

The Polyanalytic Ginibre Ensembles

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Abstract. For integers $n, q = 1, 2, 3, \dots$, let $\text{Pol}_{n,q}$ denote the \mathbb{C} -linear space of polynomials in z and \bar{z} , of degree $\leq n - 1$ in z and of degree $\leq q - 1$ in \bar{z} . We supply $\text{Pol}_{n,q}$ with the inner product structure of

$$L^2(\mathbb{C}, e^{-m|z|^2} dA), \quad \text{where } dA(z) = \frac{dx dy}{\pi}, \quad z = x + iy;$$

the resulting Hilbert space is denoted by $\text{Pol}_{m,n,q}$. Here, it is assumed that m is a positive real. We let $K_{m,n,q}$ denote the reproducing kernel of $\text{Pol}_{m,n,q}$, and study the associated determinantal process, in the limit as $m, n \rightarrow +\infty$ while $n = m + O(1)$; the number q , the degree of polyanalyticity, is kept fixed. We call these processes polyanalytic Ginibre ensembles, because they generalize the Ginibre ensemble – the eigenvalue process of random (normal) matrices with Gaussian weight. A possible interpretation is that we permit a few higher Landau levels. We consider local blow-ups of the polyanalytic Ginibre ensembles around points in the *spectral droplet*, which is here the closed unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. We obtain asymptotics for the blow-up process, using a blow-up to characteristic distance $m^{-1/2}$; the typical distance is the same both for interior and for boundary points of \mathbb{D} . This amounts to obtaining the asymptotical behavior of the generating kernel $K_{m,n,q}$. Following [2], the asymptotics of the $K_{m,n,q}$ is rather conveniently expressed in terms of the Berezin measure (and density)

$$dB_{m,n,q}^{(z)}(w) := B_{m,n,q}^{(z)}(w) dA(w), \quad B_{m,n,q}^{(z)}(w) = \frac{|K_{m,n,q}(z, w)|^2}{K_{m,n,q}(z, z)} e^{-m|z|^2}.$$

For interior points $|z| < 1$, we obtain that $dB_{m,n,q}^{(z)}(w) \rightarrow d\delta_z$ in the weak-star sense, where δ_z denotes the unit point mass at z . Moreover, if we blow up to the scale of $m^{-1/2}$ around z , we get convergence to a measure which is Gaussian for $q = 1$, but exhibits more complicated Fresnel zone behavior for $q > 1$. In contrast, for exterior points $|z| > 1$, we have instead that $dB_{m,n,q}^{(z)}(w) \rightarrow d\omega(w, z, \mathbb{D}^e)$, where $d\omega(w, z, \mathbb{D}^e)$ is the harmonic measure at z with respect to the exterior disk $\mathbb{D}^e := \{w \in \mathbb{C} : |w| > 1\}$. For boundary points, $|z| = 1$, the Berezin measure $dB_{m,n,q}^{(z)}$ converges to the unit point mass at z , like for interior points, but the blow-up to the scale $m^{-1/2}$ exhibits quite different behavior at boundary points compared with interior points. The Fresnel-type pattern appears also for boundary points when $q > 1$, but then it is not rotationally symmetric.

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1. Introduction

Notation. We will use standard notation, such as ∂X and $\text{int}(X)$ for the boundary and the interior of a subset X of the complex plane \mathbb{C} . The complex conjugate of a complex number z is usually written as \bar{z} . We write \mathbb{R} for the real line, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk, and $\mathbb{D}^e := \{z \in \mathbb{C} : |z| > 1\}$ for the open exterior (punctured) disk. The characteristic function of a set E is written 1_E . We write

$$dA(z) = \pi^{-1} dx dy, \quad \text{where } z = x + iy \in \mathbb{C},$$

for the normalized area measure in \mathbb{C} , and use the standard Wirtinger derivatives

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y), \quad \text{where } z = x + iy.$$

We also write Δ for the (quarter) Laplacian

$$\Delta_z := \partial_z \bar{\partial}_z = \frac{1}{4}(\partial_x^2 + \partial_y^2).$$

Determinantal projection processes. Given a locally compact topological space X with a Radon measure μ , a determinantal projection process (in the sequel just determinantal process) is a random configuration of n points defined by the following probability measure on X^n :

$$(1.1) \quad dP(z_1, \dots, z_n) = \frac{1}{n!} \det[K_n(z_i, z_j)]_{i,j=1}^n d\mu(z_1) \cdots d\mu(z_n).$$

Here, K_n is the integral kernel of a projection operator to an n -dimensional subspace of $L^2(X, \mu)$. It is customary to identify all the permutations of the points and think the process as a random measure $\sum_{j=1}^n \delta_{z_j}$ on X .

A general definition of a determinantal process was introduced by Macchi [17], who wanted to model fermions in quantum mechanics. Indeed, for any determinantal process, the probability density vanishes whenever any two points in the n -tuple (z_1, \dots, z_n) coincide (fermions are forbidden to be in the same state). We interpret this as saying that the points in the n -tuple repel each other. Point processes of this kind appear in several contexts, e.g., in random matrix theory and combinatorics (for general surveys, see [15], [8]; we should also mention the books [18], [9], [10], [4]).

Eigenvalues of random normal matrix ensembles. Our main motivating example comes from the theory of random normal matrices. This topic has in recent years been subject to rather active investigation by physicists as well as by mathematicians. For an introduction, see, e.g., [24]. So, we shall use $X = \mathbb{C}$ and $d\mu(z) = e^{-mQ(z)} dA(z)$, for a positive weight function Q satisfying some mild regularity and growth conditions; m is a positive real parameter, and $dA(z) = \pi^{-1} dx dy$ is the normalized area measure. Let us write $L^2(\mathbb{C}, e^{-mQ}) := L^2(X, \mu)$ in this situation. The determinantal projection process is associated with an n -dimensional subspace of $L^2(\mathbb{C}, e^{-mQ})$, and we will use the space Pol_n of all polynomials in z of degree $\leq n-1$; we write $\text{Pol}_{m,n}$ to indicate that we have supplied Pol_n with the Hilbert space structure of $L^2(\mathbb{C}, e^{-mQ})$. The density of the process is then given by the reproducing kernel $K_{m,n}$ of the space $\text{Pol}_{m,n}$. So, we are talking about the probability measure

$$(1.2) \quad dP(z_1, \dots, z_n) = \frac{1}{n!} \det[K_{m,n}(z_i, z_j)]_{i,j=1}^n e^{-m(Q(z_1) + \dots + Q(z_n))} dA(z_1) \cdots dA(z_n).$$

In terms of the *correlation kernel*

$$(1.3) \quad \mathfrak{K}_{m,n}(z, w) := K_{m,n}(z, w) e^{-\frac{1}{2}m\{Q(z)+Q(w)\}},$$

which is the an integral kernel of an orthogonal projection $L^2(\mathbb{C})$, the expression (1.2) simplifies to

$$(1.4) \quad dP(z_1, \dots, z_n) = \frac{1}{n!} \det \left[\mathfrak{K}_{m,n}(z_i, z_j) \right]_{i,j=1}^n dA(z_1) \cdots dA(z_n).$$

The process described by (1.2) and (1.4) represents the eigenvalues of a random normal matrix picked from the distribution

$$(1.5) \quad \frac{1}{Z_{m,n}} e^{-m \operatorname{tr}[Q(M)]} \operatorname{dvol}_{\operatorname{nm}(n)}(M),$$

where $\operatorname{dvol}_{\operatorname{nm}(n)}(M)$ is the natural Riemannian volume form on the $n \times n$ normal matrices inherited from the metric of \mathbb{C}^{n^2} ; $Z_{m,n}$ is the normalization constant needed to make the total mass equal to 1. We are interested in the limiting behaviour of the process as $m, n \rightarrow +\infty$ while $n = m\tau + O(1)$ for some positive real number τ . Without loss of generality, we will consider only $\tau = 1$.

Local blow-up processes. Let \mathcal{N}_+ and $\mathcal{N}_{+,0}$ be the set of points defined by

$$\mathcal{N}_+ := \{w \in \mathbb{C} : \Delta Q(w) > 0\}, \quad \mathcal{N}_{+,0} := \{w \in \mathbb{C} : \Delta Q(w) \geq 0\}.$$

In the arXiv preprint [12], which will appear later in the expanded form [13], the function \widehat{Q} was defined as a certain envelope of Q , namely the largest subharmonic function in \mathbb{C} which is $\leq Q$ everywhere and has the growth bound

$$\widehat{Q}(z) = \log |z|^2 + O(1), \quad \text{as } |z| \rightarrow +\infty.$$

It is known that $\Delta \widehat{Q} = 1_{\mathcal{S}} \Delta Q$ for some compact set \mathcal{S} (see, e.g., [13]). We assume that \mathcal{S} is the minimal compact with this property, and call \mathcal{S} a *spectral droplet*. We then have $\mathcal{S} \subset \mathcal{N}_{+,0}$. The point process (1.2) has the following property: as $m, n \rightarrow +\infty$ while $n = m + O(1)$, the points will tend to accumulate on the set \mathcal{S} with density ΔQ there. Moreover, the set $\mathcal{S} \cap \mathcal{N}_+$ is rather regular for real-analytic Q , as the Sakai theory applies (cf. [14]). Typically we then expect a real-analytic boundary, with the exception of cusps and contact (or kissing) points. Let us refer to the set $\operatorname{int}(\mathcal{S} \cap \mathcal{N}_+)$ as *the bulk*. The results of [2], [3] show that for bulk points z , the local blow-up process at z , with coordinates (ξ_1, \dots, ξ_n) ,

$$\xi_j := m^{1/2} [\Delta Q(z)]^{1/2} (z_j - z),$$

where (z_1, \dots, z_n) are from the process (1.2), converges weakly to the translation invariant Ginibre(∞) process, as $m, n \rightarrow +\infty$ while $n = m + o(1)$. The associated generating kernel is the reproducing kernel $(\xi, \eta) \mapsto e^{\xi \bar{\eta}}$ of the Bargmann-Fock space. This has the flavor of a universality result. The corresponding statement in the Gaussian unitary ensemble (GUE) case is the universality of the sine kernel for bulk. We observe here that the sine kernel is the reproducing kernel for the Paley-Wiener space (a subspace of $L^2(\mathbb{R})$ consisting of entire functions). As for the two boundary points in the GUE model, the Tracy-Widom distribution appears, which is generated by the Airy kernel. The Airy kernel is reproducing for another Hilbert space of entire functions. This suggests that for real-analytic Q and $z \in \partial \mathcal{S} \cap \mathcal{N}_+$, there should exist a local blow-up

$$\xi_j := m^\theta (z_j - z),$$

where $\theta = \theta(z)$ is a suitable positive real, such that as $m, n \rightarrow +\infty$ while $n = m + o(1)$, the process (ξ_1, \dots, ξ_n) would converge to a determinantal process whose generating kernel is the reproducing kernel of a Hilbert space of entire functions. We verify this in the context of the Ginibre ensemble (i.e., with $Q(z) = |z|^2$), and identify the associated Hilbert space with a closed subspace of the Bargmann-Fock space characterized by slow growth in a half-plane. In that case, $\theta = \frac{1}{2}$ as in the case of interior points.

The Berezin measure and the Berezin density. In [2], [3], Ameur, Hedenmalm, and Makarov study the Berezin measure $dB_{m,n}^{(z)}$ and Berezin density $\mathbb{B}_{m,n}^{(z)}$:

$$dB_{m,n}^{(z)}(w) := \mathbb{B}_{m,n}^{(z)}(w)dA(w), \quad \mathbb{B}_{m,n}^{(z)}(w) := \frac{|K_{m,n}(z, w)|^2}{K_{m,n}(z, z)}e^{-mQ(w)},$$

which arise in the study of the Berezin transform. The Berezin measure is a probability measure, which makes it more stable than the reproducing kernel $K_{m,n}$ of $\text{Pol}_{m,n}$ itself as we let $m, n \rightarrow +\infty$. In terms of the point process (1.2), $\mathbb{B}_{m,n}^{(z)}$ measures the amount of repulsion from z caused by placing one of the points at z . For bulk points z , we have the convergence $dB_{m,n}^{(z)} \rightarrow d\delta_z$ in the weak-star sense of measures, as $m, n \rightarrow +\infty$ while $n = m + o(1)$. Here, $d\delta_z$ is the Dirac point mass at z . In fact, there is a better result: the blow-up Berezin density

$$\hat{\mathbb{B}}_{m,n}^{(z)}(\xi) = \frac{1}{m\Delta Q(z)} \mathbb{B}_{m,n}^{(z)}\left(z + \frac{\xi}{\sqrt{m\Delta Q(z)}}\right)$$

converges to the standard Gaussian $e^{-|\xi|^2}$. This corresponds to the convergence of the local blow-up of the point process to the Ginibre(∞) process (cf. [2], [3]). On the other hand, for points z outside the spectral droplet, i.e., for $z \in \mathbb{C} \setminus \mathcal{S}$, the Berezin measure $dB_{m,n}^{(z)}$ converges in the weak-star sense of measures to harmonic measure at z with respect to the exterior domain $\mathbb{C} \setminus \mathcal{S}$ as $m, n \rightarrow +\infty$ while $n = m + o(1)$ (see [2] for $Q = |z|^2$, and [3] for the general result).

The local blow-up of the point process and the Berezin density. It is convenient to think of the point process (1.2) in terms of the k -point intensities

$$\det \left[\mathcal{K}_{m,n}(z_i, z_j) \right]_{i,j=1}^k = \det \left[K_{m,n}(z_i, z_j) e^{-\frac{1}{2}m[Q(z_i)+Q(z_j)]} \right]_{i,j=1}^k.$$

We notice quickly that the intensities are unchanged if the kernel changed to

$$K_{m,n}^\chi(z, w) := \chi(z)\bar{\chi}(w)K_{m,n}(z, w),$$

provided that χ is measurable with $|\chi(z)| \equiv 1$ (we can call this a “gauge transformation”). This can help in the asymptotical analysis of local blow-ups. For $k = 2$, we get the 2-point intensity

$$\begin{aligned} & \left\{ K_{m,n}(z_1, z_1)K_{m,n}(z_2, z_2) - |K_{m,n}(z_1, z_2)|^2 \right\} e^{-mQ(z_1)-mQ(z_2)} \\ &= K_{m,n}(z_1, z_1) e^{-mQ(z_1)} \left\{ K_{m,n}(z_2, z_2) e^{-mQ(z_2)} - \frac{|K_{m,n}(z_1, z_2)|^2}{K(z_1, z_1)} e^{-mQ(z_2)} \right\}, \end{aligned}$$

where we recognize the Berezin density as a correction to the product of the two 1-point densities, where the 1-point intensity is

$$K_{m,n}(z_1, z_1) e^{-mQ(z_1)}.$$

So, as far as the 2-point intensity goes, we just need the 1-point intensity and the Berezin density. Since the 1-point intensity is just the restriction to $z_1 = z_2$ of the Berezin density, the Berezin density is all we need to describe the 2-point intensity. We will be a little lazy and just work with the Berezin density in the context of local blow-ups, although the asymptotics of the k -point intensity would strictly speaking require a little more work. So, although we state our many of our assertions regarding local blow-ups in terms of the Berezin density, we maintain that they generalize to statements about the point processes (cf. [3]).

The Ginibre ensemble and its polyanalytic generalization. The case $Q(z) = |z|^2$ of (1.5) (or (1.2)) is known as the *Ginibre ensemble*. The (probability generating or reproducing) kernel is now particularly simple:

$$K_{m,n}(z, w) = m \sum_{j=0}^{n-1} \frac{(mz\bar{w})^j}{j!}.$$

Here, $S = \bar{\mathbb{D}}$, the closed unit disk. We will consider a family of generalizations of the Ginibre ensemble, the *polyanalytic Ginibre ensembles*, which are defined by the reproducing kernels $K_{m,n,q}$ of the subspaces

$$\text{Pol}_{m,n,q} := \text{span} \{z^j \bar{z}^k : 0 \leq j \leq n-1, 0 \leq k \leq q-1\}$$

supplied with the Hilbert space structure of $L^2(\mathbb{C}, e^{-m|z|^2})$. The parameter value $q = 1$ corresponds to the standard Ginibre process. In general, we now project to the nq -dimensional subspace of the *polyanalytic polynomials*, where the degree in \bar{z} is $\leq q-1$, and the degree in z is $\leq n-1$. Note that the dimension of the subspace is now nq and not n . We can think of the case $q > 1$ as permitting more Landau levels than the lowest one (there are two similar models, see [5], [19], [25]; see also [1]). We will keep the polyanalyticity degree q fixed in the process, while we let both m, n tend to infinity. We remark here that we have noticed that Tomoyuki Shirai is interested in a similar model [22].

As for the point process, the points generally repel each other, but for $q > 1$, they also tend to avoid certain geometric configurations, such as circles and lines. We have run a simulation in Figure 1.1.

Results. Macroscopically, we find that the behavior of the polyanalytic Ginibre ensemble is similar to that of the Ginibre ensemble. If we measure this in terms of the Berezin measure, we have that as $m, n \rightarrow +\infty$ while $n = m + O(1)$,

$$dB_{m,n,q}^{(z)} \rightarrow d\delta_z \text{ for } z \in \bar{\mathbb{D}}, \quad dB_{m,n,q}^{(z)} \rightarrow d\omega_z \text{ for } z \in \mathbb{D}^e,$$

where ω_z is the harmonic measure of z with respect to the exterior disk \mathbb{D}^e . Here, $K_{m,n,q}$ is the reproducing kernel for $\text{Pol}_{m,n,q}$, and we use the notation

$$dB_{m,n,q}^{(z)}(w) := \mathbb{E}_{m,n,q}^{(z)}(w) dA(w), \quad \mathbb{E}_{m,n,q}^{(z)}(w) := \frac{|K_{m,n,q}(z, w)|^2}{K_{m,n,q}(z, z)} e^{-mQ(w)},$$

for the corresponding Berezin measure and Berezin density. Interestingly, the microscopic behavior of the Berezin measure $dB_{m,n,q}^{(z)}$ is quite different for $q > 1$ compared with the Ginibre case $q = 1$. In terms of the blow-up Berezin density ($\Delta Q(z) = \Delta|z|^2 \equiv 1$ here)

$$\hat{B}_{m,n,q}^{(z)}(\xi) = m^{-1} \mathbb{E}_{m,n,q}^{(z)}(z + m^{-1/2}\xi),$$

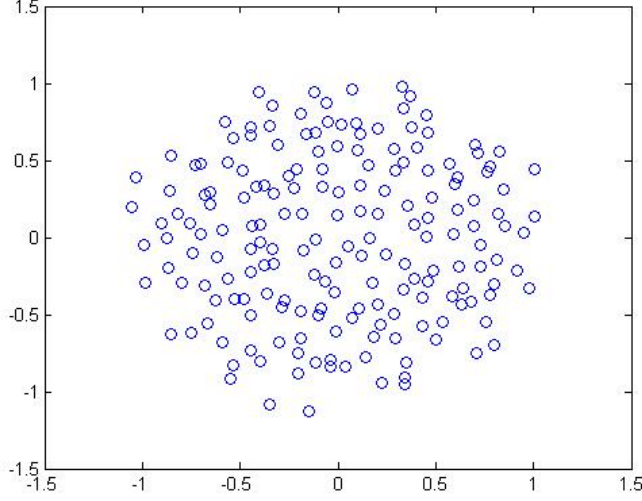


FIGURE 1.1. The polyanalytic Ginibre process with the kernel $K_{m,n,q}$ with $m = n = 61$ and $q = 3$. The simulation is based on the algorithm by Hough, Krishnapur, Peres, and Virág [15].

we have the following asymptotics as $m, n \rightarrow +\infty$ while $n = m + O(1)$:

$$\hat{B}_{m,n,q}^{(z)}(\xi) \longrightarrow q^{-1} L_{q-1}^1(|\xi|^2) e^{-|\xi|^2} \quad \text{for } z \in \mathbb{D},$$

where L_{q-1}^1 denotes the (generalized) Laguerre polynomial of degree $q - 1$ with parameter 1. It is well-known that the Laguerre polynomial L_{q-1}^1 has exactly $q - 1$ strictly positive roots, which implies that the Berezin density will exhibit a typical *Fresnel-type ring pattern*. This resembles what happens in the one-dimensional GUE case, where the zero density points for the Berezin density come from the zeros of the sine kernel. Actually, the analogy is more than a superficial similarity. If we consider rather big values of q , and scale down to local distance $(mq)^{-1/2}$, with

$$\tilde{B}_{m,n,q}^{(z)}(\xi') = (mq)^{-1} B_{m,n,q}^{(z)}\left(z + (mq)^{-1/2} \xi'\right),$$

then by the above we have, for $z \in \mathbb{D}$,

$$\hat{B}_{m,n,q}^{(z)}(\xi) \longrightarrow q^{-2} L_{q-1}^1(q^{-1} |\xi'|^2) e^{-q^{-1} |\xi'|^2},$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$. Next, if we let $q \rightarrow +\infty$, we get that

$$q^{-2} L_{q-1}^1(q^{-1} |\xi'|^2) e^{-q^{-1} |\xi'|^2} \longrightarrow \left\{ \sum_{i=0}^{+\infty} \frac{(-1)^i |\xi'|^{2i}}{i!(i+1)!} \right\}^2 = \frac{J_1(2|\xi'|)^2}{|\xi'|^2},$$

where J_1 is the standard Bessel function. The identity

$$\frac{J_1(2|\xi'|)}{|\xi'|} = \int_{\mathbb{D}} e^{2i \operatorname{Re}[\xi' \bar{\zeta}]} dA(\zeta)$$

shows that we are dealing with a planar analogue of the sine kernel (the sine kernel is the Fourier transform of the characteristic function of the interval $[-1, 1]$, the one-dimensional unit ball).

We also investigate the local behaviour of the Berezin transform when $|z| = 1$, i.e., when z is on the boundary of the bulk. Using the same blow-up scale as with an interior point, we show that the blow-up Berezin density $\hat{\mathbb{D}}_{m,n,q}^{(z)}(\xi)$ tends to a limit which can be expressed in terms of Hermite polynomials (see Theorem 5.10). Here, too, there is a ring-like pattern in the interior direction, but it is not so pronounced as it is for interior points (the bulk). We express the 1-point intensity near a boundary point in terms of a sum of squares of Hermite polynomials. The Wigner semi-circle law then gives the asymptotic behavior of the 1-point function, which tells us the intensity of the process. We find that for big q , but much bigger m, n with $m = n + O(1)$, the 1-point function is nontrivial in the annulus

$$1 - q^{1/2}m^{-1/2} \leq |z| \leq 1 + q^{1/2}m^{-1/2};$$

inside it is essentially constant, and outside it more or less vanishes. Near the outward boundary of the annulus at the scale $(mq)^{-1/2}$, we expect the statistics of the point process to be related with the well-known Airy process.

Lifting to two complex variables. Analogous results to [2] are obtained on complex manifolds by Berman [6]. We note that the polyanalytic Ginibre processes can also be viewed as processes in \mathbb{C}^2 with the rather singular weight $e^{-|z_1|^2} \delta_0(z_1 - \bar{z}_2)$, where δ_0 is unit point mass at 0. Berman considers reproducing kernels of polynomial subspaces as the total degree of the polynomials tends to infinity. In contrast, here we discuss the case where one variable has bounded degree and the degree of the other variable goes to infinity.

The polyanalytic Ginibre ensemble and weighted interpolation. It is well known in the theory of random matrices that

$$\frac{1}{n!} \det[K_{m,n,1}(z_i, z_j)]_{i,j=1}^n = \frac{1}{Z_{m,n,1}} |\Delta(z_1, \dots, z_n)|^2,$$

where $\Delta(z_1, \dots, z_n) = \prod_{i,j:i < j} (z_j - z_i)$ is the van der Monde determinant. A point configuration in a compact set which maximizes the van der Monde determinant is known to be a good choice of nodes for Lagrange interpolation [21]. Instead of considering points confined to a compact set, one can add a weight to prevent the points from going off to infinity. This leads to the same expression which arises in the context of random eigenvalues.

We turn to the polyanalytic case. One shows that

$$\frac{1}{(nq)!} \det[K_{m,n,q}(z_i, z_j)]_{i,j=1}^{nq} = \frac{1}{Z_{m,n,q}} |\Delta_q(z_1, \dots, z_{nq})|^2,$$

where $Z_{m,n,q}$ is a normalization constant

$$Z_{m,n,q} := \int_{\mathbb{C}^{nq}} |\Delta_q(z_1, \dots, z_{nq})|^2 e^{-m \sum_{j=1}^{nq} Q(z_j)} dA(z_1) \dots dA(z_{nq})$$

and

$$\Delta_q(z_1, \dots, z_{nq}) = \det \begin{pmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_{nq} \\ \vdots & \dots & \vdots \\ z_1^{n-1} & \dots & z_{nq}^{n-1} \\ \bar{z}_1 & \dots & \bar{z}_{nq} \\ \bar{z}_1 z_1 & \dots & \bar{z}_{nq} z_{nq} \\ \vdots & \dots & \vdots \\ \bar{z}_1^{q-1} & \dots & \bar{z}_{nq}^{q-1} \\ \vdots & \dots & \vdots \\ \bar{z}_1^{q-1} z_1^{n-1} & \dots & \bar{z}_{nq}^{q-1} z_{nq}^{n-1} \end{pmatrix}.$$

So, $\Delta_q(z_1, \dots, z_{nq})$ is the polyanalytic analogue of the van der Monde determinant. As in the case $q = 1$ (which gives the usual van der Monde determinant), the expression $|\Delta_q(z_1, \dots, z_{nq})|^2$ measures how good the configuration is for Lagrange interpolation by polyanalytic polynomials. So, the polyanalytic Ginibre ensemble is a way to produce random Lagrange interpolation sets, using the Gaussian weight for confinement.

Further results and open problems. In [20] and [3], the authors showed that the fluctuations of eigenvalues of random normal matrices tend to Gaussian free field. The fluctuations of the polyanalytic Ginibre process will be discussed in a separate paper – the limit is again the Gaussian free field, but the variance depends on the degree of polyanalyticity.

One could naturally address all the questions discussed in this article with a more general weight Q . We conjecture that the spectral droplet will be the same as in the analytic case. It is also likely that the blow-up of the Berezin density at a bulk point z will have a universal limit, which we here computed to be $q^{-1} L_{q-1}^{(1)}(|\xi|^2) e^{-|\xi|^2}$.

2. Polyanalytic Bargmann-Fock spaces

An orthogonal basis. We will consider the Bargmann-Fock space $A_{m,q}^2(\mathbb{C})$ of polyanalytic functions of degree $\leq q - 1$, i.e., functions of the form

$$f(z) = \sum_{r=0}^{q-1} \bar{z}^r f_r(z),$$

where all the components f_r are entire, subject to the norm integrability condition

$$\|f\|_{A_{m,q}^2(\mathbb{C})}^2 := \|f\|_{L^2(\mathbb{C}, e^{-m|z|^2})}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-m|z|^2} dA(z) < +\infty.$$

Here, as always, $m > 0$. We note that $A_{1,1}^2(\mathbb{C})$ is the standard Bargmann-Fock space. As before, let $\text{Pol}_{m,n,q}$ be the closed subspace of $A_{m,q}^2(\mathbb{C})$, defined by the condition that all the components f_r are polynomials of degree less or equal to $n - 1$. Moreover, let $K_{m,q}$ and $K_{m,n,q}$ denote the reproducing kernels for $A_{m,q}^2(\mathbb{C})$ and $\text{Pol}_{m,n,q}$, respectively. We will be concerned with the asymptotic behaviour of the kernel $K_{m,n,q}$, as $m, n \rightarrow +\infty$ while $n = m + O(1)$.

Bargmann-Fock-spaces of polyanalytic functions have been considered in, e.g., [25], where the reproducing kernels and orthonormal bases were identified. We will attempt to supply a self-contained account of these basic matters.

To begin with, we identify an orthonormal basis for the space $\text{Pol}_{m,n,q}$. Here, and later as well, we will need some standard properties of the classical orthogonal polynomials. For details, we refer the reader to [16]. We need the generalized Laguerre polynomials ($(x)_j := x(x+1)\cdots(x+j-1)$ is the Pochhammer symbol)

$$L_k^\alpha(x) := \sum_{i=0}^k (-1)^i \binom{k+\alpha}{k-i} \frac{x^i}{i!} = \sum_{i=0}^k (-1)^i \frac{(\alpha+i+1)_{k-i}}{i!(k-i)!} x^i.$$

Proposition 2.1. *For $q \leq n$, the following functions form an orthonormal basis for $\text{Pol}_{m,n,q}$ (i, r, j, k are all integer parameters):*

$$e_{i,r}^1(z) := \sqrt{\frac{r!}{(r+i)!}} m^{(i+1)/2} z^i L_r^i(m|z|^2), \quad 0 \leq i \leq n-r-1, \quad 0 \leq r \leq q-1,$$

$$e_{j,k}^2(z) := \sqrt{\frac{j!}{(j+k)!}} m^{(k+1)/2} z^k L_j^k(m|z|^2), \quad 0 \leq j \leq q-k-1, \quad 1 \leq k \leq q-1.$$

Proof. Clearly, all the above functions belong to the space $\text{Pol}_{m,n,q}$. Also, all the functions $e_{i,r}^1$ are orthogonal to each of the functions $e_{j,k}^2$ for the indicated ranges of the indices, as can be seen by integrating along circles using polar coordinates. Next, we show that the functions $e_{i,r}^1$ form an orthonormal set. Any two functions there having different parameter i are orthogonal, again by integrating along circles using polar coordinates. So, we fix i , and pick two indices r_1, r_2 . We compute the inner product of two such functions:

$$\begin{aligned} & \int_{\mathbb{C}} e_{i,r_1}^1(z) \overline{e_{i,r_2}^1(z)} e^{-m|z|^2} dA(z) \\ &= \sqrt{\frac{r_1!}{(r_1+i)!}} \sqrt{\frac{r_2!}{(r_2+i)!}} m^{i+1} \int_{\mathbb{C}} |z|^{2i} L_{r_1}^i(m|z|^2) L_{r_2}^i(m|z|^2) e^{-m|z|^2} dA(z) \\ &= \sqrt{\frac{r_1! r_2!}{(r_1+i)! (r_2+i)!}} \int_0^\infty L_{r_1}^i(t) L_{r_2}^i(t) t^i e^{-t} dt = \delta_{r_1, r_2}, \end{aligned}$$

where the delta is in Kronecker's sense. In a similar fashion, the functions $e_{j,k}^2$ form an orthonormal set. So, the functions $e_{i,r}^1, e_{j,k}^2$ together form an orthonormal set. Next, the dimension of the span equals the total number of vectors, which we calculate to nq , which equals the known dimension of the space $\text{Pol}_{m,n,q}$. The proof is complete. \square

The reproducing kernel. For $q \leq n$, we conclude that the reproducing kernel of $\text{Pol}_{m,n,q}$ equals

$$(2.1) \quad K_{m,n,q}(z, w) = m \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} (mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2) \\ + m \sum_{j=0}^{q-2} \sum_{k=1}^{q-j-1} \frac{j!}{(k+j)!} (m\bar{z}w)^k L_j^k(m|z|^2) L_j^k(m|w|^2).$$

We note that by plugging in $n = +\infty$ in Proposition 2.1 we get an orthonormal basis for the space $A_{m,q}^2(\mathbb{C})$. It follows that the same procedure of plugging in $n = +\infty$ in the above expression for $K_{m,n,q}$ gives us $K_{m,q}$, the reproducing kernel for $A_{m,q}^2(\mathbb{C})$. What is probably less obvious is that $K_{m,q}$ may be written in a much simpler form (this representation is, however, known; compare with [5], [19]).

Proposition 2.2. *We have that*

$$\begin{aligned} K_{m,q}(z, w) &= m \sum_{r=0}^{q-1} \sum_{i=0}^{\infty} \frac{r!}{(r+i)!} (mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2) \\ &\quad + m \sum_{j=0}^{q-2} \sum_{k=1}^{q-j-1} \frac{j!}{(j+k)!} (m\bar{z}w)^k L_j^k(m|z|^2) L_j^k(m|w|^2) = m L_{q-1}^1(m|z-w|^2) e^{mz\bar{w}}. \end{aligned}$$

Proof. It is clear from Proposition 2.1 that the double sum expression equals $K_{m,q}(z, w)$, so the only thing that needs attention is the last equality. We first do the case $w = 0$. Many of the terms in the double sum vanish, so that we are left with

$$K_{m,q}(z, 0) = m \sum_{r=0}^{q-1} L_r^0(m|z|^2) L_r^0(0),$$

It is well-known that $L_r^0(0) = 1$ and that $\sum_{r=0}^{q-1} L_r^\alpha = L_{q-1}^{\alpha+1}$, so that the above reduces to

$$K_{m,q}(z, 0) = m L_{q-1}^1(m|z|^2).$$

We turn to general $w \in \mathbb{C}$. It is well-known that the transformation

$$\mathbf{T}_w[f](z) := e^{-\frac{1}{2}m|w|^2 - mz\bar{w}} f(z+w)$$

acts unitarily on $A_{m,q}^2(\mathbb{C})$; its adjoint is $\mathbf{T}_w^* = \mathbf{T}_{-w}$. By the reproducing property of $K_{m,q}(z, 0)$, we have, for $f \in A_{m,q}^2(\mathbb{C})$,

$$\begin{aligned} e^{-\frac{1}{2}m|w|^2} f(w) &= \mathbf{T}_w[f](0) = \int_{\mathbb{C}} \mathbf{T}_w f(z) \overline{K_{m,q}(z, 0)} e^{-m|z|^2} dA(z) \\ &= \int_{\mathbb{C}} f(z) \overline{\mathbf{T}_w^*[K_{m,q}(\cdot, 0)](z)} e^{-m|z|^2} dA(z) \\ &= m e^{-\frac{1}{2}m|w|^2} \int_{\mathbb{C}} f(z) L_q^1(m|z-w|^2) e^{m\bar{z}w} e^{-m|z|^2} dA(z), \end{aligned}$$

The claim that $K_{m,q}(z, w) = m L_{q-1}^1(m|z-w|^2) e^{mz\bar{w}}$ now follows immediately. \square

Szegö asymptotics. We shall show that the kernel $K_{m,n,q}(z, w)$ is approximated well by $K_{m,q}(z, w)$, provided that $z, w \in \mathbb{D}$ are rather close to one another. It will be instrumental to study the asymptotic behavior of the partial sums

$$E_k(\zeta) = \sum_{j=0}^k \frac{1}{j!} \zeta^j$$

of the Taylor expansion of the exponential function. In [23], Szegö showed that

$$(2.2) \quad \frac{E_k(k\zeta)}{e^{k\zeta}} = 1 - \frac{1}{\sqrt{2\pi k}} (\zeta e^{1-\zeta})^k \frac{\zeta}{1-\zeta} (1 + \epsilon_k^1(\zeta)),$$

provided that $|\zeta e^{1-\zeta}| < 1$ and $|\zeta| < 1$, whereas

$$(2.3) \quad \frac{E_k(k\zeta)}{e^{k\zeta}} = \frac{1}{\sqrt{2\pi k}} (\zeta e^{1-\zeta})^k \frac{\zeta}{\zeta-1} (1 + \epsilon_k^2(\zeta)),$$

provided that $|\zeta e^{1-\zeta}| < 1$ and $|\zeta| > 1$. Here, we have the convergence

$$\lim_{k \rightarrow +\infty} \epsilon_k^1(z) = \lim_{k \rightarrow +\infty} \epsilon_k^2(z) = 0$$

uniformly on compact subsets of the respective domains. As for (2.3), the uniform convergence $\epsilon_k^2(\zeta) \rightarrow 0$ as $k \rightarrow +\infty$ holds also in certain unbounded subdomains; in particular, along the real line, we have uniform convergence on all intervals $[a, +\infty[$ with $a > 1$.

An elementary estimate of Laguerre polynomials. The following elementary estimate of generalized Laguerre polynomials will prove useful.

Lemma 2.3. *Suppose α is a positive real. Then, for $k = 0, 1, 2, \dots$, we have that*

$$|L_k^\alpha(x)| \leq \frac{1}{k!} (x + \alpha + k)^k, \quad x \in [0, +\infty[.$$

Moreover, with $\beta := \alpha + 2k - 2 + 2\sqrt{\frac{1}{4} + (k-1)(k+\alpha-1)}$, we have that

$$\frac{1}{k!} (x - \beta)^k \leq (-1)^k L_k^\alpha(x) \leq \frac{1}{k!} x^k, \quad x \in [\beta, +\infty[.$$

Actually, the related inequality $|L_k^\alpha(x)| \leq \frac{1}{k!} x^k$ holds for all $x \in [\frac{1}{2}\beta, +\infty[$.

Proof. We begin with the estimate ($0 \leq i \leq k$ is assumed)

$$(\alpha + i + 1)_{k-i} \leq (\alpha + k)^i.$$

Next, we note that for $x \geq 0$ and $k = 0, 1, 2, \dots$, we have

$$L_k^\alpha(x) = \sum_{i=0}^k (-1)^i \frac{(\alpha + i + 1)_{k-i}}{i!(k-i)!} x^i = \sum_{i \text{ even}} \frac{(\alpha + i + 1)_{k-i}}{i!(k-i)!} x^i - \sum_{i \text{ odd}} \frac{(\alpha + i + 1)_{k-i}}{i!(k-i)!} x^i,$$

and in view of the above estimate,

$$\sum_{i \text{ even}} \frac{(\alpha + i + 1)_{k-i}}{i!(k-i)!} \leq \sum_{i \text{ even}} \frac{(\alpha + k)^{k-i}}{i!(k-i)!} = \frac{1}{2k!} \left((\alpha + k + x)^k + (\alpha + k - x)^k \right),$$

and, analogously,

$$\sum_{i \text{ odd}} \frac{(\alpha + i + 1)_{k-i}}{i!(k-i)!} \leq \sum_{i \text{ odd}} \frac{(\alpha + k)^{k-i}}{i!(k-i)!} = \frac{1}{2k!} \left((\alpha + k + x)^k - (\alpha + k - x)^k \right).$$

By discarding alternatively the even or odd contribution, we arrive at

$$|L_k^\alpha(x)| \leq \frac{1}{2k!} \left((x + \alpha + k)^k + |\alpha + k - x|^k \right),$$

which is slightly stronger than the first estimate.

It is well-known that L_k^α is a polynomial of degree k all of whose zeros are distinct and real, and they all fall in the interval $]0, \beta[$ (see 18.16.13 of [16]). The second estimate follows immediately from this. \square

Approximation of the polynomial polyanalytic reproducing kernel. We estimate the difference $K_{m,n,q} - K_{m,q}$.

Proposition 2.4. *Let $z, w \in \mathbb{C}$ be such that $|zw| \leq \theta_1 < 1$ and $|zw|e^{1-|zw|} \leq \theta_2 < 1$. Let M be a positive real number. Then, as $m, n \rightarrow +\infty$ while $|m - n| \leq M$,*

$$|K_{m,n,q}(z, w) - K_{m,q}(z, w)|e^{-m|zw|} \leq Cm^q \theta_2^m (1 - \theta_1)^{-2} [1 + |z|^{2q} + |w|^{2q}],$$

where the constant C depends on q and M .

Proof. In view of (2.1) and Proposition 2.2, we have that for $q \leq n$,

$$K_{m,q}(z, w) - K_{m,n,q}(z, w) = m \sum_{r=0}^{q-1} \sum_{i=n-r}^{+\infty} \frac{r!}{(r+i)!} (mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2).$$

By Lemma 2.3, we get that

$$\begin{aligned} (2.4) \quad & \left| \sum_{i=n-r}^{+\infty} \frac{r!}{(r+i)!} (mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2) \right| \\ & \leq \sum_{i=n-r}^{+\infty} \frac{1}{r!(r+i)!} (m|zw|)^i (m|z|^2 + i + r)^r (m|w|^2 + i + r)^r \\ & = \frac{1}{r!} \sum_{j=0}^{+\infty} \frac{1}{(n+j)!} (m|zw|)^{n-r+j} (m|z|^2 + n + j)^r (m|w|^2 + n + j)^r \\ & \leq \frac{4^{r-1}}{r!} \sum_{j=0}^{+\infty} \frac{1}{(n+j)!} (m|zw|)^{n-r+j} [(m|z|^2)^r + (n+j)^r][(m|w|^2)^r + (n+j)^r]. \end{aligned}$$

For r confined to $0 \leq r \leq q - 1$ and for big n , say $n \geq n_0(q)$, we have

$$\frac{(n+j)^{2r}}{(n+j)!} \leq \frac{2}{(n+j-2r)!} \quad \text{and} \quad \frac{(n+j)^r}{(n+j)!} \leq \frac{2}{(n+j-r)!},$$

so that if we use the notation

$$R_k(z) = \sum_{j=k+1}^{+\infty} \frac{1}{j!} z^j = e^z - E_k(z),$$

we obtain from (2.4) that for $0 \leq r \leq q - 1$ and $n \geq n_0(q)$,

$$\begin{aligned} (2.5) \quad & \left| \sum_{i=n-r}^{+\infty} \frac{r!}{(r+i)!} (mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2) \right| \\ & \leq \frac{4^{r-1} m^r}{r!} \left[|zw|^r R_{n-1}(m|zw|) + 2(|z|^{2r} + |w|^{2r}) R_{n-r-1}(m|zw|) + 2|zw|^r R_{n-2r-1}(m|zw|) \right]. \end{aligned}$$

Next, we see from Szegő's asymptotical expansion (2.2) that for large k, l with $l = k + O(1)$, we have

$$(2.6) \quad \frac{R_k(l\zeta)}{e^{l\zeta}} = (2\pi k)^{-1/2} (l/k)^k \frac{\zeta}{k/l - \zeta} (ze^{1-l\zeta/k})^k (1 + \epsilon_k^1(l\zeta/k)),$$

as $k, l \rightarrow +\infty$ and $l = k + O(1)$. By a careful application of (2.6) to (2.5), and summing over $0 \leq r \leq q - 1$, the assertion of the proposition follows. \square

Corollary 2.5. *For $z, w \in \mathbb{D}$, let $1 - |zw| \geq \tau > 0$. Then, as $m, n \rightarrow +\infty$, while $|m - n| \leq M$,*

$$K_{m,n,q}(z, w) = K_{m,q}(z, w) + O(e^{-\frac{1}{2}m\tau^2} e^{m|zw|}),$$

where the "O" constant depends on τ, q , and M .

Proof. A Taylor series expansion of the logarithm gives that

$$t e^{1-t} < e^{-\frac{1}{2}(1-t)^2}, \quad 0 \leq t < 1,$$

and, together with the fact that the exponential function grows faster than any given power, the assertion follows from Proposition 2.4. \square

As we shall see, Proposition 2.4 implies that the Berezin density

$$\mathbb{B}_{m,n,q}^{(z)}(w) = \frac{|K_{m,n,q}(z, w)|^2}{K_{m,n,q}(z, z)} e^{-m|w|^2}$$

behaves locally near z like the Berezin density

$$(2.7) \quad \mathbb{B}_{m,q}^{(z)}(w) = \frac{|K_{m,q}(z, w)|^2}{K_{m,q}(z, z)} e^{-m|w|^2} = \frac{m^2 L_{q-1}^1(m|z-w|^2)^2 e^{2m \operatorname{Re} z \bar{w}}}{m L_{q-1}^1(0) e^{m|z|^2}} e^{-m|w|^2} \\ = \frac{m}{q} L_{q-1}^1(m|z-w|^2)^2 e^{-m|z-w|^2}.$$

To make this precise, we recall the definition of the blow-up Berezin density:

$$\hat{\mathbb{B}}_{m,n,q}^{(z)}(\xi) = m^{-1} \mathbb{B}_{m,n,q}^{(z)}(z + m^{-1/2} \xi).$$

Proposition 2.6. *Fix $z \in \mathbb{D}$. Then*

$$\int_{\mathbb{C}} |\hat{\mathbb{B}}_{m,n,q}^{(z)}(\xi) - q^{-1} L_{q-1}^1(|\xi|^2)^2 e^{-|\xi|^2}| dA(\xi) \longrightarrow 0,$$

as $m, n \rightarrow +\infty$ while $n = m + O(1)$.

Proof. By (2.7), we have

$$\hat{\mathbb{B}}_{m,q}^{(z)}(\xi) = m^{-1} \mathbb{B}_{m,q}^{(z)}(z + m^{-1/2} \xi) = q^{-1} L_{q-1}^1(|\xi|^2)^2 e^{-|\xi|^2},$$

so the comparison is with the blow-up Berezin density for $K_{m,q}$. We write $w := z + m^{-1/2} \xi$; since $z \in \mathbb{D}$ is fixed, we have $1 - |z|^2 \geq \tau > 0$ for some small τ , and we suppose that w is close to z so that $1 - |zw| \geq \tau > 0$ as well. Then, by Proposition 2.2 and Corollary 2.5,

$$\mathbb{B}_{m,n,q}^{(z)}(w) = \frac{|K_{m,n,q}(z, w)|^2}{K_{m,n,q}(z, z)} e^{-m|w|^2} = \frac{|K_{m,q}(z, w) + O(e^{-\frac{1}{2}m\tau^2} e^{m|zw|})|^2}{K_{m,q}(z, z) + O(e^{-\frac{1}{2}m\tau^2} e^{m|z|^2})} e^{-m|w|^2} \\ = \frac{|m L_{q-1}^1(m|z-w|^2) e^{mzz\bar{w}} + O(e^{-\frac{1}{2}m\tau^2} e^{m|zw|})|^2}{(mq + O(e^{-\frac{1}{2}m\tau^2})) e^{m|z|^2}} e^{-m|w|^2},$$

so that if we we expand the square using that

$$|L_{q-1}^1(m|z-w|^2)| = O(m^{q-1}),$$

where the “O” depends only on q (this follows from Lemma 2.3), and simplify the expression, we arrive at

$$\mathbb{B}_{m,n,q}^{(z)}(w) = \frac{m}{q} L_{q-1}^1(m|z-w|^2)^2 e^{-m|z-w|^2} + O(m^{2q-2} e^{-\frac{1}{2}m\tau^2} e^{-m(|z|-|w|)^2}),$$

which immediate gives that

$$\mathbb{B}_{m,n,q}^{(z)}(w) = \frac{m}{q} L_{q-1}^1(m|z-w|^2)^2 e^{-m|z-w|^2} + O(m^{2q-2} e^{-\frac{1}{2}m\tau^2}).$$

The corresponding blow-up Berezin density then has

$$(2.8) \quad \hat{\mathbb{B}}_{m,n,q}^{(z)}(\xi) = m^{-1} \mathbb{B}_{m,n,q}^{(z)}(z + m^{-1/2} \xi) = q^{-1} L_{q-1}^1(|\xi|^2)^2 e^{-|\xi|^2} + O(m^{2q-3} e^{-\frac{1}{2} m \tau^2}).$$

As exponentials grow faster than polynomials, the error terms is negligible for big m . For fixed $z \in \mathbb{D}$, the requirement on ξ so that $1 - |zw| \geq \tau$ for some fixed $\tau > 0$ is fulfilled for big m if, say, $|\xi| \leq \log m$ is required. So, (2.8) has the immediate consequence that

$$(2.9) \quad \int_{\mathbb{D}(0, \log m)} \left| \hat{\mathbb{B}}_{m,n,q}^{(z)}(\xi) - q^{-1} L_{q-1}^1(|\xi|^2)^2 e^{-|\xi|^2} \right| dA(\xi) = O\left(m^{2q-3} (\log m)^2 e^{-\frac{1}{2} m \tau^2}\right),$$

where more generally $\mathbb{D}(z_0, \rho)$ denotes the open disk of radius ρ about z_0 . Since the associated blow-up Berezin measures

$$d\hat{\mathbb{B}}_{m,n,q}^{(z)}(\xi) := \hat{\mathbb{B}}_{m,n,q}^{(z)}(w) dA(\xi), \quad d\hat{\mathbb{B}}_{m,q}^{(z)}(\xi) := q^{-1} L_{q-1}^1(|\xi|^2)^2 e^{-|\xi|^2} dA(\xi),$$

are both probability measures, the assertion of the proposition follows from (2.9) once it is noted that the right-hand side of (2.9) tends to 0 as $m \rightarrow +\infty$. \square

We note that the claimed convergence $d\mathbb{B}_{m,n,q}^{(z)} \rightarrow d\delta_z$ for $z \in \mathbb{D}$ is an immediate consequence of Proposition 2.6.

3. Berezin density asymptotics for an exterior point

Convergence to harmonic measure. We show that the Berezin measures have the convergence $d\mathbb{B}_{m,n,q}^{(z)} \rightarrow d\omega_z$ for $z \in \mathbb{D}^e$ as $m, n \rightarrow +\infty$ while $n = m + O(1)$. Here, ω_z is harmonic measure with respect to the point z and the exterior disk \mathbb{D}^e .

Concentration of the Berezin mass. We first study the concentration of the Berezin measure to neighborhoods of the closed unit disk $\bar{\mathbb{D}}$. We recall the standard notation $\mathbb{D}(z_0, \rho)$ for the open disk of radius ρ centered at z_0 .

Lemma 3.1. *Suppose $z \in \mathbb{D}^e$ and that $\rho > 1$. Then*

$$(3.1) \quad \int_{\mathbb{C} \setminus \mathbb{D}(0, \rho)} \mathbb{B}_{m,n,q}^{(z)}(w) dA(w) \rightarrow 0,$$

as $m, n \rightarrow +\infty$ while $n \leq m + O(1)$.

Proof. Let $w \in \mathbb{C} \setminus \mathbb{D}(0, \rho)$. We note that

$$(3.2) \quad n - 1 \leq m + O(1) \leq 2m$$

for sufficiently big m , and so, by Lemma 2.3,

$$(3.3) \quad \begin{aligned} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} |(mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2)| \\ \leq \sum_{i=0}^{n-r-1} \frac{1}{r!(r+i)!} (m|z\bar{w}|)^i (m|z|^2 + i + r)^r (m|w|^2 + i + r)^r \\ \leq \frac{9^r}{r!} \sum_{i=0}^{n-r-1} \frac{1}{(r+i)!} (m|z\bar{w}|)^{i+2r} \leq \frac{9^r}{r!} (m|z\bar{w}|)^r E_{n-1}(m|z\bar{w}|). \end{aligned}$$

If we plug in $n = q$ in (3.3), we obtain

$$(3.4) \quad \sum_{k=1}^{q-j-1} \frac{j!}{(j+k)!} |(m\bar{z}w)^k L_j^k(m|z|^2) L_j^k(m|w|^2)| \leq \frac{9^j}{j!} (m|zw|)^j E_{q-1}(m|zw|).$$

We may restrict to $q \leq n$; after, we are considering the limit as $n \rightarrow +\infty$. As $E_{q-1} \leq E_{n-1}$ on the positive half-axis for $q \leq n$, an application of (3.3) and (3.4) to (2.1) gives

$$(3.5) \quad |K_{m,n,q}(z, w)| \leq 2me^9 (m|zw|)^{q-1} E_{n-1}(m|zw|).$$

Next, we observe that $\text{Pol}_{m,n,1} \subset \text{Pol}_{m,n,q}$ (since $q \geq 1$), which implies that

$$K_{m,n,q}(z, z) \geq K_{m,n,1}(z, z) = mE_{n-1}(m|z|^2).$$

We conclude that the Berezin density may be estimated as follows:

$$(3.6) \quad \mathbb{B}_{m,n,q}^{(z)}(w) \leq 4m e^{18} (m|zw|)^{2q-2} \frac{E_{n-1}(m|zw|)^2}{E_{n-1}(m|z|^2)} e^{-m|w|^2}.$$

Finally, we see from Szegő's asymptotical expansion (2.3) that

$$(3.7) \quad \frac{E_k(l\zeta)}{e^{l\zeta}} = (2\pi k)^{-1/2} (l/k)^k \frac{\zeta}{z - k/l} (\zeta e^{1-l\zeta/k})^k (1 + \epsilon_k^2(l\zeta/k)).$$

$k \leq l + O(1)$, where the convergence $\epsilon_k^2(l\zeta/k) \rightarrow 0$ is uniform if ζ is real with $\zeta \geq a$ for some fixed $a > 1$. This leads to

$$\frac{E_{n-1}(m|zw|)^2}{E_{n-1}(m|z|^2)} = [2\pi(n-1)]^{-1/2} [(n-1)/m]^{n-1} e^{n-1} |w|^{2n-2} (1 + o(1)),$$

where the "o" term is uniform in the convergence. As we implement this estimate in (3.6), and integrate over $\mathbb{C} \setminus \mathbb{D}(0, \rho)$, the claim follows. \square

A principal value calculation. We follow the approach of [2], and calculate a certain principal value integral.

Lemma 3.2. Fix $z \in \mathbb{D}^e$. For any $l = 0, 1, 2, \dots$, we have

$$\text{pv} \int_{\mathbb{C}} w^{-l} \mathbb{B}_{m,n,q}^{(z)}(w) dA(w) \rightarrow z^{-l},$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$.

Proof. The case $q = 1$ was treated in [2], so we may from now on assume that $q \geq 2$. We write

$$K_{m,n,q} = K_{m,n,q}^I + K_{m,n,q}^{II},$$

where

$$K_{m,n,q}^I(z, w) := m \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} (mz\bar{w})^i L_r^i(m|z|^2) L_r^i(m|w|^2)$$

and

$$K_{m,n,q}^{II}(z, w) := m \sum_{j=0}^{q-2} \sum_{k=0}^{q-j-1} \frac{j!}{(j+k)!} (m\bar{z}w)^k L_j^k(m|z|^2) L_j^k(m|w|^2).$$

It follows that the expression $|K_{m,n,q}|^2$ decomposes accordingly:

$$(3.8) \quad |K_{m,n,q}(z, w)|^2 = |K_{m,n,q}^I(z, w)|^2 + |K_{m,n,q}^{II}(z, w)|^2 + 2 \text{Re}[K_{m,n,q}^I(z, w) \overline{K_{m,n,q}^{II}(z, w)}].$$

We first consider the following integral involving $|K_{m,n,q}^l|^2$:

$$\begin{aligned}
(3.9) \quad & \text{pv} \int_{\mathbb{C}} w^{-l} |K_{m,n,q}^l(z, w)|^2 e^{-m|w|^2} dA(w) \\
&= m^2 \sum_{r_1, r_2=0}^{q-1} \sum_{i_1=0}^{n-r_1-1} \sum_{i_2=0}^{n-r_2-1} \text{pv} \int_{\mathbb{C}} w^{-l} \frac{r_1! r_2!}{(r_1 + i_1)! (r_2 + i_2)!} (mz\bar{w})^{i_1} (m\bar{z}w)^{i_2} \\
&\quad \times L_{r_1}^{i_1}(m|z|^2) L_{r_1}^{i_1}(m|w|^2) L_{r_2}^{i_2}(m|z|^2) L_{r_2}^{i_2}(m|w|^2) e^{-m|w|^2} dA(w) \\
&= m^2 z^{-l} \sum_{r_1, r_2=0}^{q-1} \sum_{i_1=0}^{n-r_1-1} \sum_{i_2=0}^{n-r_2-1} \frac{m^{i_1+i_2} r_1! r_2!}{(r_1 + i_1)! (r_2 + i_2)!} \delta_{i_2, i_1+l} |z|^{2i_2} L_{r_1}^{i_1}(m|z|^2) L_{r_2}^{i_2}(m|z|^2) \\
&\quad \times \int_{\mathbb{C}} |w|^{2i_1} L_{r_1}^{i_1}(m|w|^2) L_{r_2}^{i_2}(m|w|^2) e^{-m|w|^2} dA(w),
\end{aligned}$$

where the delta is understood in Kronecker's sense. The identity (for $p = 1, 2, 3, \dots$)

$$(3.10) \quad L_r^{\alpha+p}(x) = \sum_{s=0}^r \binom{r-s+p-1}{p-1} L_s^{\alpha}(x)$$

plus the standard orthogonality properties of the Laguerre polynomials gives that

$$\int_0^{+\infty} t^{i_1} L_{r_1}^{i_1}(t) L_{r_2}^{i_1+l}(t) e^{-t} dt = \frac{(r_1 + i_1)! (r_2 - r_1 + l - 1)!}{r_1! (r_2 - r_1)! (l - 1)!},$$

where the right-hand side should be interpreted as 0 for $r_2 < r_1$. By polar coordinates, then, we have

$$\int_{\mathbb{C}} |w|^{2i_1} L_{r_1}^{i_1}(m|w|^2) L_{r_2}^{i_1+l}(m|w|^2) e^{-m|w|^2} dA(w) = m^{-i_1-1} \frac{(r_1 + i_1)! (r_2 - r_1 + l - 1)!}{r_1! (r_2 - r_1)! (l - 1)!},$$

and (3.9) simplifies to

$$\begin{aligned}
(3.11) \quad & \text{pv} \int_{\mathbb{C}} w^{-l} |K_{m,n,q}^l(z, w)|^2 e^{-m|w|^2} dA(w) \\
&= m z^{-l} \sum_{r_1=0}^{q-1} \sum_{r_2=r_1}^{q-1} \sum_{i_1=0}^{n-l-r_2-1} \frac{r_2! (r_2 - r_1 + l - 1)!}{(r_2 - r_1)! (l - 1)! (r_2 + i_1 + l)!} (m|z|^2)^{i_1+l} L_{r_1}^{i_1}(m|z|^2) L_{r_2}^{i_1+l}(m|z|^2),
\end{aligned}$$

provided n is so big that $q + l \leq n$. Next, we apply Lemma 2.3 and (3.2) (using that $z \in \mathbb{D}^e$) to arrive at

$$\begin{aligned}
(3.12) \quad & \sum_{i_1=0}^{n-l-r_2-1} \frac{r_1! r_2!}{(r_2 + i_1 + l)!} (m|z|^2)^{i_1+l} |L_{r_1}^{i_1}(m|z|^2) L_{r_2}^{i_1+l}(m|z|^2)| \\
&\leq \sum_{i_1=0}^{n-l-r_2-1} \frac{1}{(r_2 + i_1 + l)!} (m|z|^2)^{i_1+l} (m|z|^2 + i_1 + r_1)^{r_1} (m|z|^2 + i_1 + l + r_2)^{r_2} \\
&\leq 3^{r_1+r_2} \sum_{i_1=0}^{n-l-r_2-1} \frac{1}{(r_2 + i_1 + l)!} (m|z|^2)^{r_1+r_2+i_1+l} \leq 3^{r_1+r_2} (m|z|^2)^{r_1} E_{n-1}(m|z|^2).
\end{aligned}$$

On the other hand, by the estimate from below in Lemma 2.3,

$$(3.13) \quad K_{m,n,q}(z, z) \geq K_{m,n,q}^I(z, z) = m \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} (m|z|^2)^i L_r^i(m|z|^2)^2 \\ \geq m \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{1}{r!(r+i)!} (m|z|^2)^i [m|z|^2 - \beta(n)]^{2r}$$

provided that $m|z|^2 \geq \beta(n)$, where

$$\beta(n) := n + q - 4 + 2\sqrt{\frac{1}{4} + (q-2)(n-1)} = n + O(\sqrt{n}).$$

As we assume $z \in \mathbb{D}^e$ and $n = m + O(1)$, we must have

$$m|z|^2 - \beta(n) = m(|z|^2 - 1) + m - \beta(n) \geq \frac{1}{2}m(|z|^2 - 1)$$

for big enough m, n , and so, by (3.13),

$$(3.14) \quad K_{m,n,q}(z, z) \geq K_{m,n,q}^I(z, z) \geq m \sum_{r=0}^{q-1} \sum_{i=0}^{n-r-1} \frac{4^{-r}}{r!(r+i)!} (m|z|^2)^{i+2r} \\ \geq \frac{4^{1-q}m}{(q-1)!} (m|z|)^{q-1} [E_{n-1}(m|z|^2) - E_{q-2}(m|z|^2)] \geq \frac{4^{-q}m}{(q-1)!} (m|z|)^{q-1} E_{n-1}(m|z|^2).$$

Here, we used that the $E_{n-1}(m|z|^2)$ is much bigger than $E_{q-2}(m|z|^2)$ as m, n both grow. Let us look at the contribution from $0 \leq r_1 \leq q-2$ in the right-hand side of (3.11) using the estimate (3.12):

$$m|z|^{-l} \sum_{r_1=0}^{q-2} \sum_{r_2=r_1}^{q-1} \sum_{i_1=0}^{n-l-r_2-1} \frac{r_2!(r_2-r_1+l-1)!}{(r_2-r_1)!(l-1)!(r_2+i_1+l)!} (m|z|^2)^{i_1+l} |L_{r_1}^{i_1}(m|z|^2) L_{r_2}^{i_1+l}(m|z|^2)| \\ \leq m \frac{|z|^{-l}}{(l-1)!} \sum_{r_1=0}^{q-2} \sum_{r_2=r_1}^{q-1} \frac{(r_2-r_1+l-1)!}{r_1!(r_2-r_1)!} 3^{r_1+r_2} (m|z|^2)^{r_1} E_{n-1}(m|z|^2) \\ \leq mC_1(q, l)(m|z|^2)^{q-2} E_{n-1}(m|z|^2),$$

for an appropriate positive constant $C_1(q, l)$. As we combine this estimate with (3.14), we obtain

$$m \frac{|z|^{-l}}{K_{m,n,q}(z, z)} \sum_{r_1=0}^{q-2} \sum_{r_2=r_1}^{q-1} \sum_{i_1=0}^{n-l-r_2-1} \frac{r_2!(r_2-r_1+l-1)!}{(r_2-r_1)!(l-1)!(r_2+i_1+l)!} \\ \times (m|z|^2)^{i_1+l} |L_{r_1}^{i_1}(m|z|^2) L_{r_2}^{i_1+l}(m|z|^2)| \leq C_2(q, l)m^{-1} = O(m^{-1}) \longrightarrow 0,$$

as $m, n \rightarrow +\infty$ in the prescribed fashion. So the contribution to (3.11) which comes from $0 \leq r_1 \leq q-2$ is negligible from the point of view of the Berezin density. It remains to consider the contribution from $r_1 = q-1$. The corresponding part of the sum in the right-hand side of (3.11) equals

$$mz^{-l} \sum_{i_1=0}^{n-l-q} \frac{(q-1)!}{(q+i_1+l-1)!} (m|z|^2)^{i_1+l} L_{q-1}^{i_1}(m|z|^2) L_{q-1}^{i_1+l}(m|z|^2),$$

and we now claim that

$$(3.15) \quad m \frac{z^{-l}}{K_{m,n,q}(z,z)} \sum_{i_1=0}^{n-l-q} \frac{(q-1)!}{(q+i_1+l-1)!} (m|z|^2)^{i_1+l} L_{q-1}^{i_1}(m|z|^2) L_{q-1}^{i_1+l}(m|z|^2) \longrightarrow z^{-l},$$

as $m, n \rightarrow +\infty$ in the given fashion. We use the recurrence relation (3.10) to write

$$(3.16) \quad L_{q-1}^{i_1}(x) = L_{q-1}^{i_1+l}(x) - \sum_{s=0}^{q-2} \binom{q+l-s-2}{l-1} L_s^{i_1}(x).$$

A straightforward argument allows us to show that as insert this into (3.15), the sum that is subtracted on the right-hand side of (3.16) makes asymptotically no contribution to the sum in (3.15). Consequently, (3.15) is equivalent to having

$$(3.17) \quad m \frac{z^{-l}}{K_{m,n,q}(z,z)} \sum_{i_1=0}^{n-l-q} \frac{(q-1)!}{(q+i_1+l-1)!} (m|z|^2)^{i_1+l} L_{q-1}^{i_1+l}(m|z|^2)^2 \longrightarrow z^{-l}$$

as $m, n \rightarrow +\infty$ in the given fashion. If we insert the expression (2.1) defining $K_{m,n,q}(z,z)$, we see that (there is some cancellation of terms)

$$(3.18) \quad z^{-l} - m \frac{z^{-l}}{K_{m,n,q}(z,z)} \sum_{i_1=0}^{n-l-q} \frac{(q-1)!}{(q+i_1+l-1)!} (m|z|^2)^{i_1+l} L_{q-1}^{i_1+l}(m|z|^2) L_{q-1}^{i_1+l}(m|z|^2) \\ = \frac{z^{-l}}{K_{m,n,q}(z,z)} \left\{ m \sum_{r=0}^{q-2} \sum_{i=0}^{n-r-1} \frac{r!}{(r+i)!} (m|z|^2)^i L_r^i(m|z|^2)^2 \right. \\ \left. + m \sum_{j=1}^{q-2} \sum_{k=1}^{q-j-1} \frac{j!}{(j+k)!} (m|z|^2)^k L_j^k(m|z|^2)^2 + m \sum_{i_1=0}^{l-1} \frac{(q-1)!}{(q+i_1-1)!} (m|z|^2)^{i_1} L_{q-1}^{i_1}(m|z|^2)^2 \right\}.$$

By careful application of Lemma 2.3 to all the Laguerre polynomials in this expression, while inserting (3.13) to control the denominator, we indeed get (3.17). So, after a lot of effort, we have obtained that

$$(3.19) \quad \text{pv} \int_{\mathbb{C}} w^{-l} \frac{|K_{m,n,q}^I(z,w)|^2}{K_{m,n,q}(z,z)} e^{-m|w|^2} dA(w) \longrightarrow z^{-l}$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$. Analogous but slightly easier arguments (left to the interested reader) show that

$$\text{pv} \int_{\mathbb{C}} w^{-j} \frac{|K_{m,n,q}^{II}(z,w)|^2}{K_{m,n,q}(z,z)} e^{-|w|^2} dA(w) \longrightarrow 0$$

and

$$\int_{\mathbb{C}} w^{-j} \text{Re} \left\{ \frac{K_{m,n,q}^I(z,w) \overline{K_{m,n,q}^{II}(z,w)}}{K_{m,n,q}(z,z)} \right\} e^{-|w|^2} dA(w) \longrightarrow 0,$$

again as $m, n \rightarrow +\infty$ with $n = m + O(1)$. Finally, we put everything together based on the decomposition (3.8):

$$\text{pv} \int_{\mathbb{C}} w^{-j} B_{m,n,q}^{(z)}(w) = \text{pv} \int_{\mathbb{C}} \frac{|K_{m,n,q}(z,w)|^2}{K_{m,n,q}(z,z)} e^{-|w|^2} dA(w) \longrightarrow z^{-l},$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$. This ends the proof. \square

Convergence to harmonic measure. We now show that even in the principal value sense, the Berezin density tends to avoid the interior of the unit disk.

Lemma 3.3. *Fix a real parameter ρ with $0 < \rho < 1$ and a point $z \in \mathbb{D}^c$. Then, for $l = 0, 1, 2, \dots$, we have the convergence*

$$\text{pv} \int_{\mathbb{D}(0,\rho)} w^{-l} \mathbb{E}_{m,n,q}^{(z)}(w) dA(w) \longrightarrow 0,$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$.

Proof. In terms of the decomposition (3.8), we will focus on the term $|K_{m,n,q}^I|^2$ and leave the other two to the reader (the necessary arguments are similar but slightly easier). The analogue of (3.9) reads

$$\begin{aligned} (3.20) \quad & \text{pv} \int_{\mathbb{D}(0,\rho)} w^{-l} |K_{m,n,q}^I(z, w)|^2 e^{-m|w|^2} dA(w) \\ &= m^2 z^{-l} \sum_{r_1, r_2=0}^{q-1} \sum_{i_1=0}^{\min(n-r_1-1, n-l-r_2-1)} \frac{r_1! r_2!}{(r_1+i_1)! (r_2+i_1+l)!} m(|z|^2)^{i_1+l} L_{r_1}^{i_1}(m|z|^2) L_{r_2}^{i_1+l}(m|z|^2) \\ & \quad \times \int_{\mathbb{D}(0,\rho)} (m|w|^2)^{i_1} L_{r_1}^{i_1}(m|w|^2) L_{r_2}^{i_1+l}(m|w|^2) e^{-m|w|^2} dA(w). \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} & \int_{\mathbb{D}(0,\rho)} (m|w|^2)^{i_1} |L_{r_1}^{i_1}(m|w|^2) L_{r_2}^{i_1+l}(m|w|^2)| e^{-m|w|^2} dA(w) \\ & \leq \frac{1}{r_1! r_2!} \int_{\mathbb{D}(0,\rho)} (m|w|^2)^{i_1} (m|w|^2 + i_1 + r_1)^{r_1} (m|w|^2 + i_1 + l + r_2)^{r_2} e^{-m|w|^2} dA(w) \\ & \leq \frac{2^{r_1+r_2-2}}{r_1! r_2!} \int_{\mathbb{D}(0,\rho)} (m|w|^2)^{i_1} [(m|w|^2)^{r_1} + (i_1 + r_1)^{r_1}] [(m|w|^2)^{r_2} + (i_1 + l + r_2)^{r_2}] e^{-m|w|^2} dA(w) \\ & = \frac{2^{r_1+r_2-2}}{r_1! r_2! m} \left\{ \int_0^{m\rho^2} t^{i_1+r_1+r_2} e^{-t} dt + (i_1 + r_1)^{r_1} \int_0^{m\rho^2} t^{i_1+r_2} e^{-t} dt \right. \\ & \quad \left. + (i_1 + l + r_2)^{r_2} \int_0^{m\rho^2} t^{i_1+r_1} e^{-t} dt + (i_1 + r_1)^{r_1} (i_1 + l + r_2)^{r_2} \int_0^{m\rho^2} t^{i_1} e^{-t} dt \right\}. \end{aligned}$$

In terms of the function

$$\chi(a, b) := \frac{1}{\Gamma(a+1)} \int_0^b t^a e^{-t} dt, \quad a, b \in [0, +\infty[,$$

which takes values in the interval $[0, 1]$, the estimate becomes

$$\begin{aligned} & \int_{\mathbb{D}(0,\rho)} (m|w|^2)^{i_1} |L_{r_1}^{i_1}(m|w|^2) L_{r_2}^{i_1+l}(m|w|^2)| e^{-m|w|^2} dA(w) \\ & \leq \frac{2^{r_1+r_2-2}}{r_1! r_2! m} \left\{ (i_1 + r_1 + r_2)! \chi(i_1 + r_1 + r_2, m\rho^2) + (i_1 + r_1)^{r_1} (i_1 + r_2)! \chi(i_1 + r_2, m\rho^2) \right. \\ & \quad \left. + (i_1 + l + r_2)^{r_2} (i_1 + r_1)! \chi(i_1 + r_1, m\rho^2) + (i_1 + r_1)^{r_1} (i_1 + l + r_2)^{r_2} i_1! \chi(i_1, m\rho^2) \right\}. \end{aligned}$$

and if we use that $a \mapsto \chi(a, b)$ is decreasing for fixed b (a direct calculation involving derivatives suffices to verify this), we get

$$(3.21) \quad \int_{\mathbb{D}(0, \rho)} (m|w|^2)^{i_1} \left| L_{r_1}^{i_1} (m|w|^2) L_{r_2}^{i_1+l} (m|w|^2) \right| e^{-m|w|^2} dA(w) \\ \leq \frac{2^{r_1+r_2-2}}{r_1! r_2! m} \left\{ (i_1 + r_1 + r_2)! + (i_1 + r_1)^{r_1} (i_1 + r_2)! \right. \\ \left. + (i_1 + l + r_2)^{r_2} (i_1 + r_1)! + (i_1 + r_1)^{r_1} (i_1 + l + r_2)^{r_2} i_1! \right\} \chi(i_1, m\rho^2).$$

Next, since $z \in \mathbb{D}^e$, $n = m + O(1)$, and $i_1 + l \leq n - 1$, we may use another aspect of Lemma 2.3 to see that

$$(3.22) \quad \left| L_{r_1}^{i_1} (m|z|^2) L_{r_2}^{i_1+l} (m|z|^2) \right| \leq \frac{1}{r_1! r_2!} (m|z|^2)^{r_1+r_2},$$

provided m, n are big enough. As we combine the equality (3.20) with the estimates (3.21) and (3.22), and use some well-understood comparisons of factorials and powers, we arrive at

$$(3.23) \quad \left| \text{pv} \int_{\mathbb{D}(0, \rho)} w^{-l} |K_{m,n,q}^l(z, w)|^2 e^{-m|w|^2} dA(w) \right| \\ \leq m C_3(q, l) |z|^{-l+2q-2} \sum_{i_1=0}^{n-l-1} \frac{(m|z|^2)^{i_1+l}}{(i_1+l)!} \chi(i_1, m\rho^2),$$

for some appropriate positive constant $C_3(q, l)$. The function $\chi(i_1, m\rho^2)$ drops off exponentially quickly to 0 as i_1 exceeds $m\rho^2$ by a margin greater than $O(m^{1/2})$, as can be seen, e.g., by an application of the Central Limit Theorem (compare with the next section). This means that effectively we are summing up to $m|\rho|^2 + O(m^{1/2})$ in the right-hand side expression of (3.23), which does not permit the sum to compare with the size of $K_{m,n,q}(z, z)$; cf. (3.14). This results in the convergence

$$\text{pv} \int_{\mathbb{D}(0, \rho)} w^{-l} \frac{|K_{m,n,q}^l(z, w)|^2}{K_{m,n,q}(z, z)} e^{-m|w|^2} dA(w) \longrightarrow 0$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$. Together with the estimates which were left as an exercise to the reader, we get

$$\text{pv} \int_{\mathbb{D}(0, \rho)} w^{-l} \frac{|K_{m,n,q}(z, w)|^2}{K_{m,n,q}(z, z)} e^{-m|w|^2} dA(w) \longrightarrow 0$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$, which amounts to the assertion of the lemma. \square

As in the proof of the Theorem 2.11 in [2], we may now conclude the following.

Theorem 3.4. *Fix $z \in \mathbb{D}^e$ and a bounded continuous function g on \mathbb{C} . Then*

$$\int_{\mathbb{C}} g(w) \mathbb{E}_{m,n,q}^{(z)}(w) dA(w) \rightarrow \int_{\mathbb{C}} g(w) d\omega(w, z, \mathbb{D}^*),$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$. Here, $d\omega(w, z, \mathbb{D}^*)$ is harmonic measure with respect to the point z and the domain \mathbb{D}^e .

4. Poly-Bargmann transforms

Purpose of the section. In this section, we discuss the poly-Bargmann transforms, a generalization of the classical Bargmann transform, which are needed later when we analyze the Berezin density at a boundary point. The poly-Bargmann transforms appeared in Vasilevski's paper [25], where the basic properties were presented.

The Hermite polynomials and the Bargmann transform. We denote by H_j the j -th Hermite polynomial with respect to the Gaussian weight $e^{-\frac{1}{2}t^2}$ ("probabilistic Hermite polynomials"). The generating function identity

$$e^{tz - \frac{1}{2}z^2} = \sum_{j=0}^{+\infty} H_j(t) \frac{z^j}{j!}$$

allows us to write

$$\frac{1}{(2\pi)^{1/4}} e^{zt - \frac{1}{2}z^2 - \frac{1}{4}t^2} = \sum_{j=0}^{+\infty} \frac{H_j(t) e^{-t^2/4}}{(2\pi)^{1/4} \sqrt{j!}} \times \frac{z^j}{\sqrt{j!}}.$$

We recall the standard definition of the Bargmann transform:

$$\mathbf{B}[f](z) = \frac{1}{(2\pi)^{1/4}} \int_{\mathbb{R}} e^{zt - \frac{1}{2}z^2 - \frac{1}{4}t^2} f(t) dt, \quad f \in L^2(\mathbb{R}).$$

As the function systems

$$\left\{ \frac{1}{(2\pi)^{1/4} \sqrt{j!}} H_j(t) e^{-\frac{1}{4}t^2} \right\}_{j=0}^{+\infty} \quad \text{and} \quad \left\{ \frac{z^j}{\sqrt{j!}} \right\}_{j=0}^{+\infty}$$

form orthonormal bases for $L^2(\mathbb{R})$ and the Bargmann-Fock space $A_{1,1}^2(\mathbb{C})$ (this is $A_{m,q}^2(\mathbb{C})$ with $m = q = 1$), respectively, we obtain the following well-known fact.

Proposition 4.1. *The Bargmann transform $\mathbf{B} : L^2(\mathbb{R}) \rightarrow A_{1,1}^2(\mathbb{C})$ acts isometrically and bijectively, and for each $j = 0, 1, 2, \dots$, the basis function $(2\pi)^{-1/4} (j!)^{-1/2} H_j(t) e^{-\frac{1}{4}t^2}$ is mapped to the basis function $(j!)^{-1/2} z^j$.*

A class of auxiliary operators. Let $\partial_z, \bar{\partial}_z$ denote the standard Wirtinger differential operators

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y), \quad \text{where } z = x + iy.$$

For $r = 0, 1, 2, \dots$, we introduce the operator

$$\mathbf{T}_r[f](z) := \frac{1}{\sqrt{r!}} e^{|z|^2} \partial_z^r \{ f(z) e^{-|z|^2} \},$$

with the semi-group property $\mathbf{T}_1 \circ \mathbf{T}_{r-1} = r^{1/2} \mathbf{T}_r$. We also consider the dilated variant

$$\mathbf{T}_{m,r}[f](z) = \mathbf{T}_r[f_{m^{-1/2}}](m^{1/2}z),$$

where $f_{m^{-1/2}}(z) = f(m^{-1/2}z)$. It has the semi-group property $\mathbf{T}_{m,1} \circ \mathbf{T}_{m,r-1} = r^{1/2} \mathbf{T}_{m,r}$, and may be expressed in more concrete terms:

$$\mathbf{T}_{m,r}[f](z) = \frac{m^{-r/2}}{\sqrt{r!}} e^{m|z|^2} \partial_z^r \{ f(z) e^{-m|z|^2} \}.$$

We now study the effect of \mathbf{T}_r on the basis elements $(j!)^{-1/2} z^j$.

Proposition 4.2. For $j \geq r$, we have

$$\mathbf{T}_r \left[\frac{z^j}{\sqrt{j!}} \right] = \frac{1}{\sqrt{r!}} e^{|z|^2} \partial_z^r \left\{ e^{-|z|^2} \frac{z^j}{\sqrt{j!}} \right\} = \sqrt{\frac{r!}{j!}} z^{j-r} L_r^{j-r}(|z|^2)$$

while for $j \leq r$,

$$\mathbf{T}_r \left[\frac{z^j}{\sqrt{j!}} \right] = \frac{1}{\sqrt{r!}} e^{|z|^2} \partial_z^r \left\{ e^{-|z|^2} \frac{z^j}{\sqrt{j!}} \right\} = (-1)^{r-j} \sqrt{\frac{n!}{r!}} \bar{z}^{r-j} L_j^{r-j}(|z|^2)$$

Proof. The proof is based on an induction argument. The statement is obviously true for $r = 0$ and all j . So, by induction, we assume that the statement holds for some $r - 1 \geq 0$ and all j . In case $j \geq r$, we then have

$$\begin{aligned} \partial_z^r \{ e^{-|z|^2} z^j \} &= \partial_z \left\{ (r-1)! z^{j-r+1} L_{r-1}^{j-r+1}(|z|^2) e^{-|z|^2} \right\} \\ &= (r-1)! z^{j-r} \left\{ (j-r+1) L_{r-1}^{j-r+1}(|z|^2) - L_{r-2}^{j-r+2}(|z|^2) |z|^2 - |z|^2 L_{r-1}^{j-r+1}(|z|^2) \right\} e^{-|z|^2} \\ &= r! L_r^{j-r}(|z|^2) z^{j-r} e^{-|z|^2}, \end{aligned}$$

if we use the standard identity $rL_r^\alpha(x) = (\alpha + 1 - x)L_{r-1}^{\alpha+1}(x) - xL_{r-2}^{\alpha+2}(x)$. Next, in case $j \leq r - 1$, we have instead

$$\begin{aligned} \partial_z^r \{ e^{-|z|^2} z^j \} &= (-1)^{r-1-j} \partial \left\{ j! \bar{z}^{r-1-j} L_j^{r-1-j}(|z|^2) e^{-|z|^2} \right\} \\ &= (-1)^{r-j} j! \bar{z}^{r-j} \left\{ L_{j-1}^{r-j}(|z|^2) + L_j^{r-j-1}(|z|^2) \right\} e^{-|z|^2} = (-1)^{r-j} j! \bar{z}^{r-j} L_j^{r-j}(|z|^2) e^{-|z|^2}, \end{aligned}$$

which completes the proof. \square

Pure poly-analytic Fock spaces. Write $e_j(z) := (j!)^{-1/2} z^j$ for $j = 0, 1, 2, \dots$, which functions form the standard orthonormal basis for the space $A_{1,1}^2(\mathbb{C})$. The function $\mathbf{T}_r[e_j] \in A_{1,r+1}^2(\mathbb{C})$ is then a polynomial in z, \bar{z} , where the degree in z remains equal to j , and the degree in \bar{z} equals r . For general positive m , the functions $e_{j,m}(z) := (j!)^{-1/2} m^{\frac{1}{2}(j+1)} z^j$, with $j = 0, 1, 2, \dots$, form an orthonormal basis for the space $A_{m,1}^2(\mathbb{C})$. The functions $\mathbf{T}_{m,r}[e_{j,m}]$ are computed using Proposition 4.2 above, and we then recognize that we can identify them with the basis elements which appear in Proposition 2.1. We clearly have that

$$\text{span} \{ \mathbf{T}_{m,r}[e_{j,m}] : 0 \leq j \leq n-1, 0 \leq r \leq q-1 \} = \text{Pol}_{m,n,q},$$

and in view of the orthogonality properties in Proposition 2.1, we also must have

$$(4.1) \quad \text{span} \{ \mathbf{T}_{m,r}[e_{j,m}] : 0 \leq j \leq n-1 \} = \text{Pol}_{m,n,r+1} \ominus \text{Pol}_{m,n,r},$$

where the “ \ominus ” is with respect to the inner product in $L^2(\mathbb{C}, e^{-m|z|^2})$. Following Vasilevski [25], then, we define the pure polyanalytic Fock space of level $r + 1$:

$$(4.2) \quad \delta \text{Pol}_{m,n,r+1} := \text{Pol}_{m,n,r+1} \ominus \text{Pol}_{m,n,r}, \quad \delta A_{m,n,r+1}^2(\mathbb{C}) := A_{m,r+1}^2(\mathbb{C}) \ominus A_{m,r}^2(\mathbb{C}),$$

with the understanding that $\text{Pol}_{m,n,0} := \{0\}$ and $A_{m,0}^2(\mathbb{C}) := \{0\}$. The operator $\mathbf{T}_{m,r}$ now becomes an isometric isomorphism

$$\mathbf{T}_{m,r} : A_{m,1}^2(\mathbb{C}) \rightarrow \delta A_{m,n,r+1}^2(\mathbb{C}), \quad \mathbf{T}_{m,r} : \text{Pol}_{m,n,1} \rightarrow \delta \text{Pol}_{m,n,r+1}.$$

We obtain the orthogonal decompositions

$$\text{Pol}_{m,n,q} = \bigoplus_{r=0}^{q-1} \delta \text{Pol}_{m,n,r+1}, \quad A_{m,q}^2(\mathbb{C}) = \bigoplus_{r=0}^{q-1} \delta A_{m,r+1}^2(\mathbb{C}).$$

As a consequence, if $K_{\delta:m,n,r+1}$ and $K_{\delta:m,r+1}$ denote the reproducing kernels for the spaces $\delta \text{Pol}_{m,n,r+1}$ and $\delta A_{m,r+1}^2(\mathbb{C})$, respectively, we must have that

$$(4.3) \quad K_{m,n,q} = \sum_{r=0}^{q-1} K_{\delta:m,n,r+1}, \quad K_{m,q} = \sum_{r=0}^{q-1} K_{\delta:m,r}.$$

The poly-Bargmann transforms. For $m = 1$, the poly-Bargmann transform of level r is defined by

$$\mathbf{B}_r[f] := \mathbf{T}_r \circ \mathbf{B}[f].$$

Proposition 4.3. *We have*

$$\mathbf{B}_r[f](z) = \frac{1}{(2\pi)^{1/4} \sqrt{r!}} \int_{\mathbb{R}} f(t) H_r(t - z - \bar{z}) e^{tz - \frac{1}{2}z^2 - \frac{1}{4}t^2} dt, \quad r = 0, 1, 2, \dots$$

Proof. We proceed by induction. For $r = 0$, the formula reduces to the usual Bargmann transform (if we recall that $H_0 = 1$). By the induction hypothesis, we suppose therefore that the formula is valid for all integers up to $r - 1$. From the semi-group property $\mathbf{T}_1 \circ \mathbf{T}_{r-1} = r^{1/2} \mathbf{T}_r$, we see that

$$\mathbf{B}_r[f] = \mathbf{T}_r \circ \mathbf{B}[f] = r^{-1/2} \mathbf{T}_1 \circ \mathbf{T}_{r-1} \circ \mathbf{B}[f](z) = r^{-1/2} \mathbf{T}_1 \circ \mathbf{B}_{r-1}[f].$$

Since the formula holds for $r - 1$, the following calculation shows that it holds for r as well:

$$\begin{aligned} & r^{-1/2} e^{|z|^2} \partial_z \left([(r-1)!]^{-1/2} H_{r-1}(t - z - \bar{z}) e^{tz - \frac{1}{2}z^2 - \frac{1}{4}t^2} e^{-|z|^2} \right) \\ &= (r!)^{-1/2} \left(-H'_{r-1}(t - z - \bar{z}) + (t - z - \bar{z}) H_{r-1}(t - z - \bar{z}) \right) e^{tz - \frac{1}{2}z^2 - \frac{1}{4}t^2} \\ &= (r!)^{-1/2} H_r(t - z - \bar{z}) e^{tz - \frac{1}{2}z^2 - \frac{1}{4}t^2}; \end{aligned}$$

here, we used the standard identity $H_r(x) = xH_{r-1}(x) - H'_{r-1}(x)$. The proof is complete. \square

5. Reproducing kernel and Berezin density asymptotics for a boundary point

Purpose of the section. In this section, we will calculate the limit of the blow-up Berezin transform $\hat{\mathbf{B}}_{m,n,q}^{(z)}$ at a boundary point z , that is, $|z| = 1$. There is no loss of generality to take $z = 1$. Our strategy is to investigate the blow-up of the reproducing kernel of the space of analytic polynomials $\text{Pol}_{m,n,1}$ (with $q = 1$) first, and then use this information together with poly-Bargmann transform to lift the asymptotics to the context of the general polyanalytic spaces $\text{Pol}_{m,n,q}$.

The central limit theorem revisited. The following improvement of the central limit theorem will be needed. Let cdf_X denote the cumulative distribution function of a real-valued random variable X and let

$$\text{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}t^2} dt$$

be the error function. We shall write i.i.d. as shorthand for *independent identically distributed* in the context of random variables. The following result is from [7], [11].

Theorem 5.1 (Berry-Esséen). *Let X_1, X_2, \dots be i.i.d. real-valued random variables with $E(X_j) = 0$, $E(X_j^2) = 1$ and $E(|X_j|^3) = \rho < +\infty$ for all j . Also, let $Y_n = n^{-1/2} \sum_{j=1}^n X_j$. Then there exists an absolute constant C such that*

$$|\text{cdf}_{Y_n}(x) - \text{erf}(x)| \leq \frac{C\rho}{\sqrt{n}}, \quad x \in \mathbb{R}.$$

The Berry-Esséen theorem gives the following asymptotics for the partial Taylor sums of the exponential function.

Lemma 5.2. *We have*

$$\frac{E_{n-1}(x)}{e^x} = \text{erf}\left(\frac{n-x}{\sqrt{n}}\right) + O(n^{-1/2}),$$

as $n \rightarrow +\infty$, uniformly in $x \in [0, +\infty[$.

Proof. Let X_1, \dots, X_n be independent exponentially distributed random variables on $[0, +\infty[$ with density e^{-x} . It is well known that the sum $\sum_{j=1}^n X_j$ obeys a gamma distribution with the cumulative distribution function $1 - e^{-x}E_{n-1}(x)$. The random variables $X_1 - 1, \dots, X_n - 1$ all have zero mean and variance 1, and the third moment is finite, so by the Berry-Esséen theorem, we have

$$(5.1) \quad 1 - e^{-x}E_{n-1}(x) = \text{cdf}_{\sum_{j=1}^n X_j}(x) = \text{cdf}_{\frac{\sum_{j=1}^n X_j - n}{\sqrt{n}}}\left(\frac{x-n}{\sqrt{n}}\right) = \text{erf}\left(\frac{x-n}{\sqrt{n}}\right) + O(n^{-1/2}),$$

where the “ O ” term is uniform in x as $n \rightarrow +\infty$. \square

This allows us to blow up the function $E_{n-1}(mz\bar{w})$ when z and w are close to the point 1 and $m, n \rightarrow +\infty$ with $n = m + O(1)$.

Lemma 5.3. *Fix a positive real ε . For complex $\xi, \eta \in \mathbb{C}$, we then have*

$$\frac{E_{n-1}\left(m(1 + m^{-1/2}\xi)(1 + m^{-1/2}\bar{\eta})\right)}{e^{m + \sqrt{m}(\xi + \bar{\eta})}} = e^{\varepsilon\bar{\eta}} \text{erf}(-\xi - \bar{\eta}) + O(m^{-\frac{1}{2} + \varepsilon}),$$

as $m, n \rightarrow +\infty$ while $n = m + O(1)$. Here, the “ O ” expression on the right-hand side is uniform on compact subsets of \mathbb{C} .

Proof. If we put

$$\zeta := m(1 + m^{-1/2}\xi)(1 + m^{-1/2}\bar{\eta}) = m + m^{1/2}(\xi + \bar{\eta}) + \xi\bar{\eta},$$

then

$$\frac{n - \zeta}{\sqrt{n}} = \frac{n - m - m^{1/2}(\xi + \bar{\eta}) - \xi\bar{\eta}}{\sqrt{n}} = -\xi - \bar{\eta} + O(m^{-1/2}),$$

where the “O” term is uniform on compact subsets. If we use that E_{n-1} has only nonnegative Taylor coefficients, we get from Lemma 5.2 that

$$(5.2) \quad \frac{|E_{n-1}(\zeta)|}{e^{|\zeta|}} \leq \frac{E_{n-1}(|\zeta|)}{e^{|\zeta|}} = \operatorname{erf}\left(\frac{n-|\zeta|}{\sqrt{n}}\right) + O(n^{-1/2}),$$

with a uniform “O” term. So, for *real* ξ, η , we may deduce from (5.2) the assertion of the lemma with $\varepsilon = 0$. For general complex ξ, η , we note that

$$\begin{aligned} |\zeta| &= m|(1+m^{-1/2}\xi)(1+m^{-1/2}\bar{\eta})| \\ &= m+m^{1/2}\operatorname{Re}[\xi+\eta]+\operatorname{Re}\xi\operatorname{Re}\eta+\frac{1}{2}(\operatorname{Im}\xi)^2+\frac{1}{2}(\operatorname{Im}\eta)^2+O(m^{-1/8}) \end{aligned}$$

uniformly in the domain where $\max\{|\xi|, |\eta|\} \leq m^{1/8}$. As the right-hand side of (5.2) is $\leq \frac{3}{2}$ for big n , we see that

$$\begin{aligned} \left| \frac{E_{n-1}\left(m(1+m^{-1/2}\xi)(1+m^{-1/2}\bar{\eta})\right)}{e^{m+\sqrt{m}(\xi+\bar{\eta})+\xi\bar{\eta}}} \right| &\leq \frac{3}{2}e^{-(\operatorname{Im}\xi)(\operatorname{Im}\eta)+\frac{1}{2}(\operatorname{Im}\xi)^2+\frac{1}{2}(\operatorname{Im}\eta)^2+O(m^{-1/8})} \\ &\leq 2e^{\frac{1}{2}[\operatorname{Im}\xi-\operatorname{Im}\eta]^2} \end{aligned}$$

holds in the domain where $\max\{|\xi|, |\eta|\} \leq m^{1/8}$, provided m is big enough. We need to show that the difference

$$F_{m,n}(\xi, \eta) := \frac{E_{n-1}\left(m(1+m^{-1/2}\xi)(1+m^{-1/2}\bar{\eta})\right)}{e^{m+\sqrt{m}(\xi+\bar{\eta})+\xi\bar{\eta}}} - \operatorname{erf}(-\xi-\bar{\eta})$$

is of order $O(m^{-1/2+\varepsilon})$ uniformly as ξ, η remain confined to some compact subset of \mathbb{C} . We know that $F_{m,n}(\xi, \eta) = O(m^{-1/2})$ uniformly when $\xi, \eta \in \mathbb{R}$ with confined to $\max\{|\xi|, |\eta|\} \leq m^{1/8}$. In view of the calculation we just made, we also have a good uniform estimate of $F_{m,n}(\xi, \eta)$ when $\xi, \eta \in \mathbb{C}$ with $\max\{|\xi|, |\eta|\} \leq m^{1/8}$. By employing a standard technique involving the subharmonicity of $\xi \mapsto \log |F_{m,n}(\xi, \eta)|$, and certain classical estimates of harmonic measure, we can show that $F_{m,n}(\xi, \eta) = O(m^{\varepsilon-1/2})$ holds uniformly when $\xi, \eta \in \mathbb{C}$ belong to a compact subset of \mathbb{C} , and in addition, $\eta \in \mathbb{R}$. Here, ε is a positive number which we can get as small as we like. A similar argument with η in place ξ worsens the control to $F_{m,n}(\xi, \eta) = O(m^{2\varepsilon-1/2})$, but now the control is uniform when both ξ, η are both complex and confined to some compact subset. The proof is complete with $\varepsilon = 2\varepsilon$. \square

The reproducing kernel for a subspace of the Fock space. We shall identify both the right-hand and the left-hand side expressions appearing in Lemma 5.3 with reproducing kernels of certain Hilbert spaces of entire functions. This will be the case $r = 0$ of the proposition below.

Let us agree to identify

$$L^2(\mathbb{R}_-) = \{f \in L^2(\mathbb{R}) : f(x) = 0 \text{ for } x > 0\}.$$

Proposition 5.4. *For $r = 0, 1, 2, \dots$, the function*

$$(\xi, \eta) \mapsto \frac{e^{\xi\bar{\eta}}}{r!\sqrt{2\pi}} \int_{-\infty}^{-\xi-\bar{\eta}} H_r(t+\xi-\eta)H_r(t+\bar{\eta}-\bar{\xi})e^{-\frac{1}{2}t^2} dt$$

is the reproducing kernel for the Hilbert space $\mathbf{B}_r[L^2(\mathbb{R}_-)] \subset A_{1,r+1}^2(\mathbb{C})$.

Proof. Let $M(\xi, \eta) = M_\eta(\xi)$ be the reproducing kernel for $\mathbf{B}_r[L^2(\mathbb{R}_-)]$. This kernel has $M_\eta \in \mathbf{B}_r[L^2(\mathbb{R}_-)]$ and

$$\begin{aligned} \frac{1}{\sqrt{r!(2\pi)^{1/4}}} \int_{\mathbb{R}} f(t) H_r(t - \eta - \bar{\eta}) e^{t\eta - \frac{1}{2}\eta^2 - \frac{1}{4}t^2} dt &= \mathbf{B}_r[f](\eta) \\ &= \langle \mathbf{B}_r[f], M_\eta \rangle_{A_{1,r+1}^2(\mathbb{C})} = \langle f, \mathbf{B}_r^{-1}[M_\eta] \rangle_{L^2(\mathbb{R})} \end{aligned}$$

for all $\eta \in \mathbb{C}$ and all $f \in L^2(\mathbb{R}_-)$, which allows us to conclude that

$$\mathbf{B}_r^{-1}[M_\eta](t) = (r!)^{-1/2} (2\pi)^{-1/4} 1_{1-\infty,0}(t) H_r(t - \eta - \bar{\eta}) e^{t\bar{\eta} - \frac{1}{2}\bar{\eta}^2 - \frac{1}{4}t^2}.$$

After applying the operator \mathbf{B}_r to both sides, we see that (cf. Proposition 4.3)

$$\begin{aligned} M_\eta(\xi) &= \frac{1}{r! \sqrt{2\pi}} \int_{-\infty}^0 H_r(t - \xi - \bar{\xi}) e^{t\xi - \frac{1}{2}\xi^2 - \frac{1}{4}t^2} H_r(t - \eta - \bar{\eta}) e^{t\bar{\eta} - \frac{1}{2}\bar{\eta}^2 - \frac{1}{4}t^2} dt \\ &= \frac{e^{\bar{\eta}\xi}}{r! \sqrt{2\pi}} \int_{-\infty}^0 H_r(t - \xi - \bar{\xi}) H_r(t - \eta - \bar{\eta}) e^{-\frac{1}{2}(t - \xi - \bar{\eta})^2} dt \\ &= \frac{e^{\bar{\eta}\xi}}{r! \sqrt{2\pi}} \int_{-\infty}^{-\bar{\eta} - \xi} H_r(t + \bar{\eta} - \bar{\xi}) H_r(t + \xi - \eta) e^{-\frac{1}{2}t^2} dt. \end{aligned}$$

The proof is complete. \square

Remark 5.5. The special case $r = 0$ of the kernel in Proposition 5.4 is $e^{\xi\bar{\eta}} \operatorname{erf}(-\xi - \bar{\eta})$, which is what we have on the right-hand side in Lemma 5.3.

The blow-up of the polynomial space at the boundary point. We turn to the polyanalytic analogue of the left-hand side of Lemma 5.3.

Definition 5.6. We introduce the *blow-up space at 1*,

$$\operatorname{Pol}_{m,n,q}^{(1)} := \left\{ e^{-m^{1/2}\xi} p(1 + m^{-1/2}\xi) : p \in \operatorname{Pol}_{m,n,q} \right\},$$

which we equip with the norm

$$\left\| \xi \mapsto e^{-m^{1/2}\xi} p(1 + m^{-1/2}\xi) \right\|_{\delta\operatorname{Pol}_{m,n,q}^{(1)}} := m^{1/2} e^{\frac{1}{2}m} \|p\|_{L^2(\mathbb{C}, e^{-m|z|^2})}.$$

For $0 \leq r \leq q - 1$, we denote by $\delta\operatorname{Pol}_{m,n,r+1}^{(1)}$ the subspace

$$\delta\operatorname{Pol}_{m,n,r+1}^{(1)} := \left\{ e^{-m^{1/2}\xi} p(1 + m^{-1/2}\xi) : p \in \delta\operatorname{Pol}_{m,n,r+1} \right\},$$

equipped with the same norm.

An elementary change of variables argument allows us to identify the norm on $\operatorname{Pol}_{m,n,q}^{(1)}$ with that of $A_{m,q}^2(\mathbb{C})$:

$$(5.3) \quad \left\| \xi \mapsto e^{-m^{1/2}\xi} p(1 + m^{-1/2}\xi) \right\|_{\operatorname{Pol}_{m,n,r+1}^{(1)}} = \left\| \xi \mapsto e^{-m^{1/2}\xi} p(1 + m^{-1/2}\xi) \right\|_{A_{1,r+1}^2(\mathbb{C})}.$$

As a consequence, we may regard $\operatorname{Pol}_{m,n,q}^{(1)}$ and $\delta\operatorname{Pol}_{m,n,r+1}^{(1)}$ as norm closed subspaces of $A_{1,q}^2(\mathbb{C})$. We may read off from the definition of the norm in $\delta\operatorname{Pol}_{m,n,r+1}^{(1)}$ that the kernel on the left-hand side in Lemma 5.3 is the reproducing kernel for the space $\operatorname{Pol}_{m,n,1}^{(1)}$. So, Lemma 5.3 can be understood as saying that

$$(5.4) \quad K_{\operatorname{Pol}_{m,n,1}^{(1)}}(\xi, \eta) = K_{\mathbf{B}_0[L^2(\mathbb{R}_-)]}(\xi, \eta) + O(m^{-\frac{1}{2} + \varepsilon}),$$

where $K_{\text{Pol}_{m,n,1}}$ and $K_{\mathbb{B}_0[L^2(\mathbb{R}_-)]}$ denote the reproducing kernels of the spaces in the subscripts, and the bound is locally uniform on compact subsets. We want to generalize (5.4) beyond $r = 0$. To this end, we make use of the operators \mathbf{T}_r .

Proposition 5.7. *For $r = 1, 2, 3, \dots$, we have that*

$$r^{-1/2}\mathbf{T}_1 : \delta\text{Pol}_{m,n,r}^{(1)} \rightarrow \delta\text{Pol}_{m,n,r+1}^{(1)}$$

is an isometric isomorphism.

Proof. From the isometry properties of \mathbf{T}_{r-1} and \mathbf{T}_r together with the semi-group property $r^{-1/2}\mathbf{T}_1 \circ \mathbf{T}_{r-1} = \mathbf{T}_r$, we get that

$$r^{-1/2}\mathbf{T}_1 : \delta A_{1,r}^2(\mathbb{C}) \rightarrow \delta A_{1,r+1}^2(\mathbb{C})$$

is an isometric isomorphism. In view of (5.3), the isometry part of the assertion follows. It remains to show that the operator is onto. This is an algebraic exercise which we leave to the reader. \square

By iterating Proposition 5.7, we obtain the following.

Corollary 5.8. *For $r = 1, 2, 3, \dots$, we have that*

$$\mathbf{T}_r : \text{Pol}_{m,n,1}^{(1)} \rightarrow \delta\text{Pol}_{m,n,r+1}^{(1)}$$

is an isometric isomorphism.

The blow-up of the polynomial reproducing kernel at a boundary point. From Corollary 5.8 above, we get that

$$(5.5) \quad K_{\delta\text{Pol}_{m,n,r+1}^{(1)}}(\xi, \eta) = [\mathbf{T}_r]_{\xi}[\bar{\mathbf{T}}_r]_{\eta}(K_{\text{Pol}_{m,n,1}^{(1)}}(\xi, \eta)),$$

where the subscripts z and w are used to indicate that the operator is acting with respect to that variable, and the bar means complex conjugation of the operator. To be more precise,

$$\bar{\mathbf{T}}_r[f](z) := \frac{1}{\sqrt{r!}} e^{|z|^2} \bar{\partial}_z^r \{f(z) e^{-|z|^2}\}.$$

We would like to plug in the approximation (5.4) into (5.5). The operator \mathbf{T}_r is a sum of certain polynomials in \bar{z} of degree $\leq r$ times the differential operator ∂_z to powers $\leq r$. The Cauchy integral formula allows us to control the size of the derivatives on a compact subset in terms of the size of the functions on a slightly bigger compact subset. This means that the approximation (5.4) carries over, and we find that

$$(5.6) \quad K_{\delta\text{Pol}_{m,n,r+1}^{(1)}}(\xi, \eta) = [\mathbf{T}_r]_{\xi}[\bar{\mathbf{T}}_r]_{\eta}(K_{\mathbb{B}_0[L^2(\mathbb{R}_-)]}(\xi, \eta)) + O(m^{-\frac{1}{2}+\varepsilon}),$$

with uniform control on compact subsets. Next, as

$$\text{Pol}_{m,n,q}^{(1)} = \bigoplus_{r=0}^{q-1} \delta\text{Pol}_{m,n,r+1}^{(1)},$$

we get that

$$(5.7) \quad K_{\text{Pol}_{m,n,q}^{(1)}}(\xi, \eta) = \sum_{r=0}^{q-1} K_{\delta\text{Pol}_{m,n,r+1}^{(1)}}(\xi, \eta) \\ = \sum_{r=0}^{q-1} [\mathbf{T}_r]_{\xi} [\bar{\mathbf{T}}_r]_{\eta} (K_{\mathbf{B}_0[L^2(\mathbb{R}_-)]}(\xi, \eta)) + O(m^{-\frac{1}{2}+\varepsilon}),$$

again with uniform control on compact subsets. Next, it should be rather clear that

$$[\mathbf{T}_r]_{\xi} [\bar{\mathbf{T}}_r]_{\eta} (K_{\mathbf{B}_0[L^2(\mathbb{R}_-)]}(\xi, \eta))$$

is the reproducing kernel for the space $\mathbf{T}_r \mathbf{B}_0[L^2(\mathbb{R}_-)] = \mathbf{B}_r[L^2(\mathbb{R}_-)]$, which was identified in terms of Hermite polynomials back in Proposition 5.4. We write this down as a proposition.

Proposition 5.9. *Fix a positive real number ε . Then the reproducing kernel for $\text{Pol}_{m,n,q}^{(1)}$ has the following form:*

$$K_{\text{Pol}_{m,n,q}^{(1)}}(\xi, \eta) = \sum_{r=0}^{q-1} \frac{e^{\xi\bar{\eta}}}{r! \sqrt{2\pi}} \int_{-\infty}^{-\xi-\bar{\eta}} H_r(t + \xi - \eta) H_r(t + \bar{\eta} - \bar{\xi}) e^{-\frac{1}{2}t^2} dt + O(m^{-\frac{1}{2}+\varepsilon}),$$

as $m, n \rightarrow +\infty$ while $n = m + O(1)$, where the control is uniform on compact subsets.

If we like, we may use the classical Christoffel-Darboux identity

$$(5.8) \quad \sum_{r=0}^{q-1} \frac{1}{r!} H_r(x) H_r(y) = \frac{H_q(x) H_{q-1}(y) - H_{q-1}(x) H_q(y)}{(q-1)!(x-y)}$$

to rewrite the above sum. Also, we should note that reproducing kernel for the blow-up space $\text{Pol}_{m,n,q}^{(1)}$ is connected with the reproducing kernel $K_{m,n,q}$ for $\text{Pol}_{m,n,q}$ via the identity

$$(5.9) \quad K_{\text{Pol}_{m,n,q}^{(1)}}(\xi, \eta) = m^{-1} e^{-m^{-1/2}(\xi+\bar{\eta})} K_{m,n,q}(1 + m^{-1/2}\xi, 1 + m^{-1/2}\eta).$$

The blow-up of the 1-point intensity near a boundary point. The 1-point intensity function is

$$K_{m,n,q}(z, z) e^{-m|z|^2},$$

and the localized version with $z = 1 + m^{-1/2}\xi$ is

$$U_{m,n,q}(\xi) := m^{-1} K_{m,n,q}(1 + m^{-1/2}\xi, 1 + m^{-1/2}\xi) e^{-m|1+m^{-1/2}\xi|^2},$$

where we throw in a factor of m^{-1} to compensate for the Jacobian. In view of (5.9) together with Proposition 5.9, we obtain

$$U_{m,n,q}(\xi) = \sum_{r=0}^{q-1} \frac{1}{r! \sqrt{2\pi}} \int_{-\infty}^{-2\text{Re}\xi} H_r(t)^2 e^{-\frac{1}{2}t^2} dt + O(m^{-\frac{1}{2}+\varepsilon}).$$

So, essentially, the 1-point intensity function is determined by the density

$$t \mapsto \sum_{r=0}^{q-1} \frac{1}{r! \sqrt{2\pi}} H_r(t)^2 e^{-\frac{1}{2}t^2},$$

which corresponds to filling the lowest energy eigenstates of the harmonic oscillator. By the Wigner semi-circle law, then, we get the approximation

$$U_{m,n,q}(\xi) \approx \frac{2q}{\pi} \int_{-1}^{-q^{-1/2} \operatorname{Re} \xi} \sqrt{1 - \tau^2} d\tau,$$

valid for big m, n with $n = m + O(1)$, and big q (but much smaller than m, n). So, if we rescale to characteristic distance $q^{1/2} m^{-1/2}$ we find an interesting law in the limit.

The blow-up Berezin density at a boundary point. The blow-up Berezin density at 1 is given by

$$\hat{B}_{m,n,q}^{(1)}(\xi) = m^{-1} B_{m,n,q}^{(1)}(1 + m^{-1/2} \xi) = m^{-1} e^{-m|1+m^{-1/2}\xi|^2} \frac{|K_{m,n,q}(1 + m^{-1/2} \xi, 1)|^2}{K_{m,n,q}(1, 1)}.$$

From (5.9), we have that

$$K_{\operatorname{Pol}_{m,n,q}^{(1)}}(0, 0) = m^{-1} e^{-m} K_{m,n,q}(1, 1),$$

while Proposition 5.9 gives

$$K_{\operatorname{Pol}_{m,n,q}^{(1)}}(0, 0) = \sum_{r=0}^{q-1} \frac{1}{r! \sqrt{2\pi}} \int_{-\infty}^0 H_r(t)^2 e^{-\frac{1}{2}t^2} dt + O(m^{-\frac{1}{2}+\varepsilon}),$$

as $m, n \rightarrow +\infty$ with $n = m + O(1)$. Now, as each Hermite polynomial H_r is either even or odd,

$$\int_{-\infty}^0 H_r(t)^2 e^{-\frac{1}{2}t^2} dt = \frac{1}{2} \int_{-\infty}^{+\infty} H_r(t)^2 e^{-\frac{1}{2}t^2} dt = \frac{r! \sqrt{2\pi}}{2},$$

which leads to

$$K_{\operatorname{Pol}_{m,n,q}^{(1)}}(0, 0) = \frac{q}{2} + O(m^{-\frac{1}{2}+\varepsilon})$$

and

$$K_{m,n,q}(1, 1) = \frac{1}{2} m q e^m [1 + O(m^{-\frac{1}{2}+\varepsilon})].$$

A similar calculation gives that

$$\begin{aligned} K_{m,n,q}(1 + m^{-1/2} \xi, 1) &= m e^{m+m^{1/2}\xi} K_{\operatorname{Pol}_{m,n,q}^{(1)}}(\xi, 0) \\ &= m e^{m+m^{1/2}\xi} \left\{ \sum_{r=0}^{q-1} \frac{1}{r! \sqrt{2\pi}} \int_{-\infty}^{-\xi} H_r(t + \xi) H_r(t - \xi) e^{-\frac{1}{2}t^2} dt + O(m^{-\frac{1}{2}+\varepsilon}) \right\}. \end{aligned}$$

Putting things together, we obtain the following asymptotics for the blow-up Berezin density.

Theorem 5.10. *Fix a positive real number ε . Then the blow-up Berezin density at 1 has the following form:*

$$\hat{B}_{m,n,q}^{(1)}(\xi) = \frac{1}{\pi q} \left| \sum_{r=0}^{q-1} \frac{1}{r!} \int_{-\infty}^{-\xi} H_r(t + \xi) H_r(t - \xi) e^{-\frac{1}{2}t^2} dt \right|^2 + O(m^{-\frac{1}{2}+\varepsilon}),$$

as $m, n \rightarrow +\infty$ while $n = m + O(1)$, where the control is uniform on compact subsets.

Remark 5.11. When we make some explicit calculations based on Theorem 5.10, we see that the Fresnel zone pattern is less pronounced for a boundary point z .

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