On strict decrease of the Holevo quantity*

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Abstract

A sufficient condition for strict decreasing of the Holevo quantity of an ensemble of quantum states under action of a quantum channel in terms of properties of this channel is obtain. Several corollaries of this condition are considered.

1 Introduction

The Holevo quantity $\chi(\{\pi_i, \rho_i\})$ of an ensemble of quantum states $\{\pi_i, \rho_i\}$ provides an upper bound for accessible classical information which can be obtained by applying a quantum measurement [5]. Fundamental monotonicity property of the relative entropy implies nonincreasing of the Holevo quantity under action of an arbitrary quantum channel Φ, that is

$$\chi(\{\pi_i, \Phi(\rho_i)\}) \leq \chi(\{\pi_i, \rho_i\}). \tag{1}$$

for any ensemble of quantum states $\{\pi_i, \rho_i\}$.

Necessary and sufficient conditions for the case of equality in fundamental entropic inequalities of quantum theory have been intensively studied (see [3, 10, 12, 14] and references therein). In particular, two characterization of the equality in (1) are obtained in [3, Examples 4 and 9]. The first one derived

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from the Petz theorem characterizing the equality case in monotonicity of the relative entropy states that the equality in (1) holds if and only if

$$\rho_i = A\Phi^*(B\Phi(\rho_i)B)A, \quad A = (\bar{\rho})^{1/2},\ B = (\Phi(\bar{\rho}))^{-1/2}, \quad \forall i, \tag{2}$$

where Φ^* is a dual map to the channel Φ and $\bar{\rho}$ is the average state of the ensemble $\{\pi_i, \rho_i\}$. The second characterization is derived from characterization of the equality case in the strong subadditivity of the quantum entropy by identifying the channel Φ with a subchannel of a partial trace. This approach makes it possible to obtain a necessary and sufficient condition for the case of equality in (1), but it is not clear how to apply this condition to a given quantum channel Φ.

In this paper we derive from (2) a sufficient (not necessary) condition for the strict inequality in (1) expressed in terms of properties of the channel Φ (Theorem 1, Remark 1, Corollary 1). The main advantage of this condition consists in possibility to use it in analysis of entropic characteristics of a given quantum channel determined as extremal values of functionals depending on the Holevo quantity (such as the Holevo capacity and the related characteristics).

Several applications of the obtained condition concerning the notions of the Holevo capacity and the minimal output entropy of a quantum channel as well as properties of the quantum conditional entropy are considered (Corollaries 2-5).

2 The main results

Let $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_E$ be finite dimensional Hilbert spaces. In what follows $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is a quantum channel and $\widehat{\Phi} : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ is its complementary channel, defined uniquely up to unitary equivalence [7].

A channel Φ is called entanglement-breaking if for an arbitrary Hilbert space $\mathcal{K}$ the state $\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\omega)$ is separable for any state $\omega \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$ [9]. This notion is generalized in [1] as follows.

Definition 1. A channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is called *r-partially entanglement-breaking* (briefly r-PEB) if for an arbitrary Hilbert space $\mathcal{K}$ the Schmidt number[1] of the state $\Phi \otimes \mathrm{Id}_{\mathcal{K}}(\omega)$ does not exceed r for any state $\omega \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$.

[1] The Schmidt number $SN(\omega)$ of a state ω of a bipartite system is defined as follows:

In this notation entanglement-breaking channels are 1-PEB channels. Properties and equivalent definitions of r-partially entanglement-breaking channels are studied in [1], where it is proved, in particular, that a channel Φ is r-PEB if and only if it has the Kraus representation $\Phi(\cdot) = \sum_k V_k(\cdot)V_k^*$ such that the maximal rank of the operators V_k does not exceed r (this a natural generalization of the well known characterization of entanglement-breaking channels proved in [9]).[2]

Let $H(\rho)$ and $H(\rho\|\sigma)$ be respectively the von Neumann entropy of the state ρ and the quantum relative entropy of the states ρ and σ [8, 11].

A collection of states $\{\rho_i\}$ with the corresponding probability distribution $\{\pi_i\}$ is called *ensemble* and denoted $\{\pi_i, \rho_i\}$. The state $\bar{\rho} = \sum_i \pi_i\rho_i$ is called the *average state* of the ensemble $\{\pi_i, \rho_i\}$.

The Holevo quantity of an ensemble $\{\pi_i, \rho_i\}$ is defined as follows

$$\chi(\{\pi_i, \rho_i\}) = \sum_i \pi_i H(\rho_i\|\bar{\rho}) = H(\bar{\rho}) - \sum_i \pi_i H(\rho_i). \tag{3}$$

By monotonicity of the relative entropy for an arbitrary quantum channel Φ we have

$$\chi(\{\pi_i, \Phi(\rho_i)\}) \leq \chi(\{\pi_i, \rho_i\}). \tag{4}$$

Inequality (4) means convexity of the entropy gain $H(\Phi(\rho)) - H(\rho)$ of the channel Φ.

The results of this paper are based on the following observation.

Theorem 1. *Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel and $\widehat{\Phi} : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_E)$ be its complementary channel. If there exists an ensemble $\{\pi_i, \rho_i\}$ with the full rank average state such that $\mathrm{rank}\rho_i \leq r \in \mathbb{N}$ for all i and*

$$\chi(\{\pi_i, \Phi(\rho_i)\}) = \chi(\{\pi_i, \rho_i\}) \tag{5}$$

then $\widehat{\Phi}$ is a r-partially entanglement-breaking channel.

Remark 1. By Theorem 1 to prove the strict inequality in (4) for all ensembles $\{\pi_i, \rho_i\}$ such that $\mathrm{supp}\,\bar{\rho} = \mathcal{H}_A$ and $\mathrm{rank}\rho_i \leq r$ for all i it suffices

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$SN(\omega) = \min_{\sum_i \pi_i\omega_i=\omega, \mathrm{rank}\omega_i=1} \max_i SR(\omega_i)$, where SR is the Schmidt rank of a pure state of a bipartite system.

[2]Strictly speaking, the above characterization is proved in [1] in the case $\mathcal{H}_A = \mathcal{H}_B$, but it seems valid in general. In this paper we will use only the part "if" of this characterization which is easily verified.

to show that the channel $\widehat{\Phi}$ is not r-partially entanglement-breaking. This can be done by showing existence of a state $\omega \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{K})$ such that

$$\text{either} \quad SN(\widehat{\Phi} \otimes \mathrm{Id}_{\mathcal{K}}(\omega)) > r \quad \text{or} \quad E(\widehat{\Phi} \otimes \mathrm{Id}_{\mathcal{K}}(\omega)) > \log r, \tag{6}$$

where E is any convex entanglement monotone coinciding on the set of pure states with the entropy of a partial state, in particular, $E = EoF$ [13].

The condition $\operatorname{supp} \bar{\rho} = \mathcal{H}_A$ can be removed by considering the restriction of the channel $\widehat{\Phi}$ to the set $\mathfrak{S}(\mathcal{H}_{\bar{\rho}})$, where $\mathcal{H}_{\bar{\rho}} = \operatorname{supp} \bar{\rho}$. Thus, to prove the strict inequality in (4) for an arbitrary ensemble $\{\pi_i, \rho_i\}$ such that $\mathrm{rank}\rho_i \leq r$ for all i it suffices to show existence of a state $\omega \in \mathfrak{S}(\mathcal{H}_{\bar{\rho}} \otimes \mathcal{K})$ such that (6) holds.

By Remark 3 below the above sufficient condition for strict decreasing of the Holevo quantity is not necessary.

Proof. Let $\bar{\rho} \in \mathfrak{S}(\mathcal{H}_A)$ be the average state of the ensemble $\{\pi_i, \rho_i\}$. Without loss of generality we may assume that $\Phi(\bar{\rho})$ is a full rank state in $\mathfrak{S}(\mathcal{H}_B)$.

By condition (2) equality (5) means that $A_i = \Phi^*(B_i)$ for all i, where $A_i = \pi_i(\bar{\rho})^{-1/2}\rho_i(\bar{\rho})^{-1/2}$ and $B_i = \pi_i(\Phi(\bar{\rho}))^{-1/2}\Phi(\rho_i)(\Phi(\bar{\rho}))^{-1/2}$ are positive operators in $\mathfrak{B}(\mathcal{H}_A)$ and in $\mathfrak{B}(\mathcal{H}_B)$ correspondingly.

Let $\widehat{\Phi}(\rho) = \sum_{k=1}^n V_k \rho V_k^*$ be the Kraus representation of the channel $\widehat{\Phi}$, where $n = \dim \mathcal{H}_B$. Then (up to unitary equivalence) we have

$$\Phi(\rho) = \sum_{k,l=1}^n \mathrm{Tr} V_k \rho V_l^* |k\rangle\langle l| \quad \text{and} \quad \Phi^*(A) = \sum_{k,l=1}^n \langle l|A|k\rangle V_l^* V_k,$$

where $\{|\,k\rangle\}_{k=1}^n$ is an orthonormal basis in $\mathcal{H}_B$.

By the spectral theorem $B_i = \sum_j |\psi_{ij}\rangle\langle\psi_{ij}|$, where $\{|\psi_{ij}\rangle\}_j$ is a set of vectors in $\mathcal{H}_B$, for each i. Since $\Phi(\bar{\rho})$ is a full rank state, we have

$$\sum_{i,j} |\psi_{ij}\rangle\langle\psi_{ij}| = \sum_i B_i = I_B.$$

By Lemma 1 below $\widehat{\Phi}(\rho) = \sum_{i,j} W_{ij} \rho W_{ij}^*$, where $W_{ij} = \sum_{k=1}^n \langle\psi_{ij}|k\rangle V_k$.

Since $A_i = \Phi^*(\sum_j |\psi_{ij}\rangle\langle\psi_{ij}|)$ is an operator of rank $\leq r$ for each i and

$$\Phi^*(|\psi_{ij}\rangle\langle\psi_{ij}|) = \sum_{k,l=1}^n \langle l|\psi_{ij}\rangle\langle\psi_{ij}|k\rangle V_l^* V_k = W_{ij}^* W_{ij},$$

the family $\{W_{ij}\}$ consists of operators of rank $\leq r$. Hence $\widehat{\Phi}$ is a r-PEB channel [1]. $\square$

Lemma 1. *Let $\Phi(\rho) = \sum_{k=1}^{n} V_k \rho V_k^*$ be a quantum channel and $\{|k\rangle\}_{k=1}^{n}$ be an orthonormal basis in the n-dimensional Hilbert space $\mathcal{H}_n$. An arbitrary overcomplete system $\{|\psi_i\rangle\}$ of vectors in $\mathcal{H}_n$ generates the Kraus representation $\Phi(\rho) = \sum_i W_i \rho W_i^*$ of the channel Φ, where $W_i = \sum_{k=1}^{n} \langle\psi_i|k\rangle V_k$.*

Proof. Since $\sum_i |\psi_i\rangle\langle\psi_i| = I_{\mathcal{H}_n}$, we have

$$\sum_i W_i \rho W_i^* = \sum_{k,l=1}^{n} V_k \rho V_l^* \sum_i \langle\psi_i|k\rangle\langle l|\psi_i\rangle$$
$$= \sum_{k,l=1}^{n} V_k \rho V_l^* \sum_i \mathrm{Tr}|k\rangle\langle l||\psi_i\rangle\langle\psi_i| = \sum_{k=1}^{n} V_k \rho V_k^*. \ \square$$

The class of quantum channels complementary to entanglement-breaking channels coincides with the class of pseudo-diagonal channels described in the following definition [2, 7].

Definition 2. A channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is called *pseudo-diagonal* if it has the representation

$$\Phi(\rho) = \sum_{i,j} c_{ij} \langle\psi_i|\rho|\psi_j\rangle |i\rangle\langle j|$$

where $\|c_{ij}\|$ is a nonnegative-definite matrix, $\{|\psi_i\rangle\}$ is a collection of vectors in $\mathcal{H}_A$ satisfying the overcompleteness relation $\sum_i c_{ii}|\psi_i\rangle\langle\psi_i| = I_{\mathcal{H}_A}$ and $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_B$.

Theorem 1 with $r = 1$ implies the following observation.

Corollary 1. *Let Φ be a non-pseudo-diagonal quantum channel. Then*

$$\chi(\{\pi_i, \Phi(\rho_i)\}) < \chi(\{\pi_i, \rho_i\}) = H(\bar{\rho})$$

for any ensemble $\{\pi_i, \rho_i\}$ of pure states with the full rank average state $\bar{\rho}$.

The Holevo capacity of the channel Φ can be defined as follows

$$\bar{C}(\Phi) = \sup_{\{\pi_i, \rho_i\}} \chi(\{\pi_i, \Phi(\rho_i)\}). \tag{7}$$

Monotonicity the Holevo quantity shows that

$$\bar{C}(\Phi) \leq \log \dim \mathcal{H}_A$$

for any quantum channel $\Phi:\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$. Since the supremum in (7) is always achieved at some ensembles of pure states [15], Corollary 1 implies the following observation.

Corollary 2. *Let $\Phi:\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$ be a non-pseudo-diagonal quantum channel. Then*

$$\bar{C}(\Phi)<\log\dim\mathcal{H}_A.$$

Corollary 2 can be used to show positivity of the minimal output entropy

$$H_{\min}(\Phi)=\inf_{\rho\in\mathfrak{S}(\mathcal{H}_A)}H(\Phi(\rho))$$

for a class of quantum channels.

Corollary 3. *Let $\Phi:\mathfrak{S}(\mathcal{H}_A)\to\mathfrak{S}(\mathcal{H}_B)$, $\mathcal{H}_B=\mathcal{H}_A$, be a quantum channel covariant with respect to some irreducible representation $\{V_g\}_{g\in G}$ of a compact group G in the sense that $\Phi(V_g\rho V_g^*)=V_g\Phi(\rho)V_g^*$ for all $g\in G$. If Φ is a non-pseudo-diagonal channel then $H_{\min}(\Phi)>0$.*

Proof. It follows from the covariance condition of the corollary that $\bar{C}(\Phi)=\log\dim\mathcal{H}_A-H_{\min}(\Phi)$ [6]. By Corollary 2 we have $H_{\min}(\Phi)>0$. $\square$

Corollary 3 shows, in particular, that $H_{\min}(\Phi)>0$ for any unital qubit channel, which is not pseudo-diagonal (in particular, is not degradable[3]).

Corollary 4. *Let $\mathcal{H}_A=\mathcal{H}_B\otimes\mathcal{H}_E$ and Φ be a partial trace, that is $\Phi(\rho)=\mathrm{Tr}_{\mathcal{H}_E}\rho$. Then $\chi(\{\pi_i,\Phi(\rho_i)\})<\chi(\{\pi_i,\rho_i\})$ for any ensemble $\{\pi_i,\rho_i\}$ with the full rank average state such that $\mathrm{rank}\rho_i<\dim\mathcal{H}_E$ for all i.*

Proof. It suffices to note that the channel $\widehat{\Phi}(\rho)=\mathrm{Tr}_{\mathcal{H}_B}\rho$ is not r-PEB for $r<\dim\mathcal{H}_E$. $\square$

Remark 2. By the Stinespring representation every channel is isomorphic to a particular subchannel of the partial trace channel. Since the Holevo quantity does not strict decrease for all channels, Corollary 4 clarifies necessity of the full rank average state condition in Theorem 1. $\square$

The conditional entropy of a state ρ of a composite system AB is defined as follows

$$H_{A|B}(\rho)\doteq H(\rho)-H(\mathrm{Tr}_{\mathcal{H}_A}\rho).$$

[3]A characterization of degradable qubit channels is obtained in [2].

It is well known that the function $\rho \mapsto H_{A|B}(\rho)$ is concave [8, 11]. Corollary 4 implies the following strict concavity property of the conditional entropy.

Corollary 5. *Let ρ be a full rank state in $\mathfrak{S}(\mathcal{H}_{AB})$. Then*

$$H_{A|B}(\rho) > \sum_i \pi_i H_{A|B}(\rho_i)$$

for any ensemble $\{\pi_i, \rho_i\}$ with the average state ρ such that $\mathrm{rank}\rho_i < \dim \mathcal{H}_A$ for all i.

We complete the paper by the following remark.

Remark 3. The condition of Corollary 1 (and hence of Theorem 1) is not necessary: strict decreasing of the Holevo quantity for all ensembles of pure states for a given channel does not imply that this channel is not pseudo-diagonal. To show this consider the pseudo-diagonal channel[4]

$$\Phi(\rho) = \sum_{k=1}^{3} \langle \varphi_k|\rho|\varphi_k\rangle |k\rangle\langle k|,$$

where

$$|\varphi_1\rangle = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |\varphi_2\rangle = \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, |\varphi_3\rangle = \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

are vectors in the 2-D space $\mathcal{H}_A$ and $\{|k\rangle\}_{k=1}^{3}$ is an orthonormal basis in the 3-D space $\mathcal{H}_B$.

Suppose there exists an ensemble $\{\pi_i, \rho_i\}$ of pure states with the full rank average state $\bar{\rho}$ such that $\chi(\{\pi_i, \Phi(\rho_i)\}) = \chi(\{\pi_i, \rho_i\})$. Since $\Phi(\bar{\rho})$ is a full rank state and $\Phi^*(A) = \sum_{k=1}^{3} \langle k|A|k\rangle |\varphi_k\rangle\langle\varphi_k|$, condition (2) implies $\mathrm{rank}\Phi(\rho_i) = 1$ for any i. But this can not be valid, since it is easy to see that $\mathrm{rank}\Phi(\rho) > 1$ for any ρ. Hence $\chi(\{\pi_i, \Phi(\rho_i)\}) < \chi(\{\pi_i, \rho_i\})$ for any ensemble $\{\pi_i, \rho_i\}$ of pure states with the full rank average state. $\square$

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[4] The prototype of this q-c channel was introduced in [4].

References

[1] D.Chruscinski, A.Kossakowski "On Partially Entanglement Breaking Channels", Open Sys. Information Dyn., V.13 P.17-26, 2006; arXiv:quant-ph/0511244.

[2] T.S.Cubitt, M.B.Ruskai, G.Smith "The structure of degradable quantum channels", J. Math. Phys., V.49, 102104, 2008; arXiv:0802.1360.

[3] P.Hayden, R.Jozsa, D. Petz, A.Winter "Structure of states which satisfy strong subadditivity of quantum entropy with equality", Commun. Math. Phys., V.246, N 2, P.359-374, 2004; arXiv:quant-ph/0304007.

[4] A.S.Holevo "Information-Theoretical Aspects of Quantum Measurement", Problems Inform. Transmission, V.9, N.2, P.110-118, 1973.

[5] A.S. Holevo "Some estimates for information quantity transmitted by quantum communication channel", Probl. Inform. Transm., V.9, N3, P.3-11, 1973.

[6] A.S.Holevo "Remarks on the classical capacity of quantum channel", arXiv:quant-ph/0212025.

[7] A.S.Holevo "On complementary channels and the additivity problem", Probability Theory and Applications, V.51. N.1. P.134-143, 2006; arXiv:quant-ph/0509101.

[8] A.S.Holevo "Quantum systems, channels, information", MCCME, Moscow 2010 (in Russian).

[9] M.Horodecki, P.W.Shor, M.B.Ruskai "General Entanglement Breaking Channels", Rev. Math. Phys., V.15, P.629-641, 2003; arXiv:quant-ph/0302031.

[10] A.Jencova, M.B.Ruskai "A Unified Treatment of Convexity of Relative Entropy and Related Trace Functions, with Conditions for Equality", arXiv:0903.2895.

[11] M.A.Nielsen, I.L.Chuang "Quantum Computation and Quantum Information", Cambridge University Press, 2000.

[12] D.Petz "Monotonicity of quantum relative entropy revisited", Rev. Math. Phys., V.15, P.79-91, 2003; arXiv:quant-ph/0209053.

[13] M.B.Plenio, S.Virmani "An introduction to entanglement measures", Quantum Inf. Comput., V.7, P.1-51, 2007; arXiv:quant-ph/0504163.

[14] M.B.Ruskai, "Inequalities for Quantum Entropy: A Review with Conditions for Equality", J. Math. Phys. V.43, P.4358-4375, 2002; arXiv:quant-ph/0205064.

[15] B.Schumacher, M.D.Westmoreland "Optimal signal ensemble", arXiv:quant-ph/9912122.