

On the new translational shape invariant potentials

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Abstract. Recently, several authors have found new translational shape invariant potentials not present in classic classifications like that of Infeld and Hull. For example, Quesne on the one hand and Bougie, Gangopadhyaya and Mallow on the other have provided examples of them, consisting on deformations of the classical ones. We analyze the basic properties of the new examples and observe a compatibility equation which has to be satisfied by them. We study particular cases of such equation and give more examples of new translational shape invariant potentials.

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1. Introduction

In the late forties and early fifties of last century, Infeld and Hull [1–3] introduced the so called *factorization method*, for defining and solving problems in unidimensional Quantum Mechanics. In the mid-eighties of last century emerged the notion of *shape invariance* in Quantum Mechanics [4, 5] which has a close resemblance to that of the factorization method. In fact, it has been proved in a review article [6] the complete equivalence of both approaches. As a result, a list of shape invariant potentials have been produced, see, e.g., [6, 7]. This list has been generalized to shape invariant potentials which depend on $n > 1$ parameters transformed by translation [8].

However, very recently, new shape invariant potentials have been discovered whose parameter transforms by translation and are not present in the classifications mentioned above. The first of such examples is given by Quesne [9], work which inspired the research of Bougie, Gangopadhyaya and Mallow [10, 11]. More examples are found in the work by Odake and Sasaki [12, 13] although we will not consider them in this article. Quesne asks herself in [9] about the reason why her example is isospectral to a shape invariant potential of the ordinary type (ordinary in the sense that it belongs to the classical classifications already mentioned).

This paper studies the basic properties of the found examples and apply the results in the search of new translational shape invariant potentials using as data the classical shape invariant potentials as classified in [6], which are essentially the same potentials as in the classical Infeld and Hull [3] classification. The article is organized as follows. In the first section we recall the classic framework of shape invariance. In the second we describe the equations which satisfy the new examples of [9, 11] in order to induce a common framework for these cases. We also particularize to the specific forms of superpotentials given in [6]. In the third section we describe all the examples we have found with this technique. In the fourth section we offer some conclusions and an outlook for future research.

2. Intertwined Hamiltonians and Shape Invariance

The simplest way of generating an exactly solvable Hamiltonian \tilde{H} from a known one H is just to consider an invertible bounded operator B , with bounded inverse, and defining $\tilde{H} = BHB^{-1}$. This transformed Hamiltonian \tilde{H} has the same spectrum as the starting one H . As a generalization (see, e.g., [14]), we will say that two Hamiltonians H and \tilde{H} are intertwined or A -related when $AH = \tilde{H}A$, where A may have no inverse. In this case, if ψ is an eigenvector of \tilde{H} corresponding to the eigenvalue ϵ and $A\psi \neq 0$, at least formally $A\psi$ is an eigenvector of H corresponding to the same eigenvalue ϵ .

If A is a first order differential operator,

$$A = \frac{d}{dx} + W(x), \quad \text{and} \quad A^\dagger = -\frac{d}{dx} + W(x), \quad (1)$$

then the relation $AH = \tilde{H}A$, with

$$H = -\frac{d^2}{dx^2} + V(x), \quad \tilde{H} = -\frac{d^2}{dx^2} + \tilde{V}(x), \quad (2)$$

leads to

$$V = -2W' + \tilde{V}, \quad W(V - \tilde{V}) = -W'' - V'.$$

Taking into account the first equation, the second becomes $2WW' = W'' + V'$, which can easily be integrated giving

$$V = W^2 - W' + \epsilon, \quad (3)$$

and then,

$$\tilde{V} = W^2 + W' + \epsilon, \quad (4)$$

where ϵ is an integration constant. The important point here is that H and \tilde{H} , given by (2), are related by a first order differential operator A , given by (1), if and only if there exist a constant ϵ and a function W such that the pair of Riccati equations (3) and (4) are satisfied *simultaneously*. Moreover, this means that both Hamiltonians can be factorized as

$$H = A^\dagger A + \epsilon, \quad \tilde{H} = AA^\dagger + \epsilon. \quad (5)$$

Adding and subtracting equations (3) and (4) we obtain the equivalent pair which relates V and \tilde{V}

$$\tilde{V} - \epsilon = -(V - \epsilon) + 2W^2, \quad (6)$$

$$\tilde{V} = V + 2W'. \quad (7)$$

The function W satisfying these equations is usually called the *superpotential*, the constant ϵ is the *factorization energy* or *factorization constant* and \tilde{V} and V (resp. \tilde{H} and H) are said to be *partner potentials* (resp. Hamiltonians).

Notice that the initial solvable Hamiltonian can arbitrarily be chosen as H or \tilde{H} . In both cases the point will be to find a solution W of the corresponding Riccati equation (3) or (4) for a specific factorization energy ϵ . From this solution the expression for the (possibly) new potential follows immediately from (7).

Note that these equations have an intimate relation with what it is currently known as *Darboux transformations* of linear second-order differential equations [15, 16], or in the context of one-dimensional (or supersymmetric) quantum mechanics [17]. In fact, it is easy to prove that the equation (3) can be transformed into a Schrödinger equation $-\phi'' + (V(x) - \epsilon)\phi = 0$ by means of the change $-\phi'/\phi = W$. Likewise, by means of $\tilde{\phi}'/\tilde{\phi} = W$ (4) transforms into $-\tilde{\phi}'' + (\tilde{V}(x) - \epsilon)\tilde{\phi} = 0$. The relation between V and \tilde{V}

is given by (7). Obviously, $\phi\tilde{\phi} = 1$, up to a non-vanishing constant factor. It is also worth noting that these Schrödinger equations express that ϕ and $\tilde{\phi}$ are respective eigenfunctions of the Hamiltonians (2) for the eigenvalue ϵ .

The factorization method has been introduced by Infeld and Hull [1–3] providing the tools for solving algebraically a class of unidimensional potentials which showed a translational symmetry in one parameter. In the mid-eighties of last century emerged the notion of *shape invariance* in Quantum Mechanics [4, 5] which generalized the parameter space which is subject to a invertible transformation. Later, taken into account this generalization it has been proved that both approaches coincide, see, e.g., [6].

In essence, it is taken equations (3) and (4) as a definition of the functions V , \tilde{V} in terms of the function W and some constant ϵ . After, one can assume that W did depend on certain set of parameters a , i.e., $W = W(x, a)$, and as a consequence $V = V(x, a)$ and $\tilde{V} = \tilde{V}(x, a)$ as well. Then, the necessary condition for $\tilde{V}(x, a)$ to be essentially of the same form as $V(x, a)$, maybe for a different choice of the values of the parameters involved in V , is known as shape invariance. It amounts to assume the further relation between $V(x, a)$ and $\tilde{V}(x, a)$

$$\tilde{V}(x, a) = V(x, f(a)) + R(f(a)), \quad (8)$$

where f is an (invertible) transformation on the parameter space a and R is some function of the parameters only.

Let us remark that it is the choice of the parameter space a and of the (invertible) transformations $f(a)$ what define the different types of shape invariant potentials. Note that in principle, different types of shape invariant potentials may have members in common. Note as well that the function f may be even the identity, i.e., $f(a) = a$ for all a [18].

Just writing the a -dependence the equations (3), (4) become

$$V(x, a) - \epsilon = W^2 - W', \quad (9)$$

$$\tilde{V}(x, a) - \epsilon = W^2 + W'. \quad (10)$$

The simplest way of satisfying these equations is assuming that $V(x, a)$ and $\tilde{V}(x, a)$ are obtained from a superpotential function $W(x, a)$ by means of

$$V(x, a) - \epsilon = W^2(x, a) - W'(x, a), \quad (11)$$

$$\tilde{V}(x, a) - \epsilon = W^2(x, a) + W'(x, a). \quad (12)$$

The shape invariance property requires the further condition (8) to be satisfied, which in these terms reads

$$W^2(x, a) - W^2(x, f(a)) + W'(x, f(a)) + W'(x, a) = R(f(a)). \quad (13)$$

In practice, when searching shape invariant potentials with a given parameter space a and the transformation function f , what it is done is to (try to) find solutions for $W(x, a)$ and $R(a)$ of (13), instead of solving the pair (11), (12) and then imposing (8).

Now, we will consider the simplest but particularly important case of shape invariant potentials having only one parameter whose transformation law is a translation. In other words, this case corresponds to the whole family of factorizable problems treated in [3]. Thus, we will consider problems where the parameter space is unidimensional, and the transformation law is

$$f(a) = a - \delta, \quad \text{or} \quad f(a) = a + \delta, \quad (14)$$

where $\delta \neq 0$. In both cases we can normalize the parameter in units of δ , introducing the new parameter

$$m = \frac{a}{\delta}, \quad \text{or} \quad m = -\frac{a}{\delta}, \quad (15)$$

respectively. In each of these two possibilities the transformation law reads, with a slight abuse of the notation f ,

$$f(m) = m - 1. \quad (16)$$

and the equations which should be solved in order to find potentials in this class are

$$V(x, m) - \epsilon = W^2(x, m) - W'(x, m), \quad (17)$$

$$\tilde{V}(x, m) - \epsilon = W^2(x, m) + W'(x, m), \quad (18)$$

or the equivalent equations

$$\tilde{V}(x, m) - \epsilon = -(V(x, m) - \epsilon) + 2W^2(x, m), \quad (19)$$

$$\tilde{V}(x, m) = V(x, m) + 2W'(x, m), \quad (20)$$

as well as the shape invariance condition

$$\tilde{V}(x, m) = V(x, m - 1) + R(m - 1). \quad (21)$$

3. Properties of the new translational shape invariant potentials

In the examples of [9] and [11] the superpotential function takes the form of

$$W(x, m) = W_0(x, m) + W_{1+}(x, m) - W_{1-}(x, m), \quad (22)$$

where $W_0(x, m)$ is the superpotential of a pair of shape invariant partner potentials of the classical type, and $W_{1+}(x, m)$, $W_{1-}(x, m)$ are functions of a type described below. Substituting (22) into (17) and (18) it is observed that the final partner potentials have the form (the constant ϵ is taken as zero)

$$V(x, m) = V_0(x, m) - 2W'_{1+}(x, m), \quad (23)$$

$$\tilde{V}(x, m) = \tilde{V}_0(x, m) - 2W'_{1-}(x, m), \quad (24)$$

where $V_0(x, m)$, $\tilde{V}_0(x, m)$ is the pair of shape invariant partner potentials associated to $W_0(x, m)$. To this end, it is necessary that the following *compatibility condition* hold:

$$\begin{aligned} W_{1+}^2 + W'_{1+} + W_{1-}^2 + W'_{1-} \\ - 2W_0W_{1-} + 2W_0W_{1+} - 2W_{1-}W_{1+} = 0 \end{aligned} \quad (25)$$

(the dependence on the arguments has been omitted for brevity and clearness). The condition of shape invariance (21) reads in this case

$$\begin{aligned} \tilde{V}(x, m) - V(x, m - 1) - R(m - 1) \\ = \tilde{V}_0(x, m) - V_0(x, m - 1) - R(m - 1) \\ - 2W'_{1-}(x, m) + 2W'_{1+}(x, m - 1) = 0 \end{aligned}$$

that is equal, using (21) for the partner potentials $\tilde{V}_0(x, m)$ and $V_0(x, m)$, to

$$-2W'_{1-}(x, m) + 2W'_{1+}(x, m - 1) = 0,$$

thus we obtain a second *shape invariance condition*:

$$W'_{1-}(x, m) = W'_{1+}(x, m - 1) \quad (26)$$

3.1. Differential equation satisfied by $W_{1+}(x, m)$ and $W_{1-}(x, m)$

We will consider in this subsection superpotentials $W_0(x, m)$ of the form $W_0(x, m) = k_0(x) + mk_1(x)$, where $k_0(x)$ and $k_1(x)$ are not functions of m and has been of use in the classification of shape invariant potentials, see [3, 6].

It is further observed that in the examples of [9, 11] the functions $W_{1+}(x, m)$ and $W_{1-}(x, m)$ (in the notation of (22)) satisfy the differential equation

$$W' + W^2 - k_1(x)W = 0, \quad (27)$$

which is a Bernoulli equation with $n = 1$ (it can be regarded as a special type of Riccati equation, see, e.g., [19]). Equation (27) is explicitly solvable by two quadratures (we make two integration constants explicit):

$$W(x) = \frac{d}{dx} \log \left(c_2 + \int^x \exp \left(\int^y k_1(z) dz + \log(c_1) \right) dy \right). \quad (28)$$

Using that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ satisfy (27) in this case, the compatibility condition (25) reduces to

$$2(k_0 + mk_1)(W_{1+} - W_{1-}) + k_1(W_{1+} + W_{1-}) - 2W_{1-}W_{1+} = 0, \quad (29)$$

which is an algebraic equation to be satisfied. The method for obtaining new translational shape invariant potentials is now clear: to select solutions of (27) which satisfy (29) for each specific case of superpotential $k_0(x) + mk_1(x)$. We do it in the next section.

4. Examples

We will analyze, following the previous procedure, all cases of superpotential $k_0(x) + mk_1(x)$ present in [3, 6]. We will obtain therefore, the potentials of [9, 11] (slightly generalized) and other new potentials.

4.1. Case of $k_0(x) + mk_1(x) = \frac{b}{c} \tanh(cx) + \frac{d}{\cosh(cx)} + mc \tanh(cx)$

For this case $x \in (-\infty, \infty)$ and $k_1(x) = c \tanh(cx)$, where $c > 0$ is a constant. Then, the solutions $W_{1+}(x, m)$ and $W_{1-}(x, m)$ to (27) can be written

$$W_{1+}(x) = \frac{cc_1 \cosh(cx)}{cc_2 + c_1 \sinh(cx)}$$

$$W_{1-}(x) = \frac{cc_3 \cosh(cx)}{cc_4 + c_3 \sinh(cx)}$$

where c_1, c_2, c_3, c_4 are constants to be determined. Inserting these expressions into (29) lead to the following relations between the previous constants:

$$c_2 = \frac{c_1 (2b + 2c^2m + c^2)}{2c^2d}$$

$$c_4 = \frac{c_3 (2b + 2c^2m - c^2)}{2c^2d}$$

such that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \frac{2c^2d \cosh(cx)}{2b + c^2(2m + 1) + 2cd \sinh(cx)}$$

$$W_{1-}(x, m) = \frac{2c^2d \cosh(cx)}{2b + c^2(2m - 1) + 2cd \sinh(cx)}$$

This case generalize slightly that found in [11], with a different notation. However, it has a substantial defect, which is that both of $W_{1+}(x, m)$ and $W_{1-}(x, m)$ present singularities at some point of the domain $(-\infty, \infty)$. The reason is that when $d \neq 0$ the function $2cd \sinh(cx)$ has the whole real line as image and is strictly monotone, and the graph of vertical translations of it always crosses the horizontal axis once.

$$4.2. \text{ Case of } k_0(x) + mk_1(x) = \frac{b}{c} \coth(cx) + \frac{d}{\sinh(cx)} + mc \coth(cx)$$

For this case $x \in (0, \infty)$ and $k_1(x) = c \coth(cx)$, where $c > 0$ is a constant. Following the procedure analogous to the previous subsection we find that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \frac{2c^2 d \sinh(cx)}{2cd \cosh(cx) - 2b - c^2(2m + 1)}$$

$$W_{1-}(x, m) = \frac{2c^2 d \sinh(cx)}{2cd \cosh(cx) - 2b + c^2(1 - 2m)}$$

These functions are free of singularities, when $d > 0$ if $m < \frac{2cd - 2b - c^2}{2c^2}$, and when $d < 0$ if $m > \frac{2cd - 2b + c^2}{2c^2}$. This leads to a new case in the literature.

$$4.3. \text{ Case of } k_0(x) + mk_1(x) = \pm \frac{b}{c} + d \exp(\mp cx) \pm mc$$

For this case $x \in (-\infty, \infty)$ and $k_1(x) = \pm c$, where $c > 0$ is a constant. Following the procedure analogous to the previous subsections we find that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \pm c$$

$$W_{1-}(x, m) = \pm c$$

This leads to a new, although rather trivial, case in the literature.

$$4.4. \text{ Case of } k_0(x) + mk_1(x) = \frac{b}{2}x + \frac{d}{x} + \frac{m}{x}$$

For this case $x \in (0, \infty)$ and $k_1(x) = \frac{1}{x}$. Following the procedure analogous to the previous subsections we find that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \frac{2bx}{bx^2 - 1 - 2d - 2m}$$

$$W_{1-}(x, m) = \frac{2bx}{bx^2 + 1 - 2d - 2m}$$

These functions are free of singularities when $m < -\frac{1}{2}(1 + 2d)$. This is the case of [9] although slightly generalized.

$$4.5. \text{ Case of } k_0(x) + mk_1(x) = bx + d$$

For this case $x \in (-\infty, \infty)$ and $k_1(x) = 0$. Following the procedure analogous to the previous subsections we find no nontrivial solutions for $W_{1+}(x, m)$ and $W_{1-}(x, m)$.

4.6. Case of $k_0(x) + mk_1(x) = \frac{b}{c} \tan(cx) + \frac{d}{\cos(cx)} - mc \tan(cx)$

For this case $x \in (-\frac{\pi}{2c}, \frac{\pi}{2c})$ and $k_1(x) = -c \tan(cx)$, where $c > 0$ is a constant. Following the procedure analogous to the previous subsections we find that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \frac{2c^2 d \cos(cx)}{2cd \sin(cx) + 2b - c^2(2m + 1)}$$

$$W_{1-}(x, m) = \frac{2c^2 d \cos(cx)}{2cd \sin(cx) + 2b + c^2(1 - 2m)}$$

These functions are free of singularities, when $d > 0$ if $m > \frac{2b+c^2+2cd}{2c^2}$ or $m < \frac{2b-c^2-2cd}{2c^2}$, and when $d < 0$ if $m > \frac{2b+c^2-2cd}{2c^2}$ or $m < \frac{2b-c^2+2cd}{2c^2}$. This leads to a new case in the literature.

4.7. Case of $k_0(x) + mk_1(x) = -\frac{b}{c} \cot(cx) + \frac{d}{\sin(cx)} - mc \cot(cx)$

For this case $x \in (0, \frac{\pi}{c})$ and $k_1(x) = c \cot(cx)$, where $c > 0$ is a constant. Following the procedure analogous to the previous subsections we find that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \frac{2c^2 d \sin(cx)}{2b - c^2(2m + 1) - 2cd \cos(cx)}$$

$$W_{1-}(x, m) = \frac{2c^2 d \sin(cx)}{2b + c^2(1 - 2m) - 2cd \cos(cx)}$$

These functions are free of singularities, when $d > 0$ if $m > \frac{2b+c^2+2cd}{2c^2}$ or $m < \frac{2b-c^2-2cd}{2c^2}$, and when $d < 0$ if $m > \frac{2b+c^2-2cd}{2c^2}$ or $m < \frac{2b-c^2+2cd}{2c^2}$. This leads to a new case in the literature, related to that of the previous case by a shift in the variable x in $\frac{\pi}{2c}$.

4.8. Case of $k_0(x) + mk_1(x) = \mp i \frac{b}{c} + d \exp(\mp icx) \pm mic$

For this case $x \in (-\infty, \infty)$ and $k_1(x) = \pm ic$, where $c > 0$ is a constant. Following the procedure analogous to the previous subsections we find that $W_{1+}(x, m)$ and $W_{1-}(x, m)$ become

$$W_{1+}(x, m) = \pm ic$$

$$W_{1-}(x, m) = \pm ic$$

This leads to a new, although also trivial, case in the literature.

This is to be remarked that in all these cases $W_{1+}(x, m)$ and $W_{1-}(x, m)$ satisfy a condition slightly stronger than (26), namely

$$W_{1-}(x, m) = W_{1+}(x, m - 1) \quad (30)$$

which obviously implies (26). This means that imposing (29) is a stronger condition than (26) for the cases studied.

We also tried this procedure for the case of superpotential $W_0(x, m) = q/m + mk_1(x)$, where q is a real constant. But we have obtained no nontrivial cases.

5. Conclusions and outlook

We have studied the properties of newly discovered translational shape invariant potentials in the literature, obtained by different means: studying exceptional orthogonal polynomials in [9], and expansions in \hbar in [10, 11]. We have observed that they satisfy several equations and set up a new approach based upon them. In fact, the form of the final partner potentials obtained leads to the fulfillment of a compatibility condition.

For the special case of $W_0(x, m)$ affine in m it moreover holds that the extra terms added to $W_0(x, m)$ satisfy each the same Bernoulli equation, which is explicitly solvable. The compatibility condition becomes no longer differential but an algebraic condition. The constants of the solutions of the Bernoulli equation give enough freedom in order to satisfy the (algebraic) compatibility condition. We have found in this way the cases of [9, 11] and new ones.

For the special case of $W_0(x, m) = q/m + mk_1(x)$ no nontrivial solutions can be found (with the assumptions of Section 4). This is similar to what happened to the extension to $n > 1$ parameters transformed by translation to this form of superpotential: no new nontrivial solutions were found, see [8].

Also, it would be interesting to see whether the present approach applies to the translational shape invariant potentials found in [12, 13].

The form of the final potentials (23) and (24) recalls the ordinary Bäcklund–Darboux transformations in one-dimensional quantum mechanics, see [15, 16] for a classic treatment and [20, 21] for a geometric approach. However, the *transforming function* $W_{1-}(x, m)$ should satisfy a Riccati equation of the type $W' + W^2 = \tilde{V}_0$. Instead, it satisfies a Bernoulli equation. Maybe the work in [22] could help in understanding the problem. This implies that the description of the situation studied in this paper by means of Darboux transformations is an open question, as it is the justification of the isospectrality of the final potentials obtained with the initial pair of partner potentials.

Finally, it would be of importance to determine whether the compatibility condition (25) admits more solutions, even starting from potentials which do not conform a pair of shape invariant potentials. For that, it might be useful to consider cases where the extra terms in the superpotential do not satisfy the previous Bernoulli equation but maybe other relations. This is another open problem.

All of this will lead to interesting new research work.

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