# Families of integrable equations

Pavlos Kassotakis * Maciej Nieszporski †

June 6, 2011

**Abstract**

We present a method to obtain families of lattice equations. Specifically we focus on two of such families, which include 3-parameters and their members are connected through Bäcklund transformations. At least one of the members of each family is integrable, hence the whole family inherits some integrability properties.

**Pacs:** 02.30.Ik, 05.50.+q
**Keywords:** Integrable lattice equations, Yang-Baxter maps, Consistency around the cube
**Mathematics Subject Classification:** 82B20, 37K35, 39A05

## 1 Introduction

Discrete mathematics returned on the interest of mathematicians at the beginning of the $20^{th}$ century. Poincaré, Birkhoff, Ritt (1924) [1], Julia, Fatou (1918-23) [2],[3] and many others saw the necessity of exploring the discrete scene. Unfortunately, this trend was paused through the two big wars and only after 1960, keeping pace with the revolution caused by the discovery of soliton from Zabusky and Kruskal [4], mathematicians started to investigate discrete systems in the context of integrable systems.

It was the work of Hirota [5], as well as Ablowitz et.al [6] and separately Capel and his school [7], which introduced lattice and differential difference analogues of many integrable PDE's. The introduction of discrete versions of integrable ODE's, surprisingly , came later with the QRT family of mappings by Quispel Roberts and Thomson [8] and by the work of Papageorgiou et.al. [9, 10], where Liouville integrable maps [11] were obtained by imposing periodic staircase initial data on integrable lattices. Another way to obtain integrable mappings from an integrable lattice equation was suggested in series of papers [12, 13, 14]. Actually with this procedure one can get involutive mappings (composition of the map with itself is the identity map) which are set theoretical solutions of the quantum Yang-Baxter equation the so called Yang-Baxter maps [15, 16, 12, 17]. As in our previous work [18], we focus here on the inverse procedure, i.e. how to obtain integrable lattice equations from involutive mappings that may or may not satisfy the Yang-Baxter equation.

The main result of the paper is that the procedure we have in mind can lead to families of equations. The members of families are related by a Bäcklund transformation (see section 3) and since in considered

* *Present address:* Department of Mathematics and Statistics University of Cyprus, P.O Box: 20537, 1678 Nicosia, Cyprus; *e-mails:* `kassotakis.pavlos@ucy.ac.cy, pavlos1978@gmail.com`

† *Present address:* Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, ul. Hoża 74, 00-682 Warszawa, Poland; *e-mail:* `maciejun@fuw.edu.pl`

cases at least one of the members is integrable, the whole family inherits some properties from the distinguished member. Notion of the family of discrete integrable systems should not be confused with notion of hierarchies of integrable systems. The later notion was widely investigated in the literature whereas for the former one we can indicate only the articles that investigate family of discrete KdV equations [19] and family of discrete Boussinesq equations [20, 21, 22].

We discuss here two examples the first one is continuation of our previous paper [18]. We introduce family of difference equations associated with type III of maps discussed in [12, 13] (we introduced families related to types IV and V in [18]). Example of the map of type III is map $\mathbb{Z}^2 \ni (u, v) \mapsto (U, V) \in \mathbb{Z}^2$

$$U = v\frac{pu - qv}{qu - pv}, \quad V = u\frac{pu - qv}{qu - pv}. \tag{1}$$

and the three parameter family of equations (see section 3) reads

$$\psi_{12} = \psi + a \ln \frac{pu - qv}{qu - pv} + (p^2 - q^2)\left[b\frac{uv}{qu - pv} - c\frac{1}{pu - qv}\right] \tag{2}$$

where $u$ and $v$ are given implicitly by

$$\begin{aligned} a \ln u + p\left(bu + c\frac{1}{u}\right) &= \psi_1 + \psi, \\ a \ln v + q\left(bv + c\frac{1}{v}\right) &= \psi_2 + \psi, \end{aligned} \tag{3}$$

function $\psi$ is dependent variable on $\mathbb{Z}^2$ and we denote $\psi(m, n) =: \psi$, $\psi(m+1, n) =: \psi_1$, $\psi(m, n+1) =: \psi_2$, $\psi(m+1, n+1) =: \psi_{12}$, $p := p(m)$ and $q := q(n)$ are given functions of a single variable and $a$, $b$ and $c$ are arbitrary constants (we assume that one of the constants $a$, $b$ or $c$ is not equal to zero). All the equations within the family are consistent around the cube (for the consistency around the cube property see [23, 24, 25], notice we resign from multiaffinity assumption of paper [25]). Members of the family are Hirota's sine-Gordon equation (choice of parameters $b = 0 = c$) referred also to as lattice potential modified KdV [26, 27, 28, 29, 30, 14, 31] (see section 2 where we discuss a various forms of lattice equations)

$$p(xx_1 + x_2x_{12}) = q(xx_2 + x_1x_{12}). \tag{4}$$

lattice Schwarzian KdV [27] in a disguise, see section 2 (choice of parameters $a = 0 = b$ or $a = 0 = c$)

$$p^2(y_{12} + y_1)(y_2 + y) = q^2(y_{12} + y_2)(y_1 + y). \tag{5}$$

In the second example we go away from the maps of papers [12, 13] and consider the map

$$U = v + k\left(1 - \frac{v}{u}\right), \quad V = u + k\left(-1 + \frac{u}{v}\right). \tag{6}$$

which gives also a 3 parameter family of equations (see section 5) including Hirota's KdV lattice equation [5]

$$x_{12} - x = \kappa\left(\frac{1}{x_2} - \frac{1}{x_1}\right) \tag{7}$$

and two further bilinear equations

$$\begin{aligned} &y_1y - y_{12}y_1 = \kappa(y_{12}y + y_1y_2) \\ &z_{12}z + z_1z_2 = z_{12}z_2 + z_{12}z_1 \end{aligned} \tag{8}$$

In this case an interesting fact is that the procedure yields $\tau$-function representation of the family (see e.g. [19])

$$\begin{aligned} \tau_{112}\tau - \kappa\tau_{11}\tau_2 &= \tau_{12}\tau_1, \\ \tau_{122}\tau + \kappa\tau_{22}\tau_1 &= \tau_{12}\tau_2. \end{aligned} \tag{9}$$

In section 2, we give an overview of point transformations, Bäcklund transformations and difference substitutions and touch the issue of equivalence of lattice equations. We proceed in section 3 where we present the method that lead to families of lattice equations. In section 4 we relate our findings to some results of the papers [12, 13], followed by section 5 where we deal with Hirota's KdV lattice equation. We finish with explanation how to get Bäcklund transformation between members of the families (section 6).

## 2 Point transformations, Difference substitutions, Bäcklund transformations and Equivalence of lattice equations

Before we start we would like to give some definitions and recall some well known relations [28, 29, 30, 19, 31] between equations that appear in the article (terminology used by various authors is far from being unified). Let us consider $k$ dependent variables of $n$ independent ones: $u^i(m^1, \ldots, m^n)$, $i = 1, \ldots, k$. We denote $M \equiv (m^1, \ldots, m^n)$.

**Proposition 1 (Change of independent variables)** By change of independent variables *we understand the bijection* $f : \mathbb{Z}^n \to \mathbb{Z}^n$

$$\tilde{m}^i = f^i(M) \quad i = 1, \ldots, n$$

2D examples are $\tilde{m}^1 = m^1, \tilde{m}^2 = m^1 + m^2$, or $\tilde{m}^1 = m^1 + 2m^2, \tilde{m}^2 = m^1 + m^2$.

**Proposition 2 (Point transformations not altering independent variables)** *By* point transformation not altering independent variables *we understand an invertible map $F$ between subsets of* $\mathbb{C}^k$

$$\tilde{u}^i(M) = F^i(u^1(M), \ldots, u^k(M); M) \quad i = 1, \ldots, k.$$

**Proposition 3 (Equivalence of lattice equations)** *Two lattice equations are equivalent if and only if there exists composition of point transformation with change of independent variables which maps solutions of one equation to solutions to the second one.*

Examples of various disguises of the same equation are

- Hirota's sine-Gordon equation

  $$q \sin(\psi_{12} + \psi - \psi_1 - \psi_2) = p \sin(\psi_{12} + \psi + \psi_1 + \psi_2)$$

  turns into

  $$(H3^0): \qquad p(xx_1 + x_2x_{12}) = q(xx_2 + x_1x_{12}) \tag{10}$$

  $H3^0$ equation from ABS list [25] by means of point transformation $x = i^{m+n}e^{2i(-1)^n\psi}$. $H3^0$ in turn can be transformed into lattice potential modified KdV

  $$p(ww_1 - w_2w_{12}) = q(ww_2 - w_1w_{12})$$

  by substitution $x = i^{m+n}w$

- Schwarzian KdV equation (or cross ratio equation, or equation $Q1^0$ on ABS list)

$$\frac{(z_{12}-z_1)(z^2-z)}{(z_{12}-z_2)(z^1-z)}=\frac{q^2}{p^2}$$

under the point transformation $z=(-1)^{m+n}y$ turns into

$$(A1^0): \qquad p^2(y_{12}+y_1)(y_2+y)=q^2(y_{12}+y_2)(y_1+y) \tag{11}$$

which in the paper [25] got its own name $A1^0$.

**Proposition 4 (Difference substitutions)** *Let $j$ points $M^i$, $i=1,\dots,j$ of a lattice are given. By* difference substitution of order $j$ *we understand a transformation*

$$\tilde{u}^i(M)=F^i(u^1(M^1),\dots,u^k(M^1),\dots,u^1(M^j),\dots,u^k(M^j);M) \quad i=1,\dots,k$$

Every point transformation is difference substitutions of order 1. Standard examples of difference substitution (of order 2, 3 and 4 respectively) are

- potential relation

$$v=\frac{1}{\alpha-\beta}(u_2-u_1)$$

between lattice potential KdV

$$(u_{12}-u)(u_1-u_2)=\alpha^2-\beta^2$$

and Hirota's difference KdV

$$v_{12}-v=\frac{\alpha+\beta}{\alpha-\beta}\left(\frac{1}{v_1}-\frac{1}{v_2}\right)$$

- Miura-type transformation

$$v=\frac{\beta\psi_2-\alpha\psi_1}{(\beta-\alpha)\psi}$$

between $H3^0$ (Hirota's sine-Gordon or lattice modified potential KdV)

$$\alpha(\psi_2\psi_{12}-\psi\psi_1)=\beta(\psi_1\psi_{12}-\psi\psi_2).$$

and Hirota's difference KdV

- and finally the introduction of $\tau$ function

$$v=\frac{\tau_{12}\tau}{\tau_1\tau_2}$$

which transform every solution of the compatible system

$$\tau_{112}\tau-\kappa\tau_{11}\tau_2=\tau_{12}\tau_1, \qquad \tau_{122}\tau+\kappa\tau_{22}\tau_1=\tau_{12}\tau_2$$

to solution of Hirota's difference KdV.

To the end we propose draft definition of Bäcklund transformation which is convenient for our purposes. However we are aware that the definition is not exhaustive (some transformation that deserve this name can be not covered by the definition).

**Proposition 5 (Bäcklund transformations (in narrow sense))** *By* Bäcklund transformation *we understand here a transformation*

$$\tilde{u}_1 = f(\tilde{u}, u, u_1) \qquad \tilde{u}_2 = g(\tilde{u}, u, u_2)$$

*which is invertible to*

$$u_1 = \tilde{f}(u, \tilde{u}, \tilde{u}_1) \qquad u_2 = \tilde{g}(u, \tilde{u}, \tilde{u}_2),$$

*where functions $f$ and $g$ are fractional linear in $\tilde{u}$ and functions $\tilde{f}$, $\tilde{g}$ are function fractional linear in $u$.*

A classical example of Bäcklund transformation between

$$p(xx_1 + x_2x_{12}) - q(xx_2 + x_1x_{12}) = 0.$$

and

$$p^2(y_{12} + y_1)(y_2 + y) = q^2(y_{12} + y_2)(y_1 + y)$$

is transformation

$$y_1 + y = px_1x \qquad y_2 + y = qx_2x. \tag{12}$$

# 3 Outline of the method

Consider $\mathbb{Z}^2$ lattice together with its horizontal edges (which can be viewed as set of ordered pair of points of $\mathbb{Z}^2$ i.e. $E_h = \left\{((m,n),(m+1,n))|(m,n) \in \mathbb{Z}^2\right\}$) and vertical ones ($E_v = \left\{((m,n),(m,n+1))|(m,n) \in \mathbb{Z}^2\right\}$). We take into account function $u$ which is given on horizontal edges $u : E_h \to \mathbb{C}$ and function $v$ given on vertical ones $v : E_v \to \mathbb{C}$. Shift operators $T_1$ and $T_2$ act on horizontal edges in standard way $T_1((m,n),(m+1,n)) := ((m+1,n),(m+2,n))$, $T_2((m,n),(m+1,n)) := ((m,n+1),(m+1,n+1))$ (and similarly for vertical edges). We use convention to denote shift action on function by subscripts $T_1u := u_1$.

Now, the outline of the method we developed in [18] can be presented as follows.

## 3.1 From equations to involutive maps. Idea System

Take a function $x$ given on vertices of the lattice and which obeys $H3^0$ equation

$$p(xx_1 + x_2x_{12}) = q(xx_2 + x_1x_{12}). \tag{13}$$

Introduce fields $u$ and $v$ given on horizontal and vertical edges respectively

$$u = xx_1 \quad v = xx_2 \tag{14}$$

We get

$$\begin{aligned} &u_2u = v_1v \\ &p(u_2 + u) = q(v_1 + v) \end{aligned} \tag{15}$$

and we arrive at the system of equations

$$\begin{aligned} u_2 &= v\tfrac{pu-qv}{qu-pv} \\ v_1 &= u\tfrac{pu-qv}{qu-pv}. \end{aligned} \tag{16}$$

The main idea is to investigate system (16) rather than equation (13) itself. We dare to refer to the system (16) as to 2D Idea system III. The point is the system (16) admits, as we shall see, three parameter family of potentials $\psi$ given on vertices of the lattice. Every "potential image" of (16) we refer to as idolon (adopting Plato terminology of Ideas and idolons).

First we apply the standard procedure for reinterpretation of a map as equations on a lattice. The reinterpretation is based on identification (see Figure 1.)

$$u(m,n) = u, \quad v(m,n) = v, \quad U = u(m,n+1), \quad V = v(m+1,n) \tag{17}$$

which turns system (16) into $\mathbb{Z}^2 \to \mathbb{Z}^2$ map

$$U = v\tfrac{pu-qv}{qu-pv} \qquad V = u\tfrac{pu-qv}{qu-pv} \tag{18}$$

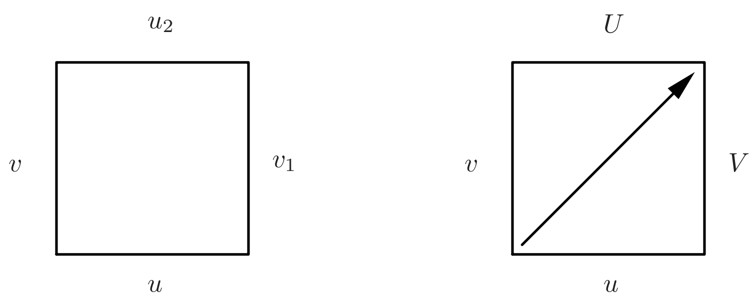


Figure 1: Variables on edges of a $\mathbb{Z}^2$ lattice (left picture) and arguments and values of a $\mathbb{C}^2 \mapsto \mathbb{C}^2$ map (right picture).

We arrive at involutive Yang-Baxter map that belongs to family of maps denoted by $F_{III}$ on list [12].

### 3.2 Finding functions such that $F(U) + G(V) = f(u) + g(v)$

The next step is to find such functions $F$ and $G$ such that for the map (18)

$$F(U) + G(V) = f(u) + g(v). \tag{19}$$

holds. Differentiation of (19) with respect $u$ and $v$ yields

$$-F''(U)qU^2(pU^2-2qUV+pV^2)+F'(U)2(qU-pV)qUVG''(V)pV^2(qU^2-2pUV+qV^2)+G'(V)2(qU-pV)pUV = 0$$

which leads to

$$F(U) + G(V) = a\ln(U/V) + b(pU - qV) + c\left(\frac{p}{U} - \frac{q}{V}\right).$$

Therefore for map (18) the following equality holds

$$a\ln(U/V) + b(pU - qV) + c\left(\tfrac{p}{U} - \tfrac{q}{V}\right) = -\left[a\ln(u/v) + b(pu - qv) + c\left(\tfrac{p}{u} - \tfrac{q}{v}\right)\right] \tag{20}$$

### 3.3 Potentials of the Idea systems. Idolons.

Returning to equations on the lattice (by means of (17)) one can rewrite (20) as

$$(T_2+1)(a\ln u+bpu+c\tfrac{p}{u}+d)=(T_1+1)(a\ln v+bqv+c\tfrac{q}{v}+d) \tag{21}$$

It means there exists function $\psi$ such that

$$\begin{array}{l} a\ln u+p(bu+c\frac{1}{u})+d=\psi_1+\psi \\ a\ln v+q(bv+c\frac{1}{v})+d=\psi_2+\psi \end{array} \tag{22}$$

where $a$, $b$, $c$ and $d$ are arbitrary constants (we assume that one of the constants $a$, $b$, $c$ is not equal zero). The constant $d$ can be always removed by redefinition $\psi\to\psi+\frac{1}{2}d$ and we neglect it. So system (22) gives rise to three parameter family of equations

$$\psi_{12}=\psi+a\ln\tfrac{pu-qv}{qu-pv}+(p^2-q^2)\left[b\tfrac{uv}{qu-pv}-c\tfrac{1}{pu-qv}\right] \tag{23}$$

As we have said in the introduction choice of parameters $b=0=c$ leads to equation $H3^0$ (10) whereas choice of parameters either $a=0=b$ or $a=0=c$ leads to equation $A1^0$ (11). Every such potential representation of the Idea system we refer to as idolon of the Idea systems. To the end let us write another idolon that can be written in explicit form. Namely, $a=0$ yields equation

$$\tfrac{\psi_2-\psi_1}{\psi_{12}-\psi}=\tfrac{p^2+q^2}{p^2-q^2}-\tfrac{pq}{p^2-q^2}\left(\tfrac{u}{v}+\tfrac{v}{u}\right) \tag{24}$$

where $u$ and $v$ are solutions of the following quadratic equations

$$\begin{array}{l} p(bu^2+c)=(\psi_1+\psi)u \\ q(bv^2+c)=(\psi_2+\psi)v. \end{array} \tag{25}$$

### 3.4 Extension to multidimension, multidimensional consistency of idolons of $I_{III}$

The system (16) can be extended to multidimension. We denote by $s^i$ (mind superscript!) function given on edges in $i$-th direction of $\mathbb{Z}^n$ lattice, by subscript we denote forward shift in indicated direction. The extension reads

$$(I_{III})\qquad s^i_j=s^j\frac{p^is^i-p^js^j}{p^js^i-p^is^j}\qquad i,j=1,\dots,n,\qquad i\neq j \tag{26}$$

where $p^i$ is given function and can depend only on $i$-th independent variable.

The crucial fact is the system is compatible

$$s^i_{jk}=s^i_{kj}. \tag{27}$$

Moreover, we have

$$(T_j+1)\left[a\ln s^i+p^i(bs^i+c\frac{1}{s^i})\right]=(T_i+1)\left[a\ln s^j+p^i(bs^j+c\frac{1}{s^i})\right] \tag{28}$$

It means that three exists scalar function $\psi$ such that

$$a\ln s^i+p^i(bs^i+c\frac{1}{s^i})=\psi_i+\psi,\qquad i=1,\dots,n \tag{29}$$

In terms of function $\psi$ system (26) reads

$$\psi_{ij} = \psi + a \ln \tfrac{p^i s^i - p^j s^j}{p^j s^i - p^i s^j} + [(p^i)^2 - (p^j)^2]\left[b \tfrac{s^i s^j}{p^j s^i - p^i s^j} - c \tfrac{1}{p^i s^i - p^j s^j}\right] \qquad i,j = 1,\dots,n, \qquad i \neq j \tag{30}$$

where $s^i$ and $s^j$ are given implicitly by means of (29). Due to (27) the system (30) is multidimensionaly consistent (compatible).

We refer to the system (26) as to n-dimensional Idea system III and that is why we have denoted it by $I_{III}$.

# 4 Maps

As we have already mentioned our inspiration was survey on Yang-Baxter maps. Our goal now is to relate our findings to some results of the papers [12, 13] and justify why it makes sense to talk about the Idea systems

$$s^i_j = s^j \frac{p^i s^i - p^j s^j}{p^j s^i - p^i s^j} \qquad i = 1,\dots,n \tag{31}$$

associated with maps of type $III$ rather than single Idea system. The Idea systems are related by point transformation.

Indeed, first we perform a cosmetic point transformation

$$s^i = p^i v^i, \qquad {p^i}^2 \to p^i$$

We get

$$v^i_j = \frac{v^j}{p^i} \frac{p^i v^i - p^j v^j}{v^i - v^j} \tag{32}$$

which in two-dimensional case after identification analogous to the one showed on the Figure 1 yields $F_{III}$ map of paper [12]

$$(F_{III}): \qquad U = \frac{v}{p}\frac{pu - qv}{u - v}, \qquad V = \frac{u}{q}\frac{pu - qv}{u - v} \tag{33}$$

In fact by $F_{III}$ we understand equivalence class of Yang-Baxter maps (c.f. [13]) the equations (31) and (33) belongs to.

Now after point transformation $v^i = u^i(-1)^{m_1+\dots+m_n}$ we get

$$u^i_j = -\frac{u^j}{p^i} \frac{p^i u^i - p^j u^j}{u^i - u^j} \tag{34}$$

associated 2D map of which is

$$(cH^A_{III}): \qquad U = -\tfrac{v}{p}\tfrac{pu-qv}{u-v}, \qquad V = -\tfrac{u}{q}\tfrac{pu-qv}{u-v} \tag{35}$$

After another point transformation $u^i = {w^i}^{(-1)^{m_1+\dots+m_n}} {p^i}^{\frac{1}{2}[(-1)^{m_1+\dots+m_n}-1]}$ we obtain

$$w^i_j = -\frac{1}{w^j} \frac{w^i - w^j}{p^i w^i - p^j w^j} \tag{36}$$

Table 1: Basic identities of the maps that leads to existence of potentials of the Idea system

| Type of the map | Example of the map | Identities |
|---|---|---|
| $F_{III}$ | $U=\frac{v}{p}\frac{pu-qv}{u-v}$ <br> $V=\frac{u}{q}\frac{pu-qv}{u-v}$ | $\frac{U}{V}=\frac{qv}{pu}$ <br> $pU-qV=-(pu-qv)$ <br> $\frac{1}{U}-\frac{1}{V}=-\left(\frac{1}{u}-\frac{1}{v}\right)$ |
| $cH^A_{III}$ | $U=-\frac{v}{p}\frac{pu-qv}{u-v}$ <br> $V=-\frac{u}{q}\frac{pu-qv}{u-v}$ | $\frac{U}{V}=\frac{qv}{pu}$ <br> $pU-qV=pu-qv$ <br> $\frac{1}{U}-\frac{1}{V}=\frac{1}{u}-\frac{1}{v}$ |
| $cH^B_{III}$ | $U=\frac{1}{v}\frac{u-v}{qv-pu}$ <br> $V=\frac{1}{u}\frac{u-v}{qv-pu}$ | $\frac{U}{V}=\frac{u}{v}$ <br> $pU+\frac{1}{U}-qV-\frac{1}{V}=pu+\frac{1}{u}-qv-\frac{1}{v}$ <br> $pU-\frac{1}{U}-qV+\frac{1}{V}=-(pu-\frac{1}{u}-qv+\frac{1}{v})$ |
| $H^A_{III}$ | $U=\frac{v(pu+qv)}{p(u+v)}$ <br> $V=\frac{u(pu+qv)}{q(u+v)}$ | $\frac{U}{V}=\frac{qv}{pu}$ <br> $pU+qV=pu+qv$ <br> $\frac{1}{U}+\frac{1}{V}=\frac{1}{u}+\frac{1}{v}$ |
| $H^B_{III}$ | $U=v\frac{quv+1}{puv+1}$ <br> $V=u\frac{puv+1}{quv+1}$ | $UV=uv$ <br> $pU+qV+\frac{1}{U}+\frac{1}{V}=pu+qv+\frac{1}{u}+\frac{1}{v}$ <br> $pU-qV-\frac{1}{U}+\frac{1}{V}=-(pu-qv-\frac{1}{u}+\frac{1}{v})$ |

and its associated map

$$(cH^B_{III}): \qquad U=-\frac{1}{v}\frac{u-v}{pu-qv}, \qquad V=-\frac{1}{u}\frac{u-v}{pu-qv} \tag{37}$$

Maps (35) and (37) are not Yang-Baxter maps but they are companions (if $f:(u,v)\mapsto(U,V)$ is involutive map then the map $(u,V)\mapsto(U,v)$ we refer to as companion of map $f$ c.f. [12]) of Yang-Baxter maps $H^A_{III}$, $H^B_{III}$ of paper [13]. The maps $H^A_{III}$, $H^B_{III}$ can be obtained in two-dimensional case by the point transformation $u^1=x$ $u^2=-y$ and $w^1=x$ $w^2=-\frac{1}{qy}$ respectively

$$x_2=\frac{y}{p}\frac{px+qy}{x+y} \qquad y_1=\frac{x}{q}\frac{px+qy}{x+y} \tag{38}$$

$$x_2=y\frac{qxy+1}{pxy+1} \qquad y_1=x\frac{pxy+1}{qxy+1} \tag{39}$$

and then by mentioned identification (see Figure 1)

$$(H^A_{III}): \qquad U=\frac{v}{p}\frac{pu-qv}{u-v}, \qquad V=\frac{u}{q}\frac{pu-qv}{u-v} \tag{40}$$

$$(H^B_{III}): \qquad U=v\frac{quv+1}{puv+1}, \qquad V=u\frac{quv+1}{puv+1} \tag{41}$$

Idea systems ($H^A_{III}$) and ($H^B_{III}$) cannot be extended to multidimension (in the sense of paper [25]).

Finally, we list in the table 4 basic identities of the maps that leads to existence of potentials of the Idea systems to illustrate how the basis changes when one changes a map.

# 5 Hirota's KdV lattice equation

As the second example we consider Hirota's KdV lattice equation [5]

$$x_{12} - x = \kappa(\frac{1}{x_2} - \frac{1}{x_1})$$

By the substitution $u = x_1 x$, $v = x_2 x$, we get

$$u_2 = v + \kappa\left(1 - \frac{v}{u}\right), \quad v_1 = u + \kappa\left(-1 + \frac{u}{v}\right). \tag{42}$$

On applying identification (17)

$$u = u(m,n), \quad v = v(m,n), \quad U = u(m,n+1), \quad V = v(m+1,n) \tag{43}$$

we obtain an involutive mapping associated to system (42)

$$U = v + \kappa\left(1 - \frac{v}{u}\right), \quad V = u + \kappa\left(-1 + \frac{u}{v}\right). \tag{44}$$

Mapping (44) satisfies (this is the outcome of searching for such functions $F$ and $G$ that $F(U) + G(V) = f(u) + g(v)$ as described in previous section):

$$\begin{aligned} &\frac{U}{V} = \frac{v}{u} \\ &(U-\kappa)(V+\kappa) = (u-\kappa)(v+\kappa) \\ &\frac{V(U-\kappa)}{U(V+\kappa)} = \frac{v(u-\kappa)}{u(v+\kappa)} \end{aligned} \tag{45}$$

hence (coming back to lattice variables (43)) we can introduce the potentials $x$, $y$ and $z$

$$\begin{aligned} &u = x_1 x, && v = x_2 x \\ &u - \kappa = y_1/y, && v + \kappa = y/y_2 \\ &\frac{u-\kappa}{u} = z_1/z, && \frac{v+\kappa}{v} = z_2/z. \end{aligned} \tag{46}$$

Eliminating $u$ and $v$ from (42) we arrive at the following lattice equations

$$\begin{aligned} x_{12} - x &= \kappa(1/x_2 - 1/x_1) \\ y_1 y - y_{12} y_1 &= \kappa(y_{12} y + y_1 y_2) \\ z_{12} z + z_1 z_2 &= z_{12} z_2 + z_{12} z_1 \end{aligned} \tag{47}$$

One can treat the equations as representatives of three-parameter family of equations on $\phi$

$$\begin{aligned} \tfrac{\phi_{12}\phi}{\phi_1\phi_2} &= [(u-\kappa)(v+\kappa)+\kappa^2]^{a(-1)^{m+n+1}-b} u^{b-c} v^{b+c} \\ \tfrac{\phi_1}{\phi} &= u^{a(-1)^{m+n}-b}(u-\kappa)^{b+c} \\ \tfrac{\phi_2}{\phi} &= v^{a(-1)^{m+n}-b}(v+\kappa)^{b-c} \end{aligned} \tag{48}$$

corresponding to choice of parameters $b = 0 = c$, $a = 0 = b$ and $a = 0 = c$ respectively.

What more important is that from (46) we infer

$$\frac{z_1}{z} = \frac{y_1}{x_1 x y}, \qquad \frac{z_2}{z} = \frac{y}{y_2 x_2 x} \tag{49}$$

Compatibility condition that guarantees existence of function $z$ reads

$$\left(\frac{x_2}{x_1}\right)^2 = \left(\frac{y_{12} y}{y_1 y_2}\right)^2, \tag{50}$$

from where we get

$$\begin{aligned} x &= \tfrac{\tau_{12}\tau}{\tau_1\tau_2} \\ y &= \tfrac{\tau_2}{\tau_1} \\ z &= \tfrac{\tau}{\tau_{12}} \end{aligned} \tag{51}$$

Eliminating $x$, $y$ and $z$ from (46) we arrive at compatible pair of bilinear forms of Hirota's KdV (cf. [19])

$$\begin{aligned} \tau_{112}\tau - \kappa\tau_{11}\tau_2 = \tau_{12}\tau_1 \\ \tau_{122}\tau + \kappa\tau_{22}\tau_1 = \tau_{12}\tau_2 \end{aligned} \tag{52}$$

## 6 Bäcklund transformations between idolons

In both presented examples one can find Bäcklund transformation between idolons. For instance eliminating $u$ and $v$ from first two lines of (46) one gets Bäcklund transformation between first two equations of (47)

$$\frac{y_1}{y} = x_1 x - k \qquad \frac{y}{y_2} = x_2 x + k.$$

Similarly in the case of $I_{III}$ one can obtain Bäcklund transformation (12).

**Acknowledgements:** We would like to thank organizers of SIDE - 9 conference in Varna for their hospitality and financial support. Special thanks to Georgi Grahovski for showing us the other side of Varna. M.N. thanks Frank Nijhoff for pointing papers [20, 21].

## References

[1] J. Ritt. Permutable rational functions. *Trans. Amer. Math. Soc.*, 25:399–448, 1923.

[2] G. Julia. Memoire sur la permutabilité des fractions rationelles. *Ann. Sci. Ecole Norm. Sup*, 39:131–152, 1922.

[3] P. Fatou. Sur l'itération analytique et les substitutions pennutables. *J. Math. Pure Appl.*, 23:1–49, 1924.

[4] N.J. Zabusky and M.D. Kruskal. Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15(6):240–243, 1965.

[5] R. Hirota. Nonlinear partial difference equation I: A difference analogue of the KdV equation. *J. Phys. Soc. Jp.*, 43:1429–1433, 1977.

[6] M.J Ablowitz and F.J. Ladik. A nonlinear difference scheme and inverse scattering. *Stud. Appl. Math.*, 55:213–229, 1976.

[7] F.W. Nijhoff, G.R.W. Quispel, and H.W. Capel. Direct linearization of nonlinear difference-difference equations. *Phys. Lett. A*, 97:125–128, 1983.

[8] G.R.W. Quispel, J.A.G Roberts, and C.J Thompson. Integrable mappings and soliton equations. *Phys. Lett. A*, 126:419–421, 1988.

[9] V.G. Papageorgiou, F.W. Nijhoff, and H.W. Capel. Integrable mappings and nonlinear integrable lattice equations. *Phys. Lett. A*, 147:106–114, 1990.

[10] F.W. Nijhoff, V.G. Papageorgiou, H.W. Capel, and G.R.W. Quispel. The lattice Gel'fand-Dikii hierarchy. *Inverse Problems*, 8:597–621, 1992.

[11] A.P Veselov. Integrable Maps. *Russ. Math. Surv.*, 46:1–51, 1991.

[12] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings. *Comm. Anal. Geom.*, 12(5):967–1007, 2004.

[13] V.G. Papageorgiou, A.G. Tongas, and A.P. Veselov. Yang-Baxter maps and symmetries of integrable equations on quad-graphs. *J. Math. Phys.*, 47:Art. no. 083502, 2006.

[14] A.I Bobenko and Y.B Suris. *Discrete differential geometry. Integrable structure*, volume 98 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.

[15] V.G. Drinfeld. On some unsolved problems in quantum group theory. In *Quantum Groups*, volume 1510 of *Lecture notes in Mathematics*, pages 1–8. Springer, New York, 1992.

[16] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett. A*, 314:214–221, 2003.

[17] V.G. Papageorgiou, Yu.B. Suris, A.G. Tongas, and A.P. Veselov. On quadrirational yang-baxter maps. *SIGMA 6*, 033:9pp, 2010.

[18] P. Kassotakis and M. Nieszporski. On non-multiaffine consistent around the cube lattice equations. arXiv:1106.0435.

[19] F. W. Nijhoff, A. Ramani, B. Grammaticos, and Y. Ohta. On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation. *Stud. Appl. Math.*, 106(3):261–314, 2001.

[20] F.W. Nijhoff. On some "schwarzian equations" and their discrete analogues. In A.S. Fokas and I.M. Gel'fand, editors, *Algebraic Aspects of Integrable Systems: In memory of Irene Dorfman*, pages 237–260. Birkhuser Verlag, 1996.

[21] F.W. Nijhoff. Discrete Painlevé equations and symmetry reduction onthe lattice. In A.I. Bobenko and R. Seiler, editors, *Discrete Integrable Geometry and Physics*, pages 209–234. Oxford Univ. Press, 1999.

[22] F.W. Nijhoff. A higher-rank version of the q3 equation. arXiv:1104.1166.

[23] F.W. Nijhoff and A.J. Walker. The discrete and continuous Painlevé VI hierarchy and the Garnier systems. *Glasgow Math. J.*, 43A:109–123, 2001.

[24] F.W. Nijhoff. Lax pair for the Adler (lattice Krichever-Novikov)system. *Phys. Lett. A*, 297:49–58, 2002.

[25] V.E. Adler, A.I. Bobenko, and Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.*, 233(3):513–543, 2003.

[26] R. Hirota. Nonlinear partial difference equations iii; discrete sine-gordon equation. *J. Phys. Soc. Jpn.*, 43:2079–2086, 1977.

[27] F. Nijhoff and H. Capel. The discrete Korteweg-de Vries equation. *Acta Appl. Math.*, 39(1-3):133–158, 1995.

[28] L. Faddeev and A. Yu. Volkov. Hirota equation as an example of an integrable symplectic map. *Lett. Math. Phys. Soc.*, 32:125–135, 1994.

[29] A. Bobenko and U. Pinkall. Discrete surfaces with constant negative Gaussian curvature and the Hirota equation. *J. Diff. Geom.*, 43(3):527–611, 1996.

[30] A. Zabrodin. Tau-function for discrete sine-gordon equation and quantum r-matrix. *arXiv ID: solv-int/9810003*, 1998.

[31] B. Grammaticos, A. Ramani, C. Scimiterna, and R. Willox. Miura transformations and the various guises of integrable lattice equations. *J. Phys. A: Math. Th.*, 44:Art. no. 152004, 2011.