avoids the use of the exact (i.e. non-asymptotic) deep results of Bismut-Gillet-Soulé for the Quillen metric, which are only known to hold under the assumption that h_X be Kähler. Accordingly the proofs are comparatively simple also in the Kähler case. A brief comparison with the tunneling effect for Witten Laplacians and large deviation principles for fermions is also made.

1. INTRODUCTION

1.1. **Setup.** Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X and let h_L be a smooth metric on the line bundle L with normalized curvature form ω (the most intersting case will be when ω is not semi-positive). The normalization is made so that ω is real and defines an integer cohomology class: $[\omega] \in H^2(X, \mathbb{Z})$. It will be convenient to use the weight notation for h_L, i.e. fixing a local trivialization of L we may locally write $\|s\|_{h_L}^2 := e^{-\phi}$ so that $\omega = dd^c\phi := \frac{i}{2\pi}\partial\bar{\partial}\phi$ where ϕ will be called a *weight* on L. We also fix a metric h_X on X and denote its volume form by dV. These metrics induce, in the standard way, Hermitian products on the space $\Omega^{0,q}(X, L)$ of smooth $(0, q)-$forms with values in L. The corresponding *analytic torsion,* introduced by Ray-Singer, is defined as

$$T(h_L, h_X) := \prod_{q=1}^{n} (\det \Delta_{\bar{\partial}}^{0,q})^{(-1)^{q+1}q}$$

where

$$\Delta_{\bar{\partial}}^{0,q} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

is the Dolbeault-Kodaira Laplacian acting on $\Omega^{0,q}(X, L)$ and $\log \det \Delta_{\bar{\partial}}^{0,1} := -\frac{\partial \zeta^{0,q}}{\partial s}_{s=0}$, where $\zeta^{0,q}(s) = \sum_i (\lambda_i^{0,q})^{-s}$ is the meromorphic continuation to $\mathbb{C}_s$ of the zeta function for the *strictly* positive eigenvalues $\{\lambda_i^{0,q}\}$ of $\Delta_{\bar{\partial}}^{0,q}$ (see [5, 17] and references

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therein). In the particular case of a Riemann surface

$$T(h_L,h_X):=(\det\Delta_{\bar{\partial}}^{0,1})=(\det\Delta_{\bar{\partial}}^{0,0})$$

Given a positive number λ we will also consider a "truncated" version $T_{]0,\lambda[}(h,\omega_0)$ of the analytic torsion $T(h_L,h_X)$ obtained by replacing the regularized determinants with the product of all positive eigenvalues strictly smaller than λ. Using functional calculus we may hence write

$$\log T_{]0,\lambda[}(h,\omega_0):=-\sum_{q=1}^{n}q(-1)^q\log\det(1_{]0,\lambda[}(\Delta_{\bar{\partial}}^{0,q})$$

We will be conserned with the limit when L is replaced by a large tensor power, written in our additive notation as kL. In this context we will say that a sequnce λ_k is *exponentially small,* if

$$\liminf_{k\to\infty}(\log\lambda_k)/k<0.$$

The asymptotics will be expressed in terms of the following bi-functional on the space of all weights on L :

$$\mathcal{E}(\phi,\phi'):=\frac{1}{(n+1)!}\int_X\sum_{j=0}^{n}(dd^c\phi)^{n-j}\wedge(dd^c\phi')^j\left(=\tilde{ch}(h,h'))\right)$$

where $\tilde{ch}(h,h')$ is (up to a multiplicative constant) the Bott-Chern class attached to the first Chern class [17].

Given a holomorphic line bundle L over X the corresponding *determinant line* $DET(L)$ is the one-dimensional complex vector space defined as the following tensor product of $\bar{\partial}-$cohomology groups:

$$\mathrm{DET}(L):=\bigotimes_{q=1}^{n}\det(H^q(X,L))^{(-1)^q},$$

where $\det(W)$ denotes the top exterior power of a complex vector space W. Given Hermitian metrics $h_L(=e^{-\phi_L})$ and h_X on L and X, respectively, the corresponding *Quillen metric* on $\mathrm{DET}(kL)$ that we will denote by $\mathcal{Q}(h_L^{\otimes k},h_X)$ is defined as the L^2-metric multiplied by the analytic torsion, where the L^2-metric is the one induced from the isomorphisms between the vector spaces $H^q(X,L)$ and the kernel of $\Delta_{\bar{\partial}}^{0,q}$.

1.2. **Statement of the main results.**

Theorem 1.1. *Let $L\to X$ be a line bundle holomorphic line bundle over a compact complex manifold (not necesseraly Kähler). Fix an Hermitian metric h_X on X and two metrics h_L and h'_L on L. Then the following asymptotic anomaly formula for the corresponding Quillen metrics on the determinant lines $DET(kL)$ holds*

$$\frac{1}{k^{n+1}}\log(\frac{\mathcal{Q}(h_L^{\otimes k},h_X)}{\mathcal{Q}(h_{L'}^{\otimes k},h_X))})\to\tilde{ch}(h_L,h'_L)$$

as $k\to\infty$.

In the case when h_X is a Kähler metric the previous theorem is a direct consequence of the *exact* anomaly formula for the Quillen metric of Bismut-Gillet-Soulé for the determinant line applied to any fixed tensor power of L (see [7], Theorem 0.3

stated in part I). However, the exact formula is not known in the non-Kähler case. As explained below the key point of the proof is to first obtain the asymptotics for the "truncated" Quillen metric obtained by replacing the analytic torsions with suitable truncations (compare Theorem 1.2 below). It should be pointed out that many applications of the exact formula of Bismut-Gillet-Soulé, notably in arithmetic (Arakelov) geometry [17], only rely on its asymptotic version. Hence, the present proof should be of interest also in the Kähler setting as it is comparitively simple.

By standard arguments the previous formula can be used to obtain an asymptotic version of Theorem 0.1 in [7] in the case when $\pi : X \to S$ is a submersion of a base S and a pair (h_L, h_X) as above is fixed. Then the curvature of the Quillen metric on $\text{DET}(kL)$, seen as a line bundle over S, converges when divided by k^{n+1} to the push-forward to S of the top exteriour power of the curvature form of h_L.

In the case when L is an ample line bundle and ϕ is a given smooth weight on L we will write $P\phi$ for the semi-positively curved weight defined as the following upper envelope:

$$P\phi = \sup \{\psi : \psi \leq \phi\}$$

where ψ ranges over all smooth weights on L with positive curvature. Then the functional $\mathcal{E}(P\phi, \phi)$ is still well-defined by basic pluripotential theory, only using that $P\phi$ is semi-positively curved and locally bounded (see [5] and references therein). Alternatively, by the regularity result in [3] $P\phi$ is locally $\mathcal{C}^{1.1}-$smooth and hence the current $dd^c(P\phi)$ has locally bounded coefficents and exists point-wise almost everywhere on X. As a consequence $\mathcal{E}(P\phi, \phi)$ can be defined as a standard Lesbegue integral over X.

Theorem 1.2. *Let $L \to X$ be an ample line bundle equipped with a smooth Hermitian metric h_L and let h_X be a fixed smooth Hermitian metric on X (which is not assumed to be Kähler). Let λ_k be a sequence such that $\lambda_k = o(k)$ and such that λ_k is not exponentially small. Then the large k limits of the corresponding analytic torsions $\frac{1}{k^{n+1}} \log T(h_L^{\otimes k}, h_X)$ and their "trunctations" $\frac{1}{k^{n+1}} \log T_{]0,\lambda_k[}(h_L^{\otimes k}, h_X)$ exist and both coincide with $\mathcal{E}(P\phi_L, \phi_L)$*

In fact, it will be clear that the only contibution to the truncated analytic torsions in the previous theorem comes from the exponentially small eigenvalues (in an appropriate sense). In the case when h_X is a Kähler metric the asymptotics of the analytic torsion in the previous theorem were deduced in [5] from the exact anomaly formula in [7] for the Quillen metric (refered to above) combined with the asymptotics for the L^2-part of the Quillen metric proved in [5].

As a consequence of the previous theorem we get the following geometric criterium for the existence of exponentially small eigenvalues of $\bar{\partial}-$Laplacians:

Corollary 1.3. *Let L be a line bundle and h_L and h_X metrics as in the previous theorem. Suppose that*

$$\mathcal{E}(P\phi_L, \phi_L) > 0$$

Then, for some q, there is a sequence of exponentially small eigenvalues of the corresponding $\bar{\partial}-$Laplacian acting on $(0, q)-$forms with values in kL. In the particular case of a line bundle of positive degree over a Riemann surface equipped with a metric h_L whose curvature is not semi-positive everywhere, the condition above is *always satisifed. More precisely,*

$$\frac{1}{k^2}\log\sum_i \lambda_i^{(k)} \to -\frac{1}{2}\left\| d(P\phi-\phi)\right\|_X^2 \tag{1.1}$$

where $(\lambda_i^{(k)})$ *are the "small" positive eigenvalues of the* $\bar{\partial}-$*Laplacian on the space of smooth sections with values in* kL *(i.e the eigenvalues are in* $]0,k^{1-\epsilon}[$*)*

Before continuing we briefly sketch the proofs of the theorems stated above. The starting pointing point is the observation that differential $\frac{1}{k^{n+1}}d_\phi \log \mathcal{Q}(e^{-k\phi},,h_X)$ of the normalized logarithm of the "truncated" Quillen metric on $\text{DET}(kL)$, seen as a functional on the space of all Hermtian metrics (or rather weights) on L, is represented by the alternating sum of the one-point measures associated to the spaces of "low-energy" $(0,q)-$forms. Using the large k asymptotics of the latter measures obtained in [2] gives

$$\frac{1}{k^{n+1}}d_\phi \log \mathcal{Q}(e^{-k\phi},,h_X) \to (dd^c\phi)^n/n!$$

and integrating between the line segement connecting ϕ_0 and ϕ_1 in the space of all weigts on L then concludes the proof of Theorem 1.1 for the "truncated" Quillen metrics. Adapting arguments in [8] to the present setting we also show that the analytic torsion coincides with a suitable truncation to the leaing order, concluding the proof of Theorem 1.1. Theorem 1.2 is then obtained by using the asymptotics for the $H^0(X,kL)-$part of the Quillen metrics for a general smooth (but possible non-positively curved) metric in [5] (or alternatively the asymptotics for the corresponding one-point (Bergman) measures in [3]).

1.3. **Comparison with the tunneling effect for Witten Laplacians.** It may be illluminating to compare the asymptotics of the truncated analytic torsions in Theorem 1.2 with the well-known results concerning the tunneling effect for Witten's deformation of the De Rham complex, which appeared in Witten's heuristic approach to Morse theory based on supersymmetric quantum mechanics [18] (see [13] for rigorous results based on semi-classical analysis). Geometrically this latter setting corresponds to letting $L \to X$ be the *trivial* line bundle over a *real* manifold X and replacing the operator $\bar{\partial}$ with the exterior derivative d. Any given global function ϕ on X induces a Hermitian metrics h_{kL} on kL represented as $h_{kL}=e^{-k\phi}$ in the standard global trivialization (i.e. $s=1$) of L, where now k makes sense for any positive number and where the semi-classical parameter $\hbar := 1/k$ corresponds to Planck's constant in the quantum mechanical picture. To this setting one associates as before a Laplacian on $(0,q)-$forms, $\Delta_k^{(q)}$, depending on $k\phi$, which may be identified with the *Witten Laplacian* (in a unitary frame). Assuming that ϕ is a Morse function we denote by $X(q)$ the finite set of all critical points of ϕ where the Hessian of ϕ has index q. As shown in [13] the set of all "small" eigenvalues $(\lambda_{i,k}^{(q)})$ of the Witten Laplacian $\Delta_k^{(q)}$ is in a one to one correspondence witht the finite set $X(q)$. Moreover, all the non-zero eigenvalues are actually *exponentially* small:

$$\lim_{k\to\infty}\frac{1}{k}\log\lambda_{i,q}^{(k)} = -c_{i,q}(\phi) \tag{1.2}$$

where the positive number $c_{i,q}(\phi)$ may be expressed in terms of an Agmond distance betweeen critical points. In the present complex geometric setting of the

$\bar{\partial}-$Laplacian the number of "small" eigenvalues $N_q^{(k)}$ depends on k and is of the order $O(k^n)$, more precisely:

$$N_q^{(k)} = k^n \int_{X(q)} (-1)^q (dd^c\phi)^n/n! + o(k^n)$$

where now $X(q)$ is the subset of X where the curvature form $dd^c\phi$ (i.e. the *complex* Hessian of ϕ) has index q (as shown by Demailly in his proof of his holomorphic Morse inequalities [11]). Hence the asymptotics of the truncated analytic torsions in Theorem 1.2 may be interpreted as an averaged version of 1.2. This analogy becomes particularly striking in the Riemannan surface case (compare the asymptotics 1.1).

The approach in [13] to study the asymptotics of the exponentially small eigenvalues, i.e. of the quantum mechanical *tunneling effect,* uses that the Witten Laplacians are *elliptic* when viewed as semi-classical elliptic differential operators. In the particular case when $q = 0$ there is also a probabilistic approach to the corresponding Witten Laplacian $\Delta_k^{(0)}$ which refines the information in 1.2 with asymptotics for pre-factors under certain genericity assumptions (see [9] and references therein for the case of Euclidean domains). The point is that the operator $\Delta_k^{(0)}$ is the Feller semigroup generator for the stochastic differential equation on the Riemannian manfiold X describing a Brownian particle in the gradient field of $-\phi$ at temperature $1/k$. The invariant probability measure of the process is

$$\mu_{k\phi} := \frac{e^{-k\phi}dV}{Z_{k\phi}} \tag{1.3}$$

and its relation to the exponentially small eigenvalues was studied in [15] in connection to simulated annealing. As explained in [9] each local minima of ϕ corresponds to a metastable state and the corresponding eigenvalue to the *inverse life time* of the state (i.e. the inverse mean exit time). See also the very recent work [16] for refinements of 1.2 for general degrees q.

There are also quantum mechanical, as well as probabilistic motivations for considering the present complex geometric setting. For example in the Riemann surface case the $\bar{\partial}-$Laplacian is the Pauli Hamiltonian for a single quantum particle with spin up ($q = 0$) and spin down ($q = 0$) subject to the magnet field with two-form $F_{kA} := kdd^c\phi$. Moreover, probabilistically the asymptotics of the corresponding truncated analytic torsions appear in the large deviation principle for the determinantal random point process on X with N_k particles defined by the following probability measure on X^{N_k} :

$$\mu_{k\phi}^{(N_k)} := \frac{|\det\Psi|^2(x_1, ...x_{N_k})e^{-k(\phi_1(x_1)+\cdots+\phi_1(x_N))}dV^{\otimes N_k}}{\mathcal{Z}_{k\phi}}$$

where N_k is the dimension of the space $H^0(X, kL)$ of global holomorphic sections of kL and $\det\Psi$ is any generator of the complex line $\det H^0(X, kL)$, identifying $\det\Psi$ with a holomorphic section over X^{N_k} [4]. Physically, this is the maximally filled fermionic many-particle ground state and $\det\Psi$ is the corresponding Slater determinant. Note that in the setting of the Witten Laplacian the role of $H^0(X, kL)$ is played by the *one*-dimensional kernel of d acting on the space smooth functions, inducing the *one* particle point-process 1.3.

The main analytical difference here is that the $\bar{\partial}-$Laplacian on $(0, q)-$forms with values in kL is *not* an elliptic operator, when viewed as a semi-classical differential operator (see [6]). There appears to be very few results concerning the tunneling effect for non-elliptic semi-classical operators (see however [14] and references therein for the study of Kramers Fokker-Planck type operators generalizing the Witten Laplacian for $q = 0$).

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2. Proofs of the main results

In the following we will fix the metric h_X on TX. Then any given weight ϕ (which we will eventually vary) induces Hermitian products $\langle \cdot, \cdot \rangle_\phi$ on the space $\Omega^{0,q}(X, L)$ using the metrics $(e^{-\phi}, h_X)$ on L and TX, respectively. Given ϕ we now denote the associated $\bar{\partial}-$Laplacians on $\Omega^{0,q}(X, L)$ by $\Delta^{0,q}$ (and sometimes by $\Delta^{0,q}_\phi$ to indicate the dependence on ϕ). We will denote by $\mathcal{H}^{0,q}_{]0,\lambda[}$ the subspace of $\Omega^{0,q}(X, L)$ spanned the eigenforms of $\Delta^{0,q}$ with eigenvalues in $]0, \lambda[$ (sometimes called the *space* of *low-energy* $(0, q)-$*forms*). To this space we associate its (non-normalized) one-point measure, i.e.

$$\mathbb{B}^{0,q}_{]0,\lambda[} := \sum_i |\Psi_i^{0,q}|^2_\phi dV, \tag{2.1}$$

where $(\Psi_i^{0,q})$ is any orthonormal base for $\mathcal{H}^{0,q}_{]0,\lambda[}$. We will also use the notation $\mathcal{H}^{0,q} := \ker \Delta^{0,q}$ and write $\mathbb{B}^{0,q}$ for the corresponding one-point measure.

2.1. **Variational formulae.** If $\mathcal{F}(\phi)$ is a (Gateaux differentiable) functional on the affine space of all weights on L we will write $d\mathcal{F}$ for its differential, which is a one-form on the space of all weights. We will identify the linear functional $d\mathcal{F}_{|\phi}$ on $\mathcal{C}^\infty(X)$ with a measure in the usual way. Conretely, this means that

$$\frac{d}{dt}\mathcal{F}(\phi_t) = \int_X (d\mathcal{F}_{|\phi_t})\frac{d\phi_t}{dt} \tag{2.2}$$

Given a pair of weights ϕ and ϕ' on L we now let

$$\mathcal{L}^{0,q}(\phi, \phi') := -\log\det(\left\langle \Psi_i^{0,q}, \Psi_j^{0,q} \right\rangle_\phi)$$

where $(\Psi_i^{0,q})_i$ is a base in $\ker \Delta^{0,q}_{\phi'}$ which is orthonormal wrt $\langle \cdot, \cdot \rangle_{\phi'}$. We then have the following basic

Lemma 2.1. *Fix a weight ϕ' on L and write $\mathcal{L}^{0,q}(\phi) := \mathcal{L}^{0,q}(\phi, \phi')$ and $\mathcal{E}(\phi) := \mathcal{E}(\phi, \phi')$. Then*

$$d\mathcal{E}_{|\phi} = \frac{(dd^c\phi)^n}{n!}, \quad d\mathcal{L}^{0,q}{}_{|\phi} = \mathbb{B}^{0,q}$$

Proof. The first point was first shown by Mabuchi (see also [5] and references therein). As for the second point it holds in a general setting where Π_ϕ is the orthogonal projection (wrt the Hermitian product $\langle \cdot, \cdot \rangle_\phi$) on a given $N-$dimensional subspace of $\mathcal{C}^\infty(X, L \otimes E)$ where E is a given Hermitian complex vector bundle (here $E = \Lambda^{0,q}(X, h_X)$). Indeed, we can then compute the time derivative of the corresponding functional $\mathcal{L}(\phi_t, \phi_{\prime})$ using the basic formula

(2.3) $$\frac{d}{dt}_{t=0}\log\det(H(t)) = \operatorname{Tr}(H(0)^{-1}\frac{d}{dt}_{t=0}H(t))$$

By the cocycle property $\mathcal{L}(\phi_0,\phi_1)+\mathcal{L}(\phi_1,\phi_2)+\mathcal{L}(\phi_2,\phi_0)=0$ we have that $\frac{d}{dt}\mathcal{L}(\phi_t,\phi_{'})$ is independant of ϕ' and hence we can set $\phi'=\phi_0$ above so that $H(0)=I$, which gives the desired formula. □

We will also have great use for the following lemma, whose proof is inspired by some arguments in [17]:

Lemma 2.2. *The differential of the functional $\phi\mapsto\tau_\lambda(\phi):=\log T_{\leq\lambda}(e^{-\phi},h_X)$ at ϕ is given by*

$$d\tau_{|\phi} = \sum_{q=1}^{n}(-1)^q\mathbb{B}^{0,q}_{]0,\lambda[}$$

for a generic number λ.

Proof. In the proof we will repeatedly use that $\bar\partial$ commutes with $\Delta(:=\Delta_{\bar\partial})$ and hence if $\Psi^{0,q}$ is an eigenform of $\Delta^{0,q}$ then $\bar\partial\Psi^{0,q}$ is an eigenform of $\Delta^{0,q+1}$ unless it vanishes identically. To fix ideas we start with

The case $n=1$:

Since, $\Delta=\bar\partial^*\bar\partial$ on $\Omega^{0,q}(X,L)$ it follows immediately from the definition of the determinant that

$$\det(1_{]0,\lambda[}(\Delta^{(0)}_{\bar\partial}) = \det(\langle\Psi_i,\Psi_j\rangle)^{-1}\det(\langle\bar\partial\Psi_i,\bar\partial\Psi_j\rangle)$$

for any given base in $\mathcal{H}^{0,q}_{]0,\lambda[}$ (with $q=0$). Next we note that, given the path ϕ_t, we may find a path $(\Psi_i^{0,q})(t)$ of bases in $\mathcal{H}^{0,q}_{]0,\lambda[}(t)$ such that $(\Psi_i^{0,q})(t)$ is orthonormal wrt ϕ_0 for $t=0$ and for any fixed index i $(\Psi_i^{0,q})(0)$ is an eigenform for Δ_{ϕ_0} and

(2.4) $$\left\langle\frac{d}{dt}_{t=0}(\Psi_i^{0,q})(t),\mathcal{H}^{0,q}_{]0,\lambda[}(0)\right\rangle_{\phi_0}=0$$

To see this we first recall that, as a well-known consequence of the ellipticity of Laplacians, for a generic λ the family $\mathcal{H}^{0,q}_{[0,\lambda[}(t)$ defines a vector bundle over $\{t\}:=]-\epsilon,\epsilon[$ (see Prop 1 on p. 123 in [17]). Moreover, since, for any t, $\ker\Delta^{0,q}_{\bar\partial}(t)$ is isomorphic to the Dolbeault cohomology group $H^{0,q}_{\bar\partial}(X,L)$ it follows that $F:=\mathcal{H}^{0,q}_{]0,\lambda[}(t)$ is also a vector bundle. Hence, we can start with an arbitary smooth curve $(s_i)(t)$ of bases in F_t satisfying the first requirements above at $t=0$. Then, for t sufficently small, $(\Psi_i^{0,q})(t):=s_i(t)-t\Pi_t(\frac{d}{dt}_{t=0}(s_i)(t))$ has the desired properties, since $\Pi_0(\frac{d}{dt}_{t=0}(\Psi_i^{0,q})(t))=0$, where Π_t denotes the orthogonal projection onto $\mathcal{H}^{0,q}_{]0,\lambda[}(t)$.

Now we can decompose

(2.5)
$$\frac{d}{dt}_{t=0}\log\det(1_{]0,\lambda[}(\Delta^{(0)}_{\bar\partial}) = -\frac{d}{dt}_{t=0}\log\det(\langle\Psi_i(t),\Psi_j(t)\rangle+\frac{d}{dt}_{t=0}\log\det(\langle\bar\partial\Psi_i(t),\bar\partial\Psi_j(t)\rangle)$$

Using formula formula 2.3 and the fact that

$$\frac{d}{dt}_{t=0}(\langle\Psi_i(t),\Psi_j(t)\rangle_{\phi_t} = \left\langle-\frac{d\phi_t}{dt}\Psi_i(t),\Psi_j(t)\right\rangle+0$$

(by Leibniz rule and 2.4) the first term in 2.5 above gives (also using that $H(0)=I$)

$$-\frac{d}{dt}_{t=0}\log\det(\langle\Psi_i(t),\Psi_j(t)\rangle=\int_X \mathbb{B}^{0,0}_{]0,\lambda[}\frac{d\phi_t}{dt}$$

The second term is computed similarly, by using that $\alpha_i:=\bar\partial\Psi_i/\left\|\bar\partial\Psi_i\right\|_{\phi_0}$ is an orthonormal base in $\mathcal{H}^{0,1}_{]0,\lambda[}(t)$ for $t=0$ and that the relation 2.4 still holds with Ψ_i replaced by α_i (indeed, since $\bar\partial$ commutes with $\frac{d}{dt}$ and $\Delta_{\bar\partial}$ we get $\Pi^{0,q+1}_{]0,\lambda[}(\frac{d}{dt}\bar\partial\Psi)=\bar\partial\Pi^{0,q}_{]0,\lambda[}(\frac{d}{dt}\Psi)=0$).

General n

By Hodge theory we have a decomposition

$$\Omega^0(X,L)=\ker\Delta_{\bar\partial}\oplus\mathrm{Im}\bar\partial\oplus\bar\partial^*$$

which is orthogonal wrt the corresponding Hermitian product. Restricting to $\mathcal{H}^{0,q}_{]0,\lambda[}$ (which by definition is in the orthogonal complement of $\ker\Delta_{\bar\partial}$ in $\Omega^0(X,L)$) gives an induced orthogonal decomposition

$$\mathcal{H}^{0,q}_{]0,\lambda[}=M_q\oplus N_q$$

Next, we note that $\bar\partial$ induces a bijection, intertwining the corresponding restriced Laplacians, such that

$$\bar\partial:\ \ N_q\to M_{q+1},\ \ \Delta_{N_q}=\bar\partial^*\bar\partial \tag{2.6}$$

and hence $\det(\Delta_{M_q})=\det(\Delta_{N_{q-1}})$ so that $\det(1_{]0,\lambda[}(\Delta^{(q)}_{\bar\partial})=\det(\Delta_{N_{q-1}})\det(\Delta_{N_q})$. The latter relation implies, since $\det(\Delta_{N_{-1}}):=1=\det(\Delta_{N_n})$ that

$$-\sum_{q=1}^n q(-1)^q\log\det(1_{]0,\lambda[}(\Delta^{(q)}_{\bar\partial})=\sum_{q=1}^n(-1)^q\log\det(\Delta_{N_q}) \tag{2.7}$$

Using the bijection 2.6 we can now repeat the arguments used above (when $n=1$) to deduce that

$$\frac{d}{dt}_{t=0}\log(\det(\Delta_{N_q})=\int_X(\mathbb{B}^{0,q}_{]0,\lambda[\cap N_q}-\sum_{q=1}^n\mathbb{B}^{0,q+1}_{]0,\lambda[\cap M_{q+1}})\frac{d\phi_t}{dt}$$

where the intersection with N_q indicates that we have replaced $\mathcal{H}^{0,q}_{]0,\lambda[}$ with its subspace N_q in the definition 2.1 (and similarly for M_{q+1}). Hence, taking the alternating sum over q and using 2.7 proves the lemma in general dimensions. □

2.2. **Asymptotics.** Next we recall the following asymptotics from [2] (which can be seen as a local version of Demailly's strong holomorphic Morse inequalities):

Proposition 2.3. *Let λ_k be a sequence of positive numbers such that $\lambda_k=O(k^{1-\epsilon})$ for some $\epsilon<1/2$. Then the sequence $k^{-n}\mathbb{B}^{0,q}_{[0,\lambda_k[}/dV$ is uniformly bounded and*

$$k^{-n}\mathbb{B}^{0,q}_{[0,\lambda_k[}\to(-1)^q1_{X(q)}(dd^c\phi)^n/n!$$

weakly as $k\to\infty$, where $X(q)$ is the subset of X where $dd^c\phi$ has exactly q negative eigenvalues. In particular,

$$\dim\mathcal{H}^{0,q}_{[0,\lambda_k[}=k^n\int_X(-1)^q1_{X(q)}(dd^c\phi)^n/n!+o(k^n).$$

Proof. As shown in [2] (Prop 5.1) the upper bound in the convergence above holds for any sequence $\lambda_k = k\mu_k$ with $\mu_k \to 0$. As for the lower bound it was was shown to hold as long as $\delta_k/\mu_k \to 0$ for δ_k a certain non-explicit sequence tending to zero (Prop 5.3). This latter sequence appears in Lemma 5.2 in [2] and the proof given there actually shows that δ_k can be taken as $\delta_k = 1/k^{1/2-\delta}$ for any $\delta > 0$ (since the radius R_k appearing in that proof is equal to $\log k$). A further refinement of this argument will be considered in Prop 3.1. □

The previous asymptotics will allow us to obtain the assumptotics of truncated analytic torsions (and hence of exponentially small eigenvalues). Using the next proposition we will then deduce the corresponding asymptotics for the usual analytic torsions.

Proposition 2.4. *The following asymptotics holds for any line bundle $L \to X$ and smooth Hermitian metrics on L and X :*

$$\log(\det(\Delta_k^{(q)}) = \log(\det(1_{]0,\lambda_k]}(\Delta_k^{(q)})) + O(k^{n+\epsilon})$$

if $\lambda_k = O(k^{1-\epsilon})$ for a given $\epsilon \in]0,1[$.

Proof. We will adapt to our setting some arguments from [8], where it was among other things shown that $\log(\det(\Delta_k^{(q)}) = O(k^n)\log k$ if the metric on L is positively curved (the point being that the smallest positive eigenvalue of $\Delta_k^{(q)}$ is then of the order k). As follows immediately from the definition (se below) the statement of the proposition to be proved is equivalent to

$$\zeta'_{1_{[\lambda_k,\infty[}(\Delta^{(q)})}(0) = O(k^{n+\epsilon}) \tag{2.8}$$

To prove the latter asymptotics we first recall some general facts about spectral zeta functions (following chapter V in [17]). If D is an Hermitian non-negative operator on a Hilber space then its spectral zeta function is defined by

$$\zeta_D(s) = \frac{1}{\Gamma(s)}\int_0^\infty (t^s(\mathrm{Tr}(e^{-tD}) - \Pi_D)\frac{dt}{t}$$

where Π_D denotes the orthogonal projection on the kernel of D (we assume that $\zeta_D(s)$ exists for s a complex number of sufficently negative real part and is then analytically contained to other values for s). In other words $\zeta_D(s)$ is the Mellin transform of the spectral theta function

$$\Theta_D(t) := (\mathrm{Tr}(e^{-tD}) - \Pi_D) = \sum_{\nu_i > 0} e^{-t\nu_i}$$

summing over the positive eigenvalues of D. Next we assume that

- $\Theta_D(t)$ converges for $t > 0$
- For every positive integer M there are real numbers $a_i = 0$ with $a_i = 0$ for $i < -n$ such that

$$\Theta_D(t) = \sum_{j=-n}^{j=M} a_j t^j + O(t^{M+1}) \tag{2.9}$$

uniformly when $t \to 0$.

Under these assumptions one obtains (see formula 11 on p. 99 in [17])

$$\zeta_D(0)=a_0, \quad \zeta_D'(0)=\int_1^\infty \Theta_D(t)\frac{dt}{t}+\left(\gamma a_0+\sum_{j<0}\frac{a_j}{j}+\int_0^1 \rho_0(t)\frac{dt}{t}\right) \tag{2.10}$$

where $\rho_0(t)$ is the $O(t^{M+1})$ term in 2.9 for $M=0$ and γ denotes Euler's constant. We also define a scaled version of ζ_D by setting $\widetilde{\zeta}_D := k^{-n}\zeta_{k^{-1}D}$ so that the derivative $\zeta_D'(0)$ at $s=0$ satisfies

$$k^{-n}\zeta_D'(0) = -\log k\widetilde{\zeta}_D(0)+\widetilde{\zeta}_D'(0) \tag{2.11}$$

(compare formula 40 in [8]).

We now come back to the present complex geometric setting. As shown in [8] (Thm 2), for any positive integer M, there is an asymptotic expansion

$$k^{-n}\mathrm{Tr}(e^{-tk^{-1}\Delta^{(q)}})\frac{dt}{t}=\sum_{j=-n}^{j=M} a_j^q(k)t^j+O(t^{M+1}) \tag{2.12}$$

when $t\to 0$ uniformly for $t\in[0,1]$ and moreover

$$a_j^q(k)=a_j^q+O(k^{-1/2}) \tag{2.13}$$

as $k\to\infty$. Next, we note that, for $t\in[0,1]$
(2.14)

$$k^{-n}\mathrm{Tr}(e^{-tk^{-1}\Delta^{(q)}})-k^{-n}\mathrm{Tr}(e^{-tk^{-1}1_{[k^{1-\epsilon},\infty[}(\Delta^{(q)})})=\int(-1)^q 1_{X(q)}(dd^c\phi)^n/n!+o(1)$$

uniformly when $k\to\infty$ (strictly speaking these asymptotics only hold when $\epsilon<1/2$ but in general the argument will show that the lhs above is uniformly bounded which is all that will be used to deduce the lemma). Indeed, the lhs above may be written and estimated from below and above as

$$e^{-tk^{-\epsilon}}k^{-n}\dim\mathcal{H}^{0,q}_{]0,\lambda_k[}\leq k^{-n}\sum_i e^{-tk^{-1}\nu_{i,k}}\leq k^{-n}\dim\mathcal{H}^{0,q}_{]0,\lambda_k[}$$

which combined with Prop 2.3 proves the previous formula. Now combining 2.12, 2.13 and 2.15 gives an expansion

$$k^{-n}\mathrm{Tr}(e^{-tk^{-1}1_{[k^{1-\epsilon},\infty[}(\Delta^{(q)})})=\sum_{j=-n}^{j=M} b_j^q t^j+O(t^{M+1})+o(1) \tag{2.15}$$

uniformly for $t\in[0,1]$ as $k\to\infty$ (where $b_j^q=a_j^q$ for $j\neq 0$). Accordingly, it follows from 2.10 that
(2.16)

$$\widetilde{\zeta}_{\Delta^{(q)}}(0)=-b_0^q(k)=-b_0^q+o(1),\ \ \widetilde{\zeta}_{1_{[\lambda_k,\infty[}(\Delta^{(q)})}(0)=-(b_0^q-\int_{X(q)}(-1)^q(dd^c\phi)^n/n!)+o(1)$$

Next, we apply the derivative formula in 2.10 to $D=k^{-1}1_{[k^{1-\epsilon},\infty[}(\Delta^{(q)}$ to get

$$\widetilde{\zeta}'_{1_{[k^{1-\epsilon},\infty[}(\Delta^{(q)})}(0)=\int_1^\infty k^{-n}\mathrm{Tr}(e^{-tk^{-1}1_{[k^{1-\epsilon},,\infty[}(\Delta^{(q)})})\frac{dt}{t}+O(1)$$

also using 2.15. From the basic estimate

$$k^{-n}\mathrm{Tr}(e^{-tk^{-1}1_{[\lambda_k,\infty[}(\Delta^{(q)})})\leq e^{-(t-1)k^{-1}\lambda_k}(k^{-n}\mathrm{Tr}(e^{-k^{-1}(\Delta^{(q)})}))$$

we deduce, since the second factor is uniformly bounded according to 2.12 applied to $t = 1$ and since we have assumed $\lambda_k = O(k^{1-\epsilon})$ the bound

$$\int_1^\infty k^{-n}\mathrm{Tr}(e^{-tk^{-1}1_{[k^{1-\epsilon},,\infty[}(\Delta^{(q)})})\frac{dt}{t} \leq C\int_1^\infty e^{-(t-1)k^{-\epsilon}}\frac{dt}{t} \leq C'k^\epsilon$$

All in all this means that $\widetilde{\zeta}'_{1_{[k^{1-\epsilon},\infty[}(\Delta^{(q)})}(0) = O(k^\epsilon)$ which combined with 2.11 and 2.16 proves 2.8 and hences finishes the proof of the proposition. □

2.3. **Proof of Theorem 1.1.** We start with a given ϕ and take a smooth path ϕ_t such that $\phi_1 = \phi$ and ϕ_0 has positive curvature. Letting

$$f_k(t) := k^{-(n+1)}(\log T_{]0,\lambda_k[}(e^{-k\phi_t}, h_X) + \sum_{q=0}^{n}(-1)^q\mathcal{L}^{0,q}(k\phi_t) \tag{2.17}$$

we have to prove that

$$\frac{1}{V}\lim_{k\to\infty}(f_k(1) - f_k(0)) = \mathcal{E}(\phi_1) - \mathcal{E}(\phi_0)(:= \mathcal{E}(\phi_1, \phi_0)) \tag{2.18}$$

To this end, note that combining Lemma 2.1 and Lemma 2.2 gives $\frac{df_k(t)}{dt} =$

$$= k^{-n}\int_X(\sum_{q=1}^{n}(-1)^q(\mathbb{B}^{0,q}_{\phi_t,]0,\lambda_k[} + \mathbb{B}^{0,1}_{\phi_t})\frac{d\phi_t}{dt} = k^{-n}\int_X(\sum_{q=1}^{n}(-1)^q\mathbb{B}^{0,q}_{\phi,[0,\lambda_k[})\frac{d\phi_t}{dt}$$

Using Prop 2.3 together with the dominated convergence theorem hence gives

$$\frac{1}{V}\lim_{k\to\infty}(f_k(1) - f_k(0)) = \lim_{k\to\infty}\int_0^1\frac{df_k(t)}{dt}dt = \frac{1}{Vn!}\int_0^1\int_X(dd^c\phi_t)^n\frac{d\phi_t}{dt}dt$$

According to the variational characterization of $\mathcal{E}$ in Lemma 2.1 (or by directly computing the integral) this finishes the proof of 2.18. Finally, by the asymptotics 2.4 this finishes the proof of the theorem.

2.4. **Proof of Theorem 1.2.** By the Kodaira vanishing theorem we have that $H^{0,q}(X, kL) = \{0\}$ for $k >> 1$ (since L is assumed ample) and hence $\mathcal{L}^{0,q}(k\phi_t) = 0$ for $q > 0$. By Theorem A in [5] $(\mathcal{L}(\phi_1) - \mathcal{L}_k(\phi_0)/k^{(n+1)}$ converges to $(\mathcal{E}(P\phi_1) - \mathcal{E}(P\phi_0)$ and hence the previous theorem (or rather its proof) gives

$$\lim_{k\to\infty}(\log T_{]0,\lambda_k[}(e^{-k\phi_1}, \omega_0) - \log T_{]0,\lambda_k[}(e^{-k\phi_0}, \omega_0)) = (\mathcal{E}(\phi_1) - \mathcal{E}(\phi_0)) - (\mathcal{E}(P\phi_1) - \mathcal{E}(P\phi_0))$$

Since, ϕ_0 is assumed positively curved we have on one hand that $P\phi_0 = \phi_0$ so that the terms involving ϕ_1 in the rhs above cancel. On the other hand, the smallest positive eigenvalue $\lambda_k^{0,q}$of $\Delta^{0,q}_{k\phi_0}$ satisfies $\lambda_k^{0,q} \geq Ck$ (see below) and hence the second term in the lhs above also vanishes. All in all this gives the convergence of the truncated analytic torsion in Theorem 1.2 and by Prop 2.4 the same asymptotics then hold for the analytic torsions.

As for the eigenvalue estimate used above it is a standard consequence of the Kodaira-Nakano identity [12] in the case when ω is Kähler. In the non-Kähler case it was shown in [8] (Theorem 1).

Remark 2.5. The eigenvalue estimate refered to above can also be deduced from the Kähler case as follows: first one notes that the first positive eigenvalue on the subspace $\ker\bar{\partial}$ of the $\bar{\partial}-$Laplacian associated to $(k\phi, h_X)$ is the supremum over all constants $C_k^{(q)}(h_X)$ such that for any f, a $\bar{\partial}-$closed $(0,1)-$form with values in L, the inhomogenous $\bar{\partial}-$equation $\bar{\partial}u = f$ for can be solved with an estimate

$$\|u\|^2_{(k\phi,h_X)} \leq \frac{1}{C_k^{(q)}(\omega)} \|f\|^2_{(k\phi,h_X)}$$

But since X is compact there is a constant A such that $A^{-1}\omega_0 \leq h_X \leq A\omega_0$ for ω_0 a fixed Kähler metric and hence replacing h_X with ω_0 only distorts $C_k^{(q)}(h_X)$ with a multiplicative constant independent of k. Finally, using the bijection 2.6 (and since $H^{0,q}(X, kL) = \{0\}$ for $k >> 1$) and the fact that $\bar{\partial}$ commutes with Δ this gives the desired lower bound for the first positive eigenvalue on all of $\Omega^{0,q}(X, kL)$ for all q.

2.5. **Proof of Cor 1.3.** Let $\lambda_k = k^{1-\epsilon}$ for some $\epsilon \in]1/2, 1[$.By Prop 2.3 the number of terms in the sum

$$k^{-(n+1)} \log \det(1_{]0,\lambda_k[}(\Delta_{\bar{\partial}}^{(q)}) = \frac{1}{k^n} \sum_i \frac{1}{k} \log \lambda_{i,k}^{(q)} \tag{2.19}$$

grows as constant times k^n. Hence, if there were no sequence of eigenvalues as in the statement of the corollary then the previous sum would converge to zero for any q contradicting the positivity assumption in the corollary (according to the previous theorem). As for the final statement we first note that, when $n = 1$:

$$\mathcal{E}(P\phi, \phi) = \frac{1}{2} \|d(P\phi - \phi)\|^2_X$$

which follows from integration by parts, using the "orthogonality relation" $\int (P\phi - \phi) dd^c(P\phi) = 0$ [5]. Hence $\mathcal{E}(P\phi, \phi) \geq 0$ with equality iff $P\phi = \phi + c$ for some constant c. But then it follows from the definition of $P\phi$ that $c = 0$, i.e. ϕ has semi-positive curvature.

3. Remarks on "small" vs. "exponentially small" eigenvalues

Fix a number $\epsilon \in]0, 1/2[$. For a given positive integer k we say that an eigenvalue ν_k of the $\bar{\partial}-$Laplacian $\Delta_{\bar{\partial}}$ associated to kL is "small" if $\nu_k \leq k^{1-\epsilon}$. By Prop 2.3 the number of "small" eigenvalues is of the order $O(k^n)$. Hence, as explained above the "small" eigenvalues which are not "exponentially small" make no contribution to the large k limit of the sum 2.19. But it is tempting to ask whether *all* "small" eigenvalues are actually "exponentially small"? As recalled in section 1.3 this is indeed the case in the setting of the Witten Laplacian under the assumption that ϕ be a Morse function. However in the present setting there are no non-degeneracy assumptions on the weight ϕ of the metric and hence it seems unlikely that the answer to the question above is yes, in general. Still when $q = 0$ the number of "small" and "exponentially" small eigenlvalues are the same to the leading order:

Proposition 3.1. *Given any $\delta > 0$ there is a constant C_δ such that the number $N_{\delta,k}$ of eigenvalues in $[0, e^{-C_\delta K}]$ of the $\bar{\partial}-$Laplacian $\Delta_{\bar{\partial}}$ acting on smooth sections of kL satisfies*

$$\int_{X(0)} (dd^c\phi)^n/n! - \delta \leq N_{\delta,k} \leq N_k \leq \int_{X(0)} (dd^c\phi)^n/n! + \delta$$

for k sufficently large, where N_k is the number of "small" eigenvalues.

Proof. Given $\delta > 0$ take K_δ a compact subset of the open set $X(0) := \{dd^c\phi > 0\}$ such that

$$\int_{X(0)} (dd^c\phi)^n/n! - \delta \leq \int_{K_\delta} (dd^c\phi)^n/n!$$

By Fatou's lemma it will be enough to prove that

$$\liminf_{k\to\infty} k^{-n}\mathbb{B}^{0,q}_{[0,e^{-C_\delta K}[}(x) \geq (dd^c\phi)^n(x)/n!$$

for any given $x \in K_\delta$. As shown in [2] (Prop 5.3) it is enough to, given a point $x \in K_\delta$, find a smooth section α_k of kL such that

$$(i)\ \liminf_{k\to\infty} k^{-n}\frac{(|\alpha_k|^2 e^{-k\phi})(x)}{\int_X |\alpha_k|^2 e^{-k\phi}dV} \geq (dd^c\phi)^n(x)/n!, \quad (ii)\ k^{-1}\left\langle \Delta_{\bar{\partial}}\alpha_k, \alpha_k\right\rangle_{k\phi} \leq \delta_k$$

for $\delta_k = e^{-2C_\delta k}$. To this end we simply take $\alpha_k = \chi s$ where χ is a smooth cut-off function which is equal to 1 close to x and s is local holomorphic frame for L on a neighbourhood x such that the corresponding local weight $\phi(z)$ is given by $\phi(z) = \sum_{i=1}^n \mu_i|z_i|^2 + O(|z|^3)$ (where $\mu_i > 0$ since $dd^c\phi > 0$ at x). Then (i) above follows from computing a Gaussian integral. Moreover,

$$\left\langle \Delta_{\bar{\partial}}\alpha_k, \alpha_k\right\rangle_{k\phi} = \left\langle \bar{\partial}\alpha_k, \bar{\partial}\alpha_k\right\rangle_{k\phi} = \int |(\partial\chi)|^2 e^{-k\phi}dV \leq e^{-2C_\delta k}$$

By the compactness of K_δ the constant C_δ can be taken to be independent of x. This finishes the proof of the lower bound on $N_{\delta,k}$ and the upper bound is a special case of Prop 2.3. □

It would be interesting to know if the analoge for $q > 0$ of the previous proposition is also valid? If one replaces e^{-kC_δ} with a sequence δ_k of the form $\delta_k = O(k^{-\infty})$, i.e. $\delta_k \leq C_m k^{-m}$ for any $m > 0$ then this is indeed case. This follows from taking α_k in the previous proof to be the given by $\Pi_k(x, \cdot)$ where $\Pi_k(x, y)$ is the local projector for $(0, q)-$forms constructed in [6], defining a local Fourier integral operator with a complex phase (in fact, only the phase function constructed in [6] is needed together with the leading term in the symbol expansion).

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