Minimal Diffeomorphisms cannot satisfy Generalized Dominated Splitting

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Abstract

We introduce a new notion called generalized dominated splitting which is weaker than classical dominated splitting. We use this notion to generalize a result of Zhang[12]: every diffeomorphism with nontrivial global generalized dominated splitting can not be minimal.

1 Introduction

Minimality is an important concept in the study of dynamical systems. It is interesting to study some nature structure of the system that incompatible with minimality. In 1980's Herman[3] constructed a (family of) C^1 diffeomorphism on a compact 4-dimensional manifold that is minimal but has positive topological entropy simultaneously and Rees[8] constructed a minimal homeomorphism with positive topological entropy on 2-torus. So positive entropy is insufficient to guarantee the non-minimality. In [5] Mañé gave an argument to locate some nonrecurrent point if the map admits some invariant expanding foliation (also see [2]). In particular this argument implies that a partially hyperbolic diffeomorphism always has some nonrecurrent point and hence can not be minimal. Recently in [12] Zhang showed that a global dominated splitting is sufficient to exclude the minimality of the system. In present paper we mainly want to generalize the result [12] to a more general assumption called generalized dominated splitting.

Let M be a compact D-dimensional smooth Riemannian manifold and let d denote the distance induced by the Riemannian metric. Denote the tangent bundle of M by TM and denote by $\text{Diff}^1(M)$ the space of C^1 diffeomorphisms of M. Denote the maximal norm of a linear map A by $\|A\|$ and denote the minimal norm of an invertible linear map A by $m(A) := \|A^{-1}\|^{-1}$. Now we introduce our new notion called generalized dominated splitting.

† Tian is supported by CAPES.

Key words and phrases: (Generalized) Dominated splitting, Minimal system, Invariant and ergodic measure, Recurrence

AMS Review: 37D30, 37C40, 37B20.

Definition 1.1. *Let $f \in \mathrm{Diff}^1(M)$ and let $S \in \mathbb{N}, \lambda > 0$. Given an $f-$invariant compact set Δ, we say a $Df-$invariant splitting $T_\Delta M = E \oplus F$ on Δ to be a* **generalized dominated splitting** *on Δ (or simply GDS), if*
(1). $T_\Delta M = E \oplus F$ is continuous on Δ;
(2). $\frac{\|Df^{kS}|_{E(x)}\|}{m(Df^{kS}|_{F(x)})} \leq \lambda,\ \forall x \in \Delta,\ \forall k \in \mathbb{N}$;
(3). there exists $x_0 \in \Delta$, $\frac{\|Df^{S}|_{E(x_0)}\|}{m(Df^{S}|_{F(x_0)})} < \lambda^{-1}$.

Note that $\|AB\| \leq \|A\|\|B\|$, $m(AB) \geq m(A)m(B)$, $\|AB\| \geq \|A\|m(B)$ and $m(AB) \leq m(A)\|B\|$. Then

$$\frac{\|Df^{[\frac{n}{S}]S}|_{E(x_0)}\|}{m(Df^{[\frac{n}{S}]S}|_{F(x_0)})} \times C^{2S} \geq \frac{\|Df^{n}|_{E(x_0)}\|}{m(Df^{n}|_{F(x_0)})} \geq \frac{\|Df^{[\frac{n}{S}]S}|_{E(x_0)}\|}{m(Df^{[\frac{n}{S}]S}|_{F(x_0)})} \times C^{-2S} \tag{1.1}$$

where $C = sup_{x\in M} \max\{\|Df(x)\|, \|Df^{-1}(x)\|\}$.

Remark 1.2. By (1.1) the second condition in Definition 1.1 implies that for all $x \in \Delta$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^{n}|_{E(x)}\|}{m(Df^{n}|_{F(x)})} = \frac{1}{S} \limsup_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^{nS}|_{E(x)}\|}{m(Df^{nS}|_{F(x)})} \leq \frac{1}{S} \min\{\log \lambda, 0\} \leq 0.$$

Recall the definition of classical dominated splitting. Let Δ be an $f-$invariant compact set. A $Df-$invariant splitting $T_\Delta M = E \oplus F$ on Δ is called (S, λ)-dominated on Δ (or simply dominated), if there exist two constants $S \in \mathbb{Z}^+$ and $0 < \lambda < 1$ such that

$$\frac{\|Df^{S}|_{E(x)}\|}{m(Df^{S}|_{F(x)})} \leq \lambda,\ \forall x \in \Delta.$$

Note that dominated splitting is always continuous (see [1]), $\lambda < 1 < \lambda^{-1}$ and

$$\frac{\|Df^{kS}|_{E(x)}\|}{m(Df^{kS}|_{F(x)})} \leq \prod_{i=0}^{k-1} \frac{\|Df^{S}|_{E(f^i(x))}\|}{m(Df^{S}|_{F(f^i(x))})} \leq \lambda^k \leq \lambda.$$

So any dominated splitting satisfies Definition 1.1(Moreover, we will give an example below which does not have global dominated splitting but admits a global GDS).

Let Δ be an $f-$invariant compact set. If $\varnothing \neq \Delta \subsetneqq M$, it is clear that f can not be minimal since the closure of every orbit in Δ is still contained in Δ. A $Df-$invariant splitting $T_\Delta M = E \oplus F$ on Δ is nontrivial if $dim(E) \cdot dim(F) \neq 0$. And we say a $Df-$invariant splitting $T_\Delta M = E \oplus F$ to be global, if $\Delta = M$. Now we state our main theorem for considering systems with global GDS.

Theorem 1.3. *Let $f \in \mathrm{Diff}^1(M)$. If there is a nontrivial global GDS $TM = E \oplus F$, then f can not be minimal.*

Remark 1.4. There exists some minimal system $f \in \mathrm{Diff}^1(M)$ such that its nontrivial global invariant splitting $TM = E \oplus F$ only satisfies the first two conditions in Definition 1.1. More precisely, for $a \in \mathbb{R}$, define the corresponding rotation $R_a : \mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto$

$x + a(mod 1)$. Clearly for the product system $f_{a_1,a_2,\cdots,a_n} := R_{a_1} \times R_{a_2} \times \cdots \times R_{a_n} : \mathbb{T}^n \to \mathbb{T}^n (a_1, a_2, \cdots, a_n \in \mathbb{R})$, its nontrivial global invariant splitting $TM = E \oplus F$ satisfies the first two conditions in Definition 1.1 for any $S \in \mathbb{N}, \lambda \geq 1$ but fails the third condition for all $S \in \mathbb{N}, \lambda \geq 1$ ($\frac{\|Df^k|_{E(x)}\|}{m(Df^k|_{F(x)})} \equiv 1, \forall x \in \Delta, \forall k \in \mathbb{N}$). It is well-known that $1, a_1, a_2, \cdots, a_n \in \mathbb{R}$ are rationally independent if and only if the product system $f_{a_1,a_2,\cdots,a_n} = R_{a_1} \times R_{a_2} \times \cdots \times R_{a_n}$ is minimal(and ergodic with respect to Lebesgue measure). This shows that both minimal and non-minimal C^∞ diffeomorphisms admit to have nontrivial global invariant splitting $TM = E \oplus F$ satisfying the first two conditions in Definition 1.1.

If a global $Df-$invariant splitting $T_\Delta M = E \oplus F$ is not GDS but the first two conditions in Definition 1.1 still hold for some S, λ, then $\lambda \geq 1$ and $\lambda^{-1} \leq \frac{\|Df^{kS}|_{E(x)}\|}{m(Df^{kS}|_{F(x)})} \leq \lambda, \forall x \in \Delta, \forall k \in \mathbb{N}$. Otherwise, there exists $x_0 \in \Delta$ and k_0, $\frac{\|Df^{k_0 S}|_{E(x_0)}\|}{m(Df^{k_0 S}|_{F(x_0)})} < \lambda^{-1}$, then $T_\Delta M = E \oplus F$ is GDS for $N := k_0 S$ and λ. Furthermore, from (1.1)

$$(C^{2S}\lambda)^{-1} \leq \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} \leq C^{2S}\lambda, \ \forall x \in \Delta, \ \forall n \in \mathbb{N}.$$

In particular, $\lim_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} = 0, \ \forall x \in \Delta$. From these analysis and Remark 1.2, the third condition in Definition 1.1 can be deduced once for some x,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} < 0.$$

Hence, we state such a corollary of Theorem 1.3 as follows.

Corollary 1.5. *Let $f \in \mathrm{Diff}^1(M)$. If there is a nontrivial global $Df-$invariant splitting $TM = E \oplus F$ satisfying the first two conditions in Definition 1.1 and there exists x_0,*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x_0)}\|}{m(Df^n|_{F(x_0)})} < 0,$$

then $TM = E \oplus F$ is a nontrivial global GDS and thus f can not be minimal.

This corollary can be as a sufficient condition to obtain GDS. By Remark 1.2, for a global (S, λ)-dominated splitting, every point x satisfies

$$\limsup_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} \leq \frac{1}{S} \log \lambda < 0.$$

This implies for any global dominated splitting, every point satisfies the assumption of Corollary 1.5 and the supreme limit can be uniformly less than 0. But Corollary 1.5 assumes only one such point and uses inferior limit. Thus the assumption in Corollary 1.5 is still weaker than dominated splitting(for instance, see Example 3.1 below).

Remark 1.6. For any *surface* diffeomorphism f with positive topological entropy, if there is a nontrivial global $Df-$invariant splitting $TM = E \oplus F$ satisfying the first two conditions in Definition 1.1, then f satisfies Corollary 1.5 and thus f can not be minimal. More precisely, by Variational Principle([11]), for any diffeomorphism with positive entropy, there exists an ergodic measure μ with positive entropy and thus by Rulle's inequality([9]) μ has both negative Lyapunov exponent($\chi_1 < 0$) and positive Lyapunov exponent($\chi_2 > 0$) simultaneously. Note that $dim(E) = dim(F) = 1$, then for μ a.e. x,

$$\lim_{n\to+\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} = \chi_1 - \chi_2 \leq -2h_\mu(f) < 0.$$

In particular, we recall that if f is a $C^{1+\alpha}$ *surface* diffeomorphism with positive topological entropy, then f always has periodic point by classical Pesin theory[4] and thus can not be minimal.

2 Proof of Theorem 1.3

Before proving Theorem 1.3 we need a result of [7] .

Lemma 2.1. *(Proposition 3.4 in [7]) Let $f : X \to X$ be a continuous map of a compact metric space. Let $a_n : X \to R, n \geq 0$, be a sequence of continuous functions such that*

$$a_{n+k}(x) \leq a_n(f^k(x)) + a_k(x) \text{ for every } x \in X, n, k \geq 0. \tag{2.2}$$

and such that there is a sequence of continuous functions $b_n : X \to R, n \geq 0$, satisfying

$$a_n(x) \leq a_n(f^k(x)) + a_k(x) + b_k(f^n(x)) \text{ for every } x \in X, n, k \geq 0. \tag{2.3}$$

If

$$\inf \frac{1}{n} \int_X a_n(x) d\mu < 0$$

for every ergodic f-invariant measure, then there is $N > 0$ such that $a_N(x) < 0$ for every $x \in X$.

Proof of Theorem 1.3 If $\lambda < 1$, then the nontrivial global GDS is a nontrivial global dominated splitting. By the result of [12], f can not be minimal. Now we assume that $\lambda \geq 1$ and we will give a proof by contradiction. Suppose f is minimal, then the nontrivial global GDS can not be a nontrivial global dominated splitting from [12]. To get a contradiction for this case, we only need to prove that the nontrivial global GDS is nontrivial global dominated splitting.

Define for $\epsilon > 0$

$$A_\epsilon := \{z \in M \mid \frac{\|Df^S|_{E(z)}\|}{m(Df^S|_{F(z)})} < -\epsilon + \lambda^{-1}\}$$

and set $A = \bigcup_{\epsilon>0} A_l$. Note that every set A_ϵ is open set and A is also open since the splitting $TM = E \oplus F$ is continuous. Clearly by assumption the point x_0 must be in A. Take $0 < \epsilon < \lambda^{-1}$ small enough such that $x_0 \in A_\epsilon$ so that $A_\epsilon \neq \varnothing$.

Since we assume that f is minimal, then for every invariant measure μ, its support $supp(\mu)$ must coincide with the whole manifold M. Otherwise, if for some μ, $supp(\mu) \subsetneqq M$. Then every point $x \in supp(\mu)$, the closure of its orbit is contained in $supp(\mu) \subsetneqq M$ since $supp(\mu)$ is always compact and invariant. This contradicts that f is minimal. So for any nonempty open set, it always has positive measure for any invariant measure. In particular, $\mu(A_\epsilon) > 0$ holds for any invariant measure μ.

Define functions for $x \in M$

$$a_n(x) := \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})}.$$

Recall that $\|AB\| \leq \|A\|\|B\|$ and $m(AB) \geq m(A)m(B)$. Then it is easy to see that a_n satisfy (2.2) of Lemma 2.1. Taking into account (2.2) we see that (2.3) holds once $a_n(x) \leq a_{n+k}(x) + b_k(f^n(x))$. This is easily verified for $b_k(x) := \log \frac{\|(Df^k|_{E(x)})^{-1}\|}{m((Df^k|_{F(x)})^{-1})}$ since

$$\frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} \leq \frac{\|Df^{n+k}|_{E(x)}\|}{m(Df^{n+k}|_{F(x)})} \times \frac{\|(Df^k|_{E(f^n(x))})^{-1}\|}{m((Df^k|_{F(f^n(x))})^{-1})}.$$

Recall that $TM = E \oplus F$ is a continuous splitting. So $a_n(x), b_n(x)$ are continuous functions. Then all assumptions of Lemma 2.1 are satisfied once $\inf \frac{1}{n} \int_M a_n(x) d\mu < 0$ holds for ergodic invariant measure μ.

Let μ be an f ergodic invariant measure. By Subadditive Ergodic Theorem(see [11]), and the ergodicity of μ, the limit function

$$a(x) := \lim_{n\to+\infty} \frac{1}{n} a_n(x)$$

is well-defined, f-invariant and can be a constant function for μ a.e x. Now we prove for μ a.e x, $a(x) < 0$ which implies $\inf \frac{1}{n} \int_M a_n(x) d\mu < 0$. Let $\Phi := \cup_{n\in\mathbb{Z}} f^n(A_\epsilon)$. Clearly it is f-invariant and from ergodicity of μ we have that $\mu(\Phi) = 1$(In fact, $\Phi = M$ since $M \setminus \Phi$ is f-invariant and closed but f is minimal). So we only need to prove $a(x) < 0$ for μ a.e $x \in A_\epsilon$ since $a(x)$ is f invariant. Define $c_n(x) := a_{nS}(x)$, then

$$c(x) := \lim_{n\to+\infty} \frac{1}{n} c_n(x) = S \lim_{n\to+\infty} \frac{1}{nS} a_{nS}(x) = Sa(x).$$

So we only need to prove $c(x) < 0$ for μ a.e. $x \in A_\epsilon$.

By Birkhoff Ergodic Theorem, the limit function

$$\chi^*_{A_\epsilon}(x) := \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{A_\epsilon}(f^{jS}(x))$$

exists for μ a.e. x and f^S-invariant. Moreover, $\int \chi^*_{A_\epsilon}(x) d\mu = \int \chi_{A_\epsilon}(x) d\mu = \mu(A_\epsilon)$. If μ is f^S ergodic, it is obvious since $\chi^*_{A_\epsilon}(x) \equiv \int \chi_{A_\epsilon}(x) d\mu = \mu(A_\epsilon) > 0$ holds for μ a.e. $x \in M$.

But we do not know whether μ is f^S ergodic or not. We only know that it is still f^S invariant. Here we recall a basic fact for recurrent times which is the claim in the proof of Proposition 3.1 [6]. This fact is that for any homeomorphism g, if ν is g-invariant and Γ is a set with $\nu(\Gamma) > 0$, then for ν a.e. $x \in \Gamma$,

$$\chi_\Gamma^*(x) := \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_\Gamma(g^j(x)) > 0$$

(Remark that in general this fact holds only for ν a.e. $x \in \Gamma$ and then $x \in \cup_{n\in\mathbb{Z}} g^n(\Gamma)$ but maybe not hold for ν a.e. $x \in M$ since ν is just g-invariant). This guarantees that for μ a.e. $x \in A_\epsilon$, $\chi_{A_\epsilon}^*(x) > 0$ since μ is f^S invariant and $\mu(A_\epsilon) > 0$. Fix such a point $x \in A_\epsilon$. Let

$$t_0(x) = 0 < t_1(x) < t_2(x) < \cdots$$

to be the all positive times such that $f^{t_i(x)S}(x) \in A_\epsilon$. Then

$$\lim_{i\to+\infty} \frac{i}{t_i(x)} = \lim_{t_i(x)\to+\infty} \frac{1}{t_i(x)} \sum_{j=0}^{t_i(x)-1} \chi_{A_\epsilon}(f^{jS}(x)) = \chi_{A_\epsilon}^*(x) > 0.$$

Recall the definition of A_ϵ and the second condition of GDS. Then

$$\frac{\|Df^S|_{E(f^{t_j(x)S}(x))}\|}{m(Df^S|_{F(f^{t_j(x)S}(x))})} \le -\epsilon + \lambda^{-1}$$

and

$$\frac{\|Df^{(t_{j+1}(x)-t_j(x)-1)S}|_{E(f^{t_j(x)S+S}(x))}\|}{m(Df^{(t_{j+1}(x)-t_j(x)-1)S}|_{F(f^{t_j(x)S+S}(x))})} \le \lambda.$$

Hence, for $n = t_i(x)$,

$$\begin{aligned} c_n(x) = \log \frac{\|Df^{nS}|_{E(x)}\|}{m(Df^{nS}|_{F(x)})} &\le \sum_{j=0}^{i-1} \log \frac{\|Df^S|_{E(f^{t_j(x)S}(x))}\|}{m(Df^S|_{F(f^{t_j(x)S}(x))})} \\ &+ \sum_{j=0}^{i-1} \log \frac{\|Df^{(t_{j+1}(x)-t_j(x)-1)S}|_{E(f^{t_j(x)S+S}(x))}\|}{m(Df^{(t_{j+1}(x)-t_j(x)-1)S}|_{F(f^{t_j(x)S+S}(x))})} \\ &\le i\log(-\epsilon + \lambda^{-1}) + i\log\lambda = i\log(1-\epsilon\lambda). \end{aligned}$$

Thus

$$c(x) = \lim_{n\to+\infty} \frac{1}{n} c_n(x) \le \lim_{i\to+\infty} \frac{i\log(1-\epsilon\lambda)}{t_i(x)} = \chi_{A_\epsilon}^*(x)\log(1-\epsilon\lambda) < 0.$$

Remark that $\chi_{A_\epsilon}^*(x) > 0$ and the estimate inequality of $c_n(x)$ play the crucial roles.

Now we can use Lemma 2.1 to get that there is $N > 0$ such that $a_N(x) < 0$ for every $x \in M$. Recall that a_N is a continuous function and M is compact. So $t := \max_{x\in M}\{a_N(x)\}$ exists and must be negative. If $\tau = e^t$, then $0 < \tau < 1$ and for any $x \in M$,

$$\frac{\|Df^N|_{E(x)}\|}{m(Df^N|_{F(x)})} = e^{a_N(x)} \le e^t = \tau.$$

So the nontrivial global GDS $TM = E \oplus F$ is a nontrivial global (N, τ)-dominated splitting. We complete the proof. $\square$

Remark 2.2. Note that the assumption of ergodicity of μ is not necessary due to the used claim in the proof of Proposition 3.1 [6]. Moreover, if we do not assume f to be minimal in the proof, it is easy to see that for any invariant (not necessarily ergodic) measure μ, $\inf \frac{1}{n}\int_M a_n(x)d\mu < 0$ if and only if $\mu(\cup_{\epsilon>0} A_\epsilon) > 0$.

Remark 2.3. This proof implies a fact that if $\inf \frac{1}{n}\int_M \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})}d\mu < 0$ holds with respect to a continuous Df-invariant splitting $TM = E \oplus F$ for all ergodic invariant measure μ, then $TM = E \oplus F$ is a global dominated splitting.

3 Difference of GDS and Dominated splitting

To further illustrate the new notion of GDS, we construct a simple example which firstly appeared in [10]. This diffeomorphism has global GDS for $S = 1, \lambda = 1$ but does not have global dominated splitting.

Example 3.1. *Let g be a $C^r (r \geq 1)$ increasing function on $[0, 1]$, satisfying:*

$$g(0) = 0,\ g(1) = 1,\ g'(0) = g'(1) = \frac{3-\sqrt{5}}{2},\ g(\frac{1}{2}) = \frac{1}{2},\ g'(\frac{1}{2}) = \frac{3+\sqrt{5}}{2}\ \textit{and}$$

$$\frac{3-\sqrt{5}}{2} \leq g'(x) \leq \frac{3+\sqrt{5}}{2}, \forall x \in [0,1],\ \ g(x) < x,\ \forall\ x \in (0, \frac{1}{2}),\ g(x) > x,\ \forall\ x \in (\frac{1}{2}, 1).$$

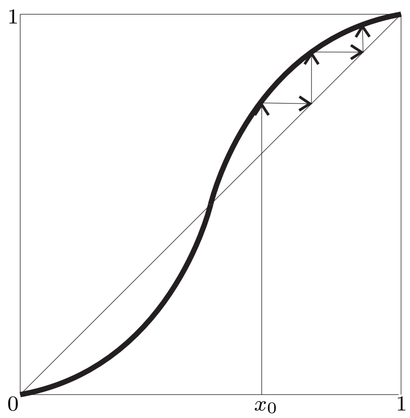

Figure 1: Graph of the function g.

And let $h : T^2 \to T^2$ be the hyperbolic Torus automorphism

$$(y, z) \mapsto (2y + z, y + z),\ \ y, z \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}.$$

Define $f = g \times h : T^3 \to T^3$. Clearly,

$$Df(x,y,z) = \begin{pmatrix} g'(x) & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

There exists naturally a continuous splitting $TT^3 = E_1 \oplus E_2 \oplus E_3$, where E_2 and E_3 are from the hyperbolic Torus automorphism h and E_1 is $g-$invariant. The forward Lyapunov exponent of E_1 is $\log \frac{3-\sqrt{5}}{2}$ over $T^3 - \{\frac{1}{2}\} \times T^2$ and the Lyapunov exponents of $E_2 \oplus E_3$ over $\{0\} \times T^2$ are $\log \frac{3-\sqrt{5}}{2}$, $\log \frac{3+\sqrt{5}}{2}$ respectively. Denote by δ_0 the point measure at point $0 \in \mathbb{S}^1$ and denote by m the Lebesgue measure on T^2, then the product measure $\mu = \delta_0 \times m$ is a hyperbolic ergodic measure of the diffeomorphism $f = g \times h$ with three nonzero Lyapunov exponents $-\log \frac{3+\sqrt{5}}{2}$, $-\log \frac{3+\sqrt{5}}{2}$, $\log \frac{3+\sqrt{5}}{2}$. Set $E = E_1 \oplus E_2$ and $F = E_3$, then $E \oplus F$ construct a continuous Df-invariant splitting of TT^3 over the whole space T^3 and $E \oplus F$ is a GDS for $S = 1, \lambda = 1$ on the whole space T^3. However, it is not a global dominated splitting since for every point $u = (\frac{1}{2}, y, z)$ $(y, z \in \mathbb{S}^1)$,

$$\frac{\|Df|_{E(u)}\|}{m(Df|_{F(u)})} = \frac{\frac{3+\sqrt{5}}{2}}{\frac{3+\sqrt{5}}{2}} = 1.$$

Similarly if $E = E_2$ and $F = E_1 \oplus E_3$, we can follow above discussion to get that the new $E \oplus F$ is also not dominated but a global GDS for $S = 1, \lambda = 1$. But if $E = E_1$ and $F = E_2 \oplus E_3$, then it is easy to see this splitting $E \oplus F$ is not a global GDS and thus is also not a global dominated splitting(even though this splitting is continuous on whole manifold). All in all, this example has nontrivial global GDS but does not admit nontrivial global dominated splitting. □

At the end of present paper, we point out a further question under a more general assumption.

Question 3.2. *Let $f \in \mathrm{Diff}^1(M)$. If there is a nontrivial global $Df-$invariant splitting $TM = E \oplus F$ satisfying*
(1). $TM = E \oplus F$ is continuous on Δ;
(2). for every $x \in M$,

$$\liminf_{n \to +\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x)}\|}{m(Df^n|_{F(x)})} \leq 0;$$

(3). there exists x_0,

$$\liminf_{n \to +\infty} \frac{1}{n} \log \frac{\|Df^n|_{E(x_0)}\|}{m(Df^n|_{F(x_0)})} < 0.$$

Then whether f can not be minimal?

Acknowledgement. The author thanks very much to Professor Marcus Bronzi, Krerley Oliveira and Fernando Pereira Micena *et al* for their help in the period of UFAL.

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