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Fuzzy Geometry via the Spinor Bundle, with Applications to Holographic Space-time and Matrix Theory

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Abstract

We present a new framework for defining fuzzy approximations to geometry in terms of a cutoff on the spectrum of the Dirac operator, and a generalization of it that we call the Dirac-Flux operator. This framework does not require a symplectic form on the manifold, and is completely rotation invariant on an arbitrary n-sphere. The framework is motivated by the formalism of Holographic Space-Time (HST), whose fundamental variables are sections of the spinor bundle over a compact Euclidean manifold. The strong holographic principle (SHP) requires the space of these sections to be finite dimensional. We discuss applications of fuzzy spinor geometry to HST and to Matrix Theory.

Contents

1 Introduction: Holographic Space-time (HST)

HST is an attempt to supply a general formalism for a theory of quantum gravity, which will reduce to string theory for space-times that are asymptotically AdS or Minkowski, but which has the flexibility to discuss cosmology, including dS space. The formalism also makes more direct contact with concepts of local physics. The Strong Holographic Principle (SHP), introduced by TB and W. Fischler is the assumption that the Covariant Entropy Bound (CEB) [1–3] implies that the Hilbert space encoding all measurements inside a causal diamond, is finite dimensional, with dimension that approaches the exponential of one quarter of the area of the holographic screen of the diamond[1]. The area is measured in Planck units and the formula is supposed to be only asymptotic for large area. In weakly coupled string theory, there is a further caveat. Here the Einstein equations break down at a length scale parametrically larger than the Planck scale and the identification of entropy and area fails unless the area is large in string units. The SHP combined with the notion of commutativity at space-like separation, encodes all of the geometrical properties of a Lorentzian space-time into quantum mechanical statements about operator algebras.

The basic idea is that space-time is only an emergent phenomenon, but that its properties reflect more basic properties of the underlying quantum theory of gravity. The kinematics of HST is a net of finite dimensional operator algebras, called diamond algebras $A(D)$, with specified intersections $O(D, D')$, which are tensor factors in both $A(D)$ and $A(D')$. $O(D, D')$ represents the set of all quantum measurements, which can be performed in the maximal area causal diamond in the intersection of the diamonds D and D'. It is clear that specifying the data in these algebras for a sufficiently rich set of diamonds, in the limit in which space-time emerges, will determine both the conformal factor and the causal structure of the Lorentzian geometry, which are thus kinematical properties of the quantum theory, rather than fluctuating quantum variables. The arguments of Jacobson [4], suggest that Einstein's

[1]The boundary of a causal diamond is a null surface, which can be foliated by space-like surfaces. The holographic screen of the diamond is the space-like surface of largest area. We abuse language and call the area of the screen the area of the diamond.

equations for the geometry will be an automatic consequence of the laws of thermodynamics, in the emergent space-time limit.

The actual quantum variables may be thought of classically as the space-time orientations of pixels on the holographic screen. Naively, a pixel is a position on the screen, through which a null ray passes, and the orientation of a bit of $d-2$ plane orthogonal to the null ray. This data is incorporated in the Cartan-Penrose (C-P) equation

$$\overline{\Psi}\gamma^{\mu}\Psi(\gamma_{\mu})_{\alpha\beta}\Psi_{\beta} = 0,$$

which forces the vector bilinear to be null, and the spinor Ψ to be a null plane spinor for that null ray:

$$\Psi = \begin{pmatrix} 0 & S_a \end{pmatrix}.$$

The C-P equation is Lorentz covariant and has a scaling symmetry. These are considered gauge equivalences. Generically, we may expect them all to be fixed in a unitary formulation of the quantum mechanics (which is all that we will consider). In fact, the scaling symmetry is explicitly broken in the quantum theory. However, there is a $\mathbb{Z}_2$ subgroup of the scaling symmetry, which is preserved and ends up playing the role of the $(-1)^F$ gauge symmetry familiar from quantum field theory. The connection between spin and statistics is automatic, as in Matrix Theory [5].

The strong holographic principle implies that a finite area holographic screen can have only a finite number of pixels, and that the algebra of variables for each pixel has a finite dimensional unitary representation. For compactifications to four dimensional space-time, the quantum commutation relations take the form

$$[(\psi^P)_i^A, (\psi^{\dagger\ Q})^j_B]_+ = \delta_i^j \delta_B^A Z^{PQ}.$$

The indices i, j run from 1 to N, A, B run from 1 to $N+1$, and P, Q run over a basis of a finite dimensional approximation to the spinor bundle over a compact manifold. We call this the *pixel algebra* of the HST model. It must be supplemented by commutation relations between the Z^{PQ} and the fermionic variables, forming a finite dimensional super-algebra. The holographic principle implies this algebra must have a finite dimensional unitary representation. We assume further that the action of the fermionic operators sweeps out the entire space of states of this representation.

An elegant choice for the finite dimensional approximation, based on A. Connes ideas about non-commutative geometry, is to put a cutoff on the spectrum of the Dirac operator. ψ and $\psi^{\dagger}$ are the two chiral spinor bundles over the fuzzy two sphere. Z^{PQ} lives in the bundle of forms over the compact manifold, fuzzified as the product of two cutoff spinor bundles. The Z^{PQ} are the analogs of wrapped brane charges in string theory.

In writing this equation, we have used the usual fuzzy quantization of the two sphere, which uses the space of sections of holomorphic line bundles to define a non-commutative approximation to the algebra of functions on the sphere and general vector bundles over this algebra. This is essentially the same as our Dirac fuzzification for the two sphere. We will show that the Dirac operator enables us to preserve rotation invariance for a space of any dimension.

2 The Dirac equation and geometry

Alain Connes [6] has made the Dirac operator the central focus of his metrical non-commutative geometry. Connes emphasis is on non-commutative geometries with infinite dimensional function algebras, while we are concerned with finite dimensional non-commutative approximations to ordinary commutative geometries. For physicists, an easy way to understand the relation between the Dirac equation and geometry is to think about the short time expansion of the heat kernel for the square of the Dirac operator

$$\langle x|e^{-tD^2}|y\rangle \to N\ t^{-\frac{d}{2}}e^{-\frac{l^2(x,y)}{4t}},$$

where d is the dimension of the manifold and $l(x,y)$ the geodesic distance between the points. The factor N is the number of geodesics of equal minimal length connecting the two points. This expression is most easily derived from the Feynman path integral representation of the heat kernel. The short time limit is a semi-classical limit for that functional integral. The heat kernel thus contains all of the geometrical information about the manifold.

Note that for this expression we need to know not only the spectrum of the Dirac operator, but also the form of its eigensections in the position representation. Geometers have long known how to describe the points of a manifold in terms of the algebraic structure of its algebra of functions. A point is equivalent to the maximal ideal of functions which vanish at that point. Alternatively, a point defines an algebra homomorphism between the algebra of functions and the complex numbers (a multiplicative linear functional). Connes shows that everything that is to be known about a manifold can be encoded in the relation between the Dirac operator and the algebra of smooth functions realized as multiplication operators on the Hilbert space of square integrable sections of the spinor bundle. He then proposes an abstract definition of the Dirac operator for a general non-commutative algebra of operators on a Hilbert space as the definition of a non-commutative Riemannian manifold.

Our aim is more modest. We simply want to recover the normal commutative geometry of manifolds as a limit of finite dimensional matrix algebras. This is relatively straightforward. For most[2] compact Riemannian manifolds of dimension d and volume V, the operator $V^{\frac{1}{d}}D$ has a spectrum that runs from $\sim \pm 1$ to $\pm\infty$. We will define a fuzzy spinor bundle over this manifold by cutting off the spectrum of this operator via the inequality $||V^{\frac{1}{d}}D|| < N$, where N is a positive integer. That is, we restrict to the space of eigensections whose eigenvalues satisfy this inequality. The dimension of this subspace of eigensections is another positive integer $K(N)$.

The algebra of $K(N)\times K(N)$ matrices is realized as a set of integral kernels

$$M_{\alpha\beta}(x,y) = \sum M_{ij}\psi^{*\ i}_{\alpha}(x)\psi^{j}_{\beta}(y),$$

on the full spinor bundle. In the limit $N\to\infty$, we restrict attention to matrices which produce kernels of the form

$$\sum M_{ij}\psi^{*\ i}_{\alpha}(x)\psi^{j}_{\beta}(y) \to f_{\alpha\beta}(x)\delta(x,y),$$

[2]To quantify the notion of most, we have to think about a moduli space of Riemannian manifolds satisfying some equations. Such moduli spaces have a natural metric on them, and although non-compact, the moduli space has finite volume. This means that extreme values of the moduli are "non-generic". Our statement will be valid in a region of moduli space that contains a large fraction of the total volume.

where $\delta(x,y)$ is the Dirac distribution on the manifold. $f_{\alpha\beta}$ belongs to the algebra of differential forms with Clifford multiplication, rather than the standard Grassmann product. The Clifford multiplication of course depends on the metric. With appropriate restrictions on the limiting form of M_{ij} we can get measurable, continuous, or smooth differential forms.

2.1 Moduli

If we have a moduli space of manifolds, then the eigenvalues and eigensections of the Dirac operator depend smoothly on the moduli. However, the spirit of non-commutative geometry and fuzzy geometry in particular is that the algebra determines the geometry. In the standard geometric quantization of the two torus, we can see that this leads to a discretization of moduli space. A square fuzzy torus is defined by the algebra of all $N \times N$ matrices, written in terms of generators U, V satisfying

$$U^N = V^N = 1,$$

$$UV = e^{\frac{2\pi i}{N}} VU.$$

The area of this torus in Planck units is $\sim N^2$. If N has a factor k, we can get a rectangular torus by restricting attention to the subalgebra generated by U^k and V, and a similar restriction produces tilted tori as well. But we only get a rational set of moduli in this manner. Continuous moduli arise, like longitudinal momenta in Matrix Theory [5] and HST, as ratios of integers, both of which are taken to infinity.

For spinor fuzzification we consider the Dirac operator with periodic boundary conditions[3]. A general 2-torus is determined by a parallelogram, parameterized in terms of three real numbers (a, b, c) with $0 < c < a$. a is the length of the horizontal segments, and b the vertical separation between them. c determines the tilt of the parallelogram. The eigenvalues and eigensections of the Dirac operator with periodic boundary conditions are determined by a two vector $\mathbf{p} = (p_1, p_2)$ with

$$p_1 = \frac{2\pi n}{a} \qquad p_2 = \frac{2\pi m}{b} - \frac{2\pi nmc}{ab}.$$

The eigenvalues are $\pm|\mathbf{p}|$ and the eigensections are

$$\psi_\pm e^{i\mathbf{p}\cdot\mathbf{x}},$$

where $\psi_\pm$ are the two eigenspinors of $\sigma_1 p_1 + \sigma_2 p_2$.

Fuzzification consists of choosing integer valued moduli $a = N$, $b = M$, $c = k \leq N$ and cutting off the values of m and n. Two natural cutoffs are $n \leq N$, $m \leq N$, and $\left(\frac{n}{N}\right)^2 + \left(\frac{m}{M} - \frac{knm}{MN}\right)^2 < K^2$, for some integer K. The first is similar to the kind of cutoff one gets from Kahler quantization, while the latter conforms to our general idea of just bounding the spectrum of the Dirac operator. For K of order 1, both methods give a number of sections of the spinor bundle that scales like MN, which is proportional to the area of the torus. If

[3]The implications of different spin structures for our program seem interesting, but we have not understood them.

we make the independent sections into independent generators of a quantum superalgebra, then the entropy of the torus will be proportional to its area.

More generally, the large eigenvalues of the Dirac equation on any smooth compact manifold are approximately like plane waves and their degeneracy grows like P^D, where P is the eigenvalue cutoff and D the dimension. Thus, the number of independent sections grows like the volume of the manifold in Planck units. Since this compact manifold is the holographic screen of a Lorentzian manifold in the HST formalism, this is precisely the right Bekenstein-Hawking entropy in the general case. That is to say, the entropy per four dimensional pixel (fixed value of i and A) will, for compact dimensions large in higher dimensional Planck units, be proportional to the volume of the internal dimensions. This is the conventional Kaluza-Klein relation between the four dimensional and higher dimensional Planck scales.

It is easy to work out the spinor fuzzification of a general torus, and we will do a general sphere in the next section. The procedure is straightforward for any manifold for which one can work out the eigenvalues and eigensections of the Dirac equation. The tensor product relation between spinor bundles and the bundles of differential forms imply that some of the topological features of the manifold are encoded in zero modes of the Dirac equation. This is familiar from the Atiyah-Singer Index theorem and its generalizations. In particular, if we have a covariantly constant spinor, $D_\mu \psi_0 = 0$, then it is also a zero mode of the Dirac equation. The non-vanishing differential forms

$$\overline{\psi}_0 \gamma_{\mu_1 \ldots \mu_k} \psi_0,$$

where the matrices are the k-fold anti-symmetrized product of tangent space Dirac matrices, contracted into the vielbein, are all elements of the cohomology of the manifold. This part of the topological information about the manifold is preserved by spinor fuzzification. Note that this is a bit different than Kahler fuzzification, where the information that is kept is a cutoff version of the Picard group and the dimensions of spaces of sections of holomorphic line bundles, as well as information about the complex structure. It is peculiar though that not all of this information is invariant information about the finite dimensional matrix algebra. For example the fuzzy square torus and the fuzzy sphere have the same algebra, and in some sense are distinguished only by the choice of a basis in this algebra (spherical harmonics vs. powers of clock and shift operators) and the way in which expansion coefficients in these bases behave in the large N limit.

We believe that the lack of some explicit topological information about the manifold in fuzzy quantization is at the root of string dualities. Highly supersymmetric compactifications of string/M theory to asymptotically flat space-times are often characterized by moduli spaces of classical background geometries. The use of classical backgrounds that are solutions of some low energy effective field theory always implies that we are working in a limit where some length scale is much larger than the Planck scale[4]. We've seen that in such limits, the discrete moduli spaces of fuzzy compactification give rise to continuous ratios of large integers[5]. The notion of continuous moduli spaces is conceptually wrong, but valid to all

[4]This can be a geometric length scale in the compactification manifold or the Compton wavelength of some quantum excitation.

[5]For example, in Kahler quantization, the Kahler moduli have to do with the direction in the Picard

orders in expansions in $\frac{L_P}{L_{Large}}$. String duality relations are derived in terms of constraints on low energy Lagrangians in two different limits, which have the same SUSY algebra.

In HST, the SUSY algebra arises in the limit of large causal diamonds in the non-compact space, with the discrete internal moduli fixed. In that limit, the pixel algebra generators become distributions, $(\psi^P)_i^A \rightarrow \psi^K \delta(\Omega, \Omega_0)$ and the anti-commutation relations become (for 4 dimensional asymptotically flat space)

$$[\psi^K, \psi^{\dagger L}]_+ = P Z^{KL}.$$

Recall that K and L label a finite dimensional basis of the space of Dirac eigensections on the internal manifold, with eigenvalue less than some bound. P is a positive real number. It arises as follows. We take the N characterizing the maximal spherical harmonic in the pixel algebra to infinity, obtaining wave functions localizable on the sphere, which deserve to be called particles penetrating the holographic screen. Now we can do this in block diagonal matrices of size $N_i \rightarrow \infty$, with $\frac{N_i}{N_j}$ fixed, obtaining continuous longitudinal fractions. We now view these fractions as ratios of dimensionfull momenta, and P is that momentum. If the internal manifold has a covariantly constant spinor, then we smear the distributional pixel algebra generators with conformal Killing spinors on the two sphere and pick K, L to both be the zero mode, we get the $\mathcal{N} = 1$ SUSY algebra with 4-momentum $P_\mu = P(1, \pm\Omega_0)$[6].

The pixel SUSY algebra will have scalar charges corresponding to BPS states if the theory has larger supersymmetry or more non-compact dimensions. However, if the internal manifold has finite volume in Planck units[7] then the eigenvalues of the charge operators are bounded. It's easy to see that the bound corresponds to the point at which a state of that charge has a mass larger than the 4D Planck mass, so that it is really a black hole. Such black holes can be made in particle collisions. It is only in extreme limits of the discrete moduli, where the dimension of the pixel algebra goes to infinity, that we can describe "all" of these black hole states as elementary objects like D-branes or Kaluza-Klein modes of compactified particles. Indeed, such limits are always characterized by a small dimensionless parameter g^2 and the non-gravitational nature of the states is only valid for values of charge less that some inverse power of g^2.

The upshot of this discussion is that we know how to describe SUSY algebras and BPS states in the HST formalism. A dual string pair corresponds to taking two different limits of the discrete parameters that characterize an HST compactification, namely the pixel algebra. We can follow states between the two limits by following their conserved charges. In the two limits, the moduli become continuous parameters and we can use the usual arguments

group in which we take fluxes to infinity at fixed ratio. Complex structure moduli have to do with choices of subalgebras of the algebra of all $N \times N$ matrices in the space of sections of the holomorphic line bundle corresponding to the chosen Picard group element. We've seen in the example of the two torus, that such sub-algebras are characterized by rational fractions $\frac{k}{N}$, where k is a divisor of N. These parameters become continuous as $N \rightarrow \infty$. The example of tori shows how a similar phenomenon arises for spinor fuzzification. The number of Dirac eigenvalues below some bound is an integer, and jumps at discreet points in torus moduli space. We can cover all possibilities in the $N \rightarrow \infty$ limit, by choosing rational values for the moduli with a maximum denominator of order the bound N.

[6] The $\pm$ ambiguity arises from a reflection ambiguity in the conformal Killing spinor equation. It has to do with incoming and outgoing particles, and we will not discuss it further.

[7] Translation: the representation space of the pixel algebra has finite dimension for fixed N.

to compare the dual formulations of the theory. One of us (TB) has been guilty on many occasions of saying that dualities proved that there were lengths smaller than the Planck scale in string theory (since *e.g.* the weak coupling IIA string limit is a zero radius circle in M-theory). This argument is specious. Every calculable limit of string moduli space, as well as limits like F-theory, which are only partially calculable, depends on having a length scale much larger than the Planck scale of the non-compact dimensions, defined by the Einstein frame Lagrangian. The expansion parameter is always a power of this ratio of scales. This is the reason that the constraints of the Holographic Principle and the fundamentally discrete nature of moduli are not apparent in these expansions.

The discreteness of moduli has profound implications for cosmology. Much of the literature on string inspired cosmology, including many papers written by one of the authors (TB), uses moduli fields as ingredients in an inflationary cosmology. Coherent fields are, from the HST point of view, an approximate way of describing states with many particles. However, the particle horizon at early times is small, and the HST formalism only admits a finite number of particles in such a region. The entropy of the particle horizon in a pre-inflationary era is roughly

$$S = \frac{K}{\rho} \sim N^2 V_I,$$

where K is a geometrical factor that depends on the details of the early history of the universe, and ρ is the energy density in Planck units. V_I is the number of independent sections in the fuzzy spinor bundle over the internal space, and N^2 is the number of spinor harmonics on the fuzzy two sphere. When we make multi-particle states using the HST variables, the number of particles scales like $N^{1/2}$ if we require the particles to be roughly localizable[8]. For unification scale inflation we have $S \sim 10^{12}$ at most. Thus, the number of particles is of order

$$10^3 V_I^{-\frac{1}{4}}.$$

Thus, the $V_I \to \infty$ limit in which the internal geometry has approximately continuous moduli, conflicts with the requirement that four dimensional field theory be a good approximation to the dynamics of the inflaton. The term *cosmological moduli* is, within the HST formalism, an oxymoron.

2.2 Flux compactifications

There has been a lot of interest over the past decade in compactifications of string theory characterized by fluxes of p-form gauge fields through non-trivial p-cycles of the compactification manifold. We would like to conjecture that the corresponding HST compactification is obtained by replacing the Dirac operator by the *flux Dirac operator*

$$D_F = D + \sum F_i^{(p)} \Gamma_p,$$

where $F_i^{(p)}$ are the fluxes and Γ_p the antisymmetrized product of Dirac matrices, contracted into the vielbein. Spinors that are covariantly constant with respect to a generalized connection, depending on the fluxes, will give zero modes of D_F, which can be used to construct

[8] In order to use the conventional field theory calculation of inflationary fluctuations, we have to consider particles that are localizable on a scale much smaller than the horizon.

SUSY generators as above. The qualitative features of the above discussion of spinor fuzzification, are unchanged by the addition of fluxes. More quantitative details of this conjecture will be addressed in future work.

3 Fuzzy spheres in any dimension

The eigenvalues and eigen-sections of the Dirac operator on the n-sphere have been worked out, for example, in [7]. For n even the eigenvalues are[9]

$$\pm(M + \frac{n}{2}),$$

where M is a non-negative integer. The degeneracy of this eigenspace is

$$D_n(M) = \frac{2^{\frac{n}{2}}(n + M - 1)!}{M!(n-1)!}.$$

The eigensections are given in terms of Jacobi polynomials. For n odd we have eigenvalues

$$\pm(M + \frac{n}{2}),$$

with degeneracy

$$\frac{2^{\frac{n-1}{2}}(n + M - 1)!}{M!(n-1)!}.$$

In both cases, the large M behavior of $\Sigma_M \equiv \sum_{m\leq M} D_n(m)$ scales like M^n, so an eigenvalues cutoff on M combined with a finite dimensional representation of the quantum algebra of variables in the spinor bundle, will have an entropy with this scaling. This suggests that M be interpreted as proportional to the radius of the sphere in Planck units.

The maximal entropy of massless particles in a region of size R in $d-1$ dimensional space, subject to the constraint that they do not collapse to form a black hole with radius $\sim R$, scales like $R^{\frac{(d-1)(d-2)}{d}}$. Now imagine that our spinor bundle variables are arranged in a $K \times L$ matrix, with $K \sim L \sim M^{\frac{d-2}{2}}$. We again try to associate particles with blocks that are roughly $P \times P$ in size. The entropy of the factor Hilbert space generated by just those block variables is of order $PM^{\frac{d-2}{2}}$. Thus if $P \sim M^{\frac{(d-2)^2}{2d}}$ and $M \sim R$ in Planck units, we reproduce the particle entropy formula coming from black hole physics. The formula for P shows that P^2 can be interpreted as the dimension of the fuzzy spinor bundle on the $d-2$ sphere, with eigenvalue cutoff $M^* \sim M^{\frac{d-2}{d}}$. This generalizes the $M^{\frac{1}{2}}$ cutoff found in [8]. Following that reference we interpret this as the cutoff on the size of the longitudinal momentum $p(1,\Omega)$ in units of the inverse radius $\frac{1}{MM_P}$ of the causal diamond.

In the four dimensions the individual K, L, or P dimensional factor spaces carry irreducible representations of the rotation group. We have not found an analog of that factorization for general d. However, the formalism is completely rotation invariant, because the spaces of $K \times L$ and (roughly) $P \times P$ matrices are all spinor bundles with an eigenvalue cutoff for the Dirac operator. Thus, the variables of our quantum theory, both the full causal diamond algebra, and the sub-algebra that describes particle-like excitations, transform as representations of the Rotation group Spin$(d-1)$.

[9] We have multiplied the Dirac operator of [7] by i, to make it Hermitian.

4 Applications to Matrix Theory

Matrix theory is an approach to a non-perturbative construction of certain super-Poincare invariant models of string/M theory. It should be thought of as a discrete light-cone quantization (DLCQ) of the underlying theory, in which only particle states with discrete, positive longitudinal momenta are kept and the total longitudinal momentum is restricted to be a positive integer N. An elegant derivation of the Matrix Theory prescription from perturbative string theory has been given in [9], following work of [10, 11]. One realizes the compact null direction as an infinite boost of a small space-like circle and uses the duality between M-theory and IIA string theory to claim that the positive momentum states are all D0-branes. The light front theory needs the *non*-relativistic D0 brane action, and with enough SUSY, this is completely determined. For four or fewer compact dimensions, preserving at least 16 supercharges, the resulting theory is a well-defined quantum field theory. From the string theory point of view, the dimensions of the compact space are small in string units, if they are $O(1)$ in 11 dimensional Planck units, so we must do T-duality transformations (Fourier-Mukhai transformations in the case of $K3$ manifolds), to get to a frame where the physics is well understood.

For five dimensions one has to deal with the poorly understood Little String Theories [12] and for six or more compact dimensions that dual theory appears to require quantum gravity and does not achieve the objective of reducing quantum gravity to a non-gravitational problem. One of the present authors (TB) has emphasized before [13] that, although the Seiberg prescription is elegant and allows us to use results of perturbative string theory, there is no such thing as a *unique* DLCQ of M-theory. DLCQ is an approximation method, and *any* approximation that gets the right results in the $N \to \infty$ limit is acceptable. Fuzzy geometry will enable us to define M-theory for all supersymmetric compactifications in terms of the large N limit of a finite matrix quantum mechanics.

The matrix Lagrangian for Matrix Theory in 11 non-compact dimensions is

$$L = \mathrm{Tr}\ \big[\frac{1}{2R}\dot{X}^2 - \theta^T\dot{\theta} - \frac{R}{4}[X^i, X^j]^2 - R\theta^T\gamma_i[\theta, X^i]\big].$$

X^i is a $9 = 11 - 2$ dimensional real transverse vector of $N \times N$ matrices and θ is an $N \times N$ matrix of 16 component real Spin (9) spinors, on which the Dirac matrices γ_i act in the usual fashion. The $U(N)$ symmetry of the Lagrangian is a gauge symmetry, and the Super-Galilean group of the light front frame acts on the gauge invariant subspace of the the Hilbert space of this theory. The Lagrangian is written in 11 dimensional Planck units and the dimensionless parameter R is the radius in Planck units of the null longitudinal circle, which determines the quantization of longitudinal momentum. The Hamiltonian is simply proportional to R. In these units, the total momentum is N. The claim is that as N and R go to infinity at fixed ratio, the states which remain at finite energy are simply supergravitons in flat 11 dimensional space-time, and the scattering matrix of those excitations along the flat directions of the quantum potential approaches the S-matrix of 11 dimensional quantum supergravity, for all momenta.

When we compactify Matrix Theory on a torus, following Seiberg's prescription, the X^I for the compact directions become covariant derivatives in a $U(N)$ gauge theory on the T-dual torus. θ, for each value of the non-compact spinor index, becomes a section of the tensor product of the spinor bundle over the T-dual torus, with the principal $U(N)$ bundle. Similarly, when we compactify Matrix Theory on a $K3$ manifold, four of the X^I are replaced by covariant derivatives on the Fourier-Mukhai dual $\widetilde{K3}$. The result is the $U(N)(2,0)$ six dimensional CFT, which is the unique maximally supersymmetric UV completion of 5 dimensional SYM, compactified on S^1 times $\widetilde{K3}$ [14].

Our proposal for Matrix Theory compactification is to take the original Seiberg proposal, which naively takes the theory to a SYM theory on the dual of the compactification manifold, and replace that manifold by its spinor fuzzification. Thus, the X^I become covariant derivative operators in a bundle of N^2 spinor fields over the manifold[10], with a cutoff on the Dirac eigenvalue that is related to the size of the dual compactification manifold in Planck units. Each non-compact component of θ_a is a section of this bundle. Each non-compact X^I is a function on this manifold. That is to say, it is in the tensor product of the algebra of $N \times N$ matrices, with the 0-form subalgebra of the Clifford algebra of forms on the fuzzy manifold.

According to this proposal, Matrix Theory compactification in *any dimension* is a supersymmetric quantum mechanics of finite dimensional matrices. The only issue one has to deal with is whether the large N limit (with the size of the compact spinor bundle fixed) defines a finite, super-Poincare invariant S-matrix. This prescription is even applicable to $G2$ compactification, or compactification on a 7-torus. Indeed, it even allows us to define a finite N version of compactification on 8 or 9 dimensional manifolds. Presumably, in those cases, the large N limit of the scattering matrix fails to exist. We hope to come back to some examples of finite Matrix Theory compactifications in a future publication.

One question left open by this proposal is what we mean by "dual" in the general case. For tori and K3 manifolds this is clear. The authors of [15] suggested that for $CY3$-folds the relevant duality is mirror symmetry. Indeed, the problem Seiberg solved with T-duality was that the description in terms of $D0$-branes on the original manifold had a manifold whose size shrank to zero in string units. The string perturbation expansion breaks down, and sometimes the duality tells us how to describe the resulting limit in an exact way. The mirror dual of a zero volume $CY3$-fold is a $CY3$-fold at its conifold singularity. In fact, our discussion of fuzzy compactification and the holographic principle suggests that when the size of the manifold is of order 1 in Planck units, the approximation of continuous moduli breaks down. The manifold and it's mirror dual are just two different $O(1)$ values of the discrete moduli.

Greg Moore has suggested to us a strategy which would obviate the need for formulating a precise notion of dual to every compactification manifold. In all known examples, the Dirac-Ramond operator, with supersymmetric boundary conditions, has a spectrum invariant under dualities of string theory that preserve $g_S = 0$. An eigenvalue cutoff on the Dirac-Ramond operator, again leads to a finite dimensional spinor bundle, so perhaps this could be used as a definition of fuzzy compactifications of Matrix Theory.

[10] In the Matrix Theory Lagrangian, we recognize that the compact X^I are the *tangent space components* of covariant derivatives, $e_a^\mu D_\mu$, so that the flat scalar product is all that is necessary.

5 Conclusions

The Strong Holographic Principle implies that a finite area holographic screen corresponds to a finite dimensional approximation to the spinor bundle over the screen. Defining this approximation by a sharp cutoff on the spectrum of the Dirac operator, preserves all isometries of the manifold, as well as SUSY. It gives a rather precise definition of compactifications of the holographic space-time formalism, as well as compactifications of Matrix Theory. The latter always correspond to a quantum system with a finite number of variables. The only question that arises is whether the large N limit of the Matrix Theory scattering matrix converges to a Super-Poincare invariant answer.

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