Semigroup modeling of confined Lévy flights

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The master equation for confined Lévy flights admits a transformation to a contractive strongly continuous semigroup dynamics. We address the ground-state (and the resultant probability density function (pdf)) reconstruction problem for generic Lévy-stable semigroups: given a priori a functional form of a semigroup potential, infer asymptotic invariant pdfs of affiliated jump-type processes, for *all* values of the stability index $\mu \in (0,2)$. We analyze a limiting (mis)behavior of solutions in the vicinity and at the boundaries 0 and 2 of the stability interval $(0,2)$.

I. CONCEPTUAL BACKGROUND

We consider a subclass of uni-variate stable probability distributions determined by a characteristic exponent $-F(p) = -|p|^\mu$ of $< \exp(ipX) >$, with $0 < \mu < 2$. The induced jump-type dynamics, $< \exp(ipX_t) >= \exp[-tF(p)]$, where $t \geq 0$, is conventionally interpreted in terms of Lévy flights and quantified by means of a pseudo-differential (fractional) equation for a corresponding time-dependent probability density function (pdf)

$$\partial_t \rho = -|\Delta|^{\mu/2} \rho = \int [w_\mu(x|y)\rho(y) - w_\mu(y|x)\rho(x)] dy \, . \tag{1}$$

The jump rate $w_\mu(x|y) \propto 1/|x-y|^{1+\mu}$ is a symmetric function, $w_\mu(x|y) = w_\mu(y|x)$. We recall that the action of a fractional operator $|\Delta|^{\mu/2}$ on a function from its domain is defined by means of the Cauchy principal value of an involved integral:

$$-\,(|\Delta|^{\mu/2} f)(x) \,=\, \frac{\Gamma(\mu+1)\sin(\pi\mu/2)}{\pi} \int \frac{f(z)-f(x)}{|z-x|^{1+\mu}}\, dz \, . \tag{2}$$

We generalize the master equation (1) to encompass non-symmetric jump rates $w_\mu(x|y) \to w_\mu^U(x|y) \neq w_\mu^U(y|x)$:

$$w_\mu^U(x|y) = w_\mu(x|y)\, \exp\left(\frac{U(y)-U(x)}{2}\right), \tag{3}$$

where $U(x)$ is a continuous function on R. With $w_\mu^U(x|y)$ replacing $w_\mu(x|y)$, (1) takes the form

$$\partial_t \rho = -[\exp(-U/2)]\, |\Delta|^{\mu/2} [\exp(U/2)\rho] + \rho \exp(U/2) |\Delta|^{\mu/2} \exp(-U/2) \, . \tag{4}$$

For a suitable (to secure normalization) choice of $U(x)$, $\rho_{eq}(x) \propto \exp[-U(x)]$ is a stationary solution of Eq. (4). The detailed balance principle of the standard form $w_U(x|y)\rho_{eq}(y) = w_U(y|x)\rho_{eq}(x)$ holds true.

The "free" fractional Fokker-Planck equation (1) has no stationary solutions. Thus, properly selected jump rates $w_U(x|y)$ surely may induce invariant pdfs. The pertinent asymptotic pdfs for confined Lévy flights may have an arbitrary, not necessarily finite, number of moments. The reference stable laws generically have no moments of order higher than one.

Let us consider the Lévy-Schrödinger Hamiltonian operator with an external potential

$$\hat{H}_\mu \equiv |\Delta|^{\mu/2} + \mathcal{V}(x) \, . \tag{5}$$

Suitable properties of $\mathcal{V}$ need to be assumed, so that $-\hat{H}_\mu$ is a legitimate generator of a dynamical semigroup $\exp(-t\hat{H}_\mu)$ and $\partial_t \Psi = -\hat{H}_\mu \Psi$ holds true for real functions $\Psi(x,0) \to \Psi(x,t)$.

Let us a priori select an invariant probability density $\rho_{eq}(x) \doteq \rho_*(x) \propto \exp[-U(x)]$ of Eq. (4). To make it an asymptotic pdf of a well defined jump-type process we address an issue of the existence of a suitable semigroup dynamics.

Looking for stationary solutions of the affiliated semigroup equation $\partial_t \Psi = -\hat{H}_\mu \Psi$, we realize that if a square root of a positive invariant pdf $\rho_*(x)$ is asymptotically to come out via the semigroup dynamics $\Psi \to \rho_*^{1/2}$, then the resulting fractional Sturm-Liouville equation $\hat{H}_\mu \rho_*^{1/2} = 0$ imposes a *compatibility condition* upon the functional form of $\mathcal{V}(x)$, that needs to be respected. Namely, the potential function and invariant pdf $\rho_*^{1/2}$ should be related as

$$\mathcal{V} = -\frac{|\Delta|^{\mu/2}\rho_*^{1/2}}{\rho_*^{1/2}}. \tag{6}$$

The resulting semigroup dynamics provides a solution for the Lévy stable *targeting problem*, with a predefined invariant pdf. We have discussed this issue in some detail in our previous publications, [1–3].

Inversely, if we choose a priori a concrete potential function $\mathcal{V}(x)$, then an ultimate functional form of an invariant pdf $\rho_*(x)$ (actually $\rho_*^{1/2}(x)$) needs to come out from the above compatibility condition. This problem we address in the present paper. For a predefined function $\mathcal{V}(x)$ we admit all stability index values $0 < \mu < 2$ in the compatibility condition (6). That assigns to $\mathcal{V}(x)$ a μ-family of inferred pdfs $\rho_{*\mu}(x)$, together with a corresponding family of jump-type processes (e.g. confined μ-stable Lévy flights) .

For clarity of discussion let us add few comments about the stochastic process in question. Let $\mathcal{V} = \mathcal{V}(x)$ be a bounded from below continuous function. Then, the integral kernel $k(y,s,x,t) = \{\exp[-(t-s)\hat{H}]\}(y,x)$, $s < t$, of the dynamical semigroup $\exp(-t\hat{H})$ is positive and jointly continuous in all variables. The semigroup dynamics reads: $\Psi(x,t) = \int \Psi(y,s)\, k(y,s,x,t)\, dy$ so that for all $0 \leq s < t$ we can reproduce the dynamical pattern of behavior, actually set by Eq. (4), but now in terms of Markovian transition probability densities $p(x,s,y,t)$: $\rho(x,t) = \rho_*^{1/2}(x)\Psi(x,t) = \int p(y,s,x,t)\rho(y,s)dy$, where $p(y,s,x,t) = k(y,s,x,t)\, \rho_*^{1/2}(x)/\rho_*^{1/2}(y)$. An asymptotic behavior of $\Psi(x,t) \to \rho_*^{1/2}(x)$ implies $\rho(x,t) \to \rho_*(x)$ as $t \to \infty$.

The spectral theory of fractional operators of the form (5) has received a broad coverage in the mathematical [4–8] and mathematical physics literature [9, 10]. Various rigorous estimates pertaining to the decay of the eigenfunctions at spatial infinities, quantify the number of moments of the associated pdfs for different classes of potential functions $\mathcal{V}(x)$. As well, fractional versions of the Feynman-Kac formula for an integral kernel of the semigroup operator have an ample coverage therein.

II. μ-FAMILY OF PDFS FOR A PREDEFINED $\mathcal{V}(x)$.

Let us have a functional form of $\mathcal{V}(x)$. The compatibility condition (6) imposes the following equation for an invariant (terminal) pdf (here we denote $\rho_*^{1/2}(x) \equiv f(x)$)

$$\mathcal{V}(x)f(x) = -|\Delta|^{\mu/2} f(x), \tag{7}$$

where $0 < \mu < 2$. Remembering that we consider $\mathcal{V}(x)$ to be a continuous and bounded from below function (may be unbounded from above), we turn over to the standard Fourier transform method with $f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{\imath kx} dx$ and $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{-\imath kx} dk$. Denoting the Fourier image of right-hand side of Eq. (7) as u_k, we obtain

$$u_k = -|k|^\mu f(k). \tag{8}$$

Equating the Fourier images of both sides of Eq. (7) yields

$$u_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}(x) f(x) e^{\imath kx} dx = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \mathcal{V}(x) e^{\imath x(k-k')} f(k') dk' dx. \tag{9}$$

In this case, the Fourier image $f(k)$ of a solution $f(x)$ to Eq. (7) is defined by following integral equation of a convolution type

$$f(k) = -\frac{1}{|k|^\mu \sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}(k-k') f(k') dk'. \tag{10}$$

Here we use the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\imath x(k-k')} dx = \delta(k-k'), \tag{11}$$

which holds everywhere on $\mathcal{R}$, except the point $k = 0$.

We now pay attention to the fact that function $\mathcal{V}_\mu(x)$ is delocalized (growing at infinities) so that its Fourier image does not exist in a regular sense so that the equation (10) could be considered as a formal one with respect to $f(k)$. However, we will show with the help of Eq. (11) that the general solution of Eq. (10) can be expressed in terms of δ-function and its derivatives.

Let us restrict further consideration to even functions $\mathcal{V}$ (in this case the terminal pdf is even function as well) and assume them to be differentiable a sufficient number of times. For such functions, the Taylor series comprise even powers of x only

$$\mathcal{V}(x) = \mathcal{V}(0) + \mathcal{V}"_\mu(0)\frac{x^2}{2!} + \mathcal{V}_\mu^{(4)}(0)\frac{x^4}{4!} + ... \tag{12}$$

The Fourier image of (12) yields

$$\begin{aligned}\mathcal{V}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\mathcal{V}(0) + \mathcal{V}"(0)\frac{x^2}{2!} + \mathcal{V}^{(4)}(0)\frac{x^4}{4!}\right]e^{\imath kx}dx \equiv \\ &\equiv \sqrt{2\pi}\left[\mathcal{V}(0)\delta(k) - \frac{\mathcal{V}"(0)}{2}\delta"(k) + \frac{\mathcal{V}^{(4)}(0)}{4!}\delta^{(4)}(k) - ...\right],\end{aligned} \tag{13}$$

i.e. it has the form of the infinite series of even derivatives of the Dirac δ - function. We note that if $\mathcal{V}(x)$ is a simple (even) polynomial of the form $\mathcal{V}(x) = ax^2 + bx^4$, the above series are finite.

Accordingly, we end up with the following differential equation of the infinite even order for the Fourier image $f(-k) = f(k)$ of $\rho_*^{1/2}(x) \equiv f(x)$:

$$\frac{\mathcal{V}"(0)}{2}\frac{d^2 f(k)}{dk^2} - \frac{\mathcal{V}^{(4)}(0)}{4!}\frac{d^4 f(k)}{dk^4} + ... = \left[k^\mu + \mathcal{V}(0)\right]f(k),\ k \geq 0. \tag{14}$$

We choose the following initial conditions for Eq. (14)

$$f(k=0) \equiv \int_{-\infty}^{\infty} f(x)dx = A,\ f^{(2n-1)}(k=0) = 0,\ n = 1, 2, 3, ... \tag{15}$$

Note that this imposes an integrability condition on $\rho_*^{1/2}(x)$ on $\mathcal{R}$. Here by $f^{(2n-1)}(k=0)$ we denote the odd derivatives of $f(k)$ at $k = 0$.

The integration constant A is not completely arbitrary and should be consistent with the normalization condition $\int_{-\infty}^{\infty} f^2(x)dx \equiv \int_{-\infty}^{\infty} \rho_*(x)dx = 1$. In view of the Parceval identity we have

$$\int_{-\infty}^{\infty} f^2(x)dx = \int_{-\infty}^{\infty} f^2(k)dk \equiv 2\int_0^{\infty} f^2(k)dk = 1. \tag{16}$$

This means that $f(k=0) = A$ must be compatible with $\int_0^\infty f^2(k)dk = 1/2$.

One should not expect an easy analytic outcome of the solution of infinite order differential equation (14). In most cases of interest the infinite series can be truncated, but most probably a numerical assistance is unavoidable. The practical strategy of finding (at worst approximately for truncated series and an arbitrary functional shape of $\mathcal{V}(x)$) an $L^2(\mathcal{R})$ integrable non-negative ground state of the semigroup, and thence the terminal pdf $\rho_*(x)$, can be summarized as follows.

- Expand $\mathcal{V}(x)$ in power series. The number of terms in the series should be chosen so as to obtain a sufficiently good approximation of the potential.
- Solve the differential equation (14) with initial conditions (15). If $\mathcal{V}(x)$ is a polynomial function, there are good chances to solve this equation analytically. Otherwise we should reiterate to numerics. Check a compatibility of $f(k=0) = A$ with the normalization condition.
- Analytically or numerically take the inverse Fourier transform to obtain (check that) a non-negative function $f(x)$, to be interpreted as $\rho_*^{1/2}(x)$.
- Square $f(x)$ and check the $L^2(\mathcal{R})$ normalization to arrive at the desired terminal pdf of a confined Lévy-stable stochastic process.

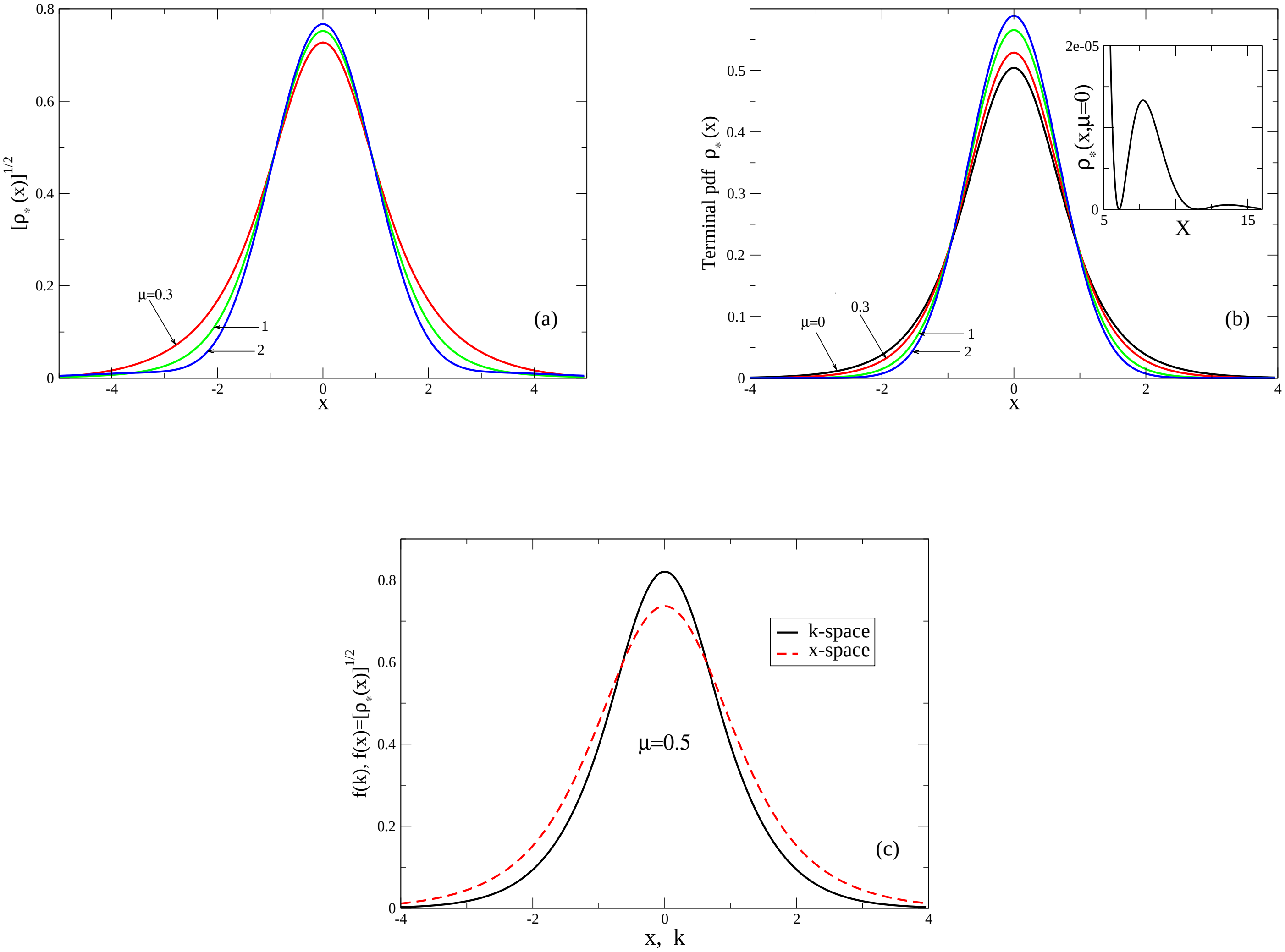

FIG. 1: Panel (a) inverted Fourier images of $[\rho_*(x)]^{1/2}$; panel (b) - terminal pdfs at different μ (figures), including $\mu = 0$ and $\mu = 2$. The inset shows an oscillatory asymptotics (for $x > 5$) of the solution corresponding to $\mu = 0$. Panel (c) compares the behavior of generic solutions in k and x-spaces, here displayed for $\mu = 0.5$.

III. LÉVY-STABLE OSCILLATORS: $\mathcal{V} = x^2/2$, $\mu \in (0,2)$

We consider an exemplary case of an analytic realization of the previously outlined procedure. Let us begin with the equation

$$\mathcal{V}(x)\rho_*^{1/2} \equiv \frac{x^2}{2}\rho_*^{1/2} = -|\Delta|^{\mu/2}\rho_*^{1/2},\ 0 < \mu < 2. \tag{17}$$

We take Fourier images of both sides of Eq.(17) to obtain

$$u_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x^2}{2}f(x)e^{\imath kx}dx = -\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\partial^2}{\partial k^2}\int_{-\infty}^{\infty}f(x)e^{\imath kx}dx \equiv -\frac{1}{2}\frac{\partial^2 f(k)}{\partial k^2} \tag{18}$$

Accordingly, we have

$$\frac{d^2 f(k)}{dk^2} = 2|k|^{\mu} f(k). \tag{19}$$

The main idea here (and possibly for the case of an arbitrary potential) is to follow the same approach as that for the Airy function, corresponding to $\mu = 1$, [2, 15]. Namely, we should find the decaying solution of the corresponding differential equation in the k-space on the positive semi-axis ($k > 0$), and an oscillatory one on the negative semi-axis ($k < 0$). Then we need to shift the obtained solution to the right so that the first maximum of the oscillatory part is at $k = 0$. After "chopping" the rest of the oscillating part one has to reflect the remaining piece about the vertical axis to get an even "bell-shaped" function. The obtained k-space solution should be Fourier-inverted and squared (keeping in mind the $L^2(R)$ normalization) to give the desired terminal pdf in the x -space.

Let us consider a pair of equations $\frac{d^2 f(k)}{dk^2} = 2 \text{ sign}k\ |k|^\mu f(k)$, substituting the previous single one, (19):

$$\begin{cases} \frac{d^2 f(k)}{dk^2} = 2k^\mu f(k), & k > 0 \\ \frac{d^2 f(k)}{dk^2} = -2(-k)^\mu f(k), & k < 0. \end{cases} \tag{20}$$

The resultant solutions have different forms for $k > 0$ and $k < 0$ respectively, [12]. Namely, for $k \geq 0$ we have

$$f(k) = \sqrt{k}\left[C_{11} I_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right) + C_{12} K_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right)\right],\ q = \frac{1}{2}(\mu + 2) \tag{21}$$

while for $k < 0$ there holds

$$f(k) = \sqrt{|k|}\left[C_{21} J_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right) + C_{22} N_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right)\right], \tag{22}$$

Here $J_\nu(x)$ and $N_\nu(x)$ are Bessel functions and $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions, see [13]. The asymptotics of $I_\nu(x)$ and $K_\nu(x)$ at $x \to \infty$ reads [13]

$$I_\nu(x) \approx \frac{e^x}{\sqrt{2\pi}\ x},\ K_\nu(x) \approx \frac{\pi}{2x}e^{-x}, \tag{23}$$

while as $k \to -\infty$ the asymptotics of the functions $J_\nu(x)$ and $N_\nu(x)$ is oscillatory [13]. This means that to have localized pdf, we should leave the term with $K_{\frac{1}{2q}}$ in (21) only so that $f(k)$ assumes following form

$$f(k) = \begin{cases} C_{12}\sqrt{k} K_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right), & k \geq 0 \\ \sqrt{|k|}\left[C_{21} J_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right) + C_{22} N_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right)\right], & k < 0. \end{cases} \tag{24}$$

As the equation (20) is of second order, we should impose the continuity conditions at $k = 0$ for a solution and its first derivative. We note, that the numerical solution of equation (20) directly involves the value of function and its first derivative at $k = 0$. That implies (see Appendix A for more details)

$$f(k) = C\sqrt{|k|}\begin{cases} K_\nu(u), & k \geq 0 \\ \frac{\pi}{2}\left[\cot\frac{\pi\nu}{2} J_\nu(u) - N_\nu(u)\right], & k < 0, \end{cases} \tag{25}$$

where $C \equiv C_{12}$,

$$\nu = \frac{1}{2q} \equiv \frac{1}{\mu + 2},\ u = \frac{\sqrt{2}}{q}|k|^q \equiv \frac{2\sqrt{2}}{\mu + 2}|k|^{1+\frac{\mu}{2}}. \tag{26}$$

We note here that for $\mu = 1$ we obtain from (25)

$$f(k) = C\sqrt{k} K_{\frac{1}{3}}\left(\frac{2\sqrt{2}}{3}k^{\frac{3}{2}}\right) = C\frac{\pi\sqrt{3}}{2^{\frac{1}{6}}}\text{Ai}\left(2^{\frac{1}{3}}\ k\right), \tag{27}$$

known from Ref. [2], where the eigenvalue problem for the Cauchy oscillator has been solved, see also Ref. [8].

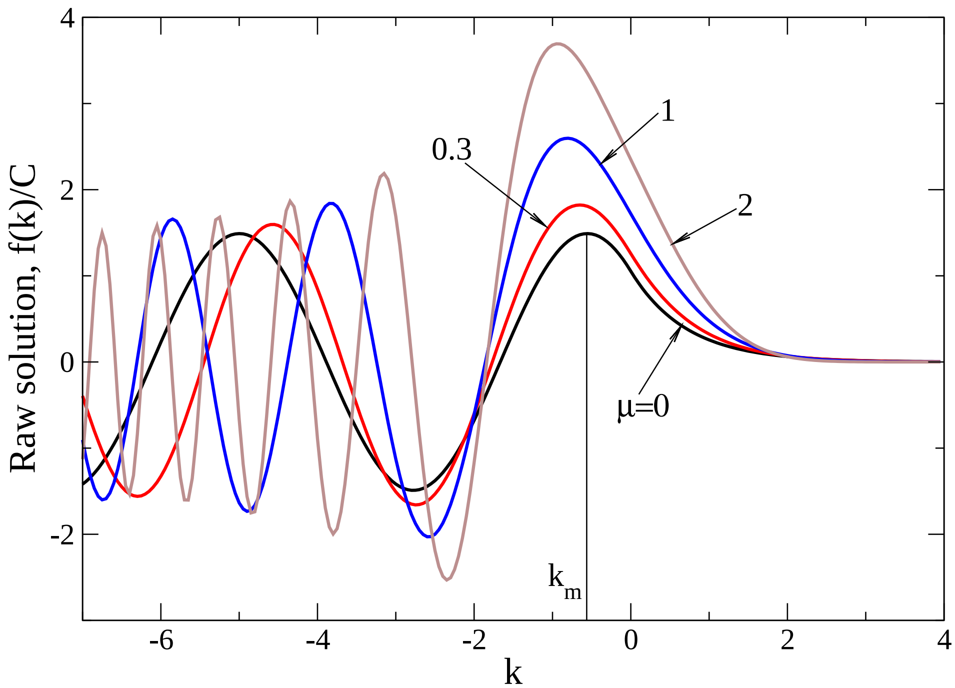

FIG. 2: Raw solutions of Eq. (25). Figures correspond to μ values. Solution for $\mu = 1$ corresponds to Airy function (27). Formal solution for $\mu = 0$ and the position of the first maximum of the oscillatory part k_m are shown as an example.

The next step is to find the position k_m of the first maximum of an oscillating part, next shift the solution to the right by k_m, reflect the solution with respect to the y axis and "chop" the rest of oscillating parts. By equating to zero the first derivative of an oscillating contribution to (25), after some algebra we get the following equation

$$N_{\nu-1}(u) - \cot\frac{\pi\nu}{2} J_{\nu-1}(u) = 0, \tag{28}$$

where ν and u are defined by (26). Solutions of this equation can be tabulated, see Table I. The "raw" solutions (25) are shown (along with the position of k_m) in Fig. 2.

The normalization condition allows us to fix the admissible values of hitherto unspecified constant C. Namely, we have

$$C^2 \int_{-\infty}^{\infty} f^2(k)dk = \; 2C^2 \int_0^{\infty} f^2(k)dk = 2C^2 \left[\int_0^{-k_m} f_1^2(k)dk + \int_{-k_m}^{\infty} f_2^2(k)dk \right] = 1, \tag{29}$$

where f_1 and f_2 denote the oscillatory and decaying parts of Eq. (25) respectively. This integration can be performed numerically and results are reproduced in the right column of the Table I.

The final step of the procedure is to invert the k-space solutions to x-space and square them to obtain the desired terminal pdf. Except for special μ cases, this procedure can be accomplished only numerically. Fig. 1 shows both the inverted functions $f(k)$, corresponding to square roots of terminal pdfs (panel a) and those pdfs themselves (panel b). We plot here the exemplary case of $\mu = 0.5$, the situation for other $\mu \in (0, 2)$ is qualitatively the same.

IV. POTENTIAL $\mathcal{V} = x^2/2$ AND THE LIMITING (MIS)BEHAVIOR AT THE BOUNDARIES OF $(0, 2) \ni \mu$.

The stability interval $\mu \in (0, 2)$ is an open set. However, since μ can be chosen to be arbitrarily close, respectively to 0 or 2, simply out of curiosity it is not useless to address a hitherto unexplored issue, of what is actually going on in the (possibly singular) limiting behaviors of $\mu \downarrow 0$ and $\mu \uparrow 2$.

We note that the operator $-|\Delta|^{\mu/2}$, as defined by Eq. (2) is a pseudo-differential (Riesz) operator and the integral there-in needs to be taken as its Cauchy principal value. On formal grounds, nothing precludes a literal setting of $\mu = 0$ or $\mu = 2$ in this formula, instead of the "normal" stability index values $\mu \in (0,2)$. The operator $-|\Delta|^{\mu/2}$ still remains a legitimate pseudodifferential operator and cannot be converted into any standard derivative, as the compelling but naive interpretation of $-|\Delta|^0$ and $-|\Delta|$ would suggest on the basis of the standard fractional derivative definition $(-\Delta)^{\mu/2} \equiv -\partial^\mu/\partial|x|^\mu$.

Concerning $\mu = 2$, we recall the eigenvalue equation $(-\Delta + \frac{x^2}{2} - E_0)\rho_*^{1/2} = 0$, where $E_0 = 1/\sqrt{2}$ is the lowest eigenvalue for a quantum harmonic oscillator in the units $\hbar^2/(2m) = 1$, $\omega^2 = m$, where m is a particle mass and ω is an oscillator frequency. That gives rise to the Gaussian ground state function. However, this equation is plainly incompatible with $(-|\Delta| + \frac{x^2}{2})\rho_*^{1/2} = 0$, see e.g. also (2). The latter equation does admit a non-Gaussian solution corresponding to the zero eigenvalue, which is derivable from (21)-(25), see also Fig. 1.

To have an insight into the $\mu \to 0$ (or μ near 0) regime, we note that the existence of Cauchy principal value for arbitrary μ can be proven by expanding $f(x+y)$ in the Taylor series with respect to small y: $f(x+y) - f(x) \approx y f'(x)$. Substituting this Taylor expansion into the integral (2), we find that it is proportional to the integral

$$\int_{-\varepsilon}^{\varepsilon} \frac{y}{|y|^{1+\mu}} dy = \int_{-\varepsilon}^{\varepsilon} |y|^{-\mu} \operatorname{sign} y \, dy \equiv 0 \tag{30}$$

as the integrand is odd for all $\mu \in (0,2)$. This property holds true for $\mu = 0$ and $\mu = 2$ as well. Therefore the limits $\mu \downarrow 0$ and $\mu \uparrow 2$ cane be approached continuously.

Let us investigate the properties of the $-|\Delta|^{\mu/2} f(x)$ in the vicinity of $\mu = 0$, by turning over to the Fourier image of $f(x)$. We employ a redefinition of Eq. (2)

$$-|\Delta|^{\mu/2} f(x) = \frac{\Gamma(1+\mu)\sin\frac{\pi\mu}{2}}{\pi} \int_{-\infty}^{\infty} dy \frac{f(x+y) - f(x)}{|y|^{1+\mu}}. \tag{31}$$

which yields

$$-|\Delta|^{\mu/2} f(x) = \frac{\Gamma(1+\mu)\sin\frac{\pi\mu}{2}}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{-\imath kx} dk \int_{-\infty}^{\infty} \frac{(e^{-\imath ky} - 1)dy}{|y|^{1+\mu}}. \tag{32}$$

The integral over dy can be calculated as follows

$$\int_{-\infty}^{\infty} \frac{(e^{-\imath ky} - 1)dy}{|y|^{1+\mu}} \equiv 2\int_0^{\infty} \frac{(\cos ky - 1)dy}{|y|^{1+\mu}} = 2|k|^\mu \Gamma(-\mu) \cos\frac{\pi\mu}{2}. \tag{33}$$

It is seen that at the limiting value $\mu = 0$, $\Gamma(0)$ is divergent, so that the integral (33) is divergent as well. However, irrespective of how close to zero the label $\mu > 0$ is, the integral (33) is convergent. Since we are interested in a continuous limiting procedure, $\mu \downarrow 0$, it is interesting to observe that the divergence of the the Fourier integral becomes compensated, if we substitute it back to Eq. (32) and next consider the limiting behavior of the result. Indeed,

$$\begin{aligned} -|\Delta|^{\mu/2} f(x) &= \frac{2\Gamma(1+\mu)\Gamma(-\mu)\sin\frac{\pi\mu}{2}\cos\frac{\pi\mu}{2}}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} |k|^\mu f(k) e^{-\imath kx} dk = \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |k|^\mu f(k) e^{-\imath kx} dk. \end{aligned} \tag{34}$$

Here we use the identity

$$\Gamma(1+\mu)\Gamma(-\mu) = -\frac{\pi}{\sin\pi\mu}. \tag{35}$$

Hence, (34) tells us that the Fourier image of the fractional operator (31) equals to $|k|^\mu$ for all μ's, including the boundary values 0 and 2. The pseudo-differential operator $-|\Delta|^{\mu/2}$ is defined via the Fourier transform of functions in its domain, valid for all $\mu \in [0,2]$. Therefore, on the Fourier transform level, we can safely pass to the boundary values 0 and 2 of the stability interval $[0,2]$. Consequently, formal symbols $-|\Delta|^0$ and $-|\Delta|$ still refer to well defined and non-trivial pseudo-differential operators.

Accounting for an additive perturbation $\mathcal{V} = x^2/2$, the eigenfunctions corresponding to the zero eigenvalue can be directly inferred. Although, as we shall prove in below, contrary to the case of $\mu = 2$ in case of $\mu = 0$ the pertinent eigenfunction of $(-|\Delta|^0 + \frac{x^2}{2})f(x) = 0$ cannot be interpreted as $\rho_*^{1/2}$ anymore.

Let us employ again Eq. (20), while setting $\mu = 0$. The "raw" solution of (20) can be either obtained from (24) or explicitly from (20). It has the form

$$f(k) = \begin{cases} C_{12}e^{-k\sqrt{2}}, & k \geq 0 \\ C_{21}\cos k\sqrt{2} + C_{22}\sin k\sqrt{2}, & k < 0. \end{cases} \tag{36}$$

The continuity condition at $k = 0$ for $f(k)$ reads

$$C_{12} = C_{21} \tag{37}$$

and for the derivative

$$C_{22} = -C_{12} = -C_{21}, \tag{38}$$

so that (we set $C_{12} \equiv C$)

$$f(k) = C\begin{cases} e^{-k\sqrt{2}}, & k \geq 0 \\ \cos k\sqrt{2} - \sin k\sqrt{2} \equiv \sqrt{2}\cos\left(k\sqrt{2} + \frac{\pi}{4}\right), & k < 0. \end{cases} \tag{39}$$

The first maximum of oscillatory part is located at

$$k_m = -\frac{\pi}{4\sqrt{2}} \approx -0.555360367, \tag{40}$$

in accordance with Table I. The "raw" solution (39) is shown (along with the position of k_m) in Fig. 2.

Now we shift the whole solution to the right and "chop" the unnecessary part of an oscillatory part

$$f(k) = C\begin{cases} \sqrt{2}\cos k\sqrt{2}, & 0 \leq k \leq -k_m \\ e^{-(k+k_m)\sqrt{2}}, & k > -k_m. \end{cases} \tag{41}$$

The normalization condition (29) reads

$$2C^2\left[2\int_0^{-k_m}\cos^2 k\sqrt{2}dk + \int_{-k_m}^{\infty} e^{-2\sqrt{2}(k+k_m)}dk\right] = 1 \tag{42}$$

so that $C = \frac{1}{\sqrt[4]{2}\sqrt{1+\frac{\pi}{4}}} \approx 0.629325$. This normalization coefficient is different from that in Table I, because the transition from Bessel functions of index 1/2 to elementary ones introduces an auxiliary coefficient $\sqrt{\pi}/2^{3/4}$. Thus, we have $C^2_{\text{table}}=(2\sqrt{2}/\pi)C^2= (2/\pi)/(1+\pi/4)$. This minor difference is insignificant with respect to the normalizability of the pertinent eigenfunction, as an overall coefficient before $f(k)$ is just C, (42).

Now we invert the Fourier transform to get

$$f(x) = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} f(k)\cos\ kx\ dk = C\sqrt{\frac{2}{\pi}}\left[\sqrt{2}\int_0^{-k_m}\cos\ k\sqrt{2}\ \cos\ kx\ dk + e^{-k_m\sqrt{2}}\int_{-k_m}^{\infty} e^{-k\sqrt{2}}\cos\ kx\ dk\right] =$$
$$= C\sqrt{\frac{2}{\pi}}\frac{4}{x^4-4}\left(x\ \sin\frac{\pi x}{4\sqrt{2}} - \sqrt{2}\cos\frac{\pi x}{4\sqrt{2}}\right),\ C = \frac{1}{\sqrt[4]{2}\sqrt{1+\frac{\pi}{4}}}. \tag{43}$$

The square of the function (43) is displayed in Fig. 1 (b) and has interesting properties. Namely, at the point $x = \pm\sqrt{2}$ function $f(x)$ looks divergent. However this is not so, because the terms in the numerator yield the same zero as in the denominator, so that the divergency is removed when one continuously approaches $x = \pm\sqrt{2}$. The function $f(x)$ decays at spatial infinities as $1/x^3$ and is positive for $|x| < 5$. For $|x| > 5$ we encounter oscillations so that both zeroes and negative values are developed. However, the square of $f(x)$, which is a probability density, does not have negative values, but has zeroes, see inset to Fig. 1 (b). This shows that $f^2(x) \equiv \rho_*(x)$ is indeed a probability density function. On the other hand, it is clear that function $f(x)$ cannot be interpreted as an (arithmetic) square root of a probability density $\rho_*(x)$ which is strictly positive and normalizable. This obstacle has not appeared in the case $\mu = 2$.

TABLE I: Roots $k_m(\mu)$ of Eq. (28) corresponding to first maximum of oscillatory part of (25) for different μ (middle column) and normalization constants $C(\mu)$ (right column).

μ	$k_m(\mu)$	$C(\mu)$
0.0	-0.55536=$-\frac{\pi}{4\sqrt{2}}$	0.597135=$\sqrt{\frac{2}{\pi}}\left(1+\frac{\pi}{4}\right)^{-1/2}$
0.2	-0.621962	0.500134
0.4	-0.679458	0.429855
0.6	-0.729002	0.376894
0.8	-0.771717	0.335701
1.0	-0.808617	0.302823
1.2	-0.840577	0.276010
1.4	-0.868346	0.253745
1.6	-0.892550	0.234970
1.8	-0.913716	0.218927
2.0	-0.932286	0.205597

V. CONCLUSIONS

To conclude, here we have presented the general formalism for how to find the terminal (at $t \to \infty$) pdf for a pre-defined semigroup potential and all stability index values $0 < \mu < 2$. We have considered the generic case of symmetric even pdfs, for which potential $\mathcal{V}(x)$is an even function. That was dicated by known spectral propeties of fractional operators associated with a symmetric stable noise. We have reduced the problem of finding a terminal pdf to that of finding a solution of the ordinary differential equation with infinite number of terms in momentum space. Such differential equation even if hard analytically, can rather easily be solved with a numerical assistance. For polynomial potentials the number of terms becomes finite and the pertinent equation can be solved analytically.

The outlined procedure has been explicitly verified for the family of Lévy stable oscillators, with a common quadratic semigroup potential $\mathcal{V}(x) = x^2/2$. In this case, the solution of corresponding differential equation in k-space has been obtained by employing a suitable continuity procedure at $k = 0$. After Fourier-inversion and squaring the result, this yields an ultimate functional form of the sought for terminal pdf of the jump-type process, for arbitrary $\mu \in (0, 2)$.

We have analyzed a limiting behavior of solutions in the vicinity and at the boundaries $\mu = 0$ and 2 of an open stability interval $(0, 2)$. We have shown thatfor $\mu = 2$ a positive $\rho_*^{1/2}(x)$ is obtained. In the case of $\mu = 0$ we have derived an explicit analytical form of the solution in k and (by Fourier inversion) in the x-spaces. The pertinent function shows a nontrivial oscillating behavior and thus is definitely not a square root of any pdf. Nonetheless, the square $f^2(x) \equiv \rho_*(x)$ of the real-valued solution $f(x)$ is an acceptable probability distribution, albeit with an oscillatory asymptotics.

Appendix A: Continuity conditions at $k = 0$

From (24), the functions at $k = 0$ read

$$C_{12}[\sqrt{k}K_\nu(u)]_{k=0} = C_{22}[\sqrt{k}N_\nu(u)]_{k=0}. \tag{A1}$$

The derivatives at $k = 0$

$$C_{21}[\sqrt{k}J_\nu(u)]'_{k=0} + C_{22}[\sqrt{k}N_\nu(u)]'_{k=0} = C_{12}[\sqrt{k}K_\nu(u)]'_{k=0}. \tag{A2}$$

Such forms of (A1) and (A2) are dictated by the following asymptotic expansions of Bessel functions near $k = 0$ in variables (26)

$$K_\nu(u) \approx \frac{\Gamma(-\nu)}{2}\sqrt{k}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu + \frac{\Gamma(\nu)}{2\sqrt{k}}\left[\frac{\mu+2}{\sqrt{2}}\right]^\nu,$$
$$N_\nu(u) \approx -\frac{\cos\pi\nu\ \Gamma(-\nu)}{\pi}\sqrt{k}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu - \frac{\Gamma(\nu)}{\pi\sqrt{k}}\left[\frac{\mu+2}{\sqrt{2}}\right]^\nu,$$
$$J_\nu(u) \approx \frac{\sqrt{k}}{\Gamma(1+\nu)}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu. \tag{A3}$$

Eq. (A3) means that $[\sqrt{k}J_\nu(u)]_{k=0} = 0$. For reference purposes, the derivatives like $[\sqrt{k}N_\nu(u)]'_{k=0}$ (we first multiply by $\sqrt{k}$ and then differentiate) read

$$[\sqrt{k}N_\nu(u)]'_{k=0} = -\frac{\cos\pi\nu\ \Gamma(-\nu)}{\pi}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu,\ [\sqrt{k}K_\nu(u)]'_{k=0} = \frac{\Gamma(-\nu)}{2}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu,$$
$$[\sqrt{k}J_\nu(u)]'_{k=0} = \frac{1}{\Gamma(1+\nu)}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu. \tag{A4}$$

Substitution of values of functions and derivatives into (A1) and (A2) yields

$$C_{22} = -\frac{\pi}{2}C_{12},\ C_{21} = \frac{\pi}{2}C_{12}\cot\frac{\pi\nu}{2}, \tag{A5}$$

which, after employing the identity $\Gamma(1+\mu)\Gamma(-\mu) = -\pi/\sin(\pi\mu)$ gives rise to Eq. (25).

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