arXiv:1106.1512v1 [math.DG] 8 Jun 2011

On Lie algebroids and Poisson algebras

Dennise García-Beltrán[†], José A. Vallejo[*†] and Yuriĭ Vorobjev[‡]

[†]Facultad de Ciencias, Universidad Autónoma de San Luis Potosí, México,

[‡]Departamento de Matemáticas, Universidad de Sonora, México.

Email: dennise.gb@alumnos.uaslp.edu.mx,jvallejo@fc.uaslp.mx,yurimv@guaymas.uson.mx

June 9, 2011

Abstract

We introduce and study a class of Lie algebroids associated to faithful modules which is motivated by the notion of cotangent Lie algebroids of Poisson manifolds. We also give a classification of transitive Lie algebroids and describe Poisson algebras by using the notions of algebroid and Lie connections.

1 Introduction

Let $(M, \{.,.\})$ be a Poisson manifold equipped with the bracket $\{.,.\}$, which is determined by a Poisson bivector $\mathbf{P} \in \Gamma\Lambda^2 TM$. It is well known that the cotangent bundle T^*M carries a natural Lie algebroid structure (see Section 2 for definitions), that is, on differential $1-$forms (sections of T^*M) the following bracket can be defined (see [2], Proposition 14.19),

$$[\![\alpha, \beta]\!] = i_{P\alpha}\mathrm{d}\beta - i_{P\beta}\mathrm{d}\alpha + \mathrm{d}(\beta(P\alpha)), \tag{1}$$

where $P : T^*M \to TM$ is the vector bundle morphism canonically induced by $\mathbf{P}$ through $\beta(P\alpha) = \mathbf{P}(\alpha, \beta)$. Moreover, the following Leibniz-like rule is satisfied, for any smooth function $f \in \mathcal{C}^\infty(M)$:

$$[\![\alpha, f\beta]\!] = f[\![\alpha, \beta]\!] + (P\alpha)(f) \cdot \beta.$$

That means that the corresponding anchor map is just P. Note that in the case when α, β in (1) are exact ($\alpha = \mathrm{d}f$ and $\beta = \mathrm{d}g$ for some $f, g \in \mathcal{C}^\infty(M)$), we have $\mathbf{P}(\alpha, \beta) = \{f, g\}$ and formula (1) reads:

$$[\![\mathrm{d}f, \mathrm{d}g]\!] = \mathrm{d}\{f, g\}.$$

Moreover, the bracket (1) of two closed forms is again closed.
In fact, the properties above characterize the Lie algebroid structure on the tangent bundle that comes from a Poisson bracket on the base manifold (see [6]). More precisely, given a Lie algebroid $(T^*M, [\![\cdot,\cdot]\!], \rho)$ on the tangent bundle T^*M, there exists a Poisson bracket $\{.,.\}$ on M such that $\rho = P$ if and only if the following conditions are satisfied:

(a) ρ is skew-symmetric, i.e. $\beta(\rho(\alpha)) = -\alpha(\rho(\beta))$, for all $\alpha, \beta \in \Omega^1(M)$.

(b) If $\alpha, \beta \in \Omega^1(M)$ are closed, then $[\![\alpha, \beta]\!]$ is also closed.

The aim of the present paper is to put the study of cotangent Lie algebroids of Poisson manifolds in an algebraic framework. In the first part (Section 3), we study and characterize a class of Lie algebroids with properties similar to (a), (b), which we call Lie algebroids of Poisson type. To this

*Corresponding author.

end, we will work in a slightly more general context than that of vector bundles over manifolds, using the notion of Lie-Rinehart algebra (see [9, 10]), also called a Lie pseudoalgebra (see [14, 17, 18, 19] and, particularly, [15] for many interesting remarks on the evolution of these notions and as a general reference), although we will continue using the denomination "Lie algebroid" for them (as they are the algebraic version of the geometric Lie algebroids). We give several examples illustrating the different situations that can appear.

In the second part (Sections 4 and 5), we deepen in the relationship between transitive algebroids and Poisson structures for a certain class of spaces, those of the form $\mathrm{Der}(\mathcal{A}) \oplus V$, where $\mathcal{A}$ is a commutative algebra and V an $\mathcal{A}-$module. We describe parametrizations of the transitive Lie algebroids on $\mathrm{Der}(\mathcal{A}) \oplus V$ following the techniques exposed in [20, 21], which are based on the use of a connection on a Lie algebroid. For completeness, we include a subsection in the preliminaries devoted to the topic of connections in an algebraic setting. Once the parametrization is given, we apply it to prove that a transitive algebroid endowed with a connection is isomorphic to one of the form $\mathrm{Der}(\mathcal{A}) \oplus V$. Finally, we obtain new classes of Poisson algebras on $\mathcal{A} \oplus V$ starting from Poisson algebras on $\mathcal{A}$.

2 Preliminaries

Throughout the paper, unless otherwise explicitly stated, $\mathcal{A}$ denotes an associative, commutative algebra with identity element $1_{\mathcal{A}}$, over a commutative ring $\mathcal{R}$ with identity element $1_{\mathcal{R}}$.

2.1 Derivations and connections in commutative algebras

In subsequent sections, we will need to introduce connections on an algebroid. In our algebraic setting, the most appropriate notion of connection is Koszul's one, which is given in terms of derivations.

Definition 2.1. A derivation of the algebra $\mathcal{A}$ over $\mathcal{R}$ is a map $X \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{A})$ satisfying the Leibniz rule

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g).$$

The set of derivations of $\mathcal{A}$ over $\mathcal{R}$ is denoted $\mathrm{Der}_{\mathcal{R}}(\mathcal{A})$ or simply $\mathrm{Der}(\mathcal{A})$ when there is no risk of confusion about the ring $\mathcal{R}$.

Remark 2.2. The definition just given can be extended to the case of derivations of $\mathcal{A}$ over $\mathcal{R}$ into an $\mathcal{A}-$module M. These are abelian groups morphisms $X : \mathcal{A} \to M$ satisfying the Leibniz rule above, and form an $\mathcal{A}-$module denoted $\mathrm{Der}_{\mathcal{R}}(\mathcal{A}, M)$ or simply $\mathrm{Der}(\mathcal{A}, M)$.

The set $\mathrm{Der}(\mathcal{A})$ has an $\mathcal{R}-$Lie algebra structure when endowed with the commutator of endomorphisms, given by $[X, Y] = X \circ Y - Y \circ X$.

Note also that, if $\mathcal{A}$ as an $\mathcal{R}-$module is faithful, then for every $X \in \mathrm{Der}(\mathcal{A})$ we have $X(1_{\mathcal{A}}) = 0$ and indeed $X(r) = 0$ for every $r \in \mathcal{R}$ viewed as a subalgebra of $\mathcal{A}$.

Definition 2.3. Let M be a unitary $\mathcal{A}-$module. A derivation law, or Koszul connection, on M is an $\mathcal{A}-$linear mapping $\nabla : \mathrm{Der}(\mathcal{A}) \to \mathrm{Hom}_{\mathcal{R}}(M, M)$ (the image of $X \in \mathrm{Der}(\mathcal{A})$ denoted ∇_X) such that

$$\nabla_X(f \cdot m) = X(f) \cdot m + f \cdot \nabla_X(m).$$

Not every $\mathcal{A}-$module M admits a connection in this sense, but it is easy to see that any free $\mathcal{A}-$module does. Of course, arbitrary $\mathcal{A}-$modules do not need to be free. So, in order to obtain a big enough class of modules for which we can guarantee the existence of a Koszul connection, we will make a brief digression on modules of differentials and Connes connections (see [3]).

Let $\Omega^1(\mathcal{A})$ be the $\mathcal{A}-$module defined by the kernel of the multiplication $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \to \mathcal{A}$. Define the map $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ by $\mathrm{d}a = 1 \otimes a - a \otimes 1$, which is a derivation of $\mathcal{A}$ over $\mathcal{R}$ with values into $\Omega^1(\mathcal{A})$.

It is clear from the definition that $\Omega^1(\mathcal{A}) = \mathrm{Span}_{\mathcal{A}}\{\mathrm{d}f : f \in \mathcal{A}\}$: Since the elements of $\Omega^1(\mathcal{A})$ lie in the kernel of the multiplication map, if $\sum a_j \otimes b_j \in \Omega^1(\mathcal{A})$, then $\sum a_j b_j = 0$ and therefore

$$\sum a_j \otimes b_j = \sum (a_j \otimes b_j - a_j b_j \otimes 1) = \sum a_j \mathrm{d} b_j.$$

In fact, $\Omega^1(\mathcal{A})$ is the submodule of $C^1(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ (the $1-$component of the differential algebra $C(\mathrm{Der}(\mathcal{A}), \mathcal{A})$ of Chevalley-Eilernberg cochains of the Lie algebra $\mathrm{Der}(\mathcal{A})$ with values in the $\mathrm{Der}(\mathcal{A})-$module $\mathcal{A}$) generated by the elements $\mathrm{d}f$, $f \in \mathcal{A}$ (see [5]). Note, in particular, that this implies $\Omega^1(\mathcal{A}) \subset \mathrm{Der}^*(\mathcal{A})$.

Definition 2.4. Let M be an $\mathcal{A}-$module. A Connes connection on M is an $\mathcal{A}-$linear map $\delta : M \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} M$ such that, for all $f \in \mathcal{A}, m \in M$,

$$\delta(fm) = f\delta(m) + \mathrm{d}a \otimes_{\mathcal{A}} m.$$

Remark 2.5. Connes' definition of a connection (see [3]) actually does not require that $\mathcal{A}$ be a commutative algebra. The definition goes back to a work by N. Katz [12].

Starting from a Connes connection, we can obtain a Koszul one. If $X \in \mathrm{Der}(\mathcal{A})$, then we define a right $\mathcal{A}-$linear pairing $\varphi : \mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{R}} \Omega^1(\mathcal{A}) \to \mathcal{A}$ by

$$(X, \sum a_j \mathrm{d} b_j) \mapsto \sum a_j X(b_j).$$

The Koszul connection ∇ associated to δ can be constructed as follows: for $X \in \mathrm{Der}(\mathcal{A})$, $\nabla_X \in \mathrm{Hom}_{\mathcal{R}}(M, M)$ is the map given by applying the connection δ and then contracting the $\Omega^1(\mathcal{A})$ component with φ. Thus, if $m \in M$ is such that $\delta(m) = \sum a_j \mathrm{d} b_j \otimes m_j$, for certain $a_j, b_j \in \mathcal{A}$ and $m_j \in M$, we have for each $f \in \mathcal{A}$ that $\delta(fm) = f \sum a_j \mathrm{d} b_j \otimes m_j + \mathrm{d}f \otimes m$, and

$$\begin{aligned} \nabla_X(fm) &= \sum \varphi(X, f a_j \mathrm{d} b_j) m_j + \varphi(X, df) m \\ &= f \sum a_j X(b_j) m_j + X(f) m \\ &= f \nabla_X(m) + X(f) m. \end{aligned}$$

A basic result obtained by J. Cuntz and D. Quillen (see [4]) is that Connes connections on an $\mathcal{A}-$module M are in bijective correspondence with $\mathcal{A}-$linear splittings of the natural action $\mathcal{A} \otimes_{\mathcal{R}} M \to M$. As a consequence, M admits a Connes connection if and only if it is projective.

As said earlier, we will need later on to work with (Koszul) connections, so we need conditions on $\mathcal{A}$ to assure their existence. From what we have seen, these connections exist on any $\mathcal{A}-$module M which is free or projective. Indeed, note that a free module is always projective, but there are projective modules which are not free. In the literature, there are several well-known conditions on $\mathcal{A}$ guaranteeing the projective character of M (for example, that $\mathcal{A}$ be semi-simple as a ring). When we talk of a connection on M, unless otherwise explicitly stated, we will mean that any one of these conditions is satisfied and that the connection is Koszul.

2.2 Lie algebroids

Definition 2.6. Let $\mathcal{F}$ be a faithful $\mathcal{A}$-module. A Lie algebroid is a triple $(\mathcal{F}, [\![\cdot,\cdot]\!], \rho)$, where $[\![\cdot,\cdot]\!]$ is a Lie bracket on $\mathcal{F}$ and $\rho : \mathcal{F} \to \mathrm{Der}(\mathcal{A})$ is a morphism of $\mathcal{A}$-modules, called the anchor map, such that:

$$[\![X, fY]\!] = f[\![X, Y]\!] + \rho(X)(f)Y,$$

for all $f \in \mathcal{A}$ and for all $X, Y \in \mathcal{F}$.

Remark 2.7. Sometimes, the condition that the anchor map be a morphism of Lie algebras is included in the definition of Lie algebroid. However, this fact is a consequence of the conditions in definition 2.6, as have been noted by J. C. Herz , Y. Kosmann-Schwarzbach, F. Magri and J. Grabowski among others (see [8, 14, 7]).

Definition 2.8. Let $(\mathcal{F}, [\![\cdot,\cdot]\!], \rho)$ and $(\mathcal{F}', [\![\cdot,\cdot]\!]', \rho')$ be Lie algebroids (over the same algebra $\mathcal{A}$ and the same ring $\mathcal{R}$). A morphism of Lie algebroids is a morphism of $\mathcal{A}$-modules $\phi : \mathcal{F} \to \mathcal{F}'$ such that

$$\rho' \circ \phi = \rho \quad \text{and} \quad \phi([\![X, Y]\!]) = [\![\phi(X), \phi(Y)]\!]',$$

for all $X, Y \in \mathcal{F}$.

Let us consider some examples. The first is the classical one.

Example 2.9. Let M be a manifold. Let $E \xrightarrow{\pi} M$ be a vector bundle over M and $\mathcal{F} = \Gamma(E)$ the $\mathcal{C}^\infty(M)$-module of sections of E (i.e. $\mathcal{A} = \mathcal{C}^\infty(M)$ and $\mathcal{R} = \mathbb{R}$). Then, the Lie algebroid structure on $\Gamma(E)$ is defined by a Lie bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(E)$ with an anchor map

$$q : \Gamma(E) \to \mathrm{Der}_{\mathbb{R}}\mathcal{C}^\infty(M) \cong \Gamma(TM),$$

such that for all $f \in \mathcal{C}^\infty(M)$ and for all $X, Y \in \Gamma(E)$:

1. $[\![X, fY]\!] = f[\![X, Y]\!] + q(X)(f)Y$,
2. $q(fX + Y) = fq(X) + q(Y)$.

In particular, if $E = T^*M$, then the Lie algebroid structure is given by the bracket (1) where the anchor is the Poisson mapping P. For $E = TM$ we have the trivial Lie algebroid, where $q = \mathrm{Id}_{TM}$.

Example 2.10. Consider the $\mathcal{R}$-algebra of dual numbers over $\mathcal{A}$,

$$\mathcal{A}' = \mathcal{A}[\theta] = \{x + y\theta : x, y \in \mathcal{A}, \theta^2 = 0\},$$

with the obvious operations. Clearly, $\mathcal{A}'$ is an $\mathcal{A}'$-module and we can endow it with the Lie algebra structure given by the bracket:

$$[\![x_1 + y_1\theta, x_2 + y_2\theta]\!] = (x_1y_2 - y_1x_2)\theta$$

for $x_1 + y_1\theta, x_2 + y_2\theta \in \mathcal{A}'$. Thus $(\mathcal{A}', [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid with anchor map

$$\begin{array}{rcl} \rho : \mathcal{A}' & \to & \mathrm{Der}(\mathcal{A}'), \\ x + y\theta & \longmapsto & \mathrm{ad}_x \end{array}$$

for $x + y\theta \in \mathcal{A}'$. Here $\mathrm{ad}_x(x_1 + y_1\theta) = [\![x, x_1 + y_1\theta]\!]$ is the adjoint map of $x = \mathrm{pr}_1(x + y\theta)$.

The next example will be relevant in Section 4.

Example 2.11. Consider the $\mathcal{A}$-module $\mathrm{Der}(\mathcal{A}) \oplus \mathcal{A}$. Denote by pr_1 the projection onto the first factor, $\mathrm{pr}_1 : \mathrm{Der}(\mathcal{A}) \oplus \mathcal{A} \to \mathrm{Der}(\mathcal{A})$, and define the following bracket:

$$[\![(D_1, a_1), (D_2, a_2)]\!] = ([D_1, D_2], D_1(a_2) - D_2(a_1))$$

for $(D_1, a_1), (D_2, a_2) \in \mathrm{Der}(\mathcal{A}) \oplus \mathcal{A}$, where $[D_1, D_2]$ is the commutator of endomorphisms. Then, $(\mathrm{Der}(\mathcal{A}) \oplus \mathcal{A}, [\![\cdot,\cdot]\!], \mathrm{pr}_1)$ is a Lie algebroid.

3 Lie algebroids of Poisson type

Definition 3.1. A Poisson algebra $(\mathcal{A}, \{\ ,\ \})$ is an associative algebra $\mathcal{A}$ together with a Lie bracket which is also a derivation for the product in $\mathcal{A}$, that is, there is an $\mathcal{R}-$bilinear operation $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ such that

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry),
2. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity),

3. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz identity),

for all $f, g, h \in \mathcal{A}$.

If $(\mathcal{A}, \{\ ,\ \})$ is a Poisson algebra, then we can define the adjoint map $\mathrm{ad} : \mathcal{A} \longrightarrow \mathrm{Der}(\mathcal{A})$ by

$$\mathrm{ad}_f(g) = \{f, g\},$$

for all $g \in \mathcal{A}$. Then, extending the mapping $df \longmapsto ad_f$ by linearity, we get a morphism $\rho : \Omega^1(\mathcal{A}) \longrightarrow \mathrm{Der}(\mathcal{A})$ uniquely defined by

$$\rho(\mathrm{d}f) = \mathrm{ad}_f, \quad \forall f \in \mathcal{A}. \tag{2}$$

Sometimes (by analogy with Poisson manifolds), ad_f is referred to as the Hamiltonian vector field corresponding to $f \in \mathcal{A}$, and denoted by X_f (we will use this notation and terminology later in Section 5).
Also, given an $\alpha \in \Omega^1(\mathcal{A})$, we can define $\mathrm{d}\alpha$ through the usual formula

$$\mathrm{d}\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]),$$

for all $X, Y \in \mathrm{Der}(\mathcal{A})$.

Theorem 3.2. *If $(\mathcal{A}, \{\ ,\ \})$ is a Poisson algebra, then $(\Omega^1(\mathcal{A}), [\ ,\ ], \rho)$ is a Lie algebroid with anchor map ρ defined by* (2) *and the Lie bracket*

$$[\![\alpha, \beta]\!] = \iota_{\rho(\alpha)}\mathrm{d}\beta - \iota_{\rho(\beta)}\mathrm{d}\alpha + \mathrm{d}(\beta(\rho(\alpha))). \tag{3}$$

Proof. Let $f \in \mathcal{A}$ and $\alpha, \beta \in \Omega^1(\mathcal{A})$. Then,

$$\begin{aligned}
[\![\alpha, f\beta]\!] &= \iota_{\rho(\alpha)}\mathrm{d}(f\beta) - \iota_{\rho(f\beta)}\mathrm{d}\alpha + \mathrm{d}(f\beta(\rho(\alpha)))\\
&= \iota_{\rho(\alpha)}(\mathrm{d}f \wedge \beta) + f\iota_{\rho(\alpha)}\mathrm{d}\beta - f\iota_{\rho(\beta)}\mathrm{d}\alpha + \mathrm{d}f\beta(\rho(\alpha)) + f\mathrm{d}(\beta(\rho(\alpha)))\\
&= f[\![\alpha, \beta]\!] + \iota_{\rho(\alpha)}(\mathrm{d}f \wedge \beta) + \mathrm{d}f\beta(\rho(\alpha))\\
&= f[\![\alpha, \beta]\!] + \mathrm{d}f(\rho(\alpha))\beta - \beta(\rho(\alpha))\mathrm{d}f + \mathrm{d}f\beta(\rho(\alpha))\\
&= f[\![\alpha, \beta]\!] + \rho(\alpha)(f)\beta.
\end{aligned}$$

The skew-symmetry and the $\mathcal{R}$-bilinearity of $[\![\cdot, \cdot]\!]$ are obvious. To verify the Jacobi identity, let us consider a system $\{\mathrm{d}f_i\}_{i\in I}$ of generators of $\Omega^1(\mathcal{A})$, so that for arbitrary elements $\alpha, \beta, \gamma \in \Omega^1(\mathcal{A})$, we have $\alpha = g_i\mathrm{d}f_i, \beta = h_j\mathrm{d}f_j, \gamma = m_k\mathrm{d}f_k$ for some $g_i, h_j, m_k \in \mathcal{A}$. Note that for any $f, g \in \mathcal{A}$, we have

$$[\![\mathrm{d}f, \mathrm{d}g]\!] = \iota_{\rho(\mathrm{d}f)}\mathrm{d}(\mathrm{d}g) - \iota_{\rho(\mathrm{d}g)}\mathrm{d}(\mathrm{d}f) + \mathrm{d}(\mathrm{d}g(\rho(\mathrm{d}f))) = \mathrm{d}\{f, g\}.$$

Then,

$$\begin{aligned}
[\![\alpha, \beta]\!] &= [\![g_i\mathrm{d}f_i, h_j\mathrm{d}f_j]\!]\\
&= h_jg_i[\![\mathrm{d}f_i, \mathrm{d}f_j]\!] - h_j\rho(\mathrm{d}f_j)(g_i)\mathrm{d}f_i + g_i\rho(\mathrm{d}f_i)(h_j)\mathrm{d}f_j\\
&= g_ih_j\mathrm{d}\{f_i, f_j\} - h_j\{f_j, g_i\}\mathrm{d}f_i + g_i\{f_i, h_j\}\mathrm{d}f_j.
\end{aligned}$$

A straightforward computation gives

$$\begin{aligned}
[\![[\![\alpha, \beta]\!], \gamma]\!] = \ & m_kh_jg_i\mathrm{d}\{\{f_i, f_j\}, f_k\} - m_k\{f_k, h_jg_i\}\mathrm{d}\{f_i, f_j\} + h_jg_i\{\{f_i, f_j\}, m_k\}\mathrm{d}f_k\\
&- m_kh_j\{f_j, g_i\}\mathrm{d}\{f_i, f_k\} + m_k\{f_k, h_j\{f_j, g_i\}\}\mathrm{d}f_i - h_j\{f_j, g_i\}\{f_i, m_k\}\mathrm{d}f_k\\
&+ m_kg_i\{f_i, h_j\}\mathrm{d}\{f_j, f_k\} - m_k\{f_k, g_i\{f_i, h_j\}\}\mathrm{d}f_j + g_i\{f_i, h_j\}\{f_j, m_k\}\mathrm{d}f_k,
\end{aligned}$$

It follows from here that the cyclic sum $\circlearrowleft [\![[\![\alpha, \beta]\!], \gamma]\!]$ is zero,

$$\begin{aligned}
\circlearrowleft [\![[\![\alpha, \beta]\!], \gamma]\!] &= m_kh_jg_i\mathrm{d} \circlearrowleft \sum(\{\{f_i, f_j\}, f_k\})\\
&+ \circlearrowleft ((-m_k\{f_k, h_jg_i\} + g_im_k\{f_k, h_j\} + h_jm_k\{f_k, g_i\})\mathrm{d}\{f_i, f_j\})\\
&+ \circlearrowleft ((h_jg_i\{\{f_i, f_j\}, m_k\} + h_j\{f_j, g_i\{f_i, m_k\}\} - h_j\{f_j, g_i\}\{f_i, m_k\}\\
&- g_i\{f_i, h_j\{f_j, m_k\}\} + g_i\{f_i, h_j\}\{f_j, m_k\})\mathrm{d}f_k)\\
&= \circlearrowleft g_ih_j((\{\{f_i, f_j\}, m_k\} + \{f_j, \{f_i, m_k\}\} - \{f_i\{f_j, m_k\}\})\mathrm{d}f_k) = 0.
\end{aligned}$$

□

Example 3.3. Let $(M, \{\ ,\ \})$ be a Poisson manifold, i.e., $(\mathcal{C}^\infty(M), \{\ ,\ \})$ is a Poisson algebra. Using Theorem 2.9 we have that $\Omega^1(\mathcal{C}^\infty(M)) = \Omega^1(M)$ is a Lie algebroid with anchor map $\rho = P$ (defined by (2)) and the bracket (1):

$$[\![\alpha, \beta]\!] = \iota_{\rho(\alpha)}\mathrm{d}\beta - \iota_{\rho(\beta)}\mathrm{d}\alpha + \mathrm{d}(\beta(\rho(\alpha))),$$

for all $\alpha, \beta \in \Omega^1(M)$. The proof is based on the fact that there exist a finite subset $\{g_1, ..., g_k\} \subset C^\infty(M)$ (with $k \leq 2\dim M + 1$) such that the $C^\infty(M)-$module $\Omega^1(M)$ is spanned by $\{\mathrm{d}g_1, ..., \mathrm{d}g_k\}$. This, in turn, is a consequence of Whitney's embedding theorem and the fact that the sheaf of germs of smooth functions is soft (see [1]).

Theorem 3.2 tells us that given a Poisson algebra we have a Lie algebroid canonically associated to it. We are interested now in the reciprocal: When does a Lie algebroid $(\Omega^1(\mathcal{A}), [\![\cdot, \cdot]\!], \rho)$ determine a Poisson structure on $\mathcal{A}$?.
Given a Lie algebroid $(\Omega^1(\mathcal{A}), [\![\cdot, \cdot]\!], \rho)$, for any $f, g \in \mathcal{A}$ we can define

$$\{f, g\} = (\mathrm{d}g)(\rho(\mathrm{d}f)) = (\rho(\mathrm{d}f))(g). \tag{4}$$

The bracket $\{\ ,\ \}$ defined in this way is clearly $\mathcal{R}$-bilinear and satisfies the Leibniz rule

$$\begin{aligned} \{fg, h\} &= (\rho(\mathrm{d}(fg)))(h) = (\rho(f\mathrm{d}g + g\mathrm{d}f))(h) = f(\rho(\mathrm{d}g))(h) + g(\rho(\mathrm{d}f))(h) \\ &= f\{g, h\} + g\{f, h\}, \end{aligned}$$

for all $f, g, h \in \mathcal{A}$.
Thus, in order to get a Poisson structure on $\mathcal{A}$ we only need to take care of the skew-symmetry and the Jacobi identity.

Definition 3.4. The anchor map ρ of a Lie algebroid $(\Omega^1(\mathcal{A}), [\![\cdot, \cdot]\!], \rho)$ is said to be skew-symmetric if

$$\alpha(\rho(\beta)) = -\beta(\rho(\alpha))$$

for all $\alpha, \beta \in \Omega^1(\mathcal{A})$.

If we assume that the anchor map is skew-symmetric, then the new operation defined by (4) is also skew-symmetric. Now, let us turn our attention to the Jacobi identity.

Theorem 3.5. *Let $(\Omega^1(\mathcal{A}), [\![\cdot, \cdot]\!], \rho)$ be a Lie algebroid, and define $Q \in \Lambda^2(\mathrm{Der}(\mathcal{A}))$ by $Q(\mathrm{d}f, \mathrm{d}g) = \mathrm{d}g(\rho(\mathrm{d}f))$ for $f, g \in \mathcal{A}$. The following conditions are equivalent:*

(i) The bracket $\{\ ,\ \}$ defined by $\{f, g\} = \rho(\mathrm{d}f)(g)$ satisfies the Jacobi identity.

(ii) $[Q, Q]_{SN} = 0$

(iii) $[\rho(\mathrm{d}f), \rho(\mathrm{d}g)] = \rho(\mathrm{d}\{f, g\})$

Here, $[\cdot, \cdot]_{SN}$ denotes the Schouten-Nijenhuis bracket on multiderivations [13].

Proof. It is based on the computation of $\frac{1}{2}[\mathcal{L}_Q, \iota_Q] = \frac{1}{2}\iota_{[Q,Q]_{SN}}$, where on the left-hand side we have the graded commutator of derivations on the graded algebra $\Lambda(\mathcal{A})$. Due to the $\mathcal{A}$-linearity we only need to apply this operator to basis elements. The direct application of the definition of the operators gives

$$\begin{aligned} \tfrac{1}{2}[\mathcal{L}_Q, \iota_q](\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) &= \tfrac{1}{2}(\mathcal{L}_Q\iota_Q - (-1)^2\iota_Q\mathcal{L}_Q)(\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) \\ &= \tfrac{1}{2}(\iota_Q \circ \mathrm{d} \circ \iota_Q + \iota_Q \circ \mathrm{d} \circ \iota_Q)(\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h) \\ &= \iota_Q \circ \mathrm{d}(\iota_Q(\mathrm{d}f \wedge \mathrm{d}g)\mathrm{d}h + \iota_Q(\mathrm{d}g \wedge \mathrm{d}h)\mathrm{d}f + \iota_Q(\mathrm{d}h \wedge \mathrm{d}f)\mathrm{d}g) \\ &= \iota_Q(dQ(\mathrm{d}f, \mathrm{d}g) \wedge \mathrm{d}h + \mathrm{d}Q(\mathrm{d}g, \mathrm{d}h) \wedge \mathrm{d}f + \mathrm{d}Q(\mathrm{d}h, \mathrm{d}f) \wedge \mathrm{d}g) \\ &= Q(\mathrm{d}Q(\mathrm{d}f, \mathrm{d}g), \mathrm{d}h) + Q(\mathrm{d}Q(\mathrm{d}g, \mathrm{d}h), \mathrm{d}f) + Q(\mathrm{d}Q(\mathrm{d}h, \mathrm{d}f), \mathrm{d}g). \end{aligned}$$

The equivalence of the conditions (i) and (ii) follows from the following identities

$$
\begin{aligned}
& Q(\mathrm{d}Q(\mathrm{d}f,\mathrm{d}g),\mathrm{d}h)+Q(\mathrm{d}Q(\mathrm{d}g,\mathrm{d}h),\mathrm{d}f)+Q(\mathrm{d}Q(\mathrm{d}h,\mathrm{d}f),\mathrm{d}g)\\
= & \{Q(\mathrm{d}f,\mathrm{d}g),h\}+\{Q(\mathrm{d}g,\mathrm{d}h),f\}+\{Q(\mathrm{d}h,\mathrm{d}f),g\}\\
= & \{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}.
\end{aligned}
$$

On the other hand, this term can be computed as follows

$$
\begin{aligned}
& -Q(\mathrm{d}h,\mathrm{d}Q(\mathrm{d}f,\mathrm{d}g))+Q(\mathrm{d}f,\mathrm{d}Q(\mathrm{d}h,\mathrm{d}g))-Q(\mathrm{d}Q(\mathrm{d}f,\mathrm{d}h),\mathrm{d}g)\\
= & -\rho(\mathrm{d}h)(Q(\mathrm{d}f,\mathrm{d}g))+\rho(\mathrm{d}f)(Q(\mathrm{d}h,\mathrm{d}g))-\rho(\mathrm{d}Q(\mathrm{d}f,\mathrm{d}h)(g)\\
= & -\rho(\mathrm{d}h)(\rho(\mathrm{d}f)(g))+\rho(\mathrm{d}f)(\rho(\mathrm{d}h)(g))-\rho(\mathrm{d}\{f,h\})(g)\\
= & [\rho(\mathrm{d}f),\rho(\mathrm{d}h)](g)-\rho(\mathrm{d}\{f,h\})(g)\\
= & ([\rho(\mathrm{d}f),\rho(\mathrm{d}h)]-\rho(\mathrm{d}\{f,h\}))(g).
\end{aligned}
$$

But, from $-\frac{1}{2}\iota_{[Q,Q]_{SN}}(\ ,\mathrm{d}f,\mathrm{d}h)(\mathrm{d}g)=\frac{1}{2}[Q,Q]_{SN}(\mathrm{d}f,\mathrm{d}g,\mathrm{d}h)$, we have the equality

$$
\left(-\frac{1}{2}\iota_{[Q,Q]_{SN}}(\ ,\mathrm{d}f,\mathrm{d}h)\right)(g)=([\rho(\mathrm{d}f),\rho(\mathrm{d}h)]-\rho(\mathrm{d}\{f,h\}))(g),
$$

which proves the equivalence between items (ii) and (iii). □

This result motivate the following definition.

Definition 3.6. A Lie algebroid $(\Omega^1(\mathcal{A}),[\![\cdot,\cdot]\!],\rho)$ is of Poisson type if:

(1) The anchor ρ is skew-symmetric,

(2) One of the equivalent conditions (ii)-(iii) in Theorem 3.5 holds.

In other words, a Lie algebroid $(\Omega^1(\mathcal{A}),[\![\cdot,\cdot]\!],\rho)$ is of Poisson type if it determines a Poisson structure on the algebra $\mathcal{A}$.

Another important issue for us is to determine the form of the bracket of a Lie algebroid of Poisson type. We know that the classical example of Poisson manifolds leads to brackets of the type (1). The following result characterizes the class of such algebroids.

Proposition 3.7. *If $(\Omega^1(\mathcal{A}),[\![\cdot,\cdot]\!],\rho)$ is a Lie algebroid for which ρ is skew-symmetric and $[\![\mathrm{d}f,\mathrm{d}g]\!]=\mathrm{d}\{f,g\}$, then the bracket of the Lie algebroid is of the form*

$$
[\![\alpha,\beta]\!]=\iota_{\rho(\alpha)}\mathrm{d}\beta-\iota_{\rho(\beta)}\mathrm{d}\alpha+\mathrm{d}(\beta(\rho(\alpha))).
$$

Proof. Note first that, if $\alpha=\mathrm{d}f$ and $\beta=\mathrm{d}g$ for some $f,g,\in\mathcal{A}$, then

$$
\iota_{\rho(\alpha)}d\beta-\iota_{\rho(\beta)}\mathrm{d}\alpha+\mathrm{d}(\beta(\rho(\alpha)))=\mathrm{d}(\mathrm{d}g(\rho(\mathrm{d}f)))=\mathrm{d}\{f,g\}=[\![\mathrm{d}f,\mathrm{d}g]\!]=[\![\alpha,\beta]\!].
$$

Since every element of $\Omega^1(\mathcal{A})$ is a linear combination of elements of the form $\mathrm{d}f_i$ ($f_i\in\mathcal{A}$), it is enough to prove the statement for $\alpha=f_1\mathrm{d}g_1$ and $\beta=f_2\mathrm{d}g_2$:

$$
\begin{aligned}
&\iota_{\rho(\alpha)}d\beta-\iota_{\rho(\beta)}\mathrm{d}\alpha+\mathrm{d}(\beta(\rho(\alpha)))\\
&\quad=\iota_{\rho(f_1\mathrm{d}g_1)}\mathrm{d}f_2\wedge\mathrm{d}g_2-\iota_{\rho(f_2\mathrm{d}g_2)}\mathrm{d}f_1\wedge\mathrm{d}g_1+f_2\mathrm{d}(\mathrm{d}g_2(f_1\rho(\mathrm{d}g_1)))+\mathrm{d}f_2\mathrm{d}g_2(f_1\rho(\mathrm{d}g_1))\\
&\quad=\mathrm{d}f_2(\rho(f_1\mathrm{d}g_1))\mathrm{d}g_2-\mathrm{d}g_2(\rho(f_1\mathrm{d}g_1))\mathrm{d}f_2-\mathrm{d}f_1(\rho(f_2\mathrm{d}g_2))\mathrm{d}g_1\\
&\qquad+\mathrm{d}g_1(\rho(f_2\mathrm{d}g_2))\mathrm{d}f_1+f_2\mathrm{d}(f_1\mathrm{d}g_2(\rho(\mathrm{d}g_1)))+f_1\mathrm{d}f_2\mathrm{d}g_2(\rho(\mathrm{d}g_1))\\
&\quad=f_1\mathrm{d}f_2(\rho(\mathrm{d}g_1))\mathrm{d}g_2-f_1\mathrm{d}g_2(\rho(\mathrm{d}g_1))\mathrm{d}f_2-f_2\mathrm{d}f_1(\rho(\mathrm{d}g_2))\mathrm{d}g_1+f_2\mathrm{d}g_1(\rho(\mathrm{d}g_2))\mathrm{d}f_1\\
&\qquad+f_1f_2\mathrm{d}(\mathrm{d}g_2(\rho(\mathrm{d}g_1)))+f_v\mathrm{d}g_2(\rho(\mathrm{d}g_1))\mathrm{d}f_1+f_1\mathrm{d}g_2(\rho(\mathrm{d}g_1))\mathrm{d}f_2\\
&\quad=f_1\mathrm{d}f_2(\rho(\mathrm{d}g_1))\mathrm{d}g_2-f_2\mathrm{d}f_1(\rho(\mathrm{d}g_2))\mathrm{d}g_1+f_2\mathrm{d}g_1(\rho(\mathrm{d}g_2))\mathrm{d}f_1\\
&\qquad+f_1f_2\mathrm{d}\{g_1,g_2\}-f_2\mathrm{d}g_1(\rho(\mathrm{d}g_2))\mathrm{d}f_1\\
&\quad=f_1\mathrm{d}f_2(\rho(\mathrm{d}g_1))\mathrm{d}g_2-f_2\mathrm{d}f_1(\rho(\mathrm{d}g_2))\mathrm{d}g_1+f_1f_2[\![\mathrm{d}g_1,\mathrm{d}g_2]\!]\\
&\quad=[\![f_1\mathrm{d}g_1,f_2\mathrm{d}g_2]\!]\\
&\quad=[\![\alpha,\beta]\!]
\end{aligned}
$$

□

The hypothesis of this Proposition can be reformulated in an alternative way.

Proposition 3.8. *If* $(\Omega^1(\mathcal{A}), [\![\cdot,\cdot]\!], \rho)$ *is a Lie algebroid with anchor* ρ *skew-symmetric, then the following assertions are equivalent:*

(i) $[\![\mathrm{d}f, \mathrm{d}g]\!] = \mathrm{d}(\mathrm{d}g(\rho(\mathrm{d}f))) = \mathrm{d}\{f, g\}$ *for all* $f, g \in \mathcal{A}$,

(ii) $\mathrm{d}\alpha = 0 = \mathrm{d}\beta$ *implies* $\mathrm{d}[\![\alpha, \beta]\!] = 0$.

Proof. First, let us assume that item (i) holds. By Proposition 3.7 we know that the bracket is of the form

$$[\![\alpha, \beta]\!] = \iota_{\rho(\alpha)}\mathrm{d}\beta - \iota_{\rho(\beta)}\mathrm{d}\alpha + \mathrm{d}(\beta(\rho(\alpha))),$$

and hence condition (ii) holds,

$$\mathrm{d}[\![\alpha, \beta]\!] = \mathrm{d}(\iota_{\rho(\alpha)}\mathrm{d}\beta - \iota_{\rho(\beta)}\mathrm{d}\alpha + \mathrm{d}(\beta(\rho(\alpha)))) = \mathrm{d}^2(\beta(\rho(\alpha))) = 0,$$

whenever $d\alpha = 0 = d\beta$.
Now let us assume that condition (ii) holds. Define $C(\alpha, \beta) = [\![\alpha, \beta]\!] - \mathrm{d}(\beta(\rho(\alpha)))$ for $\alpha, \beta \in \Omega^1(\mathcal{A})$. Notice that C is skew-symmetric and

$$\mathrm{d}C(\alpha, \beta) = \mathrm{d}([\![\alpha, \beta]\!]) - \mathrm{d}^2(\beta(\rho(\alpha)) = 0,$$

for any closed α and β.
Let us evaluate the following expression

$$\begin{aligned} C(\mathrm{d}f, h\mathrm{d}h) &= [\![\mathrm{d}f, h\mathrm{d}h]\!] - \mathrm{d}(h\mathrm{d}h(\rho(\mathrm{d}f))) \\ &= h[\![\mathrm{d}f, \mathrm{d}h]\!] + \rho(\mathrm{d}f)(\mathrm{d}h)\mathrm{d}h - \mathrm{d}h(\mathrm{d}h(\rho(\mathrm{d}f))) - h\mathrm{d}(h\mathrm{d}h(\rho(\mathrm{d}f))) \\ &= h[\![\mathrm{d}f, \mathrm{d}h]\!] - h\mathrm{d}(h\mathrm{d}h(\rho(\mathrm{d}f))) \\ &= hC(\mathrm{d}f, \mathrm{d}h). \end{aligned}$$

Now, applying the operator d and taking into account that $h\mathrm{d}h$ is closed, we get

$$0 = \mathrm{d}C(\mathrm{d}f, h\mathrm{d}h) = \mathrm{d}(hC(\mathrm{d}f, \mathrm{d}h)) = \mathrm{d}h \wedge C(\mathrm{d}f, \mathrm{d}h) + h\mathrm{d}C(\mathrm{d}f, \mathrm{d}h) = \mathrm{d}h \wedge C(\mathrm{d}f, \mathrm{d}h).$$

Interchanging the roles of f and h, we have

$$0 = \mathrm{d}C(\mathrm{d}h, f\mathrm{d}f) = \mathrm{d}f \wedge C(\mathrm{d}h, \mathrm{d}f) = -\mathrm{d}f \wedge C(\mathrm{d}f, \mathrm{d}h).$$

These relations imply that $\mathrm{d}f$, $\mathrm{d}h$ and $C(\mathrm{d}f, \mathrm{d}h)$ are linearly dependent for arbitrary f and h. In particular, if $\mathrm{d}f$ and $\mathrm{d}h$ are linearly independent, then $C(\mathrm{d}f, \mathrm{d}h) = 0$, and hence

$$[\![\mathrm{d}f, \mathrm{d}h]\!] = \mathrm{d}(\mathrm{d}h(\rho(\mathrm{d}f))) = \mathrm{d}\{f, h\}.$$

□

Let us summarize these results.

Theorem 3.9. *Let* $(\Omega^1(\mathcal{A}), [\![\cdot,\cdot]\!], \rho)$ *be a Lie algebroid. Then, there is a Poisson algebra structure* $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *on* $\mathcal{A}$ *such that*

$$\rho(\mathrm{d}f) = \mathrm{ad}_f, \quad \forall f \in \mathcal{A},$$

if and only if:

(a) ρ *is skew-symmetric*

(b) One of the following conditions holds:

(1) $\mathrm{d}\alpha = 0 = \mathrm{d}\beta$ *implies* $\mathrm{d}[\![\alpha, \beta]\!] = 0$ *;*

(2) $[\![\mathrm{d}f, \mathrm{d}g]\!] = \mathrm{d}(\mathrm{d}g(\rho(\mathrm{d}f)))$ *for all* $f, g \in \mathcal{A}$.

Under these conditions, the Lie bracket $[\![\cdot,\cdot]\!]$ *is reconstructed from* ρ *by the formula*

$$[\![\alpha, \beta]\!] = \iota_{\rho(\alpha)}\mathrm{d}\beta - \iota_{\rho(\beta)}\mathrm{d}\alpha + \mathrm{d}(\beta(\rho(\alpha))).$$

Of course, the basic example of this situation is the cotangent Lie algebroid of a symplectic manifold. A Lie algebroid structure $(\Omega^1(M), [\![\cdot,\cdot]\!], \rho)$ induces a Poisson bracket on $\mathcal{C}^\infty(M)$ if and only if ρ is skew-symmetric and, whenever $\mathrm{d}\alpha = 0 = \mathrm{d}\beta$, then $\mathrm{d}[\![\alpha, \beta]\!] = 0$. For this kind of examples, the property $[\![df, dg]\!] = d\{f, g\}$ follows directly from the injectivity of the anchor map and the fact that it is a Lie algebra morphism: $\rho[\![df, dg]\!] = [\![\rho(df), \rho(dg)]\!] = \rho(d\{f, g\})$. Let us consider the following example, where the anchor map is not injective but the property $[\![df, dg]\!] = d\{f, g\}$ still holds.

Example 3.10. Let $\mathcal{R} = \mathbb{R}$ and $\mathcal{A} = \mathbb{R}[x^1, x^2, x^3]$. Then, $\mathrm{Der}(\mathcal{A}) = \mathrm{Span}\{\partial_1, \partial_2, \partial_2\}$ and $\Omega^1(\mathcal{A}) = \mathrm{Span}\{dx^1, dx^2, dx^3\}$. Define the following bracket

$$\begin{aligned}[\![p_i \mathrm{d}x^i, q_j \mathrm{d}x^j]\!] = &((p_1(\partial_2 + \partial_3) + p_2(-\partial_1 + \partial_3) + p_3(-\partial_1 - \partial_2))(q_i) \\ &- (q_1(\partial_2 + \partial_3) + q_2(-\partial_1 + \partial_3) + q_3(-\partial_1 - \partial_2))(p_i))\,\mathrm{d}x^i,\end{aligned}$$

and the anchor map as

$$\begin{array}{rccl}\rho: & \Omega^1(\mathcal{A}) & \to & \mathrm{Der}(\mathcal{A}) \\ & p_i \mathrm{d}x^i & \mapsto & -(p_2 + p_3)\partial_1 + (p_1 - p_3)\partial_2 + (p_1 + p_2)\partial_3.\end{array}$$

Note that the matrix representation of ρ relative to the given basis in $\mathrm{Der}(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ is

$$\rho = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Therefore, ρ is skew-symmetric and of rank 2 (ρ is not injective). A long but straightforward computation shows that $(\Omega^1(\mathcal{A}), [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid.

Let us show that this Lie algebroid is of Poisson type by checking the property $[\![df, dg]\!] = d\{f, g\}$. We have for $p, q \in \mathcal{A}$:

$$\begin{aligned}\{p, q\} &= \mathrm{d}(\rho(\mathrm{d}p)(q)) = \mathrm{d}\left(\rho\left(\partial_i p \mathrm{d}x^i\right)(q)\right) \\ &= \mathrm{d}\left(-(\partial_2 p + \partial_3 p)\partial_1 q + (\partial_1 p - \partial_3 p)\partial_2 q + (\partial_1 p + \partial_2 p)\partial_3 q\right).\end{aligned}$$

The $\mathrm{d}x^1$ factor in the expansion of this expression (the other cases are similar) is:

$$\begin{aligned}&- (\partial_2 p + \partial_3 p)\partial^2_{11} q + (\partial_1 p - \partial_3 p)\partial^2_{12} q + (\partial_1 p + \partial_2 p)\partial^2_{13} q \\ &- (\partial^2_{12} p + \partial^2_{13} p)\partial_1 q + (\partial^2_{11} p - \partial^2_{13} p)\partial_2 q + (\partial^2_{11} p + \partial^2_{12} p)\partial_3 q.\end{aligned}$$

On the other hand, the Lie algebroid bracket is

$$\begin{aligned}[\![\mathrm{d}p, \mathrm{d}q]\!] &= [\![\partial_i p \mathrm{d}x^i, \partial_j q \mathrm{d}x^j]\!] \\ &= \rho(\mathrm{d}p)(\partial_k q) - \rho(\mathrm{d}q)(\partial_k p))\mathrm{d}x^k.\end{aligned}$$

For $k = 1$, we compute the coefficient of $\mathrm{d}x^1$:

$$\begin{aligned}&- (\partial_2 p + \partial_3 p)\partial^2_{11} q + (\partial_1 p - \partial_3 p)\partial^2_{12} q + (\partial_1 p + \partial_2 p)\partial^2_{13} q \\ &+ (\partial_2 q + \partial_3 q)\partial^2_{11} p - (\partial_1 q - \partial_3 q)\partial^2_{12} p - (\partial_1 q + \partial_2 q)\partial^2_{13} p,\end{aligned}$$

which is the same as above. Thus, $(\Omega^1(\mathcal{A}), [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid of Poisson type and its bracket is just given by formula (3).

Remark 3.11. The anchor map of the cotangent Lie algebroid is not injective in general. It is only true in the symplectic case.

Finally, let us consider an example of Lie algebroid of Poisson type whose bracket does not have the form (3).

Example 3.12. Let $\mathcal{R} = \mathbb{R}$ and $\mathcal{A} = \mathbb{R}[x^1, x^2, x^3]$. Define the structure of an Abelian Lie algebra with generators $\{\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\}$,

$$[\![\mathrm{d}x^i, \mathrm{d}x^j]\!] = 0, \quad i, j \in \{1, 2, 3\}.$$

Extending by $\mathcal{R}$-bilinearity and the Leibniz identity gives

$$[\![p_i \mathrm{d}x^i, q_j \mathrm{d}x^j]\!] = p_i \rho(\mathrm{d}x^i)(p_j)\mathrm{d}x^j - q_j \rho(\mathrm{d}x^j)(p_i)\mathrm{d}x^i.$$

Next, let us think of the following skew-symmetric morphism of $\mathcal{A}$-modules as the anchor map:

$$\begin{array}{rlcl} \rho: & \Omega^1(\mathcal{A}) & \to & \mathrm{Der}(\mathcal{A}), \\ & \mathrm{d}x^1 & \mapsto & x^3\partial_2, \\ & \mathrm{d}x^2 & \mapsto & -x^3\partial_1, \\ & \mathrm{d}x^3 & \mapsto & 0. \end{array}$$

With these definitions the bracket in $\Omega^1(\mathcal{A})$ can also be expressed in a form suitable for explicit computations, as

$$[\![p_i \mathrm{d}x^i, p_j \mathrm{d}x^j]\!] = x^3((p \times \nabla)_3 q_i - (q \times \nabla)_3 p_i)\mathrm{d}x^i,$$

where an element $p \in \mathcal{A}$ is viewed as a vector $p = (p_1, p_2, p_3)$, $\nabla = (\partial_1, \partial_2, \partial_3)$ and the subindex 3 denotes the third component of the cross product $p \times \nabla$.
It can be checked that $(\Omega^1(\mathcal{A}), [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid of Poisson type. However, if $p = 2x^1 + x^2$ and $q = x^1 + x^2$, then

$$[\![\mathrm{d}p, \mathrm{d}q]\!] = [\![2\mathrm{d}x^1 + \mathrm{d}x^2, \mathrm{d}x^1 + \mathrm{d}x^2]\!] = 0,$$

whereas

$$\begin{aligned} \mathrm{d}\{p, q\} = \mathrm{d}(\mathrm{d}q(\rho(\mathrm{d}p))) &= \mathrm{d}((\mathrm{d}x^1 + \mathrm{d}x^2)\rho(2\mathrm{d}x^1 + \mathrm{d}x^2)) \\ &= \mathrm{d}((\mathrm{d}x^1 + \mathrm{d}x^2)(2x^3\partial_2 - x^3\partial_1)) = \mathrm{d}x^3. \end{aligned}$$

4 Transitive Lie algebroids

To motivate the definition of transitive Lie algebroids that we will give, let us consider for a moment the geometric example of a Lie algebroid $(E, [\![\cdot,\cdot]\!], q)$, where $E \to M$ is a vector bundle over a manifold M (recall Example 2.9). If the anchor map $q : \Gamma(E) \to \Gamma(TM)$ is an epimorphism, the algebroid $(E, [\![\cdot,\cdot]\!], q)$ is said to be transitive. In this case, it is possible to construct the so-called Atiyah sequence of the algebroid, which is the short exact sequence

$$\mathfrak{g} \xhookrightarrow{\ j\ } E \xrightarrow{\ q\ } TM$$

where $\mathfrak{g} = \mathrm{Ker}q$. Thus, the existence of a section for the anchor q (equivalently, a linear connection on E) implies that, locally, $E = TM \oplus \mathfrak{g}$. Note also that the fibre of the bundle $\mathfrak{g}$ over the point $x \in M$, $\mathfrak{g}_x$ is a Lie algebra (called the isotopy Lie algebra of the algebroid E at $x \in M$) with the bracket given, for $\alpha, \beta \in \mathfrak{g}_x$, by

$$[\alpha, \beta] = [\![X, Y]\!],$$

where $X, Y \in \Gamma(E)$ are any sections such that $X(x) = \alpha$ and $Y(x) = \beta$.

Definition 4.1. A Lie algebroid $(\mathcal{F}, [\![\cdot,\cdot]\!], \rho)$ is transitive if there exist a short exact sequence of $\mathcal{A}$-modules:

$$V \xhookrightarrow{\ j\ } \mathcal{F} \xrightarrow{\ \rho\ } \mathrm{Der}(\mathcal{A})$$

Remark 4.2. There is a more general notion, the extension of a Lie-Rinehart algebra, that generalizes the transitivity condition for a geometric Lie algebroid (see [11]).

4.1 Transitive algebroids induced by connections

Let V be a unitary A-module and ∇ a connection on V.

Definition 4.3. The curvature of ∇ is the mapping $C_\nabla : \mathrm{Der}(\mathcal{A}) \times \mathrm{Der}(\mathcal{A}) \to \mathrm{Hom}(V, V)$ defined by

$$C_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Definition 4.4. If V is endowed with a Lie algebra structure $[\cdot,\cdot]_V$, then a connection ∇ is said to be a Lie connection if

$$\nabla_X[v_1, v_2]_V = [\nabla_X v_1, v_2]_V + [v_1, \nabla_X v_2]_V,$$

for all $v_1, v_2 \in V$ and for all $X \in \mathrm{Der}(\mathcal{A})$.

This subsection is devoted to the proof of the following result (see, also [17], [18]).

Theorem 4.5. *Let V be an $\mathcal{A}$-module endowed with a Lie algebra structure $[\ ,\ ]_V$ which is $\mathcal{A}$-linear (i.e., $[fX, Y]_V = f[X, Y]_V\ \forall f \in \mathcal{A},\ X, Y \in V$). Let ∇ be a Lie connection on V. If there exists a $2-$form $\mathcal{B} \in \Omega^2(\mathcal{A}; V)$ with values in V, such that, for any $X_1, X_2, X_3 \in \mathrm{Der}(\mathcal{A})$ and $v \in V$ the following conditions hold:*

(a) $[\mathcal{B}(X_1, X_2), v]_V = C_\nabla(X_1, X_2)(v)$,

(b) The cyclic sum $\circlearrowleft (\nabla_{X_1}(\mathcal{B}(X_2, X_3)) - \mathcal{B}([X_1, X_2], X_3)) = 0$,

then $(\mathrm{Der}(\mathcal{A}) \oplus V, [\![\cdot,\cdot]\!], \rho)$ *is a transitive Lie algebroid with anchor map* $\rho = \mathrm{pr}_1$ *and bracket*

$$[\![(X_1, v_1), (X_2, v_2)]\!] = ([X_1, X_2], [v_1, v_2]_V + \nabla_{X_1} v_2 - \nabla_{X_2} v_1 - \mathcal{B}(X_1, X_2)). \tag{5}$$

Moreover,

$$\iota_2(\mathcal{B}(X, Y)) = \iota_1([X, Y]) - [\![\iota_1(X), \iota_1(Y)]\!], \tag{6}$$

for $X, Y \in \mathrm{Der}(\mathcal{A})$. Here $\iota_1 : \mathrm{Der}(\mathcal{A}) \to \mathrm{Der}(\mathcal{A}) \oplus V$ and $\iota_2 : V \to \mathrm{Der}(\mathcal{A}) \oplus V$ are the inclusion maps.

Proof. It is clear that $\mathrm{Der}(\mathcal{A}) \oplus V$ is an $\mathcal{A}$-module. We must show that the bracket defined by (5) is Lie. The skew-symmetry and the $\mathcal{R}$-bilinearity are immediate. To check the Jacobi identity, let us pick $X_1, X_2, X_3 \in \mathrm{Der}(\mathcal{A})$ and $v_1, v_2, v_3 \in V$. Then, we have:

$$
\begin{array}{rl}
\circlearrowleft & ([\![[\![(X_1,v_1),(X_2,v_2)]\!],(X_3,v_3)]\!]) \\
= & ([[X_1,X_2],X_3],[[v_1,v_2]_V,v_3]_V+[\nabla_{X_1}v_2,v_3]_V-[\nabla_{X_2}v_1,v_3]_V \\
& -[\mathcal{B}(X_1,X_2),v_3]_V+\nabla_{[X_1,X_2]}v_3-\nabla_{X_3}([v_1,v_2]_V+\nabla_{X_1}v_2-\nabla_{X_2}v_1 \\
& -\mathcal{B}(X_1,X_2))-\mathcal{B}([X_1,X_2],X_3)) \\
& +([[X_2,X_3],X_1],[[v_2,v_3]_V,v_1]_V+[\nabla_{X_2}v_3,v_1]_V-[\nabla_{X_3}v_2,v_1]_V \\
& -[\mathcal{B}(X_2,X_3),v_1]_V+\nabla_{[X_2,X_3]}v_1-\nabla_{X_1}([v_2,v_3]_V+\nabla_{X_2}v_3-\nabla_{X_3}v_2 \\
& -\mathcal{B}(X_2,X_3))-\mathcal{B}([X_2,X_3],X_1)) \\
& +([[X_3,X_1],X_2],[[v_3,v_1]_V,v_2]_V+[\nabla_{X_3}v_1,v_2]_V-[\nabla_{X_1}v_3,v_2]_V \\
& -[\mathcal{B}(X_3,X_1),v_2]_V+\nabla_{[X_3,X_1]}v_2-\nabla_{X_2}([v_3,v_1]_V+\nabla_{X_3}v_1-\nabla_{X_1}v_3 \\
& -\mathcal{B}(X_3,X_1))-\mathcal{B}([X_3,X_1],X_2)) \\
= & ([[X_1,X_2],X_3]+[[X_2,X_3],X_1]+[[X_3,X_1],X_2], \\
& [[v_1,v_2]_V,v_3]_V+[[v_2,v_3]_V,v_1]_V+[[v_3,v_1]_V,v_2]_V \\
& +[\nabla_{X_1}v_2,v_3]_V+[v_2,\nabla_{X_1}v_3]_V-\nabla_{X_1}([v_2,v_3]_V) \\
& +[\nabla_{X_2}v_3,v_1]_V+[v_3,\nabla_{X_1}v_2]_V-\nabla_{X_2}([v_3,v_1]_V) \\
& +[\nabla_{X_3}v_1,v_2]_V+[v_1,\nabla_{X_2}v_3]_V-\nabla_{X_3}([v_1,v_2]_V) \\
& -[\mathcal{B}(X_1,X_2),v_3]_V+C_\nabla(X_1,X_2)(v_3) \\
& -[\mathcal{B}(X_2,X_3),v_1]_V+C_\nabla(X_2,X_3)(v_1) \\
& -[\mathcal{B}(X_3,X_1),v_2]_V+C_\nabla(X_3,X_1)(v_2) \\
& +\nabla_{X_1}\mathcal{B}(X_2,X_3)-\mathcal{B}([X_1,X_2],X_3)+\nabla_{X_2}\mathcal{B}(X_3,X_1)-\mathcal{B}([X_2,X_3],X_1) \\
& +\nabla_{X_3}\mathcal{B}(X_1,X_2)-\mathcal{B}([X_3,X_1],X_2)) \\
= & (0,0)
\end{array}
$$

Now let us show that ρ satisfies Leibniz. We have

$$
\begin{array}{rl}
& [\![(X_1,v_1),f(X_2,v_2)]\!] \\
= & ([X_1,fX_2],[v_1,fv_2]_V+\nabla_{X_1}(fv_2)-\nabla_{fX_2}(v_1)-\mathcal{B}(X_1,fX_2)) \\
= & (X_1(f)X_2+fX_1\circ X_2-fX_2\circ X_1,f[v_1,v_2]_V+\nabla_{X_1}(fv_2)-\nabla_{fX_2}(v_1)-\mathcal{B}(X_1,fX_2)) \\
= & (f[X_1,X_2]+X_1(f)X_2,f[v_1,v_2]_V+\nabla_{X_1}(fv_2)-f\nabla_{X_2}(v_1)-f\mathcal{B}(X_1,X_2)).
\end{array}
$$

On the other hand,

$$
\begin{array}{rl}
& f[\![(X_1,v_1),(X_2,v_2)]\!]+\rho(X_1,v_1)(f)(X_2,v_2) \\
= & f[(X_1,v_1),(X_2,v_2)]+X_1(f).(X_2,v_2) \\
= & (f[X_1,X_2]+X_1(f)X_2,f[v_1,v_2]_V+f\nabla_{X_1}v_2-f\nabla_{X_2}v_1-f\mathcal{B}(X_1,X_2)+X_1(f)v_2) \\
= & (f[X_1,X_2]+X_1(f)X_2,f[v_1,v_2]_V-f\nabla_{X_2}v_1-f\mathcal{B}(X_1,X_2)+\nabla_{X_1}(fv_2)).
\end{array}
$$

Thus, $[\![(X_1,v_1),f(X_2,v_2)]\!]=f[\![(X_1,v_1),(X_2,v_2)]\!]+\rho(X_1,v_1)(f)(X_2,v_2)$.
Since $\rho=\mathrm{pr}_1$ is clearly $\mathcal{A}$-linear, we have that $\mathrm{Der}(\mathcal{A})\oplus V$ is a Lie algebroid. The transitivity is obvious in view of the sequence

$$
V \overset{\iota_2}{\hookrightarrow} \mathrm{Der}(\mathcal{A})\oplus V \overset{\iota_1}{\twoheadrightarrow} \mathrm{Der}(\mathcal{A}).
$$

Moreover, a direct computation shows that

$$
\begin{array}{rcl}
\iota_1[X,Y]-[\![\iota_1(X),\iota_1(Y)]\!] & = & ([X,Y],0)-[\![(X,0),(Y,0)]\!] \\
& = & ([X,Y],0)-([X,Y],-\mathcal{B}(X,Y)) \\
& = & (0,\mathcal{B}(X,Y))=\iota_2(\mathcal{B}(X,Y)).
\end{array}
$$

□

Remark 4.6. The conditions (a) and (b) in this theorem have the following interpretation. Condition (a) states that the curvature of the connection ∇ is given by the composition $C_\nabla = \mathrm{ad}^R \circ \mathcal{B}$, where $\mathrm{ad}^R : V \to V$ is the right adjoint with respect to the bracket on V. On the other hand, (b) expresses a modified Bianchi identity (see also [17], [18]).

4.2 Parametrization of transitive algebroids on $\mathrm{Der}\mathcal{A} \oplus V$

The following result says that the converse of Theorem 4.5 is also true.

Theorem 4.7. *Let* $(\mathrm{Der}(\mathcal{A}) \oplus V, [\![\cdot,\cdot]\!], \rho)$ *be a transitive Lie algebroid with* $\rho = \mathrm{pr}_1$. *Then:*

(i) The bracket on V, *defined for* $v_1, v_2 \in V$ *by*

$$[v_1, v_2]_V = \mathrm{pr}_2([\![\iota_2(v_1), \iota_2(v_2)]\!]), \tag{7}$$

is an $\mathcal{A}$*-linear Lie bracket.*

(ii) The mapping $\nabla : \mathrm{Der}(\mathcal{A}) \to \mathrm{Hom}(V, V)$ *given by*

$$\nabla_X(v) = \mathrm{pr}_2([\![\iota_1(X), \iota_2(v)]\!]),$$

for $v \in V$ *and* $X \in \mathrm{Der}(\mathcal{A})$, *is a Lie connection on* V.

(iii) The mapping $\mathcal{B} : \mathrm{Der}(\mathcal{A}) \times \mathrm{Der}(\mathcal{A}) \to V$ *defined by*

$$\mathcal{B}(X, Y) = \mathrm{pr}_2([\![\iota_1(X), \iota_1(Y)]\!] - \iota_1([X, Y])),$$

is $\mathcal{A}$*-bilinear, skew-symmetric and it satisfies conditions* (a) *and* (b) *of Theorem 4.5.*

Proof. First let us show that $[\ ,\ ]_V$ defined by (7) is a Lie bracket and $\mathcal{A}$-linear. The skew-symmetry and $\mathcal{R}-$bilinearity are inherited from $[\![\cdot,\cdot]\!]$ and ι_2. Let $v_1, v_2, v_3 \in V$, and $f \in \mathcal{A}$. For the Jacobi identity we have:

$$\begin{aligned} \circlearrowleft [[v_1, v_2]_V, v_3]_V &= \circlearrowleft (\mathrm{pr}_2[\![\iota_2(\mathrm{pr}_2[\![\iota_2(v_1), \iota_2(v_2)]\!]), \iota_2(v_3)]\!]) \\ &= \circlearrowleft (\mathrm{pr}_2[\![[\![\iota_2(v_1), \iota_2(v_2)]\!], \iota_2(v_3)]\!]) = 0. \end{aligned}$$

And for the $\mathcal{A}$-linearity:

$$\begin{aligned} [fv_1, v_2]_V &= \mathrm{pr}_2[\![\iota_2(fv_1), \iota_2(v_2)]\!] \\ &= \mathrm{pr}_2[\![f\iota_2(v_1), \iota_2(v_2)]\!] \\ &= \mathrm{pr}_2(f[\![\iota_2(v_1), \iota_2(v_2)]\!] - \rho(\iota_2(v_2))(f)\iota_2(v_1)) \\ &= f\mathrm{pr}_2([\![\iota_2(v_1), \iota_2(v_2)]\!] \\ &= f[v_1, v_2]_V. \end{aligned}$$

Now, let us show that ∇ is a Lie connection on V. Let $X, Y, Z \in \mathrm{Der}(\mathcal{A}), f \in \mathcal{A}, v, v_1, v_2 \in V$ and $s \in R$. First, we observe that ∇ is $\mathcal{A}$-linear:

$$\begin{aligned} \nabla_{fX+Y}(v) &= \mathrm{pr}_2([\![\iota_1(fX + Y), \iota_2(v)]\!]) \\ &= \mathrm{pr}_2([f\iota_1(X), \iota_2(v)]\!] + [\![\iota_1(Y), \iota_2(v)]\!]) \\ &= \mathrm{pr}_2(f[\![\iota_1(X), \iota_2(v)]\!] - \rho(\iota_2(v))(f)(\iota_1(X)) + [\![\iota_1(Y), \iota_1(v)]\!]) \\ &= f\mathrm{pr}_2([\![\iota_1(X), \iota_2(v)]\!] + [\![\iota_1(Y), \iota_2(v)]\!]) \\ &= f\nabla_X(v) + \nabla_Y(v). \end{aligned}$$

Also, ∇_X is $\mathcal{R}-$linear,

$$\begin{aligned} \nabla_X(av_1 + v_2) &= \mathrm{pr}_2([\![\iota_1(X), \iota_2(av_1 + v_2)]\!]) \\ &= \mathrm{pr}_2([\![\iota_1(X), a\iota_2(v_1) + \iota_2(v_2)]\!]) = a\nabla_X(v_1) + \nabla_X(v_2). \end{aligned}$$

The Leibniz rule can be verified as follows

$$\begin{array}{rcl}\nabla_X(fv) & = & \mathrm{pr}_2([\![\iota_1(X),\iota_2(fv)]\!]) \\ & = & \mathrm{pr}_2([\![\iota_1(X),f\iota_2(v)]\!]) \\ & = & \mathrm{pr}_2(f[\![\iota_1(X),\iota_2(v)]\!]+\rho(\iota_1(X))(f)(\iota_2(v)) \\ & = & \mathrm{pr}_2(f[\![\iota_1(X),\iota_2(v)]\!]+X(f)(\iota_2(v)) \\ & = & f\mathrm{pr}_2([\![\iota_1(X),\iota_2(v)]\!])+X(f)\mathrm{pr}_2(\iota_2(v)) \\ & = & f\nabla_X(v)+X(f)v.\end{array}$$

Also, ∇ has the Lie property,

$$\begin{array}{rcl}\iota_2([\nabla_X v_1,v_2]_V+[v_1,\nabla_X v_2]_V) & = & \iota_2(\mathrm{pr}_2([\![\iota_2(\nabla_X v_1),\iota_2(v_2)]\!]) \\ & & +\mathrm{pr}_2([\![\iota_2(v_1),\iota_2(\nabla_X v_2)]\!])) \\ & = & \iota_2(\mathrm{pr}_2([\![\iota_2(\mathrm{pr}_2([\![\iota_1(X),\iota_2(v)]\!])),\iota_2(v_2)]\!]) \\ & & +\mathrm{pr}_2([\![\iota_2(v_1),\iota_2(\mathrm{pr}_2([\![\iota_1(X),\iota_2(v_2)]\!]))]\!])) \\ & = & [[\![\iota_1(X),\iota_2(v_1)]\!],\iota_2(v_2)]\!] \\ & & +[\![\iota_2(v_1),[\![\iota_1(X),\iota_2(v_2)]\!]]\!] \\ & = & [\![\iota_1(X),[\![\iota_2(v_1),\iota_2(v_2)]\!]]\!] \\ & = & [\![\iota_1(X),\iota_2([v_1,v_2]_V)]\!] \\ & = & \iota_2(\mathrm{pr}_2([\![\iota_1(X),\iota_2([v_1,v_2]_V)]\!])) \\ & = & \iota_2(\nabla_X([v_1,v_2]_V)),\end{array}$$

which implies that $\nabla_X[v_1,v_2]_V=[\nabla_X v_1,v_2]_V+[v_1,\nabla_X v_2]_V$.
Let us now check the properties of $\mathcal{B}$: Let $X_1,X_2,X_3\in\mathrm{Der}(\mathcal{A})$ and $f\in\mathcal{A}$. Then $\mathcal{B}$ is skew-symmetric,

$$\begin{array}{rcl}\mathcal{B}(X_1,X_2) & = & \mathrm{pr}_2([\![\iota_1(X_1),\iota_1(X_2)]\!]-\iota_1([X_1,X_2])) \\ & = & -(\mathrm{pr}_2([\![\iota_1(X_2),\iota_1(X_1)]\!]-\iota_1([X_2,X_1]))) \\ & = & -R(X_2,X_1),\end{array}$$

and $\mathcal{A}$-linear,

$$\begin{array}{rcl}\mathcal{B}(fX_1+X_2,X_3) & = & \mathrm{pr}_2([\![\iota_1(fX_1+X_2),\iota_1(X_3)]\!]-\iota_1([fX_1+X_2,X_3])) \\ & = & \mathrm{pr}_2([f\iota_1(X_1),\iota_1(X_3)]\!]+[\![\iota_1(X_2),\iota_1(X_3)]\!] \\ & & -\iota_1([fX_1,X_3])+[X_2,X_3])) \\ & = & \mathrm{pr}_2(f[\![\iota_1(X_1),\iota_1(X_3)]\!]-\rho(\iota_1(X_3))(f)\iota_1(X_1) \\ & & +[\![\iota_1(X_2),\iota_1(X_3)]\!]-\iota_1(f[X_1,X_3] \\ & & -X_3(f)X_1+[X_2,X_3])) \\ & = & \mathrm{pr}_2(f[\![\iota_1(X_1),\iota_1(X_3)]\!]+[\![\iota_1(X_2),\iota_1(X_3)]\!] \\ & & -\iota_1(f[X_1,X_3]-X_3(f)X_1+[X_2,X_3])) \\ & = & f\mathcal{B}(X_1,X_3)+\mathcal{B}(X_2,X_3).\end{array}$$

Finally, it remains to see that conditions (a) and (b) of Theorem 4.5 are satisfied. For (a), we have:

$$\begin{array}{rcl}\iota_2(C_\nabla(X,Y)(v)) & = & \iota_2([\nabla_x,\nabla_Y](v)-\nabla_{[X,Y]}(v)) \\ & = & \iota_2(\nabla_X(\nabla_Y)(v))-\iota_2(\nabla_Y(\nabla_X)(v)) \\ & & -[\![\iota_1([X,Y]),\iota_2(v)]\!] \\ & = & [\![\iota_1(X),\iota_2(\nabla_Y(v))]\!]-[\![\iota_1(Y),\iota_2(\nabla_X(v))]\!] \\ & & -[\![\iota_1([X,Y]),\iota_2(v)]\!] \\ & = & [\![\iota_1(X),[\![\iota_1(Y),\iota_2(v)]\!]]\!] \\ & & +[\![\iota_1(Y),[\iota_2(v),\iota_1(X)]\!]]\!] \\ & & -[\![\iota_1([X,Y]),\iota_2(v)]\!] \\ & = & -[\iota_2(v),[\![\iota_1(X),\iota_1(Y)]\!]]\!] \\ & & -[\![\iota_1([X,Y]),\iota_2(v)]\!] \\ & = & [\![([\![\iota_1(X),\iota_1(Y)]\!]-\iota_1([X,Y]),\iota_2(v)]\!] \\ & = & [\![\iota_2(\mathrm{pr}_2([\![\iota_1(X),\iota_1(Y)]\!]-\iota_1([X,Y])),\iota_2(v)]\!] \\ & = & \iota_2([\mathrm{pr}_2([\![\iota_1(X),\iota_1(Y)]\!]-\iota_1([X,Y]),v]_V) \\ & = & \iota_2([(\mathcal{B}(X,Y)),v]_V).\end{array}$$

Hence, $[(\mathcal{B}(X,Y)),v]_V = C_\nabla(X,Y)(v)$. Next,

$$\begin{array}{rcl}\iota_2(\circlearrowleft\{\nabla_X(\mathcal{B}(Y,Z)) - \mathcal{B}([X,Y],Z)\}) & = & \circlearrowleft\{[\![\iota_1(X),\iota_2(\mathcal{B}(Y,Z))]\!] - \iota_2(\mathcal{B}([X,Y],Z))\} \\ & = & \circlearrowleft\{[\![\iota_1(X),\iota_1[Y,Z]]\!] \\ & - & [\![\iota_1(X),[\![\iota_1(Y),\iota_1(Z)]\!]]\!] \\ & - & \iota_1[[X,Y],Z] + [\![\iota_1[X,Y],\iota_1(Z)]\!]\} \\ & = & \circlearrowleft\{[\![\iota_1(X),\iota_1[Y,Z]]\!] + [\![\iota_1[X,Y],\iota_1(Z)]\!]\} \\ & = & 0.\end{array}$$

Thus, $\circlearrowleft\{\nabla_X(\mathcal{B}(Y,Z)) - \mathcal{B}([X,Y],Z)\} = 0$. □

As a consequence of Theorems 4.5 and 4.7, we have the following.

Corollary 4.8. *Let V be an $\mathcal{A}$-module. There is a one to one correspondence between transitive Lie algebroids $(\mathrm{Der}(\mathcal{A}) \oplus V, [\ ,\ ]\!], \rho)$ with anchor $\rho = \mathrm{pr}_1$ and triples $([\cdot,\cdot]_V, \nabla, \mathcal{B})$ consisting of:*

(i) A $\mathcal{A}$-linear Lie bracket $[\cdot,\cdot]_V$ in V;

(ii) A Lie connection ∇ on V;

(iii) A $2-$form $\mathcal{B} \in \Omega^2(\mathcal{A};V)$ satisfying conditions (a) *and* (b) *of Theorem 4.5.*

4.3 Algebroid connections

In this subsection, we first generalize Theorem 4.7 to the case of a transitive Lie algebroid which is not necessarily of the form $\mathrm{Der}(\mathcal{A}) \oplus V$, but we require that it be endowed with a Lie algebroid connection (see [16, 18]), considered as a section of the anchor map. Then, in Theorem 4.11 we construct a Lie algebroid structure on $\mathrm{Der}(\mathcal{A}) \oplus V$ which is isomorphic to the given algebroid.

Definition 4.9. Let $(\mathcal{F}, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid. A morphism of $\mathcal{A}$-modules $\gamma : \mathrm{Der}(\mathcal{A}) \to \mathcal{F}$ is called a Lie algebroid connection (on $\mathcal{F}$) if it is a section of ρ, that is, $\rho \circ \gamma = \mathrm{Id}_{\mathrm{Der}(\mathcal{A})}$.

Theorem 4.10. *Let*

$$V \overset{\iota}{\hookrightarrow} \mathcal{F} \overset{\rho}{\twoheadrightarrow} \mathrm{Der}(\mathcal{A}) \tag{8}$$

be a transitive Lie algebroid and $\gamma : \mathrm{Der}(\mathcal{A}) \to \mathcal{F}$ a Lie algebroid connection for $\mathcal{F}$. Then:

(i) The bracket $[\cdot,\cdot]_V$ defined for $v_1, v_2 \in V$ by

$$[v_1,v_2]_V = [\![\iota(v_1),\iota(v_2)]\!], \tag{9}$$

is an $\mathcal{A}$-linear Lie bracket.

(ii) The mapping $\nabla : \mathrm{Der}(\mathcal{A}) \to \mathrm{Hom}(V,V)$ given by

$$\nabla_X(v) = [\![\gamma(X),\iota(v)]\!],$$

is a Lie connection on V (called the adjoint connection).

(iii) The mapping $\mathcal{B} : \mathrm{Der}(\mathcal{A}) \times \mathrm{Der}(\mathcal{A}) \to V$, defined by

$$\mathcal{B}(X,Y) = [\![\gamma(X),\gamma(Y)]\!] - \gamma([X,Y]),$$

is $\mathcal{A}$-bilinear, skew-symmetric and satisfies the conditions (a) *and* (b) *of Theorem 4.5.*

Proof. First, note that the exactness of (8) allows us to identify V with its image under ι in $\mathcal{F}$, and that $\mathrm{Im}\iota = \ker\rho$ in $\mathcal{F}$. Then, since ρ is a morphism of Lie algebras, we have

$$\rho[\![\iota(v_1), \iota(v_2)]\!] = [\rho\iota(v_1), \rho\iota(v_2)] = 0.$$

Thus, the bracket $[\cdot,\cdot]_V$ is well-defined. By a similar argument it can be proved that ∇ and $\mathcal{B}$ are well-defined. Now, let us show that $[\ ,\ ]_V$ defined by (9) is a Lie bracket and $\mathcal{A}$-linear. The $\mathcal{R}-$bilinearity and skew-symmetry are again inherited from $[\![\cdot,\cdot]\!]$. For the rest of properties, let $v_1, v_2, v_3 \in V$ and $f \in \mathcal{A}$. The Jacobi identity results from

$$\begin{aligned} \circlearrowleft [[v_1, v_2]_V, v_3]_V &= \circlearrowleft [\![\iota([\![\iota(v_1), \iota(v_2)]\!]), \iota(v_3)]\!] \\ &= \circlearrowleft [\![[\![\iota(v_1), \iota(v_2)]\!], \iota(v_3)]\!] = 0. \end{aligned}$$

The $\mathcal{A}$-linearity can be checked as follows:

$$\begin{aligned} [fv_1, v_2]_V &= [\![\iota(fv_1), \iota(v_2)]\!] \\ &= [\![f\iota(v_1), \iota(v_2)]\!] \\ &= f[\![\iota(v_1), \iota(v_2)]\!] - \rho(\iota(v_2))(f)\iota(v_1)) \\ &= f[\![\iota(v_1), \iota(v_2)]\!] \\ &= f[v_1, v_2]_V. \end{aligned}$$

Next, we show that ∇ is Lie connection in V. The $\mathcal{R}-$linearity is inherited from ι and $[\![\cdot,\cdot]\!]$. If $X, Y, Z \in \mathrm{Der}(\mathcal{A})$, $f \in \mathcal{A}$, and $v, v_1, v_2 \in V, s \in R$, the $\mathcal{A}$-linearity of ∇ follows from the computation

$$\begin{aligned} \nabla_{fX+Y}(v) &= [\![\gamma(fX+Y), \iota(v)]\!] \\ &= [\![f\gamma(X), \iota(v)]\!] + [\gamma(Y), \iota(v)]\!] \\ &= f[\![\gamma(X), \iota(v)]\!] - \rho(\iota(v))(f)(\gamma(X)) + [\gamma(Y), \gamma(v)]\!] \\ &= f[\![\gamma(X), \iota(v)]\!] + [\gamma(Y), \iota(v)]\!] \\ &= f\nabla_X(v) + \nabla_Y(v). \end{aligned}$$

For a fixed $X \in \mathrm{Der}(\mathcal{A})$, ∇_X satisfies the Leibniz identity,

$$\begin{aligned} \nabla_X(fv) &= [\![\gamma(X), \iota(fv)]\!] \\ &= [\![\gamma(X), f\iota(v)]\!] \\ &= f[\![\gamma(X), \iota(v)]\!] + \rho(\gamma(X))(f)(\iota(v)) \\ &= f[\![\gamma(X), \iota(v)]\!] + X(f)(\iota(v)) \\ &= f[\![\gamma(X), \iota(v)]\!] + X(f)k(\iota(v)) \\ &= f\nabla_X(v) + X(f)v, \end{aligned}$$

and ∇ is a Lie connection,

$$\begin{aligned} \iota([\nabla_X v_1, v_2]_V + [v_1, \nabla_X v_2]_V) &= \iota([[\![\gamma(X), \iota(v_1)]\!], v_2]_V + [v_1, [\![\gamma(X), \iota(v_2)]\!]]_V) \\ &= [\![[\![\gamma(X), \iota(v_1)]\!], \iota(v_2)]\!] + [\![\iota(v_1), [\![\gamma(X), \iota(v_2)]\!]]\!] \\ &= [\![\gamma(X), [\![\iota(v_1), \iota(v_2))]\!]]\!] \\ &= [\![\gamma(X), \iota([v_1, v_2)]_V)]\!] \\ &= \iota(\nabla_X[v_1, v_2]). \end{aligned}$$

This implies that $\nabla_X[v_1, v_2]_V = [\nabla_X v_1, v_2]_V + [v_1, \nabla_X v_2]_V$.
By similar computations it can be shown that $\mathcal{B} \in \Omega^2(\mathcal{A}; V)$. The proof of the property which relates $\mathcal{B}$ to the curvature of ∇ also follows the same guidelines,

$$\begin{aligned} \iota(C_\nabla(X,Y)(v)) &= \iota([\nabla_x, \nabla_Y](v) - \nabla_{[X,Y]}(v)) \\ &= \iota(\nabla_X(\nabla_Y)(v)) - \iota(\nabla_Y(\nabla_X)(v)) - [\![\gamma([X,Y]), \iota(v)]\!] \\ &= [\![\gamma(X), \iota(\nabla_Y(v))]\!] - [\![\gamma(Y), \iota(\nabla_X(v))]\!] - [\![\gamma([X,Y]), \iota(v)]\!] \\ &= [\![\gamma(X), [\![\gamma(Y), \iota(v)]\!]]\!] + [\![\gamma(Y), [\![\iota(v), \gamma(X)]\!]]\!] - [\![\gamma([X,Y]), \iota(v)]\!] \\ &= -[\![\iota(v), [\![\gamma(X), \gamma(Y)]\!]]\!] - [\![\gamma([X,Y]), \iota(v)]\!] \\ &= [\![([\![\gamma(X), \gamma(Y)]\!] - \gamma([X,Y]), \iota(v)]\!] \\ &= [\![\iota([\![\gamma(X), \gamma(Y)]\!] - \gamma([X,Y])), \iota(v)]\!] \\ &= \iota([[\![\gamma(X), \gamma(Y)]\!] - \gamma([X,Y]), v]_V) \\ &= \iota([\mathcal{B}(X,Y), v]_V). \end{aligned}$$

Therefore, $[\mathcal{B}(X,Y),v]_V = C_\nabla(X,Y)(v)$. Finally, we check the modified Bianchi identity,

$$\begin{array}{rcl} \iota(\circlearrowleft\{\nabla_X(\mathcal{B}(Y,Z)) - \mathcal{B}([X,Y],Z)\}) & = & \circlearrowleft\{[\![\gamma(X),\iota(\mathcal{B}(Y,Z))]\!] - \iota(\mathcal{B}([X,Y],Z))\} \\ & = & \circlearrowleft\{[\![\gamma(X),\gamma([Y,Z])]\!] \\ & & -[\![\gamma(X),[\![\gamma(Y),\gamma(Z)]\!]]\!] \\ & & -\gamma([[X,Y],Z]) + [\![\gamma([X,Y]),\gamma(Z)]\!]\} \\ & = & \circlearrowleft\{[\![\gamma(X),\gamma([Y,Z])]\!] + [\![\gamma([X,Y]),\gamma(Z)]\!]\} = 0. \end{array}$$

Thus, $\circlearrowleft\{\nabla_X(\mathcal{B}(Y,Z)) - \mathcal{B}([X,Y],Z)\} = 0$. □

Theorem 4.11. *Let*

$$V \xhookrightarrow{\iota} \mathcal{F} \xrightarrow{\rho} \mathrm{Der}(\mathcal{A})$$

be a transitive Lie algebroid and $\gamma : \mathrm{Der}(\mathcal{A}) \to \mathcal{F}$ *a Lie algebroid connection on* $\mathcal{F}$*. Then* $(\mathcal{F}, [\![\cdot,\cdot]\!], \rho)$ *is isomorphic to* $(\mathrm{Der}(\mathcal{A}) \oplus V, \langle\cdot,\cdot\rangle, \mathrm{pr}_1)$*, where the bracket is defined through*

$$[(X_1,v_1),(X_2,v_2)] = ([X_1,X_2],[v_1,v_2]_V + \nabla_{X_1}(v_2) - \nabla_{X_2}(v_1) - \mathcal{B}(X_1,X_2)),$$

and $[\cdot,\cdot]_V, \nabla_X$*,* $\mathcal{B}$ *are given in Theorem 4.10.*

Proof. We know that $(\mathrm{Der}(\mathcal{A}) \oplus V, \langle\cdot,\cdot\rangle, \mathrm{pr}_1)$ is a Lie algebroid by Theorem 4.5. Moreover,

$$\begin{array}{rcl} \phi : \mathrm{Der}(\mathcal{A}) \oplus V & \longrightarrow & \mathcal{F} \\ (X,v) & \longmapsto & \gamma(X) + \iota(v) \end{array}$$

is an $\mathcal{A}$-module isomorphism such that

$$(\rho \circ \phi)(X,v) = \rho(\gamma(X) + \iota(v)) = X = \mathrm{pr}_1(X,v)$$

and

$$\begin{array}{rcl} \phi(\langle(X_1,v_1),(X_2,v_2)\rangle) & = & \phi([X_1,X_2],[v_1,v_2]_V + \nabla_{X_1}(v_2) - \nabla_{X_2}(v_1) - \mathcal{B}(X_1,X_2)) \\ & = & \phi([X_1,X_2],[\![\iota(v_1),\iota(v_2)]\!] + [\![\gamma(X_1),\iota(v_2)]\!] \\ & & -[\![\gamma(X_2),\iota(v_1)]\!] - [\![\gamma(X_1),\gamma(X_2)]\!] - \gamma([X_1,X_2])) \\ & = & \gamma([X_1,X_2]) + \iota([\![\iota(v_1),\iota(v_2)]\!]) + \iota([\![\gamma(X_1),\iota(v_2)]\!]) \\ & & -\iota([\![\gamma(X_2),\iota(v_1)]\!]) - \iota([\![\gamma(X_1),\gamma(X_2)]\!]) - \gamma([X_1,X_2]) \\ & = & [\![\iota(v_1),\iota(v_2)]\!] + [\![\gamma(X_1),\iota(v_2)]\!] - [\![\gamma(X_2),\iota(v_1)]\!] + [\![\gamma(X_1),\gamma(X_2)]\!] \\ & = & [\![\gamma(X_1)+\iota(v_1),\gamma(X_1)+\iota(v_1)]\!] \\ & = & [\![\phi(X_1,v_1),\phi(X_2,v_2)]\!]. \end{array}$$

□

5 Poisson algebras on $\mathcal{A} \oplus V$

Recall that, if $(\mathcal{A}, \{\ ,\ \})$ is a Poisson algebra, then we define for every $f \in \mathcal{A}$ the Hamiltonian derivation $X_f \in \mathrm{Der}(\mathcal{A})$ as the adjoint map $X_f = \{f,\cdot\}$. The following standard properties will be used below:

(i) $X_{f_1 f_2} = f_1 X_{f_2} + f_2 X_{f_1}$,

(ii) The mapping $f \mapsto X_f$ is a Lie algebra morphism, i.e, $X_{\{f_1,f_2\}} = [X_{f_1}, X_{f_2}]$.

Now, on $\mathcal{A} \oplus V$ we can define a product given by

$$(f_1,v_1)\cdot(f_2,v_2) = (f_1 f_2, f_1 v_2 + f_2 v_1), \tag{10}$$

for $f_1, f_2 \in \mathcal{A}$ and $v_1, v_2 \in V$. This makes $\mathcal{A} \oplus V$ a commutative ring. Under certain conditions, this ring is also a Poisson algebra, as shown by the following result.

Theorem 5.1. *Let $(\mathcal{A}, \{,\}_{\mathcal{A}})$ be a Poisson algebra. Suppose we have a transitive Lie algebroid*

$$V \xhookrightarrow{\iota_2} \mathrm{Der}(\mathcal{A}) \oplus V \xrightarrow{\rho} \mathrm{Der}(\mathcal{A}),$$

with anchor $\rho = \mathrm{pr}_1$. Then, $\mathcal{A} \oplus V$ is also a Poisson algebra with product defined by (10) *and Poisson bracket*

$$\{(f_1, v_1), (f_2, v_2)\} = (\{f_1, f_2\}_{\mathcal{A}}, \nabla_{X_{f_1}}(v_2) - \nabla_{X_{f_2}}(v_1) + [v_1, v_2]_V + \mathcal{B}(X_{f_1}, X_{f_2})), \qquad (11)$$

where $[\cdot,\cdot]_V$, ∇ and $\mathcal{B}$ are given in Theorem 4.7.

Proof. Let $f_1, f_2, f_3 \in \mathcal{A}$ and $v_1, v_2, v_3 \in V$. The skew-symmetry of the bracket is immediate

$$\begin{aligned}
\{(f_1, v_1), (f_2, v_2)\} &= (\{f_1, f_2\}_{\mathcal{A}}, \nabla_{X_{f_1}}(v_2) - \nabla_{X_{f_2}}(v_1) + [v_1, v_2]_V + \mathcal{B}(X_{f_1}, X_{f_2})) \\
&= (-\{f_2, f_1\}_{\mathcal{A}}, \nabla_{X_{f_1}}(v_2) - \nabla_{X_{f_2}}(v_1) - [v_2, v_1]_V - \mathcal{B}(X_{f_2}, X_{f_1})) \\
&= -\{(f_2, v_2), (f_1, v_1)\}.
\end{aligned}$$

Next we verify the Leibniz identity, (where $[\![\cdot,\cdot]\!]$ denotes the Lie bracket on $\mathrm{Der}(\mathcal{A}) \oplus V$)

$$\begin{aligned}
& \{(f_1, v_1), (f_2, v_2) \cdot (f_3, v_3)\} \\
= & \{(f_1, v_1), (f_2 f_3, f_2 v_3 + f_3 v_2)\} \\
= & (\{f_1, f_2 f_3\}_{\mathcal{A}}, \nabla_{X_{f_1}}(f_3 v_2 + f_2 v_3) - \nabla_{X_{f_2 f_3}}(v_1) + [v_1, f_3 v_2 + f_2 v_3]_V + \mathcal{B}(X_{f_1}, X_{f_2 f_3})) \\
= & (f_2\{f_1, f_3\}_{\mathcal{A}} + f_1\{f_2, f_3\}_{\mathcal{A}}, X_{f_1}(f_3) v_2 \\
& + f_3 \nabla_{X_{f_1}}(v_2) + X_{f_1}(f_2) v_3 + f_2 \nabla_{X_{f_1}}(v_3) \\
& - \nabla_{X_{f_2 f_3}}(v_1) + f_2 [v_1, v_3]_V + f_3 [v, v_2]_V \\
& + \mathrm{pr}_2([\![\iota_1(X_{f_1}), \iota_1(X_{f_2 f_3})]\!] - \iota_1([X_{f_1}, X_{f_2 f_3}])) \\
= & (f_2\{f_1, f_3\}_{\mathcal{A}} + f_1\{f_2, f_3\}_{\mathcal{A}}, f_2 \mathrm{pr}_2([\![\iota_1(X_{f_1}), \iota_2(v_3)]\!]) \\
& + f_2 \mathrm{pr}_2([\![\iota_2(v_1), \iota_1(X_{f_3})]\!]) + f_2 [v_1, v_3]_V + f_2 \mathrm{pr}_2([\![\iota_1(X_{f_1}), \iota_1(X_{f_3})]\!]) \\
& - f_2 \iota_1([X_{f_1}, X_{f_3}]) + \{f_1, f_3\} v_2 \\
& + f_3 \mathrm{pr}_2([\![\iota_1(X_{f_1}), \iota_2(v_2)]\!]) + f_3 \mathrm{pr}_2([\![\iota_2(v_1), \iota_1(X_{f_2})]\!]) + f_3 [v_1, v_2]_V \\
& + f_3 \mathrm{pr}_2([\![\iota_1(X_{f_1}), \iota_1(X_{f_2})]\!]) - f_3 \mathrm{pr}_2(\iota_1([X_{f_1}, X_{f_2}]) + \{f_1, f_2\} v_3) \\
= & (f_2, v_2)\{(f_1, v_1), (f_3, v_3)\} + (f_3, v_3)\{(f_1, v_1), (f_2, v_2)\},
\end{aligned}$$

Here we have used that the bracket $[\ ,\ ]_V$ is $\mathcal{A}$-linear. Finally, for the Jacobi identity we have

$$\begin{aligned}
& \{\{(f_1, v_1), (f_2, v_2)\}, (f_3, v_3)\} \\
= & \{(\{f_1, f_2\}_{\mathcal{A}},\ \nabla_{X_{f_1}}(v_2) - \nabla_{X_{f_2}}(v_1) + [v_1, v_2]_V + \mathcal{B}(X_{f_1}, X_{f_2})), (f_3, v_3)\} \\
= & (\{\{f, f_2\}_{\mathcal{A}}, f_3\}_{\mathcal{A}}, \\
& \mathrm{pr}_2([\![\iota_1([X_{f_1}, X_{f_2}]), \iota_2(v_3)]\!] \\
& - [\![\iota_1(X_{f_3}), [\![\iota_1(X_{f_1}), \iota_2(v_2)]\!]]\!] \\
& + [\![\iota_1(X_{f_3}), [\![\iota_1(X_{f_2}), \iota_2(v_1)]\!]]\!] \\
& - [\![\iota_1(X_{f_3}), [\![\iota_2(v_1), \iota_2(v_2)]\!]]\!] \\
& - [\![\iota_1(X_{f_3}), [\![\iota_1(X_{f_1}), \iota_1(X_{f_2})]\!]]\!] \\
& + [\![\iota_1(X_{f_3}), \iota_1([X_{f_1}, X_{f_2}])]\!] \\
& + [[\![\iota_1(X_{f_1}), \iota_2(v_2)]\!], \iota_2(v_3)]\!] \\
& - [[\![\iota_1(X_{f_2}), \iota_2(v_1)]\!], \iota_2(v_3)]\!] \\
& + [[\![\iota_1(X_{f_1}), \iota_1(X_{f_2})]\!], \iota_2(v_3)]\!] \\
& - [\![\iota_1([X_{f_1}, X_{f_2}]), \iota_2(v_3)]\!] \\
& + [\![\iota_1([X_{f_1}, X_{f_2}]), \iota_1(X_{f_3})]\!] \\
& - \iota_1([[X_{f_1}, X_{f_2}], X_{f_3}])) \\
& + [[v_1, v_2]_V, v_3]_V).
\end{aligned}$$

It follows that

$$\begin{array}{rl}\circlearrowleft & \{\{(f_1,v_1),(f_2,v_2)\},(f_3,v_3)\}\\ = & (\{\{f_1,f_2\}_{\mathcal{A}},f_3\}_{\mathcal{A}}+\{\{f_2,f_3\}_{\mathcal{A}},f_1\}_{\mathcal{A}}+\{\{f_3,f_1\}_{\mathcal{A}},f_2\}_{\mathcal{A}},\\ & +\circlearrowleft \mathrm{pr}_2(-[\![\iota_1(X_{f_3}),[\![\iota_1(X_{f_1}),\iota_2(v_2)]\!]]\!]\\ & -[\![\iota_1(X_{f_1}),[\![\iota_1(X_{f_2}),\iota_2(v_3)]\!]]\!]\\ & -[\![\iota_1(X_{f_2}),[\![\iota_1(X_{f_3}),\iota_2(v_1)]\!]]\!])\\ & +\circlearrowleft \mathrm{pr}_2(-[\![\iota_1(X_{f_3}),[\![\iota_2(v_1),\iota_2(v_2)]\!]]\!]\\ & -[\![\iota_1(X_{f_1}),[\![\iota_2(v_2),\iota_2(v_3)]\!]]\!]\\ & -[\![\iota_1(X_{f_2}),[\![\iota_2(v_3),\iota_2(v_1)]\!]]\!])\\ & +\circlearrowleft \mathrm{pr}_2(-[\![\iota_1(X_{f_3}),[\![\iota_1(X_{f_1}),\iota_1(X_{f_2})]\!]]\!])\\ & +\circlearrowleft \mathrm{pr}_2(-\iota_1([[X_{f_1},X_{f_2}],X_{f_3}]))\\ & +\circlearrowleft([[v_1,v_2]_V,v_3]_V))=(0,0).\end{array}$$

□

Corollary 5.2. *V is an ideal of $\mathcal{A}\oplus V$ with respect to the Poisson bracket defined by* (11).

The proof of this result follows from the fact $\{0,f\}_{\mathcal{A}}=0$.

Corollary 5.3. *If $(\mathcal{A},\{,\}_{\mathcal{A}})$ is a Poisson algebra and we have a transitive Lie algebroid*

$$V \xhookrightarrow{\iota_2} \mathrm{Der}(\mathcal{A})\oplus V \xrightarrow{\rho} \mathrm{Der}(\mathcal{A}),$$

endowed with an algebroid connection γ, then, $\mathcal{A}\oplus V$ is a Poisson algebra.

Proof. By Theorem 4.11, we know that $(\mathcal{F},[\![\cdot,\cdot]\!],\rho)$ is isomorphic to $(Der(\mathcal{A})\oplus V,\langle\cdot,\cdot\rangle,\mathrm{pr}_1)$. The statement follows from Theorem 5.1. □

Acknowledgements

JAV and DGB express their gratitude to the members of the Department of Mathematics of the University of Sonora (where part of this work was done), particularly to R. Flores-Espinoza and G. Dávila-Rascón, for their warm hospitality. JAV also thanks the National Council of Science and Technology in Mexico (CONACyT), for the research grant JB2-CB-78791.

References

[1] C. J. Atkin, J. Grabowski: *Homomorphisms of the Lie algebras associated with a symplectic manifold*. Comp. Math. **76** (1990) 315-349.

[2] R. Abraham, J. Marsden: *Foundations of Mechanics*. W. A. Benjamin (1967), Menlo Park, CA.

[3] A. Connes: *Non-commutative differential geometry*. Inst. Hautes Etudes Sci. Publ. Math. **62** (1985), 257-360.

[4] J. Cuntz, D. Quillen: *Algebra extensions and nonsingularity*. J.Amer. Math. Soc. **8** (1995), 251-289.

[5] M. DuBois-Violette, P. W. Michor: *Dérivations et calcul différentiel non commutatif II*. Compt. Rend. Acad. Sci. Paris **319** Serie I (1994) 927-931.

[6] J. P. Dufour, N. Tien-Zung: *Poisson Structures and Their Normal Forms*. Birkhauser (2005), Basel.

[7] J. Grabowski: *Quasi-derivations and QD-algebroids*. Rep. Math. Phys. **52** 3 (2003), 445-451.

[8] J. C. Herz: *Pseudo-algebres de Lie*. C. R. Acad. Sci. Paris. **236** (1953), 1935-1937 and 2289-2291.

[9] J. Huebschmann: *Poisson cohomology and quantization*. J. reine angew. Math. **408** (1990), 57-113.

[10] J. Huebschmann: *Lie-Rinehart algebras, Gerstenhaber algebras, and Batalin- Vilkovisky algebras*. Ann. de l'Institut Fourier **48** (1998) 425-440.

[11] J. Huebschmann: *Extensions of Lie-Rinehart algebras and the Chern-Weil construction*. In: *Festschrift in honour of Jim Stasheff's 60'th anniversary*. Contemp. Math. **227** (1999), 145-176.

[12] N. M. Katz: *Nilpotent connections and the monodromy theorem; applications of a result of Turritin*. Publ. Math. Inst. Hautes Etudes Sci. **39** (1970) 175-232.

[13] I. Kolar, J. Slovak, and P. W. Michor: *Natural operations in differential geometry*. Springer-Verlag (1993) Berlin.

[14] Y. Kosmann-Schwarzbach, F. Magri: *Poisson-Nijenhuis structures*. Ann. Inst. Henri Poincaré, Phys. Théor. **53** (1990), 35-81.

[15] Y. Kosmann-Schwarzbach, K. Mackenzie: *Differential operators and actions of Lie algebroids*. In: *Quantization, Poisson Brackets and Beyond*. Contemp. Math. **315** (2002) 213-233.

[16] J. Kubarski: *The Chern-Weil homomorphism of regular Lie algebroids*. Publications du Département de Mathématiques, Université Claude Bernard Lyon 1, nouvelle série, (1991), 1-70

[17] K. Mackenzie: *Lie algebroids and Lie pseudoalgebras*. Bull. London Math. Soc. **27** (1995), 97-147.

[18] K. Mackenzie: *General theory of Lie groupoids and Lie algebroids*. Cambridge UP. London Math. Soc. Lect. Note Ser. **212** (2005) Cambridge.

[19] G. Rinehart: *Differential forms for general commutative algebras*. Trans. Amer. Math. Soc. **108** (1963), 195-222.

[20] Yu. Vorobjev: *Coupling tensors and Poisson geometry near a single symplectic leaf*. In *Lie Algebroids*, Banach Center Publ. **54** (2001) 249-274.

[21] Yu. Vorobjev: *On Poisson realizations of transitive Lie algebroids*. J. of Nonlinear Math. Phys. **11**, Suppl. (2004) 43-48.