THE GARDNER EQUATION AND THE STABILITY OF MULTI-KINK SOLUTIONS OF THE MKDV EQUATION

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ABSTRACT. Multi-kink solutions of the defocusing, modified Korteweg-de Vries equation (mKdV) found by Grosse [19, 20] are shown to be globally H^1-stable, and asymptotically stable. Stability in the one-kink case was previously established by Zhidkov [47] and Merle-Vega [37]. The proof uses transformations linking the mKdV equation with focusing, Gardner-like equations, where stability and asymptotic stability in the energy space are known. We generalize our results by considering the existence, uniqueness and the dynamics of generalized multi-kinks of defocusing, non-integrable gKdV equations, showing the inelastic character of the kink-kink collision in some regimes.

1. INTRODUCTION AND MAIN RESULTS

In this paper we continue our work on stability of multi-soliton solutions for some well-known, dispersive equations, started in a joint work with M.A. Alejo and L. Vega [3]. In this opportunity, we consider the nonlinear H^1-stability, and asymptotic stability, of the *multi-kink* solution of the *defocusing*, modified Korteweg-de Vries (KdV) equation

$$u_t + (u_{xx} - u^3)_x = 0. \tag{1.1}$$

Here $u = u(t, x)$ is a real valued function, and $(t, x) \in \mathbb{R}^2$. Solutions $u = u(t, x)$ of (1.1) are invariant under space and time translations, and under suitable scaling properties. Indeed, for any $t_0, x_0 \in \mathbb{R}$, and $c > 0$, both $u(t - t_0, x - x_0)$ and $c^{1/2}u(c^{3/2}t, c^{1/2}x)$ are solutions of (1.1). Finally, $u(-t, -x)$ and $-u(t, x)$ are also solutions.

From a mathematical point of view, equation (1.1) is an *integrable model* [2], with a Lax pair structure and infinitely many conservation laws. Moreover, equation (1.1) has non-localized *solitons* solutions, called *kinks*, namely solutions of the form

$$u(t, x) = \varphi_c(x + ct + x_0), \quad \varphi_c(s) := \sqrt{c}\, \varphi(\sqrt{c}s), \quad c > 0,\ x_0 \in \mathbb{R}, \tag{1.2}$$

and

$$\varphi(s) := \tanh(\frac{s}{\sqrt{2}}), \tag{1.3}$$

which solves

$$\varphi'' + \varphi - \varphi^3 = 0, \quad \text{in } \mathbb{R}, \quad \varphi(\pm\infty) = \pm 1, \quad \varphi' > 0. \tag{1.4}$$

On the one hand, since the Cauchy problem associated to (1.1) is locally well posed in $\varphi_c(\cdot + ct) + H^1(\mathbb{R})$ (cf. Merle-Vega [37, Prop. 3.1]), each solution is indeed global in time thanks to the conservation of Energy:

$$E[u](t) := \frac{1}{2} \int_{\mathbb{R}} u_x^2(t, x) dx + \frac{1}{4} \int_{\mathbb{R}} (u^2 - c)^2(t, x) dx = E[u](0). \qquad (1.5)$$

A simple inspection reveals that this is a non-negative quantity.

On the other hand, the standard Cauchy problem for initial data in the Sobolev space $H^s(\mathbb{R})$ is locally well-posed for $s \geq \frac{1}{4}$ (Kenig-Ponce-Vega [24]), and globally well-posed for $s > \frac{1}{4}$ (Colliander et al. [9]). This result is almost sharp since for $s < \frac{1}{4}$ the solution map has been shown to not be uniformly continuous, see Christ-Colliander-Tao [8] (see also Kenig-Ponce-Vega [25] for an early result in the focusing case).

It is also important to stress that (1.1) has in addition another less regular conserved quantity, called mass:

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} (c - u^2(t, x)) dx = M[u](0). \qquad (1.6)$$

Of course this quantity is well-defined for solutions $u(t)$ such that $(u^2(t) - c)$ has enough decay at infinity. In particular, one has $M[\varphi_c] < +\infty$.

Now we focus on the study of suitable perturbations of kinks solutions of the form (1.2). This question leads to the introduction of the concepts of *orbital and asymptotic stability*. In particular, since the energy (1.5) is a conserved quantity –in other words, it is a Lyapunov functional–, well defined for solutions at the H^1-level, it is natural to expect that kinks are (orbitally) stable under small perturbations in the energy space. Indeed, H^1-stability of mKdV kinks has been considered initially by Zhidkov [47], see also Merle-Vega [37] for a complete proof, including an adapted well-posedness theory. We recall that their proof is strongly based in the **non-negative** character of the energy (1.5) around a kink solution φ_c, which balances the bad behavior of the mass (1.6) under general H^1-perturbations of a kink solution.

For additional purposes, to be explained later, we recall that in [37], the main objective of Merle and Vega was to prove that solitons of the KdV equation

$$u_t + (u_{xx} + u^2)_x = 0, \qquad (1.7)$$

were L^2-stable, by using the *Miura transform*

$$M[v](t, x) := \frac{3}{2}c + \left[\frac{3}{\sqrt{2}} v_x - \frac{3}{2} v^2 \right](t, x - 3ct). \qquad (1.8)$$

This nonlinear $H^1 - L^2$ transformation links solutions of (1.1) with solutions of the KdV equation (1.7). In particular, the image of the family of kink solutions (1.2) under the transformation (1.8) is the well-known soliton of KdV, with scaling $2c$ (cf. [37]):

$$M[\varphi_c(x + ct + x_0)] = Q_{2c}(x - 2ct + x_0).$$

Therefore, by proving the H^1-stability of single kinks –a question previously considered by Zhidkov [47]–, and (1.8), they obtained a form of L^2-stability for the KdV soliton. Additionally, a simple form of asymptotic stability for the kink solution

was proved. Related asymptotic results for soliton-like solutions can also be found e.g. in [12, 43, 41, 26, 30, 31].

Kinks are also present in other nonlinear models, such as the sine-Gordon (SG) equation, the ϕ^4-model, and the Gross-Pitaevskii (GP) equation [1, 13]. In each case, it has been proved that their are stable for small perturbations in a suitable space, cf. [23, 18, 47, 15, 6]. Let us also recall that the SG and GP equations are integrable models in one dimension [1, 13].

Let us come back to the equation (1.1). In addition to the previously mentioned kink solution (1.2), mKdV has *multi-kink* solutions, as a consequence of the integrability property, and the Inverse Scattering method. This result, due to Grosse [19, 20], can be obtained by a different approach using the Miura transform (1.8), see Gesztesy-Schweinger-Simon [16, 17], or the monograph by Thaller [44]. According to Gesztesy-Schweinger-Simon [17], there are at least two different forms of multi-kink solutions for (1.1), which we describe below (cf. Definitions 1.1 and 1.4). Moreover, they proved that the Miura transform sends these solutions towards a well defined family of multi-soliton solutions of the KdV equation, provided a *criticality* property is satisfied (see [18, 17] for such an assumption).

In order to present multi-kinks from a different point of view, we need some preliminaries.

In [3] (see e.g. [40] for a short review), Alejo-Muñoz-Vega showed the L^2-stability of KdV multi-solitons following the Merle-Vega approach and the Gesztesy-Schweinger-Simon property [17], above described. However, H^1-stability of multi-kinks was not known at that moment. Even worse, according to our knowledge, there was no result involving stability of several kink solutions, for any type of dispersive equation with stable kinks. Instead, we avoided this problem and followed a different approach, based in the use of the **Gardner transform**

$$M_\beta[v] := v - \frac{3}{2}\sqrt{2\beta}v_x - \frac{3}{2}\beta v^2. \qquad (1.9)$$

This nonlinear map links solutions of the KdV equation and the *Gardner equation* [38, 14],

$$v_t + (v_{yy} + v^2 - \beta v^3)_y = 0, \ \text{ in } \ \mathbb{R}_t \times \mathbb{R}_y, \quad \beta > 0. \qquad (1.10)$$

In particular, the Gardner transform sends Gardner solitons towards KdV solitons (see [3] for further details).

However, we have realized that the existence, uniqueness and stability of multi-kinks is closely related to the solitons of the Gardner equation, and more generally, *dynamical properties of defocusing gKdV equations are closely related to those of suitable focusing counterparts*. In particular, as a consequence of our results, we provide the first proof of stability for multi-kinks solutions of the mKdV equation. This result can be also considered a first step towards the understanding of the dynamics of several SG and ϕ^4 kinks.

In order to explain in more detail this relationship, let us recall that the Gardner equation is also an integrable model [14], with soliton solutions of the form

$$v(t,y) := Q_{c,\beta}(y - ct),$$

and[1]

$$Q_{c,\beta}(s) := \frac{3c}{1 + \rho \cosh(\sqrt{c}s)}, \quad \text{with} \quad \rho := (1 - \frac{9}{2}\beta c)^{1/2}, \quad 0 < c < \frac{2}{9\beta}. \quad (1.11)$$

In particular, in the formal limit $\beta \to 0$, we recover the standard KdV soliton. On the other hand, the Cauchy problem associated to (1.10) is globally well-posed under initial data in the energy class $H^1(\mathbb{R})$ (cf. [24]), thanks to the mass and energy conservation laws (see [3] for more details).

The first, striking connection is well-known in the mathematical physics literature, and it was in part used in the recent paper [3]. Indeed, let $v = v(t, y) \in C(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (1.10). Then

$$u(t, x) := b - \sqrt{\beta}\, v\big(t, x + \frac{t}{3\beta}\big), \quad b := \frac{1}{3\sqrt{\beta}}, \quad (1.12)$$

solves the mKdV equation (1.1).[2] In terms of the Miura and Gardner transform, it reads as follows
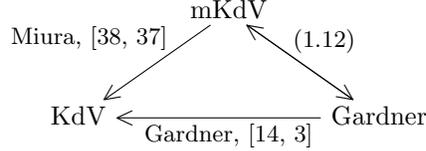
FIGURE 1. Transformation (1.12) in terms of Miura (1.8) and Gardner transforms.

Note that, for t fixed, (1.12) is a diffeomorphism which **preserves regularity**, a key difference with respect to the Miura and Gardner transforms. Note in addition that u in (1.12) is an L^∞-function with nonzero limits at infinity. This will allow to consider the first class of multi-kink solutions of (1.1), characterized by the same positive limit $(= b)$ as $x \to \pm\infty$. Moreover, since $-u(t, x)$ is also a solution of (1.1), one can easily construct a solution with negative limits at infinity. This analysis motivates the following alternative approach for the multi-kink solution:

Definition 1.1 (*Even* multi-kink solutions, see also [19, 20, 17, 44]).
Let $\beta > 0$, scaling parameters $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ and $x_1^-, \ldots, x_N^- \in \mathbb{R}$ be fixed numbers. We say that a solution $U_e(t) := U_e(t; c_1^0, \ldots, c_N^0; x_1^-, \ldots, x_N^-)$ of (1.1) is an even multi-kink *if it satisfies*

$$\lim_{t \to -\infty} \big\|U_e(t) - b + \sqrt{\beta} \sum_{j=1}^{N} Q_{c_j^0, \beta}(\cdot + \tilde{c}_j t + x_j^-)\big\|_{H^1(\mathbb{R})} = 0, \quad (1.14)$$

$$\lim_{t \to +\infty} \big\|U_e(t) - b + \sqrt{\beta} \sum_{j=1}^{N} Q_{c_j^0, \beta}(\cdot + \tilde{c}_j t + x_j^+)\big\|_{H^1(\mathbb{R})} = 0, \quad (1.15)$$

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[1]See e.g. [39, 3] and references therein for a more detailed description of solitons and integrability for the Gardner equation.

[2]Note that

$$u(t, x) := -b + \sqrt{\beta}\, v\big(t, x + \frac{t}{3\beta}\big), \quad b = \frac{1}{3\sqrt{\beta}}, \quad \beta > 0, \quad (1.13)$$

is also a solution of (1.1).

with $b = \frac{1}{3\sqrt{\beta}}$, $\tilde{c}_j := \frac{1}{3\beta} - c_j^0 > 0$, $x_j^+ \in \mathbb{R}$ *depending only on* (x_k^-) *and* (c_k^0), *and* $Q_{c,\beta}$ *being solitons of the Gardner equation (1.10).*

Remarks.

1. Let us emphasize that $\tilde{c}_N < \tilde{c}_{N-1} < \ldots < \tilde{c}_1$, which means that bigger Gardner solitons are actually slower than the smaller ones. Note also that they move from the right to the left, as time evolves. In conclusion, as time goes to $\pm\infty$, the Gardner components of the multi-kink solution are ordered in the **inverse** sense compared with the usual solitons of the Gardner equation, or any *focusing* gKdV equation.

2. The denomination **multi-kink** above comes from the fact the these solutions can be seen asymptotically as the *sum* of several kinks $\pm\varphi_c$ of the form (1.2). For instance, with our notation, given $\beta > 0$ and $0 < c < \frac{2}{9\beta}$, an expression for the 2-kink solution is given by [17, p. 505] (see also [44, p. 273])

$$U_e(t,x) = b - [\varphi_{c/2}(x + \tilde{c}t + 2x_0) - \varphi_{c/2}(x + \tilde{c}t)], \qquad (1.16)$$

with

$$b := \frac{1}{3\sqrt{\beta}}, \quad \tilde{c} := \frac{1}{3\beta} - c, \quad x_0 := \frac{1}{2\sqrt{c}} \log\left(\frac{\sqrt{2} + 3\sqrt{\beta c}}{\sqrt{2} - 3\sqrt{\beta c}}\right) > 0,$$

and φ_c as in (1.2). Note that both kinks $\pm\varphi_{c/2}$ have the **same velocity** $\tilde{c}$, a key difference with the SG and ϕ^4 models. After a quick computation, using the identity

$$\tanh(a + k) - \tanh a = 2\tanh(k)\left[1 + \frac{\cosh(2a + k)}{\cosh(k)}\right]^{-1},$$

one can see that (1.16) can be written, as in (1.14)-(1.15), using the Gardner soliton (1.11):

$$U_e(t,x) = b - \sqrt{\beta}Q_{c,\beta}(x + \tilde{c}t + x_0),$$

which will be helpful for our purposes. From this fact one can say that in general, the function U_e represents a $2N$-**kink solution**. In terms of our point of view, it will represent N different Gardner solitons attached to the non-zero constant b.

The existence of a solution U_e satisfying (1.14) is a simple consequence of (1.12) and the behavior of the N-soliton solution of the Gardner equation (see also [19, 20, 17] and [44, pp. 272-273] for the standard deduction). Indeed, from the integrable character of this last equation, given parameters $\beta > 0$, $0 < c_1^0 < \ldots < c_N^0 < \frac{2}{9\beta}$, and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$, it is well-known that there exists a N-soliton solution of the form (see e.g. Maddocks-Sachs [28] for a similar structure)

$$V^{(N)}(t,x) := V^{(N)}(x; c_j^0; x_j^0 - c_j^0 t) \qquad (1.17)$$

of (1.10), and which satisfies

$$\lim_{t \to -\infty} \left\| V^{(N)}(t) - \sum_{j=1}^{N} Q_{c_j^0,\beta}(\cdot - c_j^0 t - x_j^-) \right\|_{H^1(\mathbb{R})} = 0, \qquad (1.18)$$

$$\lim_{t \to +\infty} \left\| V^{(N)}(t) - \sum_{j=1}^{N} Q_{c_j^0,\beta}(\cdot - c_j^0 t - x_j^+) \right\|_{H^1(\mathbb{R})} = 0, \qquad (1.19)$$

for some $x_j^\pm \in \mathbb{R}$, uniquely depending on the set of parameters (c_k^0, x_k^0). Moreover, note that $V^{(N)}(t)$ is **unique** in the sense described by Martel in [29]:

Asymptotic uniqueness: Given $\beta > 0$, $0 < c_1^0 < \ldots < c_N^0 < \frac{2}{9\beta}$, and $x_1^-, \ldots, x_N^- \in \mathbb{R}$, the corresponding multi-soliton $V^{(N)}$ given in (1.17) is the unique $C(\mathbb{R}, H^1(\mathbb{R}))$-solution of (1.10) satisfying (1.18).

From (1.12) we can define

$$U_e(t) := b + \sqrt{\beta}V^{(N)}(t, \cdot + \frac{t}{3\beta}). \tag{1.20}$$

Note that we **do not need** the criticality assumption required in [17, 44]. Following the notation of Maddocks and Sachs [28] and (1.17), we may think U_e as a function of three independent set of variables:

$$U_e(t, x) := U_e(x; c_j^0, x_j^0 + \tilde{c}_j t),$$

with $\tilde{c}_j$ given in Definition 1.1. Therefore, as a conclusion of the preceding analysis, and using (1.12), we get the uniqueness of the corresponding solution U_e.

Theorem 1.2 (Uniqueness of even multi-kink solutions)**.**
Let $\beta > 0$, $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ and $x_1^-, \ldots, x_N^- \in \mathbb{R}$ be fixed numbers. Then the associated even multi-kink U_e defined in (1.20) is the unique solution of (1.1) satisfying (1.14).

Proof. See Section 2. □

The second problem that we want to consider is the **stability** of the multi-kink U_e. First of all, we recall some important literature.

In [36, 35], Martel, Merle and Tsai have showed the stability and asymptotic stability of the *sum of N solitons* of some generalized KdV equations, *well decoupled* at the initial time, in the energy space $H^1(\mathbb{R})$. We say that such an initial data is **well-prepared**. Their approach is based on the construction of N *almost conserved quantities*, related to the mass of each solitary wave, plus the total energy of the solution. Although the proof for general nonlinearities is not present in the literature, it is a direct consequence of [36] (see also Section 5 in [35].). An important remark to stress is that their proof applies even for non-integrable cases, provided they have **stable solitons**, in the sense of Weinstein [45]. In the particular case of the Gardner equation, this condition reads

$$\partial_c \int_{\mathbb{R}} Q_{c,\beta}^2(s) ds > 0, \quad \text{for } c < \tfrac{2}{9\beta}. \tag{1.21}$$

This inequality is directly verifiable in the case of Gardner solitons, see (B.5). From this result and Definition 1.1 we claim the following

Theorem 1.3 (Stability of *even* multi-kink solutions)**.**
The family of multi-kink solutions $U_e(t)$ from Definition 1.1 and (1.20) is global-in-time H^1-stable, and asymptotically stable as $t \to \pm\infty$.

In Section 2 we give a precise, ε-δ formulation of this result. See Theorem 2.3.

There is a second type of multi-kink solutions for (1.1), which is actually the best known one. Here, the standard kink φ_c in (1.2) and the Gardner equation play once again a crucial and surprising rôle. Indeed, let $\beta > 0$ be a fixed parameter

and suppose that one has a solution of (1.1) of the form (the reader may compare with (1.12) and (1.13))

$$u(t,x) := \varphi_c(x + ct) + \sqrt{\beta}\tilde{u}(t, x + \frac{t}{3\beta}), \quad c := \frac{1}{9\beta}, \tag{1.22}$$

and $\tilde{u}(t) \in H^1(\mathbb{R})$. Then $\tilde{u}(t, y)$ satisfies the equation

$$\tilde{u}_t + (\tilde{u}_{yy} + \tilde{u}^2 - \beta\tilde{u}^3)_y = 3((\varphi_c^2 - c)\tilde{u} + \sqrt{\beta}(\varphi_c + \sqrt{c})\tilde{u}^2)_y, \tag{1.23}$$

with $\varphi_c = \varphi_c(y - 2ct)$. In particular, if the support of $\tilde{u}(t)$ is mainly localized in the region where $\varphi_c \sim -\sqrt{c}$, namely $y \ll 2ct$, then the right hand side above is a **small perturbation** of the left hand side, a Gardner equation with parameter $\beta > 0$. As an admissible function $\tilde{u}$, we can take e.g. a sum of Gardner solitons:

$$\tilde{u}(t,y) \sim \sum_{j=1}^{N-1} Q_{c_j,\beta}(y - c_j t), \qquad 0 < c_1 < c_2 \ldots < c_{N-1} < \frac{2}{9\beta} = 2c,$$

with support localized in the region $c_1 t \lesssim y \lesssim c_{N-1}t$, for $t \gg 1$. In particular, one has $c_{N-1}t \ll 2ct$ for $t \gg 1$, which is a necessary condition for the existence of a solution of the form (1.22). Note in addition that Figure 1 can be adapted in the following way:

$$\text{mKdV}$$
$$\text{Miura} \diagup \qquad \diagdown \text{(1.22)}$$
$$\text{KdV} \longleftarrow_{(1.24)} - \sim \text{Gardner (1.23)}$$
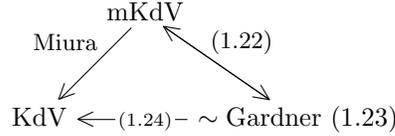
FIGURE 2. The generalized diffeomorphism (1.22) linking (1.23) and mKdV.

The transformation linking (1.23) and KdV is nothing but

$$\tilde{u}(t,x) \mapsto Q_{2c}(x - 2ct) - 3\sqrt{\beta}\varphi_c(x - 2ct)\tilde{u}(t,x) + \frac{3}{2}\sqrt{2\beta}\tilde{u}_x(t,x) - \frac{3}{2}\beta\tilde{u}^2(t,x), \tag{1.24}$$

(compare with the Gardner transform (1.9)).

Finally, the same argument can be done in the case of a solution of the form $u(t,x) := \varphi_c(x + ct) - \sqrt{\beta}\hat{u}(t, x + \frac{t}{3\beta})$, and the equation for $\hat{u}(t, y)$,

$$\hat{u}_t + (\hat{u}_{yy} + \hat{u}^2 - \beta\hat{u}^3)_y = 3((\varphi_c^2 - c)\hat{u} + \sqrt{\beta}(\sqrt{c} - \varphi_c)\hat{u}^2)_y,$$

provided $\hat{u}$ is supported mainly in the region $\{\varphi_c \sim \sqrt{c}\}$. These two new ideas allow us to consider the following definition of a multi-kink solution, from the point of view of the Gardner equation:

Definition 1.4 (*Odd* multi-kink solutions, [17, 44])**.**
   Let $N \geq 2$, $\beta > 0$, scaling parameters $0 < c_1^0 < c_2^0 < \ldots < c_{N-1}^0 < \frac{2}{9\beta}$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$ be fixed numbers. We say that a solution $U_o(t) := U_o(t; c_j^0; x_j^0)$ of (1.1) is an odd multi-kink solution if it satisfies

$$\lim_{t \to -\infty} \left\| U_o(t) - \varphi_{c_N^0}(\cdot + c_N^0 t + x_N^-) + \sqrt{\beta}\sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j^-) \right\|_{H^1(\mathbb{R})} = 0, \tag{1.25}$$

$$\lim_{t \to +\infty} \left\| U_o(t) - \varphi_{c_N^0}(\cdot + c_N^0 t + x_N^+) - \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j^+) \right\|_{H^1(\mathbb{R})} = 0, \quad (1.26)$$

with $c_N^0 := \frac{1}{9\beta}$, $\tilde{c}_j := \frac{1}{3\beta} - c_j^0 > 0$ and $x_j^{\pm} \in \mathbb{R}$ depending only on (c_k^0). Finally, φ_c is a kink solution (1.2) with scaling c, and $Q_{c,\beta}$ is a soliton solution of the Gardner equation (1.10).

One can also say that U_o is composed by $(2N-1)$ single kinks, in other words, it is a $(2N-1)$-**kink solution**. Additionally, as above mentioned, one may think this solution as function composed of three different class of parameters:

$$U_o(t, x) = U_o(x; c_j^0; \tilde{c}_j t + x_j^0), \quad (\tilde{c}_N := c_N^0).$$

From the point of view of the Gardner equation, this solution represents a big kink, solution of mKdV, with attached $(\pm)$Gardner solitons ordered according their corresponding velocities $\tilde{c}_j$. Note finally that solitons move from the right to the left.
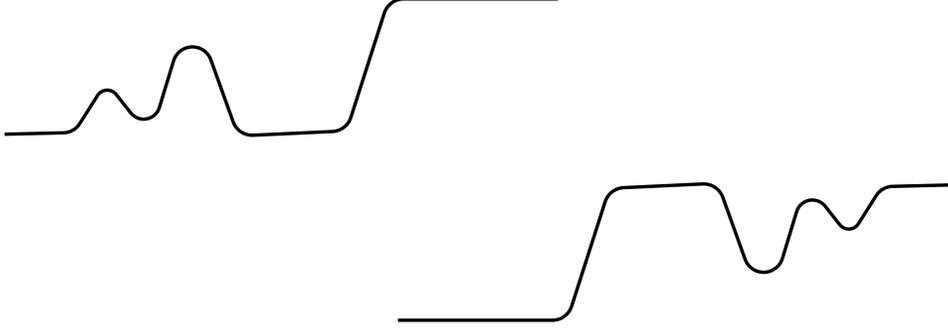
FIGURE 3. A schematic design of the evolution in time of a 5-kink solution of mKdV, composed of a big kink $\varphi_{c_3^0}$ and two Gardner solitons, $Q_{c_1^0,\beta}$ and $Q_{c_2^0,\beta}$, $0 < c_1^0 < c_2^0 < 2c_3^0$, and $\beta = \frac{1}{9c_3^0}$. Below, the behavior as $t \to -\infty$; above, the behavior as $t \to +\infty$. Each part is ordered according to their respective velocity $-\tilde{c}_j := c_j^0 - 3c_3^0 < 0$, $j = 1, 2$. Note that $-\tilde{c}_1 < -\tilde{c}_2 < -c_3^0$, which means that the smallest soliton $Q_{c_1^0,\beta}$ is actually the fastest one.

The proof of the existence of this family is not direct, although it can be explicitly obtained from the solutions found by Grosse [19, 20], using the Inverse Scattering method (see also [17], or [44, pp. 270-272] for an alternative procedure involving the inverse Miura transform). In this paper we present a third proof, which gives in addition a uniqueness property, uniform estimates and does not require the criticality property considered in [17, 44]. The uniqueness is, of course, modulo the $2N$-parameter family (c_j^0, x_j^-).

Theorem 1.5 (Existence and uniqueness of odd multi-kink solutions)**.**

Let $N \geq 2$, $\beta > 0$, $c_N^0 = \frac{1}{9\beta}$, $0 < c_1^0 < c_2^0 < \ldots < c_{N-1}^0 < \frac{2}{9\beta}$ and $x_1^-, \ldots, x_N^- \in \mathbb{R}$ be fixed numbers. There exists a unique solution $U_o(t)$ of (1.1) satisfying (1.25).

Proof. See Section 5. $\square$

The next result is a positive answer to the open question of stability of odd multi-kinks.

Theorem 1.6 (Stability of odd multi-kink solutions)**.**
The family of multi-kink solutions $U_o(t)$ from Definition 1.4 is global-in-time H^1-stable.

We prove this result in Section 4. In particular, we give a more precise statement in Theorem 4.1.

General nonlinearities. We point out that our results, starting from transformations (1.12)-(1.22), and the Zhidkov theory developed in [47], can be made even more general and include a wide range of non-integrable, defocusing gKdV equations.

Indeed, in the next paragraphs we first introduce the notion of **generalized, even and odd multi-kink solutions**. Of course these objects have to match with those considered in Definitions 1.1 and 1.4, for the special case of the integrable mKdV model. Second, we study the existence, uniqueness and stability of these new solutions in the case of *well-prepared* initial data. Next, we consider some particular collision problems, in the spirit of [32, 33, 39] (note that the collision problem makes sense since we consider non-integrable equations). The method used is the same as in the previous results, so *we will skip most of the proofs*. We emphasize that the main idea is to exploit the properties contained in the following figure:

$$u \in \text{ defocusing gKdV (even)} \longleftarrow (u \sim b + \tilde{u}) \longrightarrow \tilde{u} \in \text{focusing gKdV}$$

$$u \in \text{ defocusing gKdV (odd)} \longleftarrow (u \sim \varphi_c + \tilde{u}) \longrightarrow \tilde{u} \in \text{focusing gKdV}$$

FIGURE 4. The generalized transformations (1.12)-(1.22) linking a defocusing gKdV with a focusing gKdV equation.

Let us consider the *generalized, defocusing* KdV equation

$$u_t + (u_{xx} - f(u))_x = 0, \quad u = u(t, x) \in \mathbb{R}. \tag{1.27}$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a non-linear term, with enough regularity, to be specified below. From now on, we will assume the following hypotheses:

(a) There exists $b_0 \in \mathbb{R}$ such that $f^{(k)}(b_0) \neq 0$, for some $k \in \{2, 3, 4\}$.

Let $k_0 \in \{2, 3, 4\}$ be the first integer k satisfying this property. We assume f is of class $C^{k_0 + 1}(\mathbb{R})$.

(b) If $f^{(2)}(b_0) = 0$ and $f^{(3)}(b_0) \neq 0$, then $f^{(3)}(b_0) < 0$.

The reader may compare e.g. with the integrable case $f(s) = s^3$, where for $b_0 \neq 0$ one has $f''(b_0) = 6b_0 \neq 0$. Another important example is the *cubic-quintic* nonlinearity $f(s) := s^3 + \mu s^5$, $\mu \in \mathbb{R} - \{0\}$ fixed, for which

$$f''(b_0) = 2b_0(3 + 10\mu b_0^2) \neq 0,$$

provided $b_0 \neq 0$ and $b_0^2 \neq -\frac{3}{10\mu}$ in the case $\mu < 0$. Otherwise, we are in a degenerate case and $f''(b_0) = 0$. Computing the third derivative, one has

$$f^{(3)}(b_0) = 6(1 + 10\mu b_0^2).$$

For any $\mu \neq 0$ one has that $b_0 = 0$ does not satisfy **(b)** above and therefore is not allowed. In the case $\mu < 0$, $b_0^2 = -\frac{3}{10\mu}$, one has $f^{(3)}(b_0) = -12 < 0$, which is an admissible case.

Under these two assumptions and using [29, Remark 2], we prove existence and uniqueness of generalized even multi-kinks for (1.27), satisfying the equivalent of (1.15) (cf. Definition 1.1).

Theorem 1.7 (Existence and uniqueness of generalized, even multi-kinks).
  Let $N \geq 2$ and $b_0 \in \mathbb{R}$ be such that **(a)**-**(b)** *above are satisfied. There exists $c^* := c^*(f, b_0) > 0$ such that, for all $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < c^*$, the following holds. There exists a unique solution $U_e = U_e(t)$ of (1.27) such that $U_e(t) - b_0 \in C(\mathbb{R}, H^1(\mathbb{R}))$, and it satisfies*

$$\lim_{t \to +\infty} \left\| U_e(t) - b_0 - b_1 \sum_{j=1}^{N} Q_{c_j^0}(\cdot + \tilde{c}_j t) \right\|_{H^1(\mathbb{R})} = 0, \tag{1.28}$$

with[3]

$$b_1 := \left(\frac{-k!}{f^{(k_0)}(b_0)} \right)^{\frac{1}{k_0-1}}, \qquad \tilde{c}_j := f'(b_0) - c_j^0 \in \mathbb{R}, \tag{1.29}$$

*and $\tilde{u}(t, y) = Q_{c_j}(y - c_j t)$ is a soliton solution of the **focusing** gKdV equation*

$$\tilde{u}_t + (\tilde{u}_{yy} + \tilde{f}(\tilde{u}))_y = 0, \tag{1.30}$$

where

$$\tilde{f}(s) := -\frac{1}{b_1} \big[f(b_0 + b_1 s) - f(b_0) - b_1 f'(b_0)s \big]. \tag{1.31}$$

Proof. See Section 3. □

Remarks.
1. Let $b_0 \in \mathbb{R} - \{0\}$, and $f_{b_0}(s) := s^5 - 5b_0 s^4 + 10b_0^2 s^3 - 10b_0^3 s^2$. Then $c^*(f_{b_0}, b_0) = +\infty$ and $\tilde{f}_{b_0}(s) = s^5$. Therefore, according to [29, Theorem 1], the conclusion of the above Theorem are still valid for the L^2-critical case. Even better, the above result can be adapted in the case of L^2-supercritical nonlinearities, as a consequence of Côte-Martel-Merle [11] and Combet [10].
2. Note that from $k_0 \geq 2$ and **(a)** above one has at least $\tilde{f} \in C^3(\mathbb{R})$, a sufficient condition to obtain global well-posedness for (1.30)-(1.31) in $H^1(\mathbb{R})$ [24, 35].

Following the Martel-Merle-Tsai's paper [36], we say that an initial configuration $u_0 \in b_0 + H^1(\mathbb{R})$ for (1.27) is **well-prepared** if for $L, \alpha > 0$ and $0 < c_1^0 < \ldots < c_N^0 < c^*$ (c^* given by Theorem 1.7), and $x_1^0 < x_2^0 < \ldots < x_N^0 \in \mathbb{R}$, one has

$$\left\| u_0 - b_0 - b_1 \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j^0) \right\|_{H^1(\mathbb{R})} \leq \alpha, \qquad x_j^0 > x_{j-1}^0 + L, \quad j = 2, \ldots, N, \tag{1.32}$$

with b_1 defined in (1.29), and $Q_{c_j^0}$ solitons of (1.30)-(1.31). Our next result states that this configuration is preserved for positive times.

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[3] In the case $k_0 = 2$ both b_1 and $-b_1$ are admissible values.

Theorem 1.8 (Stability of the even multi-kinks)**.**
    There exist $L_0, \alpha_0 > 0$ such that for all $L > L_0$ and $\alpha \in (0, \alpha_0)$, a well-prepared initial configuration u_0 for (1.27), satisfying (1.32), is H^1-stable for all positive times.

Proof. The proof is similar to that of Theorem 1.3. We skip the details.   □

    Another striking consequence of (3.1) is the fact that we can describe the **interaction** among even kinks in some regimes, in the spirit of [32, 33, 39]. Indeed, one has the following

Theorem 1.9 (Inelastic interaction of even 4-kinks)**.**
    Let $b_0 \in \mathbb{R}$ such that (a)-(b) are satisfied, and suppose in addition that f is of class $C^{k_1+1}(\mathbb{R})$, where

$$f^{(k_1)}(b_0) \neq 0 \quad \text{for some } k_1 \geq 4. \tag{1.33}$$

Let $c^* = c^*(f, b_0) > 0$ be the corresponding threshold for the existence and stability of single solitons for (1.30)-(1.31). Consider $0 < c_1^0 \ll c_2^0 \ll c^*$, and let $U_e(t)$ be the unique 4-kink solution of (1.27) satisfying (1.28). Then $U_e(t)$ is global-in-time H^1-stable, but it is not pure as $t \to -\infty$.

Remarks.
1. The condition (1.33) allows us to rule out the integrable cases $f(s) = \alpha s^2 + \beta s^3$, $\alpha, \beta \in \mathbb{R}$.
2. By *not pure as* $t \to -\infty$ in Theorem 1.9 we mean that (1.14) cannot happen: for any $x_j^- \in \mathbb{R}$,

$$\liminf_{t \to -\infty} \left\| U_e(t) - b_0 - \sqrt{\beta} \sum_{j=1}^{2} Q_{c_j^0, \beta}(\cdot + \tilde{c}_j t + x_j^-) \right\|_{H^1(\mathbb{R})} > 0.$$

3. The collision problem has been recently considered in the case of the NLS equation: see e.g. Holmer-Marzuola-Zworski [21, 22], Perelman [42], and references therein.

Proof of Theorem 1.9. We prove Theorem 1.9 in Section 3. The main idea is that condition (1.33) is the key point to invoke [32] and our result [39, Theorem 1.3] to equations (1.30)-(1.31), classifying the nonlinearities for which the 2-soliton collision is inelastic.   □

    Another collision result is the following remarkable consequence of the recent Martel-Merle's papers describing the interaction of (i) two very different [32], and (ii) two nearly equal solitons of the quartic gKdV equation [34].

Corollary 1.10 (Inelastic interaction of even 4-kink solutions, quartic case)**.**
    Let $b_0 \in \mathbb{R}$, and $f_{b_0}(s) := s^4 - 4b_0 s^3 + 6b_0^2 s^2$. Then f_{b_0} satisfies (a)-(b) above, one has $c^*(f_{b_0}, b_0) = +\infty$ and for **any** $0 < c_1^0 \ll c_2^0$, the corresponding 4-kink $U_e(t)$ constructed in Theorem 1.7, and pure as $t \to +\infty$, is globally H^1-stable, but it is not pure as $t \to -\infty$. The same result is valid in the regime $0 < c_1^0 < c_2^0$, with $|c_1^0 - c_2^0| \ll 1$.

Remark. The last result is a consequence of the fact that from (1.30) and (1.31), one has $\tilde{f}_{b_0}(s) = s^4$, for which solitons Q_c exist for any $c > 0$. Note in addition that for any $b_0 \neq 0$, the corresponding nonlinearity f_{b_0} does not allow to perform the standard transformation $u \to -u$, which links the defocusing and focusing quartic equations. Moreover, the quadratic term in f_{b_0} is always of defocusing nature. Therefore, the above result is completely new for $b_0 \neq 0$.

Finally, we consider the case of generalized, odd multi-kink solutions. First of all, we have to recall some important facts. For more details, the reader may consult the monograph of Zhidkov [47].

Let $\varphi^-, \varphi^+ \in \mathbb{R}$, with $\varphi^- < \varphi^+$, and let $c > 0$, $x_0 \in \mathbb{R}$ be fixed numbers. Let f be the nonlinearity considered in (1.27). Suppose that the following hypotheses hold:

(**c**) one has
$$c\varphi^- - f(\varphi^-) = c\varphi^+ - f(\varphi^+);$$

(**d**) the function $F(s) := \int_{\varphi^-}^{s} (c\sigma - f(\sigma) - c\varphi^- + f(\varphi^-))d\sigma$ satisfies

$$F(\varphi^+) = 0, \quad \text{and} \quad F(s) < 0, \quad \text{for all } s \in (\varphi^-, \varphi^+);$$

(**e**) $f'(\varphi^\pm) > c$ (non degeneracy condition).[4]

Then there exists a monotone, generalized kink solution of (1.27), of the form

$$u(t,x) := \varphi_c(x + ct + x_0), \qquad \lim_{s \to \pm\infty} \varphi_c(s) = \varphi^\pm, \qquad (1.34)$$

and φ_c satisfies

$$\varphi_c'' + c\varphi_c - f(\varphi_c) = c\varphi^- - f(\varphi^-), \quad \varphi_c' > 0, \; \varphi_c' \in H^1(\mathbb{R}).$$

Moreover, this solution satisfies, for some constants $K, \gamma > 0$, the following estimates

$$|\varphi_c(s) - \varphi^\pm| + |\varphi_c'(s)| \le Ke^{-\gamma|s|}.$$

Finally, but not least important, the condition $\varphi_c' > 0$ implies that (1.34) is H^1-stable (Zidkhov [47, p. 91], Merle-Vega [37]).

Remarks.
1. For the sake of clarity, let us mention that in the integrable case $f(s) = s^3$, given $\varphi^- \in \mathbb{R}$ and $c > 0$, one has that for $\varphi^+ > \varphi^-$, conditions (**c**)-(**d**) and (**e**) lead to necessary conditions $\varphi^- = -\sqrt{c}$ and $\varphi^+ = \sqrt{c}$, namely (1.2).

2. It is important to point out that the multi-kink solution U_e constructed in Theorem 1.7 cannot be decomposed as the sum of several kinks of the form (1.34), at least in a general situation (compare e.g. with (1.16)). Therefore, we believe that (1.28) and (1.35) below are the correct ways to define generalized multi-kink solutions, in the case of defocusing gKdV equations.

3. The Cauchy problem associated to (1.27) with initial condition satisfying $u(0) - \varphi_c \in H^1(\mathbb{R})$, is locally well-posed in the class $\varphi_c(\cdot + ct) + H^1(\mathbb{R})$. This result is consequence of the analysis carried out by Merle and Vega in [37] and the fact that

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[4]This condition ensures that the continuous spectrum of the linearized operator $\mathcal{L} := -\partial_x^2 - c + f'(\varphi_c)$ is bounded from below, away from zero, and the kernel of $\mathcal{L}$ is spawned by its ground state $\varphi_c' > 0$.

f is regular enough. In what follows, we will only consider stable solutions, then globally well defined.

Our next objective is to generalize the Zhidkov's results to the case of $(2N-1)$-kinks, as follows:

Theorem 1.11 (Existence and uniqueness of generalized odd multi-kinks)**.**
Let $N \geq 2$ and $\varphi^-, \varphi^+ \in \mathbb{R}$ be such that (**a**)-(**b**) hold with $b_0 := \varphi^-$, and (**c**)-(**e**) above are satisfied. There exists $c^* := c^*(f, \varphi^-) > 0$ such that, for all $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < c^*$, and $x_1^-, x_2^-, \ldots, x_N^- \in \mathbb{R}$, the following holds. There exists a unique solution $U_o = U_o(t)$ of (1.27) such that $u(t) - \varphi_c(\cdot + ct) \in H^1(\mathbb{R})$, and it satisfies

$$\lim_{t \to +\infty} \left\| U_o(t) - \varphi_c(\cdot + ct + x_N^-) - b_1 \sum_{j=1}^N Q_{c_j^0}(\cdot + \tilde{c}_j t + x_j^-) \right\|_{H^1(\mathbb{R})} = 0, \qquad (1.35)$$

with $b_1, \tilde{c}_j$ as in (1.29), and $\tilde{u}(t, y) = Q_{c_j}(y - c_j t)$ is a soliton of the focusing gKdV equation (1.30)-(1.31).

The proof of this result follows the lines of the proof of Theorem 1.5, see also Remark 2 in [29]. Let us recall that in this situation, and following the notation of (1.30)-(1.31), equation (1.23) now becomes

$$\tilde{u}_t + (\tilde{u}_{yy} + \tilde{f}(\tilde{u}))_y = [F(t, y)]_y, \quad \tilde{u} \in H^1(\mathbb{R}),$$

with

$$\begin{aligned}
F(t, y) &:= \frac{1}{b_1}[f(\varphi_c + b_1 \tilde{u}) - f(\varphi_c) - f(\varphi^- + b_1 \tilde{u}) + f(\varphi^-)] \\
&= \frac{1}{b_1} \int_0^{b_1 \tilde{u}} \int_{\varphi^-}^{\varphi_c} f''(t + s) \, ds \, dt.
\end{aligned}$$

Therefore, since $u \in L^\infty(\mathbb{R})$, one has $F(t, y) = O(|\varphi_c - \varphi^-||\tilde{u}|)$, which is enough to conclude.

Finally, we say that an initial configuration u_0, perturbation of a kink solution φ_c, is **well-prepared** if for $L, \alpha > 0$, $0 < c_1^0 < \ldots < c_{N-1}^0 < c^*$, and $x_1^0 < x_2^0 < \ldots < x_N^0$, one has

$$\left\| u_0 - \varphi_c(\cdot - x_N^0) - b_1 \sum_{j=1}^N Q_{c_j^0}(\cdot - x_j^0) \right\|_{H^1(\mathbb{R})} \leq \alpha, \quad x_j^0 > x_{j-1}^0 + L, \quad j = 2, \ldots, N,$$

$$(1.36)$$

with b_1 defined in (1.29). In addition, by taking c^* smaller if necessary, we assume that each soliton $Q_{c_j^0}$ is stable in the sense of Weinstein (1.21).

Theorem 1.12 (Stability of the odd multi-kinks)**.**
There exist $L_0, \alpha_0 > 0$ such that for all $L > L_0$ and $\alpha \in (0, \alpha_0)$, a well-prepared initial data u_0 for (1.27), satisfying (1.36), is H^1-stable for all positive times.

This result is proved following the lines of the proof of Theorem 1.6, using in addition that single solitons are stable. We skip the details.

Final remarks.
1. We recall that the collision problem in the case of odd multi-kink solutions remains an interesting open question. In addition, we believe that our approach

introduces new ideas to deal with the dynamics of kink solutions in the L^2-critical and supercritical setting, by using a suitable focusing counterpart.

2. Let us mention that a similar transformation to (1.22) can be introduced in the cases of the ϕ^4 and sine-Gordon models, with similar results. For the first equation,

$$u_{tt} - u_{xx} = u(1 - u^2), \quad u(t,x) \in \mathbb{R}, \tag{1.37}$$

it is well know that $u(t,x) = \varphi(x)$, with φ given in (1.3), is a stationary kink solution. However, the transformation $u(t,x) := \varphi(x) + \frac{2}{3}\tilde{u}(\sqrt{2}t, \sqrt{2}x)$ leads to the following equation for $\tilde{u}(s,y)$

$$\tilde{u}_{ss} - \tilde{u}_{yy} + \tilde{u} + \tilde{u}^2 + \frac{2}{9}\tilde{u}^3 = \frac{3}{2}(1 - \varphi^2)\tilde{u} + (1 - \varphi)\tilde{u}^2.$$

Looking for an approximate, localized, stationary solution, we arrive to study the elliptic equation associated to the Gardner nonlinearity:

$$\tilde{u}_{yy} - \tilde{u} - \tilde{u}^2 + \frac{2}{9}\tilde{u}^3 = 0, \quad \tilde{u} \in H^1(\mathbb{R}).$$

Using the transformation $\tilde{u} = -\hat{u}$, we get a Gardner elliptic equation with parameters $c := 1$ and $\beta := -\frac{2}{9}$ (cf. (B.1)), therefore it is possible to attach to the kink solution of (1.37) suitable soliton-like structures of the form (1.11). For instance, we may take a sum of boosted Gardner profiles:

$$\tilde{u}(s,y) \sim -\sum_{j=1}^{N} Q(\gamma_j(y - v_j s)), \qquad 0 < v_1 < v_2 < \ldots < v_N, \quad \gamma_j := (1 - v_j^2)^{-1/2}.$$

and $Q(x) := Q_{1,-\frac{2}{9}}(x) = 3(1 + \sqrt{2}\cosh(x))^{-1}$ from (1.11).

Now we perform the same analysis in the case of the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0, \quad u(t,x) \in \mathbb{R}, \tag{1.38}$$

and its kink solution $\varphi(x) := 4\arctan e^x$. Indeed, using the transformation $u(t,x) := \varphi(x) + \tilde{u}(t,x)$, we arrive to the following perturbed, sine-Gordon equation

$$\tilde{u}_{tt} - \tilde{u}_{xx} + \sin\tilde{u} = (1 - \cos\varphi)\sin\tilde{u} + (1 - \cos v)\sin\varphi,$$

where the right hand side is small if we consider $\tilde{u}$ as a localized solution of (1.38) in the region where $\varphi \sim 0$. We can put for instance, a sum of *breather* solutions, provided this solution is stable, which is an open problem. We expect to consider some of these problems in a forthcoming publication.

Idea of the proofs. Theorems 1.2, 1.3, 1.7 and 1.8 can be deduced from Martel [29] and Martel-Merle-Tsai [36, 35]. We recall that, without using transformation 1.12, these results were unable to be tackled down by using any direct method.

We prove Theorem 1.6 in Section 4. The proof is based in the approach introduced in [36] in order to describe the stability in $H^1(\mathbb{R})$ of N decoupled solitons. However, in this opportunity we face several new problems since the kink solution and the Gardner solitons are in strong interaction through the dynamics. Moreover, the mass (1.6) cannot be used to control the Gardner solitons, as has been done in [36]. This means that Theorem 1.6 cannot be deduced from the standard Zhidkov [47] and Martel-Merle-Tsai [36] results, and we need new ideas. In that sense, the transformation (1.22) is the first step –and the more important one– to understand the interaction among kinks as actually localized, soliton-like interactions.

Let us be more precise. Using the energy (see (1.5)) of the solution $u(t)$, one can control with no additional difficulties the kink solution. This is a consequence of the non negative character of the linearized operator around the kink solution, see [47, 37] for more details. However, this quantity is far from being enough to control the behavior of the Gardner solitons. We overcome this difficulty by using the transformation (1.22), which introduces a new function $\tilde{u}(t)$, almost solution of a Gardner-like equation (cf. (1.23)). It turns out that the perturbative terms on the right hand side of (1.23) can be controlled provided the solitons are far from the center of the main kink solution, which holds true if we assume that the initial configuration is well prepared (see Proposition 4.2). Additionally, we introduce a new, **almost conserved mass** (see (4.25)) for the portion on the left of the solution $\tilde{u}$, which allows to control each Gardner soliton by separated. Using this, we avoid the problem of using the natural mass (1.6), which is very bad behaved for $H^1(\mathbb{R})$ perturbations. This approach is completely general and can be adapted to prove Theorem 1.12. No additional hypotheses are needed, only the single stability of each generalized soliton component of the multi-kink solution. The proof of the asymptotic stability property generalizes the argument used [35], this time to the function $\tilde{u}$.

Finally, concerning Theorem 1.5 –proved in Section 5–, we extend the result of Martel [29]. Most of the proof is similar to the proof of Theorem 1.6, but estimates are easier to carry out since we do not need to control the scaling parameters of each Gardner solitons.

2. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Let $\tilde{U}$ be another solution of (1.1) satisfying (1.14). Then, from (1.12),

$$\tilde{V}(t,y) := \frac{1}{\sqrt{\beta}}\Big[b - \tilde{U}(t, y - \frac{t}{3\beta})\Big],$$

is solution of the Gardner equation (1.10) and satisfies (1.18). From the uniqueness of $V^{(N)}$ [29], one has $\tilde{V} \equiv V^{(N)}$, and therefore $\tilde{U} \equiv U_e$.

Proof of Theorem 1.3. First of all, let us recall the Martel-Merle-Tsai's stability result [36]:

Theorem 2.1 (H^1-stability of the sum of N-Gardner solitons, [36, 35]).
Let $N \geq 2$, $\beta > 0$ and $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ be such that (1.21) holds for all $j = 1, \ldots, N$. There exists $\tilde{\alpha}_0, \tilde{A}_0, \tilde{L}_0, \tilde{\gamma} > 0$ such that the following is true. Let $v_0 \in H^1(\mathbb{R})$, and assume that there exists $\tilde{L} > \tilde{L}_0$, $\tilde{\alpha} \in (0, \tilde{\alpha}_0)$ and $\tilde{x}_1^0 < \tilde{x}_2^0 < \ldots < \tilde{x}_N^0$, such that

$$\Big\|v_0 - \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde{x}_j^0)\Big\|_{H^1(\mathbb{R})} \leq \tilde{\alpha}, \qquad \tilde{x}_j^0 > \tilde{x}_{j-1}^0 + \tilde{L}, \quad j = 2, \ldots, N. \quad (2.1)$$

Then there exists $\tilde{x}_1(t), \ldots \tilde{x}_N(t)$ such that the solution $v(t)$ of the Cauchy problem associated to (1.10), with initial data v_0, satisfies

$$v(t) = S(t) + w(t), \quad S(t) := \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde{x}_j(t)),$$

and

$$\sup_{t\geq 0}\left\{\|w(t)\|_{H^1(\mathbb{R})} + \sum_{j=1}^{N}|\tilde{x}'_j(t) - c^0_j|\right\} \leq \tilde{A}_0(\tilde{\alpha} + e^{-\tilde{\gamma}\tilde{L}}). \qquad (2.2)$$

Moreover, there exist $c^\infty_j > 0$ such that $\lim_{+\infty}\tilde{x}'_j(t) = c^\infty_j$ and

$$\lim_{t\to +\infty}\left\|v(t) - \sum_{j=1}^{N}Q_{c^\infty_j,\beta}(\cdot - \tilde{x}_j(t))\right\|_{H^1(x>\frac{c^0_1}{10}t)} = 0. \qquad (2.3)$$

It is important to stress that the well-preparedness restriction on the initial data (2.1) is by now necessary since there is no satisfactory collision theory for the non-integrable cases.[5]

However, as explained in [36] for the KdV case, the above argument can be extended to a global-in-time stability result, thanks to the continuity of the Gardner flow in $H^1(\mathbb{R})$ [24], and the fact that the Gardner equation (1.10) is an integrable model, with explicit N-soliton solutions (see (1.18)-(1.19)), given by the family $V^{(N)}$ above described. Therefore, a direct consequence of this property and the invariance of the equation under the transformation $u(t,x) \mapsto u(-t,-x)$ is the following

Corollary 2.2 (H^1-stability of Gardner multi-solitons, [36, 35]).
Let $\delta > 0$, $N \geq 2$, $0 < c^0_1 < \ldots < c^0_N$ and $x^0_1,\ldots,x^0_N \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$, then the following holds. Let $v(t)$ be a solution of (1.10) such that

$$\|v(0) - V^{(N)}(\cdot;c^0_j,x^0_j)\|_{L^2(\mathbb{R})} \leq \alpha,$$

with $V^{(N)}$ the N-soliton satisfying (1.18)-(1.19). Then there are $x_j(t) \in \mathbb{R}$, $j = 1,\ldots,N$, such that

$$\sup_{t\in\mathbb{R}}\left\|v(t) - V^{(N)}(\cdot;c^0_j,x_j(t))\right\|_{H^1(\mathbb{R})} \leq \delta. \qquad (2.4)$$

Moreover, there exist $c^\infty_j > 0$ such that

$$\lim_{t\to +\infty}\left\|v(t) - V^{(N)}(\cdot;c^\infty_j,x_j(t))\right\|_{H^1(x>\frac{c^0_1}{10}t)} = 0, \qquad (2.5)$$

and $x_j(t)$ are C^1 for all $|t|$ large enough, with $x'_j(t) \to -c^\infty_j \sim -c^0_j$ as $t \to +\infty$. A similar result holds as $t \to -\infty$, with the obvious modifications.

Remark. Let us emphasize that the proof of this result requires the *existence* and the *explicit behavior* of the multi-soliton solution $V^{(N)}$ of the Gardner equation, and therefore the integrable character of the equation. In particular, we do not believe that a similar result is valid for a completely general, non-integrable gKdV equation, unless one considers some perturbative regimes (cf. [32, 34] for some global H^1-stability results in the non-integrable setting.)

Therefore, using (1.12) and the previous result one has the following more precise version of Theorem 1.3.

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[5]See [32, 33, 34, 39] for some recent results describing the collision of two solitons for gKdV equations in some particular regimes and with general nonlinearities, beyond the integrable cases.

Theorem 2.3 (Stability of *even* multi-kink solutions)**.**

   The family of multi-kink solutions $U_e(t)$ from Definition 1.1 and (1.20) is global-in-time H^1-stable, and asymptotically stable as $t \to \pm\infty$. More precisely, let $\beta, \delta > 0$, $N \geq 2$, $0 < c_1^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$, then the following holds. Let $u(t)$ be a solution of (1.1) such that

$$\|u(0) - U_e(\cdot; c_j^0, x_j^0)\|_{H^1(\mathbb{R})} \leq \alpha, \tag{2.6}$$

with U_e the $2N$-kink solution defined in (1.20). Then there exist $x_j(t) \in \mathbb{R}$, $j = 1, \ldots, N$, such that

$$\sup_{t \in \mathbb{R}} \left\| u(t) - U_e(\cdot; c_j^0, x_j(t)) \right\|_{H^1(\mathbb{R})} \leq \delta. \tag{2.7}$$

Moreover, there exist $c_j^\infty > 0$ such that

$$\lim_{t \to +\infty} \left\| u(t) - U_e(\cdot; c_j^\infty, x_j(t)) \right\|_{H^1(x > (\frac{1}{10} c_1^0 - 3c)t)} = 0, \tag{2.8}$$

and $x_j(t)$ are C^1 for all $|t|$ large enough, with $x_j'(t) \to c_j^\infty \sim \tilde{c}_j$ as $t \to +\infty$. A similar result holds as $t \to -\infty$, with the obvious modifications.

Remark. Let us recall, for the sake of completeness, that estimate (2.8) is deduced from (2.5) by using the transformation (1.12).

3. Proof of Theorems 1.7 and 1.9

Proof of Theorem 1.7. Thanks to **(a)-(b)**, there exists a generalized transformation of the form (1.12), such that

$$u(t, x) = b_0 + b_1 \tilde{u}(t, x + f'(b_0)t), \tag{3.1}$$

with b_1 given by (1.29), and such that $\tilde{u}(t, y)$ satisfies (1.30)-(1.31). Moreover, note that a Taylor expansion gives us that $\tilde{f}$ is a subcritical perturbation of the pure power nonlinearity:

$$\begin{aligned} \tilde{f}(s) &= -\frac{1}{k_0!} b_1^{k_0-1} f^{(k_0)}(b_0) s^{k_0} - \frac{1}{(k_0+1)!} b_1^{k_0} f^{(k_0+1)}(\xi) s^{k_0+1} \\ &= s^{k_0} + \tilde{f}_{b_0}(s), \quad k_0 \in \{2, 3, 4\}, \end{aligned} \tag{3.2}$$

for some ξ in between b_0 and $b_0 + b_1 s$. Note in addition that

$$\lim_{s \to 0} \frac{\tilde{f}_{b_0}(s)}{|s|^{k_0}} = 0.$$

According to Berestycki and Lions [5], $\tilde{f}$ is an admissible nonlinearity for the **existence of small solitons**, in the sense that there exists $c^* > 0$ (depending on f and b_0 fixed), such that for all $0 < c < c^*$, there exists a solution $\tilde{u} = \tilde{u}(t, y)$ of (1.30)-(1.31), of the form

$$\tilde{u}(t, y) = Q_c(y - ct),$$

and such that $Q_c = Q_c(s)$ satisfies

$$Q_c'' - cQ_c + \tilde{f}(Q_c) = 0, \quad Q_c > 0, \quad Q_c \in H^1(\mathbb{R}).$$

Moreover, Q_c can be chosen even and exponentially decreasing as $s \to \pm\infty$.

   In addition, for $0 < c < c^*$ small, solitons satisfy the corresponding Weinstein condition (1.21) (cf. Martel-Merle [35]), which implies orbital stability in the energy

space $H^1(\mathbb{R})$. From Theorem in [29], given $0 < c_1^0 < \ldots < c_N^0 < c^*$, $N \geq 2$, there exists a unique solution $\tilde{U} \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1.30)-(1.31), satisfying

$$\lim_{t \to +\infty} \left\| \tilde{U}(t) - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - c_j^0 t) \right\|_{H^1(\mathbb{R})} = 0.$$

The final conclusion follows after applying the transformation (3.1). Note that $\tilde{c}_j$ defined in (1.29) can be either zero, positive or negative, depending on b_0 and c^*; however one always has $\tilde{c}_N < \tilde{c}_{N-1} < \ldots < \tilde{c}_1$.

Proof of Theorem 1.9. Let us consider the transformation (3.1), which leads to the focusing gKdV equation (1.30)-(1.31). Let $k_1 \geq 4$ be the first integer satisfying $f^{(k_1)}(b_0) \neq 0$. Note that from (1.29) and (3.2), one has

$$\tilde{f}(s) = \begin{cases} s^2 - \dfrac{1}{3!} b_1^2 f^{(3)}(b_0) s^3 - \dfrac{1}{k_1!} b_1^{k_1-1} f^{(k_1)}(b_0) s^{k_1} + \tilde{f}_1(s), & k_0 = 2; \\ s^3 - \dfrac{1}{k_1!} b_1^{k_1-1} f^{(k_1)}(b_0) s^{k_1} + \tilde{f}_1(s), & k_0 = 3; \\ s^4 + \tilde{f}_1(s), & k_0 = 4. \end{cases}$$

Note that in each case one has

$$\lim_{s \to 0} \frac{\tilde{f}_1(s)}{|s|^{k_1}} = 0.$$

Since by hypothesis $f^{(k_1)}(b_0) \neq 0$, one has that $\tilde{f}$ is a nontrivial perturbation of the integrable models $\tilde{f}(s) = s^2, s^3$ and $\tilde{f}(s) = s^2 + \beta s^3$.

Therefore, from the classification theorem for the regime $0 < c_1 \ll c_2 \ll c^*$ showed in [32, 33, 39], one can conclude that the 2-soliton structure is globally H^1-stable, but the solution U_e constructed in Theorem 1.7 is **never pure** as $t \to -\infty$. The final conclusion follows after applying (3.1). The proof is complete.

4. PROOF OF THEOREM 1.6

In this section we prove Theorem 1.6. First of all, we state a more detailed version of this result.

Theorem 4.1 (Stability of odd multi-kink solutions).

The family of multi-kink solutions $U_o(t)$ from Definition 1.4 is global-in-time H^1-stable, and asymptotically stable as $t \to \pm\infty$. More precisely, let $\delta, \beta > 0$, $N \geq 2$, $0 < c_1^0 < \ldots < c_{N-1}^0 < \frac{2}{9\beta}$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$, then the following holds. Let $u(t)$ be a solution of (1.1) such that

$$\|u(0) - U_o(\cdot; c_j^0, x_j^0)\|_{H^1(\mathbb{R})} \leq \alpha, \quad c_N^0 := \frac{1}{9\beta} \tag{4.1}$$

with U_o the $(2N-1)$-kink solution from Definition 1.4. Then there exist $x_j(t) \in \mathbb{R}$, $j = 1, \ldots, N$, such that

$$\sup_{t \in \mathbb{R}} \left\| u(t) - U_o(\cdot; c_j^0, x_j(t)) \right\|_{H^1(\mathbb{R})} \leq \delta. \tag{4.2}$$

It turns out that the proof of Theorem 4.1 follows as a consequence of the following

Proposition 4.2 (H^1-stability of the one kink and $(N-1)$ Gardner solitons)**.**

Let $N \geq 2$, $\beta > 0$, $c_N^0 = \frac{1}{9\beta}$, and $0 < c_1^0 < c_2^0 < \ldots < c_{N-1}^0 < \frac{2}{9\beta}$ be such that (1.21) holds for all $j = 1, \ldots, N$. There exists $\alpha_0, A_0, L_0, \sigma_0 > 0$ such that the following is true. Let $u_0 \in H^1(\mathbb{R})$, and assume that there exists $L > L_0$, $\alpha \in (0, \alpha_0)$ and $x_1^0 < x_2^0 < \ldots < x_N^0$, such that

$$\left\|u_0 - \varphi_{c_N^0}(\cdot + x_N^0) - \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + x_j^0)\right\|_{H^1(\mathbb{R})} \leq \alpha, \quad x_{j-1}^0 < x_j^0 - L, \quad (4.3)$$

for $j = 2, \ldots, N$. Then there exists $x_1(t), \ldots, x_N(t)$ such that the solution $u(t)$ of the Cauchy problem associated to (1.1), with initial data u_0, satisfies

$$u(t) = S(t) + w(t), \qquad S(t) := \varphi_{c_N^0}(\cdot + c_N^0 t + x_N(t)) + \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j(t)),$$

and

$$\sup_{t \geq 0}\left\{\|w(t)\|_{H^1(\mathbb{R})} + \sum_{j=1}^N |x_j'(t)|\right\} \leq A_0(\alpha + e^{-\sigma_0 L}). \quad (4.4)$$

Proof of Theorem 4.1. From Proposition 4.2, the proof of Theorem 4.1 follows directly from the integrable character of the mKdV equation and the existence of a suitable multi-kink solution satisfying (1.25)-(1.26). See e.g. [3, Corollary 1.2] for a similar, detailed proof.

Therefore, we are left to prove Proposition 4.2.

Proof of Proposition 4.2.

Stability. Let us assume the hypotheses of Proposition 4.2. Let σ_0 satisfying

$$0 < \sigma_0 \leq \frac{1}{2}\min(c_2^0 - c_1^0, c_3^0 - c_2^0, \ldots, 2c_N^0 - c_{N-1}^0), \quad (4.5)$$

a measure of the minimal difference among the scaling parameters. This quantity may change from one line to another, but always satisfies (4.5).

Note that a simple continuity argument, using the local Cauchy theory developed in [37] shows that there exists $t_0 > 0$ such that

$$\sup_{t \in [0,t_0]} \left\|u(t) - \varphi_{c_N^0}(\cdot + c_N^0 t + \tilde{x}_N(t)) - \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + \tilde{x}_j(t))\right\|_{H^1(\mathbb{R})} \leq 2(\alpha + e^{-\sigma_0 L}), \quad (4.6)$$

for some $\tilde{x}_j(t) \in \mathbb{R}$, $j = 1, \ldots, N$. Therefore, given $K^* > 2$, we can define the following quantity

$$T^* := \sup\{T > 0, \text{ for all } t \in [0,T], (4.6) \text{ is satisfied with 2 replaced by } K^*,$$
$$\text{and for some } \tilde{x}_j(t) \in \mathbb{R}.\}. \quad (4.7)$$

Our objective is to show that for some $K^* > 0$ large enough, one has $T^* = +\infty$. Following a contradiction argument, we will assume $T^* < +\infty$. This allows to prove the following modulation property.

Lemma 4.3 (Modulation)**.**

Possibly taking $\alpha_0 > 0$ and $\frac{1}{L_0}$ smaller, there exists $K > 0$ independent of K^*, such that if $L > L_0$ and $0 < \alpha < \alpha_0$, the following holds. There exist unique

C^1 *functions* $c_j : [0, T^*] \to (0, +\infty)$, $j = 1, \ldots, N-1$, *and* $x_j : [0, T^*] \to \mathbb{R}$, $j = 1, \ldots, N$, *such that*

$$z(t,x) := u(t,x) - \varphi_{c_N^0}(x + c_N^0 t + x_N(t)) - \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j,\beta}(x + \tilde{c}_j t + x_j(t)), \quad (4.8)$$

satisfies, for all $j = 1, \ldots, N$, *and for all* $t \in [0, T^*]$,

$$\int_{\mathbb{R}} z(t,x) \varphi'_{c_N^0}(x + c_N^0 t + x_N(t)) dx = 0, \tag{4.9}$$

and

$$\int_{\mathbb{R}} z(t,x) Q'_{c_j,\beta}(x + \tilde{c}_j t + x_j(t)) dx = \int_{\mathbb{R}} z(t,x) Q_{c_j,\beta}(x + \tilde{c}_j t + x_j(t)) dx = 0. \tag{4.10}$$

Moreover, there exists $K > 0$ *such that for all* $t \in [0, T^*]$,

$$\|z(t)\|_{H^1(\mathbb{R})} + \sum_{j=1}^{N-1} |c_j(t) - c_j(0)| \le K K^*(\alpha + e^{-\sigma_0 L}), \tag{4.11}$$

and

$$\|z(0)\|_{H^1(\mathbb{R})} + \sum_{j=1}^{N-1} |c_j(0) - c_j^0| \le K\alpha. \tag{4.12}$$

Proof. The proof of this result is a standard exercise of Implicit Function Theorem, see e.g. Lemma in [36] for a similar proof. Note that in this opportunity we have modulated the translation parameter associated to the kink solution (cf. (4.9)). From [47, 37], there is no need to modulate the scaling parameter, see e.g. Lemma A.1 in Appendix A. □

In what follows, we introduce some useful notation. Let us consider

$$R(t,x) := \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j,\beta}(x + \tilde{c}_j t + x_j(t)), \quad (= \text{the Gardner solitons}) \tag{4.13}$$

and $\tilde{u}(t,y)$ defined by the relation

$$u(t,x) := \varphi_c(x + ct + x_N(t)) + \sqrt{\beta}\tilde{u}(t, x + \frac{t}{3\beta}), \tag{4.14}$$

where, for the sake of clarity, we have defined $c := c_N^0$. In particular,

$$
\begin{aligned}
\tilde{u}(t,y) &= \frac{1}{\sqrt{\beta}}(R+z)(t, y - \frac{t}{3\beta}) \\
&= \sum_{j=1}^{N-1} Q_{c_j,\beta}(y - c_j t + x_j(t)) + \frac{1}{\sqrt{\beta}} z(t, y - \frac{t}{3\beta}) \\
&=: \tilde{R}(t,y) + \tilde{z}(t,y).
\end{aligned}
\tag{4.15}
$$

A simple computation shows that $\tilde{u} = \tilde{u}(t,y)$ satisfies the modified Gardner equation (compare with (1.23))

$$\tilde{u}_t + (\tilde{u}_{yy} + \tilde{u}^2 - \beta\tilde{u}^3)_y = 3[(\varphi_c^2 - c)\tilde{u} + (\sqrt{c} + \varphi_c)\tilde{u}^2]_y + \frac{x_N'(t)}{\sqrt{\beta}}\varphi_c'. \tag{4.16}$$

In this last equation φ_c is a function of the variable y in the sense that $\varphi_c(x + ct + x_N(t)) = \varphi_c(y - 2ct + x_N(t))$. The following result gives an explicit expansion of the energy of $u(t)$.

Lemma 4.4 (Expansion of the energy).
Consider the energy $E[u](t)$ defined in (1.5), for $c = c_N^0 = \frac{1}{9\beta}$. Then, for any $t \in [0, T^*]$, one has the following decomposition

$$E[u](t) = E[\varphi_c] + \frac{2}{3} \sum_{j=1}^{N-1} c_j^{3/2}(t) + \mathcal{F}(t) + O(\|z(t)\|_{H^1(\mathbb{R})}^3) + O(e^{-\sigma_0 L}), \quad (4.17)$$

with $\mathcal{F}(t)$ the following second order functional

$$\mathcal{F}(t) := \frac{1}{2} \int_{\mathbb{R}} (z_x^2(t) + 2cz^2(t) - 3(c - \varphi_c^2)z^2(t) - 6\sqrt{c}Rz^2(t) + 3R^2 z^2(t)). \quad (4.18)$$

Proof. From (4.8) and (4.13), one has:

$$
\begin{aligned}
E[u](t) &= E[\varphi_c + R + z](t) \\
&= E[\varphi_c](t) + \frac{1}{2}\int_{\mathbb{R}} R_x^2 + c\int_{\mathbb{R}} R^2 - \sqrt{c}\int_{\mathbb{R}} R^3 + \frac{1}{4}\int_{\mathbb{R}} R^4 \\
&\quad + \int_{\mathbb{R}}(\varphi_c + \sqrt{c})R^3 + \int_{\mathbb{R}}(\varphi_c)_x R_x + \frac{3}{2}\int_{\mathbb{R}}(\varphi_c^2 - c)R^2 + \int_{\mathbb{R}}\varphi_c(\varphi_c^2 - c)R \\
&\quad - \int_{\mathbb{R}} z(\varphi_c'' + c\varphi_c - \varphi_c^3) - \int_{\mathbb{R}} z(R_{xx} - 2cR + 3\sqrt{c}R^2 - R^3) \\
&\quad + 3\int_{\mathbb{R}}(\varphi_c^2 - c)Rz + \frac{1}{2}\int_{\mathbb{R}}(z_x^2 + 2cz^2 - 3(c - \varphi_c^2)z^2 - 6\sqrt{c}Rz^2 + 3R^2 z^2) \\
&\quad + 3\int_{\mathbb{R}}(\varphi_c + \sqrt{c})Rz^2 + 3\int_{\mathbb{R}} R^2(\varphi_c + \sqrt{c})z + \int_{\mathbb{R}}\varphi_c z^3 + \int_{\mathbb{R}} Rz^3 + \frac{1}{4}\int_{\mathbb{R}} z^4.
\end{aligned}
$$

First of all, note that the term $E[\varphi_c](t)$ actually does not depend on t. Additionally, from (1.2) and (1.4) one has

$$\varphi_c'' + c\varphi_c - \varphi_c^3 = 0. \quad (4.19)$$

In order to obtain some estimates of the above quantities, we need the following

Lemma 4.5 (Identities for $R(t)$).
Let R be the sum of N decoupled Gardner solitons defined in (4.13). Then one has the following identities:

$$R_{xx} - 2cR + 3\sqrt{c}R^2 - R^3 = -\sqrt{\beta}\sum_{j=1}^{N-1}(2c - c_j)Q_{c_j,\beta} + O_{H^1(\mathbb{R})}(e^{-\sigma_0 L}). \quad (4.20)$$

and

$$\frac{1}{2}\int_{\mathbb{R}} R_x^2 + c\int_{\mathbb{R}} R^2 - \sqrt{c}\int_{\mathbb{R}} R^3 + \frac{1}{4}\int_{\mathbb{R}} R^4 = \frac{2}{3}\sum_{j=1}^{N-1} c_j^{3/2}(t) + O(e^{-\sigma_0 L}). \quad (4.21)$$

Proof. Let us prove (4.20). From (4.13) and the fact that $c = \frac{1}{9\beta}$, one has

$$
\text{l.h.s. of (4.20)} = \sqrt{\beta} \sum_{j=1}^{N-1} (Q''_{c_j,\beta} - 2cQ_{c_j,\beta}) + 3\beta\sqrt{c}\Big(\sum_{j=1}^{N-1} Q_{c_j,\beta}\Big)^2 - \beta^{3/2}\Big(\sum_{j=1}^{N-1} Q_{c_j,\beta}\Big)^3
$$

$$
= \sqrt{\beta} \sum_{j=1}^{N-1} (Q''_{c_j,\beta} - 2cQ_{c_j,\beta} + Q^2_{c_j,\beta} - \beta Q^3_{c_j,\beta})
$$

$$
+ \sqrt{\beta} \sum_{i \neq j}^{N-1} Q_{c_i,\beta} Q_{c_j,\beta} - \beta^{3/2}\Big[\Big(\sum_{j=1}^{N-1} Q_{c_j,\beta}\Big)^3 - \sum_{j=1}^{N-1} Q^3_{c_j,\beta}\Big].
$$

Using the equation for $Q_{c,\beta}$ (cf. (B.1)), one has

$$
\text{l.h.s. of (4.20)} = -\sqrt{\beta} \sum_{j=1}^{N-1} (2c - c_j) Q_{c_j,\beta} + O_{H^1(\mathbb{R})}(e^{-\sigma_0 L}),
$$

as desired.

Now we consider (4.21). From (4.13) and (B.4), one has

$$
\text{l.h.s. of (4.21)} = \beta \sum_{j=1}^{N-1} \int_{\mathbb{R}} \Big\{ \frac{1}{2} Q'^2_{c_j,\beta} + cQ^2_{c_j,\beta} - \sqrt{\beta}cQ^3_{c_j,\beta} + \frac{\beta}{4} Q^4_{c_j,\beta} \Big\} + O(e^{-\sigma_0 L})
$$

$$
= \beta \sum_{j=1}^{N-1} \Big\{ E_\beta[Q_{c_j,\beta}] + 2cM[Q_{c_j,\beta}] \Big\} + O(e^{-\sigma_0 L})
$$

$$
= \frac{2}{3} \sum_{j=1}^{N-1} c_j^{3/2}(t) + O(e^{-\sigma_0 L}).
$$

The proof is complete. □

Let us come back to the proof of Lemma 4.4. From the above results, the orthogonality conditions (4.10), (4.18) and (4.19) we have

$$
E[u](t) = E[\varphi_c] + \frac{2}{3} \sum_{j=1}^{N-1} c_j^{3/2}(t) + \mathcal{F}(t) + O(\|z(t)\|^3_{H^1(\mathbb{R})})
$$

$$
- \int_{\mathbb{R}} (\varphi_c - \sqrt{c}) R^3 - \int_{\mathbb{R}} (\varphi_c)_x R_x + \frac{3}{2} \int_{\mathbb{R}} (\varphi_c^2 - c) R^2 \qquad (4.22)
$$

$$
- \int_{\mathbb{R}} \varphi_c(\varphi_c^2 - c)R - 6 \int_{\mathbb{R}} (\varphi_c^2 - c)Rz - 3 \int_{\mathbb{R}} (\varphi_c - \sqrt{c})Rz^2. (4.23)
$$

Finally, the last two lines in the above identity, namely (4.22)-(4.23), are exponentially small. Indeed, one has e.g.

$$
\Big| \int_{\mathbb{R}} (\varphi_c - \sqrt{c}) R^3 \Big| \leq Ke^{-\sigma_0 L}. \qquad (4.24)
$$

The other terms can be bounded in a similar fashion. From these estimates, (4.17) follows directly. □

In the next step, we introduce a modified mass, almost monotone in time, which allows to control the Gardner solitons. Let $Q(s) := (\cosh(s))^{-1}$,

$$\phi(x) := mQ(\sqrt{\sigma_0}x/2), \quad \psi(x) := \int_{-\infty}^{x} \phi(s)ds, \quad \text{where } m := \Big[\frac{2}{\sqrt{\sigma_0}}\int_{-\infty}^{\infty} Q\Big]^{-1}.$$

Note that, for all $x \in \mathbb{R}$, $\psi'(x) > 0$, $0 < \psi(x) < 1$, and $\lim_{x \to -\infty} \psi(x) = 0$, $\lim_{x \to +\infty} \psi(x) = 1$.

Finally, let, for $j = 1, \ldots, N-1$, the **modified mass**

$$M_j(t) := \frac{1}{2}\int_{\mathbb{R}} \tilde{u}^2(t,y)(1 - \psi_j(t,y))\,dy, \qquad \psi_j(t,x) := \psi(y - \sigma_j(t)), \qquad (4.25)$$

with $\sigma_j(t) := \frac{1}{2}(c_{j-1}^0 t + c_j^0 t + x_{j-1}^0 + x_j^0)$, and $\tilde{u}$ defined in (4.14). Note that this quantity considers the mass **on the left** of each soliton, which represents the main difference, compared with the standard arguments included in [36, 31].

Lemma 4.6 (Almost monotonicity of the mass, see also [36])**.**
There exist $K > 0$ and $L_0 > 0$ such that, for all $L > L_0$, the following is true. For all $t \in [0, T^*]$ one has

$$M_j(t) - M_j(0) \geq -K\,e^{-\sigma_0 L}. \qquad (4.26)$$

Proof. The proof is similar to [29, 36], so we sketch the main steps. Let $j \in \{1, \ldots, N-1\}$. Using equation (4.16) and integrating by parts several times, we have

$$\frac{d}{dt}M_j(t) =$$

$$= -\frac{1}{2}\int_{\mathbb{R}}\Big[-3\tilde{u}_x^2 - (c_{j-1}^0 + c_j^0)\tilde{u}^2 + \frac{4}{3}\tilde{u}^3 - \frac{3}{2}\beta\tilde{u}^4\Big]\psi_j' - \frac{1}{2}\int_{\mathbb{R}}\tilde{u}^2\psi_j^{(3)} \quad (4.27)$$

$$+\frac{3}{2}\int_{\mathbb{R}}\tilde{u}(1 - \psi_j)[(\varphi_c^2 - c)\tilde{u} + (\sqrt{c} + \varphi_c)\tilde{u}^2]_y \qquad (4.28)$$

$$+\frac{1}{2\sqrt{\beta}}x_N'(t)\int_{\mathbb{R}}\tilde{u}(1 - \psi_j)\varphi_c'. \qquad (4.29)$$

Let us consider the term (4.27). By definition of ψ, $|\psi^{(3)}| \leq \frac{\sigma_0}{4}\psi'$, so that

$$\Big|\int_{\mathbb{R}}\tilde{u}^2\psi_j^{(3)}\Big| \leq \frac{1}{4}(c_{j-1}^0 + c_j^0)\int_{\mathbb{R}}\tilde{u}^2\psi_j'. \qquad (4.30)$$

In order to bound the term $\int_{\mathbb{R}}(\frac{4}{3}\tilde{u}^3 - \frac{3}{2}\beta\tilde{u}^4)\psi_j'$, one follows the argument of [36], splitting the real line in two different regions according to the position of each soliton, and the rest. Following that argument, one finds

$$\Big|\int_{\mathbb{R}}(\frac{4}{3}\tilde{u}^3 - \frac{3}{2}\beta\tilde{u}^4)\psi_j'\Big| \leq Ke^{-\sigma_0(t+L)} + \frac{1}{4}\sigma_0\int_{\mathbb{R}}\tilde{u}^2\psi_j'.$$

Now we consider the term (4.28). Note that one has

$$\big|(1 - \psi_j)(\varphi_c^2 - c)\big| \leq Ke^{-\sigma_0(t+L)},$$

and a similar estimate is valid for the term $|(1 - \psi_j)(\sqrt{c} + \varphi_c)|$. Therefore

$$\Big|\frac{3}{2}\int_{\mathbb{R}}\tilde{u}(1 - \psi_j)[(\varphi_c^2 - c)\tilde{u} + (\sqrt{c} + \varphi_c)\tilde{u}^2]_y\Big| \leq Ke^{-\sigma_0(t+L)},$$

Let us consider the term (4.29). In this case, it is enough to recall that

$$\|(1 - \psi_j)\varphi_c'\|_{L^2(\mathbb{R})} \leq K e^{-\sigma_0(t+L)},$$

and $|x_N'(t)| \leq K\alpha$. Finally, we obtain for some $K > 0$,

$$\frac{d}{dt} M_j(t) \geq -K e^{-\sigma_0(t+L)}.$$

Thus, by integrating between 0 and t, we get the conclusion. Note that K and L are chosen independently of t. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.7 (Quadratic control of the variation of $c_j(t)$)**.**
    There exists $K > 0$ independent of K^*, such that for all $t \in [0, T^*]$,

$$\sum_{j=1}^{N-1} |c_j(t) - c_j(0)| \leq K(\|z(t)\|_{H^1(\mathbb{R})}^2 + \|z(0)\|_{H^1(\mathbb{R})}^2 + e^{-\sigma_0 L}). \qquad (4.31)$$

Proof. We proceed in several steps, following the proof given in [36].

1. Note that from (4.17), and using a Taylor expansion of the function

$$f(M[Q_{c_j(t),\beta}]) := \frac{2}{3} c_j^{3/2}(t)$$

around the point $s_0 := M[Q_{c_j(0),\beta}]$,[6] one has for some $K > 0$,

$$\left| \sum_{j=1}^{N-1} \beta(2c - c_j(0))(M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}]) \right| \leq$$

$$\leq K(\|z(t)\|_{H^1(\mathbb{R})}^2 + \|z(0)\|_{H^1(\mathbb{R})}^2) + K e^{-\sigma_0 L} + K \sum_{j=1}^{N-1} (M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}])^2.$$

Note in addition that for α_0 small and $L_0 > 0$ large, from (B.5),

$$\left| M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}] \right| = \left. \partial_c M[Q_{c,\beta}] \right|_{c=c_j(0)} (c_j(t) - c_j(0)) + O(|c_j(t) - c_j(0)|^2)$$

$$= \frac{c_j^{1/2}(0)}{\beta(2c - c_j(0))}(c_j(t) - c_j(0)) + O(|c_j(t) - c_j(0)|^2).$$

Since $2c - c_j(0) \geq 2c - c_{N-1}(0) > \sigma_0 > 0$, one has

$$\left| \sum_{j=1}^{N-1} \beta(2c - c_j(0))(M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}]) \right|$$

$$\leq K(\|z(t)\|_{H^1(\mathbb{R})}^2 + \|z(0)\|_{H^1(\mathbb{R})}^2 + e^{-\sigma_0 L}) + K \sum_{j=1}^{N-1} |c_j(t) - c_j(0)|^2.$$

$$(4.32)$$

In other words, the left hand side above is of *quadratic variation in z*.

---

[6]In particular, a simple computation using (B.5) and $c = \frac{1}{9\beta}$ shows that

$$f'(s_0) = \left. \frac{c^{1/2}}{\partial_c M[Q_{c,\beta}]} \right|_{c=c_j(0)} = \frac{1}{9c}(2c - c_j(0)) = \beta(2c - c_j(0)).$$

2. Let

$$d_j(t) := \beta \sum_{k=1}^{j} M[Q_{c_k(t),\beta}], \quad j = 1, \ldots, N-1. \quad (= \text{the mass on the left}) \quad (4.33)$$

We claim that there exists $K > 0$ such that, for all $j = 1, \ldots, N-1$,

$$|d_j(t) - d_j(0)| \le (d_j(t) - d_j(0)) + K\|z(0)\|_{L^2(\mathbb{R})}^2 + K\|z(t)\|_{L^2(\mathbb{R})}^2 + Ke^{-\sigma_0 L}. \quad (4.34)$$

Let us prove this last identity. Suppose $j \in \{1, \ldots, N-1\}$. First of all, if $d_j(t) - d_j(0)$ is nonnegative, there is nothing to prove. Let us assume that $d_j(t) - d_j(0) < 0$, therefore we have to show that

$$d_j(0) - d_j(t) \le K\|z(0)\|_{L^2(\mathbb{R})}^2 + K\|z(t)\|_{L^2(\mathbb{R})}^2 + Ke^{-\sigma_0 L}.$$

Recall that from Lemma 4.6, one has $M_j(0) \le M_j(t) + Ke^{-\sigma_0 L}$. On the other hand, from (4.10), (4.13), (4.14) and (B.3),

$$\begin{aligned}
M_j(t) &= \frac{1}{2} \int_{\mathbb{R}} (\tilde{R}^2 + 2\tilde{R}\tilde{z} + \tilde{z}^2)\tilde{\psi}_j(t) \\
&= \frac{1}{2} \sum_{k=1}^{N-1} \int_{\mathbb{R}} Q_{c_k,\beta}^2 \tilde{\psi}_j(t) + \frac{1}{2} \int_{\mathbb{R}} \tilde{z}^2 \tilde{\psi}_j(t) + O(e^{-\sigma_0 L}) \\
&= \frac{1}{\beta} \left[\beta \sum_{k=1}^{j} M[Q_{c_k(t),\beta}] + \frac{1}{2} \int_{\mathbb{R}} z^2 \psi_j(t) \right] + O(e^{-\sigma_0 L}).
\end{aligned}$$

Therefore,

$$d_j(t) - d_j(0) = -\frac{1}{2} \int_{\mathbb{R}} (z^2 \psi_j(t) - z^2 \psi_j(0)) + \beta(M_j(t) - M_j(0)) + O(e^{-\gamma_0 L}). \quad (4.35)$$

Using Lemma 4.6, (4.34) follows easily.

3. Conclusion. From the definition of $d_j(t)$ in (4.33),

$$\sum_{j=1}^{N-1} \beta(2c - c_j(0))(M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}]) =$$

$$= (2c - c_1(0))(d_1(t) - d_1(0)) + \sum_{j=2}^{N-1} (2c - c_j(0))[d_j(t) - d_{j-1}(t) - (d_j(0) - d_{j-1}(0))]$$

$$= (2c - c_{N-1}(0))(d_{N-1}(t) - d_{N-1}(0)) + \sum_{j=1}^{N-2} (c_{j+1}(0) - c_j(0))(d_j(t) - d_j(0)). \quad (4.36)$$

Therefore, by (4.32) and (4.36),

$$(4.36) \le K\|z(t)\|_{H^1(\mathbb{R})}^2 + K\|z(0)\|_{H^1(\mathbb{R})}^2 + Ke^{-\sigma_0 L} + K \sum_{j=1}^{N-1} |c_j(t) - c_j(0)|^2. \quad (4.37)$$

Since $2c - c_{N-1}(0) \geq \sigma_0$ and $c_{j+1}(0) - c_j(0) \geq \sigma_0$, by (4.34), one has

$$
\begin{aligned}
\sigma_0 \sum_{j=1}^{N-1} |d_j(t) - d_j(0)| \ &\leq \ (2c - c_{N-1}(0))|d_{N-1}(t) - d_{N-1}(0)| \\
&\qquad + \sum_{j=1}^{N-2} (c_{j+1} - c_j)(0)|d_j(t) - d_j(0)| \\
&\leq \ (4.36) + K\|z(0)\|_{L^2(\mathbb{R})}^2 + K\|z(t)\|_{L^2(\mathbb{R})}^2 + Ke^{-\sigma_0 L}.
\end{aligned}
$$

Thus, by (4.37),

$$
\sum_{j=1}^{N-1} |d_j(t) - d_j(0)| \leq K\|z(t)\|_{H^1(\mathbb{R})}^2 + K\|z(0)\|_{H^1(\mathbb{R})}^2 + Ke^{-\gamma_0 L} + K \sum_{j=1}^{N-1} |c_j(t) - c_j(0)|^2.
\tag{4.38}
$$

Since for $j \geq 2$ one has

$$
\begin{aligned}
|c_j(t) - c_j(0)| \ &\leq \ K|M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}]| \\
&= \ K|d_j(t) - d_{j-1}(t) - d_j(0) + d_{j-1}(0)| \\
&\leq \ K|d_j(t) - d_j(0)| + K|d_{j-1}(t) - d_{j-1}(0)|,
\end{aligned}
$$

we obtain from (4.38)

$$
\sum_{j=1}^{N-1} |c_j(t) - c_j(0)| \leq K\|z(t)\|_{H^1(\mathbb{R})}^2 + K\|z(0)\|_{H^1(\mathbb{R})}^2 + Ke^{-\sigma_0 L} + K \sum_{j=1}^{N-1} |c_j(t) - c_j(0)|^2.
$$

Choosing a smaller α_0 and a larger L_0, depending on K^*, we can assume $K|c_j(t) - c_j(0)| \leq 1/2$ and so

$$
\sum_{j=1}^{N-1} |c_j(t) - c_j(0)| \leq K\|z(t)\|_{H^1(\mathbb{R})}^2 + K\|z(0)\|_{H^1(\mathbb{R})}^2 + Ke^{-\sigma_0 L}.
\tag{4.39}
$$

The proof is complete. $\square$

Lemma 4.8 (Bootstrap)**.**
   There exists $K > 0$, independent of K^*, such that for all $t \in [0, T^*]$,

$$
\|z(t)\|_{H^1(\mathbb{R})}^2 \leq K(\|z(0)\|_{H^1(\mathbb{R})}^2 + e^{-\sigma_0 L}).
$$

Proof. By (4.17),

$$
\begin{aligned}
\mathcal{F}(t) \; = \; & \mathcal{F}(0) - \frac{2}{3} \sum_{j=1}^{N-1} (c_j^{3/2}(t) - c_j^{3/2}(0)) + O(\|z(t)\|_{H^1(\mathbb{R})}^3) + O(e^{-\sigma_0 L}) \\
= \; & -\sum_{j=1}^{N-1} \beta(2c - c_j(0))(M[Q_{c_j(t),\beta}] - M[Q_{c_j(0),\beta}]) \\
& + O\Big(\|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3 + \sum_{j=1}^{N-1} |c_j(t) - c_j(0)|^2 + e^{-\sigma_0 L} \Big) \\
= \; & -(2c - c_{N-1}(0))(d_{N-1}(t) - d_{N-1}(0)) - \sum_{j=1}^{N-2} (c_{j+1}(0) - c_j(0))(d_j(t) - d_j(0)) \\
& + O\Big(\|z(0)\|_{H^1(\mathbb{R})}^2 + \|z(t)\|_{H^1(\mathbb{R})}^3 + \sum_{j=1}^{N-1} |c_j(t) - c_j(0)|^2 + e^{-\sigma_0 L} \Big).
\end{aligned}
$$

On the other hand, note that from (4.32) and Lemma 4.6,

$$
d_j(t) - d_j(0) \geq -\frac{1}{2} \int_{\mathbb{R}} (z^2 \psi_j(t) - z^2 \psi_j(0)) - K e^{-\sigma_0 L}, \quad j = 1, \dots, N-1.
$$

Therefore

$$
\tilde{\mathcal{F}}(t) \leq K \|z(0)\|_{H^1(\mathbb{R})}^2 + K \|z(t)\|_{H^1(\mathbb{R})}^3 + K \sum_{j=1}^{N-1} (c_j(t) - c_j(0))^2 + e^{-\sigma_0 L}, \quad (4.40)
$$

with $\tilde{\mathcal{F}}(t)$ given by the formula

$$
\begin{aligned}
\tilde{\mathcal{F}}(t) := \\
= \; & \mathcal{F}(t) - \frac{1}{2} \sum_{j=1}^{N-2} ((c_{j+1}(0) - 2c) + (2c - c_j(0))) \int_{\mathbb{R}} z^2 \psi_j(t) \\
& - \frac{1}{2}(2c - c_{N-1}(0)) \int_{\mathbb{R}} z^2 \psi_{N-1}(t) \\
= \; & \mathcal{F}(t) - \frac{1}{2} \sum_{j=2}^{N-1} (2c - c_j(0)) \int_{\mathbb{R}} z^2 (\psi_j(t) - \psi_{j-1}(t)) - \frac{1}{2}(2c - c_1(0)) \int_{\mathbb{R}} z^2 \psi_1(t) \\
= \; & \frac{1}{2} \int_{\mathbb{R}} \big\{ z_x^2(t) + c(t,x) z^2(t) - 3(c - \varphi_c^2) z^2(t) - 6\sqrt{c} R z^2(t) + 3R^2 z^2(t) \big\}, \; (4.41)
\end{aligned}
$$

with

$$
\begin{aligned}
c(t,x) \; := \; & 2c - \sum_{j=2}^{N-1} (2c - c_j(0))(\psi_j - \psi_{j-1})(t) - (2c - c_1(0))\psi_1(t) \\
= \; & 2c \Big[1 - \sum_{j=2}^{N-1} (\psi_j - \psi_{j-1})(t) - \psi_1(t) \Big] \\
& + \sum_{j=2}^{N-1} c_j(0)(\psi_j - \psi_{j-1})(t) + c_1(0)\psi_1(t). \quad (4.42)
\end{aligned}
$$

We prove in Appendix A that this quadratic form is coercive, in the sense that there exists $\lambda_0 > 0$ independent of t and K^* such that, thanks to (4.9) and (4.10),

$$\tilde{\mathcal{F}}(t) \geq \lambda_0 \|z(t)\|^2_{H^1(\mathbb{R})}. \tag{4.43}$$

Therefore, from (4.40), (4.31), and taking α_0 smaller if necessary, we obtain

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq K\|z(0)\|^2_{H^1(\mathbb{R})} + K\|z(t)\|^3_{H^1(\mathbb{R})} + e^{-\sigma_0 L},$$

and so

$$\|z(t)\|^2_{H^1(\mathbb{R})} \leq K\|z(0)\|^2_{H^1(\mathbb{R})} + Ke^{-\sigma_0 L},$$

for some constant $K > 0$, independent of K^*. Thus, the proof of Lemma 4.8 is complete. $\qquad\square$

We conclude the proof of Proposition 4.2. From (4.11), Lemmas 4.7 and 4.8, we have

$$\left\| u(t) - \varphi_c(\cdot + ct + x_N(t)) + \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j^0, \beta}(\cdot + \tilde{c}_j t + x_j(t)) \right\|_{H^1(\mathbb{R})}$$

$$\leq \|z(t)\|_{H^1(\mathbb{R})} + \sqrt{\beta} \left\| \sum_{j=1}^{N-1} [Q_{c_j, \beta}(\cdot + \tilde{c}_j t + x_j(t)) - Q_{c_j^0, \beta}(\cdot + \tilde{c}_j t + x_j(t))] \right\|_{H^1(\mathbb{R})}$$

$$\leq \|z(t)\|_{H^1(\mathbb{R})} + K \sum_{j=1}^{N-1} |c_j(t) - c_j^0|$$

$$\leq \|z(t)\|_{H^1(\mathbb{R})} + K \sum_{j=1}^{N-1} |c_j(t) - c_j(0)| + K \sum_{j=1}^{N-1} |c_j(0) - c_j^0|$$

$$\leq \|z(t)\|_{H^1(\mathbb{R})} + K\|z(0)\|^2_{H^1(\mathbb{R})} + Ke^{-\sigma_0 L} + K\alpha \quad \leq K(\alpha + e^{-\sigma_0 L}),$$

where $K > 0$ is a constant independent of K^*. Finally, choosing $K^* = 4K$, we get the desired contradiction. The proof is complete.

5. Sketch of proof of Theorem 1.5

The proof follows the lines of [29, Theorem 1] and the proof of Proposition 4.2 from the previous section. Let us assume the hypotheses of Theorem 1.5. Let $T_n \to +\infty$ be an increasing sequence, and

$$R(t) := \varphi_{c_N^0}(\cdot + c_N^0 t) + \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j^0, \beta}(\cdot + \tilde{c}_j t), \quad \tilde{c}_j = 3c_N^0 - c_j^0.$$

It is clear that $R(t) - \varphi_{c_N^0}(\cdot + c_N^0 t)$ is uniformly bounded in any $H^s(\mathbb{R})$, $s \geq 0$.

Now we consider the following Cauchy problem

$$(u_n)_t + ((u_n)_{xx} - u_n^3)_x = 0, \qquad u_n(t, x) \in \mathbb{R}, \tag{5.1}$$

$$u_n(T_n) = R(T_n). \tag{5.2}$$

From [37], one has global existence of a unique solution $u_n(t)$ for (5.1), satisfying $u_n - \varphi_{c_N^0} \in C(\mathbb{R}, H^1(\mathbb{R}))$, with conserved energy (1.5). The main part of the proof is to establish the following uniform estimates:

Proposition 5.1 (Uniform estimates).

There exist $K, n_0 > 0$ such that for all $n \geq n_0$, and for all $t \in [T_{n_0}, T_n]$, one has

$$\|u_n(t) - R(t)\|_{H^s(\mathbb{R})} \leq K_s e^{-\sigma_0 t}, \quad s \in [0, 1^+], \tag{5.3}$$

with $\sigma_0 > 0$ defined in (4.5).

The proof of this result is similar to the proof of Proposition 4.2, but it is easier since we do not need to modulate the scaling parameters $c_j(t)$ in Lemma 4.3. In particular, the term $\sim \sum_j c_j^{3/2}(t)$ in (4.17) is constant. Lemma 4.6 holds with no modifications. In order to control the directions $Q_{c_j, \beta}$ in (4.43), we only use Lemma 4.6, so Lemma 4.7 is not needed. The reader may consult [29] for a detailed proof.

As a consequence of the above estimate, one has, up to a subsequence, and for all $t \geq T_{n_0}$,

$$u_n(t) - \varphi_{c_N^0}(\cdot + c_N^0 t) \rightharpoonup u_0 \quad \text{in } H^1(\mathbb{R}),$$

$$u_n(t) - \varphi_{c_N^0}(\cdot + c_N^0 t) \to u_0 \quad \text{in } L^2(K),$$

for all $K \subset \mathbb{R}$ compact. On the other hand, note that from (1.22) and (1.23) the function

$$\tilde{u}_n(t, y) := \frac{1}{\sqrt{\beta}}(u_n(t, y - 3c_N^0 t) - \varphi_{c_N^0}(y - 2c_N^0 t)), \quad c_N^0 = \frac{1}{9\beta},$$

satisfies the equation (1.23). Arguing as in [29, eqn. (14)], one has the following

Lemma 5.2 (Egorov estimate).

There exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the following holds. There exists $R_0 = R_0(\varepsilon) > 0$ such that for all $n \geq n_0$,

$$\|u_n(T_{n_0}) - \varphi_{c_N^0}(\cdot + c_N^0 T_{n_0})\|_{L^2(|x| > R_0)} \leq \varepsilon. \tag{5.4}$$

From (5.4) one has that

$$u_n(T_{n_0}) - \varphi_{c_N^0}(\cdot + c_N^0 T_{n_0}) \to u_0 \quad \text{in } L^2(\mathbb{R}),$$

and by interpolation and (5.3), the convergence is in $H^1(\mathbb{R})$. Let U be the unique solution of (1.1) such that $U(T_{n_0}) = \varphi_{c_N^0}(\cdot + c_N^0 T_{n_0}) + u_0$ (cf [37, Proposition 3.1]). From the uniform $H^1(\mathbb{R})$ continuity of the mKdV flow on compact sets of time, one has

$$u_n(t) \to U(t) \quad \text{in } H^1(\mathbb{R}),$$

for all $t \geq T_{n_0}$. Therefore, $\|u_n(t) - U(t)\|_{H^1(\mathbb{R})} \to 0$ as $n \to +\infty$, for all $t \geq T_{n_0}$. Finally, passing to the limit in (5.3), we get the desired existence conclusion.

Uniqueness. Using once again the transformation (1.22), and the equation (1.23), we claim that from [29], one has the following

Lemma 5.3 (Exponential decay).

Let $v \in \varphi_{c_N^0}(\cdot + c_N^0 t) + C(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (1.1) satisfying (1.26). Then there exists $K, T_0 > 0$ such that

$$\sup_{t \geq T_0} \|v(t) - \varphi_{c_N^0}(\cdot + c_N^0 t + x_N^+) - \sqrt{\beta} \sum_{j=1}^{N-1} Q_{c_j, \beta}(\cdot + \tilde{c}_j t + x_j^+)\|_{H^1(\mathbb{R})} \leq Ke^{-\sigma_0 t}.$$

Using this property, the uniqueness result is just a consequence of the analysis carried out in [29]. We skip the details.

Appendix A. Proof of (4.43)

In this section we sketch the proof of (4.43). See e.g. [36] for a detailed, similar proof. First of all, note that from (4.41), (4.13) and (4.15) one has

$$\tilde{\mathcal{F}}(t) = \frac{1}{2}\int_{\mathbb{R}}(z_x^2 + c(t,x)z^2 - 3(c - \varphi_c^2)z^2 - (2\tilde{R} - 3\beta\tilde{R}^2)z^2),$$

with $c(t,x)$ given in (4.42).

1. We recall the following well-known result.

Lemma A.1 (Positivity of the Zhidkov functional, see [47, 37]).
  There exists $\lambda_0 > 0$ such that for all $z \in H^1(\mathbb{R})$, with $\int_{\mathbb{R}} z\varphi_c' = 0$, one has

$$\frac{1}{2}\int_{\mathbb{R}}(z_x^2 + 2cz^2 - 3(c - \varphi_c^2)z^2) \geq \lambda_0 \int_{\mathbb{R}}(z_x^2 + z^2). \tag{A.1}$$

2. Let $\Phi \in C^2(\mathbb{R})$, with $\Phi(s) = \Phi(-s)$, $\Phi' \leq 0$ on $\mathbb{R}^+$, and

$$\Phi(s) = 1 \text{ on } [0,1]; \quad \Phi(s) = e^{-s} \text{ on } [2,+\infty), \quad e^{-s} \leq \Phi(s) \leq 3e^{-s} \quad \text{on } \mathbb{R}^+.$$

Finally, let $\Phi_B(s) := \Phi(\frac{s}{B})$.

Lemma A.2 (Localized coercivity, see e.g. [36, 46]).
  There exists $B_0, \lambda_0 > 0$ such that, for all $B > B_0$, if $z \in H^1(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} Q_c z = \int_{\mathbb{R}} Q_c' z = 0,$$

then

$$\int_{\mathbb{R}} \Phi_B(z_x^2 - (2Q_c^2 - 3\beta Q_c^2)z^2 + cz^2) \geq \lambda_0 \int_{\mathbb{R}} \Phi_B(z_x^2 + z^2). \tag{A.2}$$

Note that a similar argument can be followed in order to prove a localization property for the Zhidkov functional considered in Lemma A.1. We skip the details.

3. Now we perform a localization argument, as in [36]. One has from (A.1), $\tilde{\mathcal{F}}(t) = \sum_{j=1}^{N}\mathcal{F}_j(t) + (\tilde{\mathcal{F}}(t) - \sum_{j=1}^{N}\mathcal{F}_j(t))$, with

$$\mathcal{F}_j(t) := \frac{1}{2}\int_{\mathbb{R}}\Phi_{B,j}(z_x^2 + c_j(t)z^2 - (2Q_{c_j,\beta} - 3\beta Q_{c_j,\beta}^2)z^2), \quad \Phi_{B,j} := \Phi_B(x + \tilde{c}_j t + x_j(t)),$$

for all $j = 1, \ldots, N-1$, and

$$\mathcal{F}_N(t) := \frac{1}{2}\int_{\mathbb{R}}\Phi_{B,N}(z_x^2 + 2cz^2 - 3(c - \varphi_c^2)z^2), \quad \Phi_{B,N} := \Phi_B(x + ct + x_N(t)).$$

From Lemma A.2, one has for B large enough,

$$\mathcal{F}_j(t) \geq \lambda_0 \int_{\mathbb{R}} \Phi_{B,j}(z_x^2 + z^2)(t,x)dx,$$

for all $j = 1, \ldots, N$. On the other hand,

$$
\tilde{\mathcal{F}}(t) - \sum_{j=1}^{N} \mathcal{F}_j(t) \;=\; \int_{\mathbb{R}} \Big(1 - \sum_{j=1}^{N} \Phi_{B,j} \Big) (z_x^2 + c(t,x)z^2)
$$

$$
+ \sum_{j=1}^{N-1} \int_{\mathbb{R}} \Phi_{B,j}(c(t,x) - c_j(t))z^2 + \int_{\mathbb{R}} \Phi_{B,N}(c(t,x) - 2c)z^2
$$

$$
- \frac{1}{2} \int_{\mathbb{R}} \Big(2\tilde{R} - 3\beta\tilde{R}^2 - \sum_{j=1}^{N-1} \Phi_{B,j}(2Q_{c_j,\beta} - 3\beta Q_{c_j,\beta}^2) \Big) z^2
$$

$$
- \frac{3}{2} \int_{\mathbb{R}} (1 - \Phi_{B,N})(c - \varphi_c^2)z^2
$$

Each term above can be treated following the lines of the proof of Lemma 4 in [36], and it is proved that for all B large, the above terms can be estimated by

$$
\geq -\frac{1}{16}\lambda_0 \int_{\mathbb{R}} (z_x^2 + z^2).
$$

The final conclusion is that for B large enough, but independent of z,

$$
\tilde{\mathcal{F}}(t) \geq \frac{1}{8}\lambda_0 \sum_{j=1}^{N} \int_{\mathbb{R}} \Phi_{B,j}(z_x^2 + z^2) \geq \frac{1}{16}\lambda_0 \int_{\mathbb{R}} (z_x^2 + z^2),
$$

for some $\lambda_0 > 0$ independent of $z(t)$ and B. Thus the proof of (4.43) is complete.

APPENDIX B. PROOF OF SOME IDENTITIES

Lemma B.1 (Identities)**.**
  Let $Q_{c,\beta}$ be the Gardner soliton from (1.11). Then one has
  (1) Basic identities.

$$
Q_{c,\beta}'' = cQ_{c,\beta} - Q_{c,\beta}^2 + \beta Q_{c,\beta}^3, \qquad Q_{c,\beta}'^2 = cQ_{c,\beta}^2 - \frac{2}{3}Q_{c,\beta}^3 + \frac{\beta}{2}Q_{c,\beta}^4. \tag{B.1}
$$

  (2) Integrals.

$$
\beta \int_{\mathbb{R}} Q_{c,\beta}^3 = -c \int_{\mathbb{R}} Q_{c,\beta} + \int_{\mathbb{R}} Q_{c,\beta}^2, \qquad \beta \int_{\mathbb{R}} Q_{c,\beta}^4 = -\frac{4}{3}c \int_{\mathbb{R}} Q_{c,\beta}^2 + \frac{10}{9} \int_{\mathbb{R}} Q_{c,\beta}^3, \tag{B.2}
$$

  and

$$
\int_{\mathbb{R}} Q_{c,\beta} = \frac{3}{2}\beta \int_{\mathbb{R}} Q_{c,\beta}^2 + 6\sqrt{c}. \tag{B.3}
$$

  (3) Energy. Let

$$
E_\beta[Q_{c,\beta}] := \frac{1}{2} \int_{\mathbb{R}} Q_{c,\beta}'^2 - \frac{1}{3} \int_{\mathbb{R}} Q_{c,\beta}^3 + \frac{\beta}{4} \int_{\mathbb{R}} Q_{c,\beta}^4.
$$

  Then one has

$$
E_\beta[Q_{c,\beta}] = \frac{2}{3\beta}c^{3/2} - \frac{1}{9\beta} \int_{\mathbb{R}} Q_{c,\beta}^2. \tag{B.4}
$$

  (4) Weinstein's condition. For $c < \frac{2}{9\beta}$,

$$
\partial_c \frac{1}{2} \int_{\mathbb{R}} Q_{c,\beta}^2 = \frac{9c^{1/2}}{2 - 9\beta c}. \tag{B.5}
$$

Proof. The first identity in (B.1) is just the elliptic equation for $Q_{c,\beta}$, obtained by replacing in (1.10). The second one follows from the first identity in (B.1), after multiplication by $Q'_{c,\beta}$ and integration in space.

On the other hand, the first identity in (B.2) follows after integration of (B.1). In the same form, the second identity in (B.2) is a consequence of the first identity in (B.1) and the integration of the second one in (B.1) against $Q_{c,\beta}$.

Let us prove (B.3). Note that from (B.1),

$$\left(\frac{Q'_{c,\beta}}{Q_{c,\beta}}\right)' = -\frac{1}{3}Q_{c,\beta} + \frac{\beta}{2}Q^2_{c,\beta}.$$

Using the definition of $Q_{c,\beta}$ from (1.11), and integrating, one gets

$$-2\sqrt{c} = -\frac{1}{3}\int_{\mathbb{R}} Q_{c,\beta} + \frac{\beta}{2}\int_{\mathbb{R}} Q^2_{c,\beta},$$

namely (B.3). Now we prove (B.4). From (B.1) and (B.2)

$$
\begin{aligned}
E_\beta[Q_{c,\beta}] &= \frac{1}{2}\int_{\mathbb{R}} Q'^2_{c,\beta} - \frac{1}{3}\int_{\mathbb{R}} Q^3_{c,\beta} + \frac{\beta}{4}\int_{\mathbb{R}} Q^4_{c,\beta} \\
&= \frac{1}{2}c\int_{\mathbb{R}} Q^2_{c,\beta} - \frac{2}{3}\int_{\mathbb{R}} Q^3_{c,\beta} + \frac{\beta}{2}\int_{\mathbb{R}} Q^4_{c,\beta} \\
&= -\frac{1}{6}c\int_{\mathbb{R}} Q^2_{c,\beta} - \frac{1}{9}\int_{\mathbb{R}} Q^3_{c,\beta} = \frac{c}{9\beta}\int_{\mathbb{R}} Q_{c,\beta} - \left(\frac{c}{6} + \frac{1}{9\beta}\right)\int_{\mathbb{R}} Q^2_{c,\beta}.
\end{aligned}
$$

Using (B.3), we obtain (B.4), as desired.

Finally, let us prove (B.5). From the definition (1.11), one has

$$\partial_c Q_{c,\beta}(s) = \frac{1}{c}\left[(1 + \frac{9\beta c}{4\rho^2})Q_{c,\beta} - \frac{3\beta}{4\rho^2}Q^2_{c,\beta} + \frac{1}{2}sQ'_{c,\beta}(s)\right];$$

therefore, using (B.2) and (B.3),

$$
\begin{aligned}
\partial_c \frac{1}{2}\int_{\mathbb{R}} Q^2_{c,\beta} &= \int_{\mathbb{R}} Q_{c,\beta}\partial_c Q_{c,\beta} = \frac{3}{4c}\int_{\mathbb{R}} Q_{c,\beta}\left[(1 + \frac{3\beta c}{\rho^2})Q_{c,\beta} - \frac{\beta}{\rho^2}Q^2_{c,\beta}\right] \\
&= \frac{3}{4c}\left[(1 + \frac{3\beta c}{\rho^2} - \frac{1}{\rho^2})\int_{\mathbb{R}} Q^2_{c,\beta} + \frac{c}{\rho^2}\int_{\mathbb{R}} Q_{c,\beta}\right] \\
&= \frac{3}{4c}\left[(1 + \frac{3\beta c}{\rho^2} - \frac{1}{\rho^2} + \frac{3\beta c}{2\rho^2})\int_{\mathbb{R}} Q^2_{c,\beta} + \frac{6c^{3/2}}{\rho^2}\right] = \frac{9c^{1/2}}{2\rho^2}.
\end{aligned}
$$

$\square$

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