Genus-1 Virasoro conjecture along quantum volume direction

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Abstract

In this paper, we show that the derivative of the genus-1 Virasoro conjecture for Gromov-Witten invariants along the direction of quantum volume element holds for all smooth projective varieties. This result provides new evidence for the Virasoro conjecture.

1 Introduction

The Virasoro conjecture predicts that the generating functions of the Gromov-Witten invariants of smooth projective varieties are annihilated by a sequence of differential operators which form a half branch of the Virasoro algebra. This conjecture was proposed by Eguchi-Hori-Xiong [EHX] and modified by S. Katz [CX]. In case the underlying manifold is a point, this conjecture is equivalent to Witten's conjecture [W], proved by Kontsevich [K], that the generating function of intersection numbers on the moduli spaces of stable curves is a τ-function of the KdV hierarchy. Together with Tian, we proved that the genus-0 part of the Virasoro conjecture holds for all compact symplectic manifolds (cf. [LT]). For manifolds with semisimple quantum cohomology, the genus-1 part of this conjecture was proved by Dubrovin and Zhang [DZ]. Without assuming semisimplicity, the genus-1 Virasoro conjecture was studied in [L1] and [L2]. Among other results, it was proved in [L1] that the genus-1 Virasoro conjecture can be reduced to the the L_1-constraint. Using the genus-1 topological recursion relation, it was also proved that Virasoro constraints can be reduced to equations on the *small phase space*, i.e. the space of cohomology classes of the underlying manifold. Compatibility conditions for Virasoro conjectures were studied in [L2]. Despite these efforts, the general case of the genus-1 Virasoro conjecture is still largely open. In this paper, we give more evidence to the genus-1 Virasoro conjecture without any assumption on the quantum cohomology of the underlying manifold.

Let M be a smooth projective variety. Choose a basis $\{\gamma_\alpha \mid \alpha = 1, \ldots, N\}$ of the space of cohomology classes $H^*(M; \mathbb{C})$. For simplicity, we assume $H^{\text{odd}}(M; \mathbb{C}) = 0$. We choose the basis in such a way that γ_1 is the identity of the cohomology ring and $\gamma_\alpha \in H^{p_\alpha, q_\alpha}(M)$ for some integers p_α and q_α. Let $\{t^1, \ldots, t^N\}$ be the coordinates on $H^*(M; \mathbb{C})$ with respect to this basis. We can identify each γ_α with the vector field $\frac{\partial}{\partial t^\alpha}$ and further identify each cohomology class with a constant vector field on $H^*(M; \mathbb{C})$. Let

$$b_\alpha = p_\alpha - \frac{1}{2}(d-1) \tag{1}$$

where d is the complex dimension of M. Then the *Euler vector field* (on the small phase space) is defined to be

$$E := c_1(M) + \sum_\alpha (b_1 + 1 - b_\alpha) t^\alpha \gamma_\alpha.$$

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We refer to [LiT] [RT] for definitions of Gromov-Witten invariants. In genus-1 case, it suffices to study only primary Gromov-Witten invariants since all genus-1 descendant invariants can be reduced to primary invariants due to the genus-1 topological recursion relation. Therefore we only consider primary Gromov-Witten invariants in this paper. Let F_g be the generating function of genus-g primary Gromov-Witten invariants of M. The *k-point function* is defined to be

$$\langle\langle\, v_1 \cdots v_k \,\rangle\rangle_g := \sum_{\alpha_1,\ldots,\alpha_k} f^1_{\alpha_1} \cdots f^k_{\alpha_k} \, \frac{\partial^k F_g}{\partial t^{\alpha_1} \cdots \partial t^{\alpha_k}},$$

for vector fields $v_i = \sum_\alpha f^i_\alpha \gamma_\alpha$ where f^i_α are functions on $H^*(M;\mathbb{C})$. Note that F_g and $\langle\langle \cdots \rangle\rangle_g$ in this paper corresponds to F^s_g and $\langle\langle \cdots \rangle\rangle_{g,s}$ in [L1]. Let $\eta_{\alpha\beta} = \int_M \gamma_\alpha \cup \gamma_\beta$ be the intersection form on $H^*(M,\mathbb{C})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. For example $\gamma^\alpha := \eta^{\alpha\beta}\gamma_\beta$ where repeated indices should be summed over entire range. We recall that the *quantum product* of two vector fields v_1 and v_2 is defined by

$$v_1 \circ v_2 := \langle\langle\, v_1 \, v_2 \, \gamma^\alpha \,\rangle\rangle_0 \, \gamma_\alpha.$$

Define

$$\Psi := \langle\langle\, E^2 \,\rangle\rangle_1 + \frac{1}{24} \sum_\alpha \langle\langle E E \gamma_\alpha \gamma^\alpha \rangle\rangle_0 - \frac{1}{2} \sum_\alpha \left(b_\alpha(1-b_\alpha) - \frac{b_1+1}{6} \right) \langle\langle \gamma_\alpha \gamma^\alpha \rangle\rangle_0 \qquad (2)$$

where $E^2 = E \circ E$ is the quantum square of the Euler vector field. It was proved in [L1] that, for any smooth projective variety M, the *genus-1 Virasoro conjecture* can be reduced to a single equation on $H^*(M;\mathbb{C})$:

$$\Psi = 0. \qquad (3)$$

Moreover, since $E\Psi = \Psi$ (cf. [L1, Lemma 6.3]), the genus-1 Virasoro conjecture holds if and only if

$$E\Psi = 0. \qquad (4)$$

Therefore, to prove the genus-1 Virasoro conjecture, it suffices to show that $v\Psi = 0$ for all vector field v on $H^*(M;\mathbb{C})$. It follows from the string equation that $\gamma_1\Psi = 0$ where $\gamma_1 = E^0$ is the identity of the ordinary cohomology ring. In this paper we will give another vector field which always annihilates Ψ.

Define the vector field

$$\Delta := \gamma^\alpha \circ \gamma_\alpha. \qquad (5)$$

If, in the definition of Δ, we replace the quantum product "$\circ$" by the ordinary cup product, we get a vector field proportional to the volume element. Therefore we call Δ the *quantum volume element*. The main result of this paper is the following

Theorem 1.1 *For all smooth projective varieties,*

$$\Delta\Psi = 0.$$

This result provides a new evidence for the genus-1 Virasoro conjecture.

2 Properties of Euler vector fields

We first recall some basic properties of the Euler vector field E. We start with the *quasi-homogeneity equation*

$$\langle\langle E\rangle\rangle_g = (3-d)(1-g)F_g + \frac{1}{2}\delta_{g,0}\sum_{\alpha,\beta}\mathcal{C}_{\alpha\beta}t_0^\alpha t_0^\beta - \frac{1}{24}\delta_{g,1}\int_M c_1(M)\cup c_{d-1}(M).$$

This equation is a consequence of the divisor equation. Define the grading operator G by

$$G(v) := \sum_\alpha b_\alpha f_\alpha \gamma_\alpha$$

for any vector field $v = \sum_\alpha f_\alpha\gamma_\alpha$. Derivatives of quasi-homogeneity equation has the form

$$\begin{aligned}\langle\!\langle\, E\, v_1\, \cdots\, v_k \,\rangle\!\rangle_g &= \sum_{i=1}^k \langle\!\langle\, v_1\, \cdots\, G(v_i)\, \cdots\, v_k\,\rangle\!\rangle_g \\ &\quad -(2g+k-2)(b_1+1)\,\langle\!\langle\, v_1\, \cdots\, v_k\,\rangle\!\rangle_g \\ &\quad +\delta_{g,0}\nabla^k_{v_1,\cdots,v_k}\left(\frac{1}{2}\mathcal{C}_{\alpha\beta}t_0^\alpha t_0^\beta\right) \end{aligned} \tag{6}$$

where $\mathcal{C}_{\alpha\beta}$ is defined by $c_1(M)\cup\gamma_\alpha = \mathcal{C}_\alpha^\beta\gamma_\beta$, and ∇ is the trivial connection on $H^*(M;\mathbb{C})$ defined by $\nabla\gamma_\alpha = 0$ for all α. In particular,

$$\langle\!\langle\, E\, v_1\, v_2\, \gamma^\alpha\,\rangle\!\rangle_0\, \gamma_\alpha = G(v_1)\circ v_2 + v_1\circ G(v_2) - G(v_1\circ v_2) - b_1 v_1\circ v_2. \tag{7}$$

Combining with [L1, Lemma 4.2], we can obtain

$$\begin{aligned}\nabla_{E^k}\Delta &= \left\langle\!\!\left\langle\, E^k\, \gamma^\alpha\, \gamma_\alpha\, \gamma^\beta\,\right\rangle\!\!\right\rangle_0 \gamma_\beta \\ &= (k-b_1)E^{k-1}\circ\Delta - G(E^{k-1}\circ\Delta) - \sum_{i=1}^{k-1}\,\Delta\circ E^{i-1}\circ G(E^{k-i}) \\ &\quad -\sum_{i=1}^{k-1} G(\Delta\circ E^{i-1})\circ E^{k-i}\end{aligned} \tag{8}$$

for $k\geq 1$. Covariant derivative of E is given by

$$\nabla_v E = -G(v) + (b_1+1)v. \tag{9}$$

Using the fact that

$$\nabla_w(v_1\circ v_2) = (\nabla_w v_1)\circ v_2 + v_1\circ(\nabla_w v_2) + \langle\!\langle\, w\, v_1\, v_2\, \gamma^\alpha\,\rangle\!\rangle_0\,\gamma_\alpha,$$

we can also show that

$$\nabla_\Delta E^2 = \Delta\circ G(E) - G(\Delta)\circ E - G(\Delta\circ E) + (b_1+2)\Delta\circ E. \tag{10}$$

Combining equations (8) and (10), we have

$$[E^2,\Delta] = -2b_1 E\circ\Delta - 2G(E)\circ\Delta.$$

3 Proof of the main theorem

For any vector fields $v_1, \ldots v_4$ on the small phase space, we define

$$\begin{aligned} G_0(v_1, v_2, v_3, v_4) &= \sum_{g \in S_4} \sum_{\alpha,\beta} \left\{ \frac{1}{6} \langle\langle v_{g(1)} v_{g(2)} v_{g(3)} \gamma^\alpha \rangle\rangle_0 \left\langle\left\langle \gamma_\alpha v_{g(4)} \gamma_\beta \gamma^\beta \right\rangle\right\rangle_0 \right. \\ &\quad + \frac{1}{24} \langle\langle v_{g(1)} v_{g(2)} v_{g(3)} v_{g(4)} \gamma^\alpha \rangle\rangle_0 \left\langle\left\langle \gamma_\alpha \gamma_\beta \gamma^\beta \right\rangle\right\rangle_0 \\ &\quad \left. - \frac{1}{4} \left\langle\left\langle v_{g(1)} v_{g(2)} \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta v_{g(3)} v_{g(4)} \rangle\rangle_0 \right\}, \end{aligned}$$

and

$$\begin{aligned} G_1(v_1, v_2, v_3, v_4) &= \sum_{g \in S_4} 3 \langle\langle \{v_{g(1)} \circ v_{g(2)}\}\{v_{g(3)} \circ v_{g(4)}\} \rangle\rangle_1 \\ &\quad - \sum_{g \in S_4} 4 \langle\langle \{v_{g(1)} \circ v_{g(2)} \circ v_{g(3)}\} v_{g(4)} \rangle\rangle_1 \\ &\quad - \sum_{g \in S_4} \sum_\alpha \langle\langle \{v_{g(1)} \circ v_{g(2)}\} v_{g(3)} v_{g(4)} \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \rangle\rangle_1 \\ &\quad + \sum_{g \in S_4} \sum_\alpha 2 \langle\langle v_{g(1)} v_{g(2)} v_{g(3)} \gamma^\alpha \rangle\rangle_0 \langle\langle \{\gamma_\alpha \circ v_{g(4)}\} \rangle\rangle_1 . \end{aligned}$$

Note that G_0 is completely determined by genus-0 data, while each term in G_1 contains genus-1 information. These two tensors are connected by Getzler's equation (cf. [Ge]):

$$G_0 + G_1 = 0. \tag{11}$$

Theorem 1.1 is obtained by applying this equation to $v_1 = v_2 = E$, $v_3 = \gamma^\alpha$, $v_4 = \gamma_\alpha$, and summing over α.

We first consider the genus-1 part of Equation (11).

Lemma 3.1

$$\sum_\alpha G_1(E, E, \gamma^\alpha, \gamma_\alpha) = 24 \Delta \left\langle\left\langle\, E^2 \,\right\rangle\right\rangle_1 .$$

Proof: We will use the convention that repeated indices should be summed over their entire range. Therefore we will omit $\sum_\alpha$ in the left hand of this formula. To compute $G_1(E, E, \gamma^\alpha, \gamma_\alpha)$, we notice that

$$\begin{aligned} \langle\langle\, \{E \circ \gamma_\alpha\}\, \{\gamma^\alpha \circ E\} \,\rangle\rangle_1 &= \left\langle\left\langle\, E \gamma_\alpha \gamma^\beta \,\right\rangle\right\rangle_0 \langle\langle\, E \gamma^\alpha \gamma^\mu \,\rangle\rangle_0 \langle\langle\, \gamma_\beta \gamma_\mu \,\rangle\rangle_1 \\ &= \langle\langle\, E E \gamma_\alpha \,\rangle\rangle_0 \left\langle\left\langle\, \gamma^\alpha \gamma^\beta \gamma^\mu \,\right\rangle\right\rangle_0 \langle\langle\, \gamma_\beta \gamma_\mu \,\rangle\rangle_1 \\ &= \left\langle\left\langle\, \{E^2 \circ \gamma^\mu\}\, \gamma_\mu \,\right\rangle\right\rangle_1 = \left\langle\left\langle\, \{E^2 \circ \gamma^\alpha\}\, \gamma_\alpha \,\right\rangle\right\rangle_1 . \end{aligned}$$

In the second equality, we have used the associativity of the quantum product. This observation enables us to simplify the formula for $G_1(E, E, \gamma^\alpha, \gamma_\alpha)$ and obtain

$$\begin{aligned}
& G_1(E, E, \gamma^\alpha, \gamma_\alpha) \\
= \ & 24 \langle\langle E^2 \Delta \rangle\rangle_1 - 48 \langle\langle \{E \circ \Delta\} E \rangle\rangle_1 - 4 \left\langle\left\langle E^2 \gamma^\alpha \gamma_\alpha \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& -16 \left\langle\left\langle \{E \circ \gamma^\alpha\} \gamma_\alpha E \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 - 4 \left\langle\left\langle \Delta E E \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
& +24 \left\langle\left\langle E E \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \{\gamma_\alpha \circ \gamma_\beta\} \rangle\rangle_1 + 24 \left\langle\left\langle E \gamma_\alpha \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \{\gamma_\beta \circ E\} \rangle\rangle_1 . \qquad (12)
\end{aligned}$$

We now use formulas in Section 2 to compute each term on the right hand side of this equation. Using equation (10), we have

$$\begin{aligned}
\langle\langle E^2 \Delta \rangle\rangle_1 &= \Delta \langle\langle E^2 \rangle\rangle_1 - \langle\langle \{\nabla_\Delta E^2\} \rangle\rangle_1 \\
&= \Delta \langle\langle E^2 \rangle\rangle_1 - \langle\langle \{\Delta \circ G(E) - G(\Delta) \circ E - G(\Delta \circ E) + (b_1 + 2)\Delta \circ E\} \rangle\rangle_1 .
\end{aligned}$$

Since $\langle\langle E \rangle\rangle_1$ is a constant due to the quasi-homogeneity equation, by equation (9), we have

$$\begin{aligned}
\langle\langle \{E \circ \Delta\} E \rangle\rangle_1 &= \{E \circ \Delta\} \langle\langle E \rangle\rangle_1 - \langle\langle \{\nabla_{E \circ \Delta} E\} \rangle\rangle_1 \\
&= \langle\langle \{G(E \circ \Delta) - (b_1 + 1)E \circ \Delta\} \rangle\rangle_1 .
\end{aligned}$$

By equation (8), we have

$$\left\langle\left\langle E^2 \gamma^\alpha \gamma_\alpha \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 = \langle\langle \{(2 - b_1)E \circ \Delta - G(E \circ \Delta) - G(E) \circ \Delta - E \circ G(\Delta)\} \rangle\rangle_1 .$$

By equation (7), we have

$$\begin{aligned}
& \left\langle\left\langle \{E \circ \gamma^\alpha\} \gamma_\alpha E \gamma^\beta \right\rangle\right\rangle_0 \langle\langle \gamma_\beta \rangle\rangle_1 \\
= \ & \langle\langle \{G(E \circ \gamma^\alpha) \circ \gamma_\alpha + E \circ \gamma^\alpha \circ G(\gamma_\alpha) - G(E \circ \Delta) - b_1 E \circ \Delta\} \rangle\rangle_1 .
\end{aligned}$$

As a convention, we arrange the basis $\{\gamma_1, \ldots, \gamma_N\}$ of $H^*(M, \mathbb{C})$ in such a way that the degree $p_\alpha + q_\alpha$ of $\gamma_\alpha \in H^{p_\alpha, q_\alpha}$ is non-decreasing with respect to α and if two cohomology classes have the same dimension, we also require that the holomorphic dimension p_α is non-decreasing. Under this convention, we have

$$G(\gamma^\alpha) = (1 - b_\alpha)\gamma^\alpha$$

for all α, and

$$G(\gamma^\alpha) \circ \gamma_\alpha = \Delta - \gamma^\alpha \circ G(\gamma_\alpha).$$

On the other hand,

$$G(\gamma^\alpha) \circ \gamma_\alpha = \eta^{\alpha\beta} G(\gamma_\beta) \circ \gamma_\alpha = G(\gamma_\beta) \circ \gamma^\beta = \gamma^\alpha \circ G(\gamma_\alpha).$$

So we must have

$$G(\gamma^\alpha) \circ \gamma_\alpha = \gamma^\alpha \circ G(\gamma_\alpha) = \frac{1}{2} \Delta. \qquad (13)$$

Hence

$$\begin{aligned}
G(E \circ \gamma^\alpha) \circ \gamma_\alpha &= \left\langle\left\langle E \gamma^\alpha \gamma^\beta \right\rangle\right\rangle_0 G(\gamma_\beta) \circ \gamma_\alpha = G(\gamma_\beta) \circ (E \circ \gamma^\beta) \\
&= \frac{1}{2} E \circ \Delta. \qquad (14)
\end{aligned}$$

Therefore we obtain

$$\left\langle\!\!\left\langle \{E \circ \gamma^\alpha\} \, \gamma_\alpha \, E \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 \;\; = \;\; \langle\!\langle \{(1-b_1) E \circ \Delta - G(E \circ \Delta)\} \rangle\!\rangle_1 \, .$$

Similarly,

$$\left\langle\!\!\left\langle \Delta \, E \, E \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 \;\; = \;\; \langle\!\langle \{G(\Delta) \circ E + \Delta \circ G(E) - G(\Delta \circ E) - b_1 \Delta \circ E\} \rangle\!\rangle_1 \, ,$$

and

$$\begin{aligned} & \left\langle\!\!\left\langle E \, E \, \gamma^\alpha \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \langle\!\langle \{\gamma_\alpha \circ \gamma_\beta\} \rangle\!\rangle_1 \\ = \;\; & \langle\!\langle \{\gamma_\alpha \circ (G(E) \circ \gamma^\alpha + E \circ G(\gamma^\alpha) - G(E \circ \gamma^\alpha) - b_1 E \circ \gamma^\alpha)\} \rangle\!\rangle_1 \\ = \;\; & \langle\!\langle \{G(E) \circ \Delta - b_1 E \circ \Delta\} \rangle\!\rangle_1 \, . \end{aligned}$$

To compute the last term in equation (12), we first compute

$$\begin{aligned} \left\langle\!\!\left\langle E \, \gamma_\alpha \, \gamma^\alpha \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \gamma_\beta \;\; &= \;\; G(\gamma_\alpha) \circ \gamma^\alpha + \gamma_\alpha \circ G(\gamma^\alpha) - G(\Delta) - b_1 \Delta \\ &= \;\; (1-b_1)\Delta - G(\Delta) \end{aligned} \tag{15}$$

by equation (13). So the last term in equation (12) is

$$\left\langle\!\!\left\langle E \, \gamma_\alpha \, \gamma^\alpha \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \langle\!\langle \{\gamma_\beta \circ E\} \rangle\!\rangle_1 \;\; = \;\; \langle\!\langle \{(1-b_1) E \circ \Delta - E \circ G(\Delta)\} \rangle\!\rangle_1 \, .$$

After plugging the above formulas into equation (12), all terms on the right hand side cancel except the term $24\Delta \left\langle\!\left\langle \, E^2 \, \right\rangle\!\right\rangle_1$. The lemma is thus proved. $\Box$

Now we consider the genus-0 part of Equation (11). Let

$$\Phi := -\frac{1}{24} \sum_\alpha \langle\langle E E \gamma_\alpha \gamma^\alpha \rangle\rangle_0 + \frac{1}{2} \sum_\alpha \left(b_\alpha (1 - b_\alpha) - \frac{b_1 + 1}{6} \right) \langle\langle \gamma_\alpha \gamma^\alpha \rangle\rangle_0 \, . \tag{16}$$

Then

$$\Psi = \left\langle\!\left\langle \, E^2 \, \right\rangle\!\right\rangle_1 - \Phi \tag{17}$$

and the genus-1 Virasoro conjecture can be reduced to

$$\left\langle\!\left\langle \, E^2 \, \right\rangle\!\right\rangle_1 = \Phi.$$

Lemma 3.2

$$\sum_\alpha G_0(E, E, \gamma^\alpha, \gamma_\alpha) = -24 \Delta \Phi.$$

Proof: Again we will assume that repeated indices will be summed over their entire range. First, by definition of G_0, we have

$$\begin{aligned} G_0(E, E, \gamma^\alpha, \gamma_\alpha) \;\; = \;\; & 2 \left\langle\!\!\left\langle E \, E \, \gamma^\alpha \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_\beta \, \gamma_\alpha \, \gamma^\mu \, \gamma_\mu \rangle\!\rangle_0 + 2 \left\langle\!\!\left\langle E \, \gamma^\alpha \, \gamma_\alpha \, \gamma^\beta \right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_\beta \, E \, \gamma^\mu \, \gamma_\mu \rangle\!\rangle_0 \\ & + \langle\!\langle E \, E \, \gamma^\alpha \, \gamma_\alpha \, \Delta \rangle\!\rangle_0 - 2 \left\langle\!\!\left\langle E \, E \, \gamma^\beta \, \gamma^\mu \right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_\beta \, \gamma_\mu \, \gamma^\alpha \, \gamma_\alpha \rangle\!\rangle_0 \\ & - 4 \left\langle\!\!\left\langle E \, \gamma^\alpha \, \gamma^\beta \, \gamma^\mu \right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_\beta \, \gamma_\mu \, E \, \gamma_\alpha \rangle\!\rangle_0 \, . \end{aligned} \tag{18}$$

Note that the first and the fourth terms on the right hand side are canceled with each other. Applying equation (15) to the second term, we have

$$\left\langle\!\!\left\langle E\,\gamma^{\alpha}\,\gamma_{\alpha}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\beta}\,E\,\gamma^{\mu}\,\gamma_{\mu}\rangle\!\rangle_0 \;=\; \langle\!\langle \{(1-b_1)\Delta - G(\Delta)\}\;E\,\gamma^{\mu}\,\gamma_{\mu}\rangle\!\rangle_0 . \tag{19}$$

Using equation (15) again, we obtain

$$\begin{aligned}\langle\!\langle \Delta\,E\,\gamma^{\mu}\,\gamma_{\mu}\rangle\!\rangle_0 &= \left\langle\!\!\left\langle E\,\gamma^{\mu}\,\gamma_{\mu}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\beta}\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 \\ &= \langle\!\langle \{(1-b_1)\Delta - G(\Delta)\}\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 .\end{aligned}$$

Moreover,

$$\begin{aligned}\langle\!\langle G(\Delta)\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 &= b_{\mu}\left\langle\!\!\left\langle \gamma^{\beta}\,\gamma_{\beta}\,\gamma^{\mu}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\mu}\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 \\ &= \left\langle\!\!\left\langle \gamma^{\beta}\,\gamma_{\beta}\,\{\gamma^{\mu} - G(\gamma^{\mu})\}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\mu}\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 \\ &= \langle\!\langle \Delta\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 - \left\langle\!\!\left\langle \gamma^{\beta}\,\gamma_{\beta}\,G(\Delta)\right\rangle\!\!\right\rangle_0 .\end{aligned}$$

Moving the second term on the right hand to the left hand, we obtain

$$\langle\!\langle G(\Delta)\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 = \frac{1}{2}\;\langle\!\langle \Delta\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 . \tag{20}$$

Hence, we have

$$\langle\!\langle \Delta\,E\,\gamma^{\mu}\,\gamma_{\mu}\rangle\!\rangle_0 \;=\; \left(\frac{1}{2} - b_1\right)\langle\!\langle \Delta\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 . \tag{21}$$

By equation (19), we have

$$\begin{aligned}&\left\langle\!\!\left\langle E\,\gamma^{\alpha}\,\gamma_{\alpha}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\beta}\,E\,\gamma^{\mu}\,\gamma_{\mu}\rangle\!\rangle_0 \\ =\;& (1-b_1)\left(\frac{1}{2} - b_1\right)\langle\!\langle \Delta\,\gamma^{\alpha}\,\gamma_{\alpha}\rangle\!\rangle_0 - \langle\!\langle G(\Delta)\,E\,\gamma^{\mu}\,\gamma_{\mu}\rangle\!\rangle_0 .\end{aligned} \tag{22}$$

To compute the last term on the right hand side of equation (18), we set

$$f := \left\langle\!\!\left\langle E\,\gamma^{\alpha}\,\gamma^{\beta}\,\gamma^{\mu}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\beta}\,\gamma_{\mu}\,E\,\gamma_{\alpha}\rangle\!\rangle_0 .$$

Applying equation (7), we have

$$\begin{aligned}f &= \langle\!\langle \{G(\gamma^{\alpha})\circ\gamma^{\mu} + \gamma^{\alpha}\circ G(\gamma^{\mu}) - G(\gamma^{\alpha}\circ\gamma^{\mu}) - b_1\gamma^{\alpha}\circ\gamma^{\mu}\}\;\gamma_{\mu}\,E\,\gamma_{\alpha}\rangle\!\rangle_0 \\ &= (2-b_{\alpha}-b_{\mu}-b_1)\,\langle\!\langle \{\gamma^{\alpha}\circ\gamma^{\mu}\}\;\gamma_{\mu}\,E\,\gamma_{\alpha}\rangle\!\rangle_0 - \langle\!\langle G(\gamma^{\alpha}\circ\gamma^{\mu})\,\gamma_{\mu}\,E\,\gamma_{\alpha}\rangle\!\rangle_0 \\ &= (2-2b_{\alpha}-b_1)\left\langle\!\!\left\langle \gamma^{\alpha}\,\gamma^{\mu}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 \langle\!\langle \gamma_{\beta}\,\gamma_{\mu}\,E\,\gamma_{\alpha}\rangle\!\rangle_0 - \left\langle\!\!\left\langle \gamma^{\alpha}\,\gamma^{\mu}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 \langle\!\langle G(\gamma_{\beta})\,\gamma_{\mu}\,E\,\gamma_{\alpha}\rangle\!\rangle_0 .\end{aligned}$$

Switching α and β in the last term, we have

$$f = (2-b_1)\,\langle\!\langle E\,\gamma_{\alpha}\,\gamma_{\beta}\,\gamma^{\mu}\rangle\!\rangle_0 \left\langle\!\!\left\langle \gamma_{\mu}\,\gamma^{\alpha}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 - 3\,\langle\!\langle E\,G(\gamma_{\alpha})\,\gamma_{\beta}\,\gamma^{\mu}\rangle\!\rangle_0 \left\langle\!\!\left\langle \gamma_{\mu}\,\gamma^{\alpha}\,\gamma^{\beta}\right\rangle\!\!\right\rangle_0 . \tag{23}$$

Applying equation (7) again, we have

$$\begin{aligned}
&\langle\!\langle\, E\,\gamma_\alpha\,\gamma_\beta\,\gamma^\mu\,\rangle\!\rangle_0 \left\langle\!\!\left\langle\, \gamma_\mu\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 \\
= &\left\langle\!\!\left\langle\, \{G(\gamma_\alpha)\circ\gamma_\beta + \gamma_\alpha\circ G(\gamma_\beta) - G(\gamma_\alpha\circ\gamma_\beta) - b_1\gamma_\alpha\circ\gamma_\beta\}\;\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 .
\end{aligned}$$

By the associativity of the quantum product and equation (13),

$$\left\langle\!\!\left\langle\, \{G(\gamma_\alpha)\circ\gamma_\beta\}\;\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 = \left\langle\!\!\left\langle\, \{G(\gamma_\alpha)\circ\gamma^\alpha\}\;\gamma_\beta\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 = \frac{1}{2}\,\langle\!\langle\,\Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0 \tag{24}$$

and

$$\begin{aligned}
\left\langle\!\!\left\langle\, G(\gamma_\alpha\circ\gamma_\beta)\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 &= \langle\!\langle\,\gamma_\alpha\,\gamma_\beta\,\gamma^\mu\,\rangle\!\rangle_0 \left\langle\!\!\left\langle\, G(\gamma_\mu)\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 \\
&= \langle\!\langle\,\gamma_\alpha\;\{G(\gamma_\mu)\circ\gamma^\alpha\}\;\gamma^\mu\,\rangle\!\rangle_0 = \frac{1}{2}\,\langle\!\langle\,\Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0 .
\end{aligned} \tag{25}$$

So we have

$$\langle\!\langle\, E\,\gamma_\alpha\,\gamma_\beta\,\gamma^\mu\,\rangle\!\rangle_0 \left\langle\!\!\left\langle\, \gamma_\mu\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 = \left(\frac{1}{2} - b_1\right)\langle\!\langle\,\Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0 . \tag{26}$$

Moreover

$$\begin{aligned}
&\langle\!\langle\, E\,G(\gamma_\alpha)\,\gamma_\beta\,\gamma^\mu\,\rangle\!\rangle_0 \left\langle\!\!\left\langle\, \gamma_\mu\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 \\
= &\left\langle\!\!\left\langle\, \{G(G(\gamma_\alpha))\circ\gamma_\beta + G(\gamma_\alpha)\circ G(\gamma_\beta) - G(G(\gamma_\alpha)\circ\gamma_\beta) - b_1 G(\gamma_\alpha)\circ\gamma_\beta\}\;\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 .
\end{aligned}$$

Since

$$\begin{aligned}
&\left\langle\!\!\left\langle\, \{G(\gamma_\alpha)\circ G(\gamma_\beta)\}\;\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 \\
= &\left\langle\!\!\left\langle\, \{G(\gamma_\alpha)\circ\gamma^\alpha\}\;G(\gamma_\beta)\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 = \frac{1}{2}\left\langle\!\!\left\langle\, \{\gamma_\alpha\circ\gamma^\alpha\}\;G(\gamma_\beta)\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 \\
= &\frac{1}{2}\left\langle\!\!\left\langle\, \gamma_\alpha\,\gamma^\alpha\,\left\{G(\gamma_\beta)\circ\gamma^\beta\right\}\,\right\rangle\!\!\right\rangle_0 = \frac{1}{4}\,\langle\!\langle\,\Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\left\langle\!\!\left\langle\, G(G(\gamma_\alpha)\circ\gamma_\beta)\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 \\
= &\langle\!\langle\, G(\gamma_\alpha)\,\gamma_\beta\,\gamma^\mu\,\rangle\!\rangle_0 \left\langle\!\!\left\langle\, G(\gamma_\mu)\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 = \langle\!\langle\,\{G(\gamma_\alpha)\circ\gamma^\mu\}\;G(\gamma_\mu)\,\gamma^\alpha\,\rangle\!\rangle_0 \\
= &\langle\!\langle\,\{G(\gamma_\alpha)\circ\gamma^\alpha\}\;G(\gamma_\mu)\,\gamma^\mu\,\rangle\!\rangle_0 = \frac{1}{2}\,\langle\!\langle\,\{\gamma_\alpha\circ\gamma^\alpha\}\;G(\gamma_\mu)\,\gamma^\mu\,\rangle\!\rangle_0 \\
= &\frac{1}{2}\,\langle\!\langle\,\gamma_\alpha\,\gamma^\alpha\;\{G(\gamma_\mu)\circ\gamma^\mu\}\,\rangle\!\rangle_0 = \frac{1}{4}\,\langle\!\langle\,\Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0 ,
\end{aligned} \tag{28}$$

together with equation (24), we have

$$\langle\!\langle\, E\,G(\gamma_\alpha)\,\gamma_\beta\,\gamma^\mu\,\rangle\!\rangle_0 \left\langle\!\!\left\langle\, \gamma_\mu\,\gamma^\alpha\,\gamma^\beta\,\right\rangle\!\!\right\rangle_0 = \left(b_\alpha^2 - \frac{1}{2}b_1\right)\langle\!\langle\,\Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0 . \tag{29}$$

Combining results of equations (23), (26), and (29), we obtain that the last term on the right hand side of equation (18) is

$$f = (-3b_\alpha^2 + b_1^2 - b_1 + 1)\,\langle\!\langle\, \Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0\,. \tag{30}$$

Together with equation (22), we can simplify equation (18) as

$$\begin{aligned} G_0(E, E, \gamma^\alpha, \gamma_\alpha) \;&=\; \langle\!\langle\, E\,E\,\gamma^\alpha\,\gamma_\alpha\,\Delta\,\rangle\!\rangle_0 - 2\,\langle\!\langle\, G(\Delta)\,E\,\gamma^\mu\,\gamma_\mu\,\rangle\!\rangle_0 \\ &\quad +(12b_\alpha^2 - 2b_1^2 + b_1 - 3)\,\langle\!\langle\, \Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0\,. \end{aligned} \tag{31}$$

On the other hand, by the definition of Φ in equation (16), we have

$$\begin{aligned} 24\Delta\Phi \;&=\; -\,\langle\!\langle\, \Delta\,E\,E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 - 2\,\langle\!\langle\, \{\nabla_\Delta E\}\;E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 \\ &\quad +12\left(b_\alpha(1 - b_\alpha) - \frac{b_1 + 1}{6}\right)\langle\!\langle\, \Delta\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0\,. \end{aligned}$$

By equations (9) and (21), we have

$$\begin{aligned} \langle\!\langle\, \{\nabla_\Delta E\}\;E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 \;&=\; \langle\!\langle\, \{-G(\Delta) + (b_1 + 1)\Delta\}\;E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 \\ &=\; -\,\langle\!\langle\, G(\Delta)\,E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 + (b_1 + 1)\left(\frac{1}{2} - b_1\right)\langle\!\langle\, \Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0\,. \end{aligned}$$

Moreover,

$$b_\alpha\,\langle\!\langle\, \Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0 = \langle\!\langle\, \Delta\,\gamma^\alpha\,G(\gamma_\alpha)\,\rangle\!\rangle_0 = \frac{1}{2}\,\langle\!\langle\, \Delta\,\gamma^\alpha\,\gamma_\alpha\,\rangle\!\rangle_0\,.$$

So we have

$$\begin{aligned} 24\Delta\Phi \;&=\; -\,\langle\!\langle\, \Delta\,E\,E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 + 2\,\langle\!\langle\, G(\Delta)\,E\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0 \\ &\quad + \left(-12b_\alpha^2 + 2b_1^2 - b_1 + 3\right)\langle\!\langle\, \Delta\,\gamma_\alpha\,\gamma^\alpha\,\rangle\!\rangle_0\,. \end{aligned}$$

Comparing with equation (31), we obtain

$$G_0(E, E, \gamma^\alpha, \gamma_\alpha) = -24\Delta\Phi.$$

The lemma is thus proved. □

Proof of Theorem 1.1: Since $\Psi = \left\langle\!\left\langle\, E^2\,\right\rangle\!\right\rangle_1 - \Phi$, this theorem follows from Lemmas 3.1, 3.2 and Equation (11). □

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