Unharnessing the power of Schrijver's permanental inequality (The Very First Draft - More To Follow)

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Abstract

Let $A \in \Omega_n$ be doubly-stochastic $n \times n$ matrix. Alexander Schrijver proved in 1998 the following remarkable inequality

$$per(\widetilde{A}) \geq \prod_{1 \leq i,j \leq n} (1 - A(i,j)); \widetilde{A}(i,j) =: A(i,j)(1 - A(i,j)), 1 \leq i,j \leq n \tag{1}$$

We prove in this paper the following generalization (or just clever reformulation) of (1):
For all pairs of $n \times n$ matrices (P, Q), where P is nonnegative and Q is doubly-stochastic

$$\log(per(P)) \geq \sum_{1 \leq i,j \leq n} \log(1 - Q(i,j))(1 - Q(i,j)) - \sum_{1 \leq i,j \leq n} Q(i,j) \log(\frac{Q(i,j)}{P(i,j)}) \tag{2}$$

The main corrollary of (2) is the following inequality for doubly-stochastic matrices:

$$\frac{per(A)}{F(A)} \geq 1; F(A) =: \prod_{1 \leq i,j \leq n} (1 - A(i,j))^{1 - A(i,j)}.$$

We present explicit doubly-stochastic $n \times n$ matrices A with the ratio $\frac{per(A)}{F(A)} = \sqrt{2}^n$ and conjecture that

$$\max_{A \in \Omega_n} \frac{per(A)}{F(A)} \approx \left(\sqrt{2}\right)^n.$$

If true, it would imply a deterministic poly-time algorithm to approximate the permanent of $n \times n$ nonnegative matrices within the relative factor $\left(\sqrt{2}\right)^n$.

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1 The permanent

Recall that a $n \times n$ matrix A is called doubly stochastic if it is nonnegative entry-wise and its every column and row sum to one. The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n. Let $\Lambda(k,n)$ denote the set of $n \times n$ matrices with nonnegative integer entries and row and column sums all equal to k . We define the following subset of rational doubly stochastic matrices: $\Omega_{k,n} = \{k^{-1}A : A \in \Lambda(k,n)\}$.

Recall that the permanent of a square matrix A is defined by

$$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A(i, \sigma(i)).$$

1.1 Schrijver-Valiant Conjecture and (main) Schrijver's permanental inequality

Define

$$\lambda(k,n) = \min\{per(A) : A \in \Omega_{k,n}\} = k^{-n} \min\{per(A) : A \in \Lambda(k,n)\};$$

$\theta(k) = \lim_{n \to \infty} (\lambda(k,n))^{\frac{1}{n}}$.

It was proved in [2] (also earlier in [1]) that, using our notations, $\theta(k) \leq G(k) =: (\frac{k-1}{k})^{k-1}$ and conjectured that $\theta(k) = G(k)$. Though the case of $k = 3$ was proved by M. Voorhoeve in 1979 [15] , this conjecture was settled only in 1998 [3] (17 years after the published proof of the Van der Waerden Conjecture). The main result of [3] (as many people, including myself, wrongly thought) is the remarkable **(Schrijver-bound)** :

$$\min\{per(A) : A \in \Omega_{k,n}\} \geq \left(\frac{k-1}{k}\right)^{(k-1)n} \tag{3}$$

The bound (3) is a corollary of another inequality for doubly-stochastic matrices:

$$per(\widetilde{A}) \geq \prod_{1 \leq i,j \leq n} (1 - A(i,j)); A \in \Omega_n; \widetilde{A}(i,j) =: A(i,j)(1 - A(i,j)), 1 \leq i,j \leq n. \tag{4}$$

The proof of (4) in [3] is, in the words of its author, "highly complicated". Surprisingly, the only known to me application of (4) is the bound (3), which applies only to "very" rational doubly-stochastic matrices. The main goal of this paper is to show the amazing power of (4), which has been overlooked for 13 years.

2 A Generalization of Schrijver's permanental inequality

We prove in this section the following theorem, stated in [9] in a rather cryptic way.Fortunatelly, the paper cites [10] and M. Chertkov is my colleague in Los Alamos.
The statement in the current paper has been communicated to me by Misha Chertkov, to whom I am profoundly grateful.

Theorem 2.1: *Define for a pair (P, Q) of non-negative matrices the following functional:*

$$CW(P,Q) =: \sum_{1 \leq i,j \leq n} \log(1 - Q(i,j))(1 - Q(i,j)) - \sum_{1 \leq i,j \leq n} Q(i,j) \log\left(\frac{Q(i,j)}{P(i,j)}\right). \quad (5)$$

(Note that for fixed P the functional $CW(P,Q) = \sum_{1 \leq i,j \leq n} F_{i,j}(Q(i,j))$ and $F_{i,j}(0) = 0$. The functional $CW(P,Q)$ is concave in P; it is in general neither concave nor convex in Q.)

If $Per(P) > 0$ then $max_{Q \in \Omega_n} CW(P,Q)$ is attained and

$$\log(Per(P)) \geq max_{Q \in \Omega_n} CW(P,Q) \quad (6)$$

(It is assumed that $0^0 = 1$.)
An equivalent statement of this theorem is

$$\log(Per(P)) \geq \sum_{1 \leq i,j \leq n} \log(1 - Q(i,j))(1 - Q(i,j)) - \sum_{1 \leq i,j \leq n} Q(i,j) \log\left(\frac{Q(i,j)}{P(i,j)}\right) : P \geq 0, Q \in \Omega_n \quad (7)$$

Proof: We will prove, to avoid trivial technicalities, just the positive case, i.e when $P(i,j) > 0, 1 \leq i,j \leq n$.
We compute first partial derivatives:

$$\frac{\partial}{\partial Q} CW(P,Q) = \{-2 - \log(1 - Q(i,j)) - \log(Q(i,j)) + \log(P(i,j)) : 1 \leq i,j \leq n\} \quad (8)$$

In the positive case, i.e. for the fixed positive P, the functional $CW(P,Q)$ is bounded and continuous on Ω_n. Therefore the maximum exists. Let $V \in \Omega_n$ be one of argmaximums, i.e.

$$CW(P,V) = max_{Q \in \Omega_n} CW(P,Q).$$

Then, after some column/row permutations

$$V = \begin{pmatrix} V_{1,1} & 0 & \ldots & 0 \\ 0 & V_{2,2} & 0 & \ldots 0 \\ . & . & . & . \\ 0 & \ldots & 0 & V_{k,k} \end{pmatrix};$$

$$P = \begin{pmatrix} P_{1,1} & . & ... & 0 \\ . & P_{2,2} & . & .... \\ . & . & . & . \\ . & ... & . & P_{k,k} \end{pmatrix};$$

The diagonal blocks $V_{i,i}$ are indecomposable doubly-stochastic $d_i \times d_i$ matrices; $\sum_{1 \le i \le k} d_i = n$ and $1 \le k \le n$. Clearly,

$$CW(P, V) = \sum_{1 \le i \le k} CW(P_{i,i}, V_{i,i}).$$

As $\log(per(P)) \ge \sum_{1 \le i \le k} \log(per(P_{i,i}))$ it is sufficient to prove that

$$\log(Per(P_{i,i})) \ge CW(P_{i,i}, V_{i,i}); 1 \le i \le k.$$

For blocks of size one, the inequality is trivial: $(1-1)^{1-1} - 1\log(\frac{1}{a}) = \log(a)$.
Consider a (indecomposable) block $V_{i,i}$ of size $d_i \ge 2$ and define its support

$$Supp(V_{i,i}) = \{(k, l) : V_{i,i}(k, l) > 0\}.$$

We will write the local extremality condition not on full Ω_{d_i} but rather on its compact convex subset of doubly-stochastic matrices which are zero outside of $Supp(P_{i,i})$.
Using (8) and doing standard Lagrange multipliers respect to variables $V_{i,i}(k, l), (k, l) \in Supp(V_{i,i})$, we get that there exists real numbers $(\alpha_k; \beta_l)$ such that

$$-2 - \log(1 - V_{i,i}(k, l)) - \log(V_{i,i}(k, l)) + \log(P_{i,i}(k, l)) = \alpha_k + \beta_l : (k, l) \in Supp(V_{i,i}).$$

Which gives for some positive numbers a_k, b_l the following scaling :

$$P_{i,i}(k, l) = a_k b_l V_{i,i}(k, l)(1 - V_{i,i}(k, l)); (k, l) \in Supp(V_{i,i}). \tag{9}$$

It follows from the definition of the support that

1. $$P_{i,i} \ge Diag(a_k)\widetilde{V_{i,i}}Diag(b_l); \widetilde{V_{i,i}}(k, l) = V_{i,i}(k, l)(1 - V_{i,i}(k, l)). \tag{10}$$

2. $$CW(P_{i,i}, V_{i,i}) = \sum \log(a_k) + \sum \log(b_l) + \sum_{(k,l) \in Supp(V_{i,i})} \log(1 - V_{i,i}(k, l)). \tag{11}$$

Finally it follows from (11) and Schriver's permanental inequality (4) that

$$\log(per(Diag(a_k)\widetilde{V_{i,i}}Diag(b_l)) \ge CW(P_{i,i}, V_{i,i});$$

and that

$$\log(per(P_{i,i})) \ge \log(per(Diag(a_k)\widetilde{V_{i,i}}Diag(b_l)) \ge CW(P_{i,i}, V_{i,i}).$$

∎

3 Corollaries

1. Schrijver's permanental inequality (4) is a particular case of (7). Indeed
$$CW(\widetilde{V}, V) = \sum_{1 \le i,j \le n} \log(1 - V(i,j)) : V \in \Omega_n.$$

2. Let $P \in \Omega_n$ be doubly-stochastic $n \times n$ matrix. Then
$$log(per(P)) \ge CW(P,P) = \sum_{1 \le i,j \le n} \log(1 - P(i,j))(1 - P(i,j)).$$
We get the following important inequality:
$$\frac{per(P)}{F(P)} \ge 1; F(P) =: \prod_{1 \le i,j \le n} (1 - P(i,j))^{1 - P(i,j)}; P \in \Omega_n \tag{12}$$
(To say more on this). The lower bound (12) suggests the importance of the following quantity:
$$UB(n) =: max_{P \in \Omega_n} \frac{per(P)}{F(P)}.$$
It is easy to show that the limit
$$UB =: \lim_{n \to \infty} (UB(n))^{\frac{1}{n}}$$
exists and $1 \le UB \le e$. There is obvious deterministic poly-time algorithm to approximate the permanent of nonnegative matrices within relative factor $UB(n)$. The current best rate is e^n. Therefore proving that $UB < e$ is of major algorithmic importance.

Example 3.1: I. Let $P = aJ_n + bI_n, a = \frac{1}{2(n-1)}, b = \frac{n-2}{2(n-1)}$, i.e. the diagonal $P(i,i) = \frac{1}{2}, 1 \le i \le n$ and the off-diagonal entries are equal to $\frac{1}{2(n-1)}$.
It is easy to see that for these (a,b):
$$2^{-n+1} \le per(aJ_n + bI_n) = n!a^{-n} \sum_{0 \le i \le n} \frac{1}{i!} \left(\frac{b}{a}\right)^i \le n!a^{-n} exp\left(\frac{b}{a}\right).$$
Non-difficult calculations show that for this $P \in \Omega_n$
$$\frac{per(P)}{F(P)} \approx \left(\sqrt{\frac{e}{2}}\right)^n \tag{13}$$
II. Let $P \in \Omega_2 = \frac{1}{2}J_2$ be 2×2 "uniform" doubly-stochastic matrix. The direct inspection gives that
$$CW(P,Q) \equiv -2\log(2) = F(P), Q \in \Omega_n.$$

Consider now the direct sum $P_{2n} \in \Omega_{2n} = \frac{1}{2}J_2 \oplus ... \oplus \frac{1}{2}J_2$. Then

$$max_{Q \in \Omega_{2n}} CW(P_{2n}, Q) = \log(F(P_{2n})) = -2n \log(2). \tag{14}$$

Therefore in this case

$$\frac{per(P_{2n})}{F(P_{2n})} = 2^n. \tag{15}$$

Which gives the following lower bound on $UB(k)$ for even k:

$$UB(k) \geq (\sqrt{2})^k. \tag{16}$$

As $max_{Q \in \Omega_{2n}} CW(P_{2n}, Q) = \log(F(P_{2n}))$, this class of matrices also provides a counter-example to the non-trivial part of **Conjecture 15** in [9].
Is the bound (16) sharp? ∎

3. Recall the main function from [7]:

$$G(x) = \left(\frac{x-1}{x}\right)^{x-1}, x \geq 1.$$

Note that for $P \in \Omega_n$ the column product

$$CPR(j) =: \prod_{1 \leq i \leq n} (1 - P(i,j))^{1-P(i,j)} \geq G(n). \tag{17}$$

Define C_j as the number of non-zero entries in the jth column then

$$CPR(j) =: \prod_{1 \leq i \leq n} (1 - P(i,j))^{1-P(i,j)} \geq G(C_j). \tag{18}$$

The inequality (17) gives a slightly weaker version of the celebrated Falikman-Egorychev-Van der Waerden lower bound $vdw(n) =: \frac{n!}{n^n}$:

$$per(P) \geq \prod_{1 \leq j \leq n} CPR(j) \geq \left(\frac{n-1}{n}\right)^{n(n-1)}$$

The inequality (18) gives a non-regular real-valued version of **(Schrijver-bound)**:

$$per(P) \geq \prod_{1 \leq j \leq n} CPR(j) \geq \prod_{1 \leq j \leq n} G(C_j) \tag{19}$$

In the worst case, my bound from [7] is better:

$$per(P) \geq \prod_{1 \leq j \leq n} G\left(min(j, C_j)\right)$$

Perhaps, it is true that

Conjecture 3.2:

$$per(P) \geq \prod_{1 \leq j \leq n} G\left(min(j, EC_j)\right)?$$

where the effective real-valued degree $EC_j = G^{-1}(CPR(j))$. ∎

4 Some historical remarks

The column products $CPR(j) =: \prod_{1 \leq i \leq n}(1 - P(i,j))^{1-P(i,j)} \geq G(C_j)$ have appeared in the permanent context before. Let $P = [a|b,..,|b] \in \Omega_n$ be doubly-stochastic matrix with 2 distinct columns. Then (Proposition 2.2 in [13])

$$Per(P) \geq CPR(1) vdw(n-1). \tag{20}$$

Let us recall a few notations from [7] and [5]:

1. The linear space of homogeneous polynomials with real (complex) coefficients of degree n and in m variables is denoted $Hom_R(m,n)$ ($Hom_C(m,n)$).
   We denote as $Hom_+(m,n)$ ($Hom_{++}(n,m)$) the closed convex cone of polynomials $p \in Hom_R(m,n)$ with nonnegative (positive) coefficients.
2. For a polynomial $p \in Hom_+(n,n)$ we define its **Capacity** as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, \ldots, x_n) = \inf_{x_i > 0} \frac{p(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n} x_i}. \tag{21}$$

3. The following product polynomial is associated with $n \times n$ matrix P:

$$Prod_P(x_1, \ldots, x_n) =: \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} P(i,j) x_j. \tag{22}$$

4. $$q_{(j)} =: \frac{\partial}{\partial x_j} Prod_P(x_1, \ldots, x_n) : x_j = 0.$$

   Note that the polynomials $q_{(j)} \in Hom_+(n-1, n-1)$
   For example, $q_{(n)} = \frac{\partial}{\partial x_n} Prod_P(x_1, \ldots, x_{n-1}, 0)$.

The following lower bound, which is much stronger than (20), was proved in [5]:

$$Cap(q_{(j)}) \geq CPR(j), 1 \leq j \leq n. \tag{23}$$

Combining results from [7] (i.e. $Per(P) \geq vdw(n-1) Cap(q_{(j)}), 1 \leq j \leq n$) and (23) gives a different version of (12)

$$Per(P) \geq \left(\prod_{1 \leq j \leq n} CPR(j) \right)^{\frac{1}{n}} vdw(n-1), P \in \Omega_n. \tag{24}$$

Perhaps, it is even true that

Conjecture 4.1:

$$Per(P) \geq \prod_{1 \leq j \leq n} Cap(q_{(j)})?$$

∎

5 Credits

The Definition (5) apparently has rich and important stat-physics meaning centered around so called *Bethe Approximation*. Although this stat-physics background was not used in the current paper, it and its developers(to be named in the final version) deserve a lot of praise: don't forget that many very good mathematicians have completely overlooked seemingly simple Theorem 2.1. It would be fantastic to have a rigorous and readable proof of Theorem 2.1 based on new(age) methods. I am a bit sceptical at this point: any such proof would essentially reprove very hard Schrijver's permanental bound. The other avenue is to better understand the original Schrijver's proof, perhaps it has some deep stat-physics meaning.It is possible that one can use higher order approximation(the Bethe Approximation being of order two, it involves marginals of subsets of cardinality two). Luckily, this order two case is covered by Schrijver's lower bound (4). The higher order cases will probably need new lower bounds (involving subpermanents?). It looks like a beginning of a beautiful(and hard) new line of research.

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