A New Proposal for the Picture Changing Operators in the Minimal Pure Spinor Formalism

Oscar A. Bedoya[*,1] and Humberto Gomez[†,2]

[*]*Instituto de Física-Universidade de São Paulo*
Caixa Postal 66.318 CEP 05314-970 São Paulo, SP, Brazil

[†]*Instituto de Física Teórica UNESP - Universidade Estadual Paulista*
Caixa Postal 70532-2 01156-970 São Paulo, SP, Brazil

Abstract

Using a new proposal for the "picture lowering" operators, we compute the tree level scattering amplitude in the minimal pure spinor formalism by performing the integration over the pure spinor space as a multidimensional Cauchy-type integral. The amplitude will be written in terms of the projective pure spinor variables, which turns out to be useful to relate rigorously the minimal and non-minimal versions of the pure spinor formalism. The natural language for relating these formalisms is the Čech-Dolbeault isomorphism. Moreover, the Dolbeault cocycle corresponding to the tree-level scattering amplitude must be evaluated in $SO(10)/SU(5)$ instead of the whole pure spinor space, which means that the origin is removed from this space. Also, the Čech-Dolbeault language plays a key role for proving the invariance of the scattering amplitude under BRST, Lorentz and supersymmetry transformations, as well as the decoupling of unphysical states. We also relate the Green's function for the massless scalar field in ten dimensions to the tree-level scattering amplitude and comment about the scattering amplitude at higher orders. In contrast with the traditional picture lowering operators, with our new proposal the tree level scattering amplitude is independent of the constant spinors introduced to define them and the BRST exact terms decouple without integrating over these constant spinors.

[1]abedoya@fma.if.usp.br
[2]humgomzu@ift.unesp.br

Contents

1 Introduction **3**

2 Review of Minimal Pure Spinor Formalism **6**

3 The New Picture Changing Operators **8**

3.1 The New Proposal for the PCO's 9

3.2 Some Examples for the Constant Spinors C^I_α 11

4 The Tree Level Scattering Amplitude and Čech-Dolbeault Equivalence in the Pure Spinor Formalism **14**

4.1 Integration Contours 14

4.2 The Scattering Amplitude as an Integral over the Projective Pure Spinor Space . . 15

4.2.1 Contours for the Amplitude in the Projective Pure Spinor Coordinates . . . 15

4.2.2 The Tree Level Scattering Amplitude in the Projective Pure spinor Space . . 16

4.3 Čech and Dolbeault Language 19

4.3.1 Čech-Dolbeault Isomorphism 21

4.4 Equivalence Between the Minimal and Non-Minimal Formalism 23

4.4.1 The Projective Pure Spinor Degree 25

4.4.2 The Dolbeault Cocycle 26

4.4.3 A Particular Regulator 29

5 Symmetries of the Scattering Amplitude **30**

5.1 BRST Invariance 30

5.2 Lorentz and Supersymmetry Invariance 34

5.2.1 Lorentz Invariance 34

5.2.2 Invariance under Supersymmetry 35

6 Independence of the Constant Spinors C^I_α's **37**

6.1 Pure Spinors in $d = 4$: A Simple Example 37

6.2 Pure Spinors in $d = 10$ 38

7 Relation with Twistor Space **39**

8 Comments About the Loop-Level **41**

8.1 Product of Čech Cochains 41

8.2 The b-ghost 43

8.2.1 The Ghost Number Bidegree 44

9 Conclusions **46**

A Some Simple Examples **47**

A.1 The Pure Spinor Condition in the $U(5)$ decomposition 47

A.2 Another Cover For The Pure Spinor Space 48

A.3 The Čech-Dolbeault Correspondence for Pure Spinor in $d = 4$ 50

A.4 Global Integrals 52

B Proof of the Identity $[\mathbf{d}\tilde{\lambda}] = \mathbf{d}u_{12} \wedge ... \wedge \mathbf{d}u_{45}$. **53**

1 Introduction

For more than a decade a manifestly super-Poincaré covariant formulation for the superstring, known as the pure spinor formalism [1], has shown to be a powerful framework in two branches. The first one is the computation of scattering amplitudes and the second one is the quantization of the superstring in curved backgrounds which can include Ramond-Ramond flux. The strength of the pure spinor formalism resides precisely in the fact that it can be quantized in a manifestly super-Poincaré manner, so this covariance is not lost neither in the scattering amplitudes computation nor in the quantization of the superstring in curved backgrounds.

Since the present paper is about the first branch, we will give a brief description of what has been done in scattering amplitudes, not attempting to give a complete list of references.
One key ingredient in this formalism is a bosonic ghost λ^α, constrained to satisfy Cartan's pure spinor condition in 10 space-time dimensions [2] [3]. The prescription for computing multiloop am-

[3]Even before pure spinor were incorporated in the description for the superstring, Howe showed that integrability

plitudes was given in [4], where as in the RNS formalism, it was necessary to introduce picture changing operators (PCO's) in order to absorb the zero-modes of the pure spinor variables. Up to two-loops, various amplitudes were computed in [5], [6] and [7]. Later on, by introducing a set of non-minimal variables $\bar{\lambda}_\alpha$ and r_α, an equivalent prescription for computing scattering amplitudes was formulated in [8] and [9]. This last superstring description is known as the "non-minimal" pure spinor formalism, in order to distinguish it from the former "minimal" pure spinor formalism. With the non-minimal formalism, also were computed scattering amplitudes up to two-loops [10], [11]. Because of its topological nature, in the non-minimal version it is not necessary to introduce PCO's. Nevertheless, it is necessary to use a regulator. The drawback of having to introduce this regulator appears beyond two-loops, since it gets more complicated due to the divergences coming from the poles contribution of the b ghost [12] [13].

In this paper we will make a new proposal for the lowering picture changing operators, so in the following, we will discuss some facts which led us to them. First of all, the pure spinor condition defines a space, also called the pure spinor cone. In the geometric treatment by Nekrasov [14] it was found that the pure spinor space has non-vanishing first Pontryagin class, as well as non-vanishing first Chern class; leading to anomalies in the pure spinor space diffeomorphism and worldsheet conformal symmetry respectively. Nevertheless, the careful analysis in [14] shows that these anomalies are cancelled by removing the tip of the cone i.e the point $\lambda^\alpha = 0$. Therefore, in order to have a well defined theory, one should remove this point from the pure spinor space. Secondly, according to Berkovits' prescription for computing scattering amplitudes [4], in order to match the 11 pure spinor zero-modes in the minimal formalism, one should introduce 11 lowering PCO's defined by $Y^I_{Old} = C^I_\alpha \theta^\alpha \delta(C^I_\alpha \lambda^\alpha)$, for $I = 1 \ldots 11$, where C^I_α are constant spinors. In this definition, θ^α are the fermionic superspace coordinates. With the measure element also given in [4], the integration over the pure spinor zero-modes is performed without removing the point $\lambda^\alpha = 0$[4]. A third consideration that suggests for another treatment for the PCO's comes from the higher dimensional twistor transform using pure spinor; which allowed to obtain higher-dimensional scalar Green's functions [15], [16]. As shown in [16], in order to integrate over the projective pure spinor space when $d > 6$, it was necessary to develop integration techniques because of the non-linearity of the pure spinor conditions. Those integrations are always integrations over cycles. These three considerations lead us to define a new lowering PCO, given by $Y^I_{New} = \frac{C^I_\alpha \theta^\alpha}{C^I_\alpha \lambda^\alpha}$. In this way the integration over the pure spinor zero-modes is performed as a multidimensional Cauchy integral, where the integration contours go around the anomalous point $\lambda^\alpha = 0$. As we will discuss in this paper, the new PCO fulfill our requirement and as a bonus, allows to establish elegant relationships

along pure spinor lines allowed to find the super Yang-Mills and supergravity equations of motion in ten dimensions [3].

[4]Here is worth to mention that the geometric treatment of [14] was posterior to the multiloop scattering amplitude prescription of [4].

between the minimal and non-minal formalisms, as well as between the minimal formalism and the twistor space. Furthermore, as was shown explicitly by tree and one-loop computations, given the distributional definition of the PCO's Y^I_{Old}, the scattering amplitudes depends on the constant spinors C^I; so for some choices of these C^I's, the theory is non-Lorentz invariant and the unphysical states do not decouple [17]. These issues were solved by integrating over the C^I's [17], [18]. In contrast, with our PCO's proposal there is no need to integrate over them. We will also formally prove that at tree level the unphysical states decouple and that the scattering amplitude does not depend on the constant spinors C^I's.
Although we only consider tree-level scattering amplitudes in this paper, we hope to make some progress at the loop level in the future, by also redefining the raising PCO's.

The organization of this paper is as follows. In section 2 we briefly review the minimal pure spinor formalism, where we focus in introducing the basic notation in order to write down the tree-level scattering amplitude prescription of [4].
In section 3 we make our proposal for the new set of PCO's and discuss the restriction that must be imposed in order to have a well defined multidimensional Cauchy-type integral, which will result in the condition that the integration cycles go around the anomalous point of the theory $\lambda^\alpha = 0$. It happens that this condition is related to the specific choice of the constant spinors C^I's; so we will give two examples, one where the C^I's choice does not allows to define contours around the origin and another one which does. It turns out that the first choice is the same made in [17], which will allow to make some comparisons.
In section 4 we will compute the tree-level scattering amplitude. We start by formally defining the integration contours. Then, we proceed to write the amplitude using the projective pure spinor coordinates. Using these coordinates we analyze the poles structure and express the result of the scattering amplitude in terms of the *degree* of the projective pure spinor space, which is useful to relate the minimal and non-minimal formalism. Although in [8] was argued that taking the large scale limit for a regulator of the non-minimal pure spinor formalism, the scattering amplitude behaves like the scattering amplitude in the minimal formalism using the old PCO's and in [19] was shown that fixing the gauge of a topological theory of gravity coupled to the worldsheet, the old PCO's are equivalent to a particular regulator in the non-minimal side, we present here a rigorous equivalence at tree level, in which the PCO's do not correspond to any particular regulator in the non-minimal side. Computations of the kinematical factors in one and two-loops [10] give evidence of the equivalence of the scattering amplitudes prescription for the two formulations, as well as the equivalence obtained in [19], so it will be interesting to generalize the arguments presented in this paper at the loop level. The relationship between the minimal and non-minimal formalisms will be established using the Čech-Dolbeault language; for that reason we include a subsection about this subject.
In section 5 we show that the scattering amplitude is invariant under BRST, Lorentz and super-

symmetry transformations. Also we show the decoupling of unphysical states. The Dolbeault formulation will be extremely useful, both for proving the invariances as well as the decoupling of unphysical states.
In section 6 we prove that the scattering amplitude is independent of the constant spinors C^I's. First we consider the simplest non-trivial case, i.e pure spinor in four dimensions. Then, we proceed to consider the ten dimensional case. The two cases are studied differently; in four dimensions is straightforward and it teaches us what should be done. Extending the four dimensional proof to ten dimensions would be difficult, so we present a more elegant demonstration using the Čech-Dolbeault language.
In section 7 we will establish a direct relation between pure spinor scattering amplitudes and Green's functions for massless scalar fields in ten dimensions.
In section 8 we will comment about what should be done in order to have a genus g formulation for the scattering amplitude. In particular, we define a product for Čech cochains which would allow to get a well defined scattering amplitude from the Čech point of view.
Finally, we present some conclusions. The appendix contains several simple examples cited through the paper, as well as some demonstration of statements.

2 Review of Minimal Pure Spinor Formalism

In this section we will review the tree-level N-point amplitude prescription given in [4]. As noted in [17], the picture changing operators are not BRST closed inside the correlators, leading to a more careful treatment for decoupling the unphysical states.

In the pure spinor formalism, the type IIB superstring action is given by

$$S = \frac{1}{2\pi}\int_{\Sigma_g} \mathrm{d}^2 z \ \left(\frac{1}{2}\partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial}\theta^\alpha + \hat{p}_\alpha \partial\hat{\theta}^\alpha - \omega_\alpha \bar{\partial}\lambda^\alpha - \hat{\omega}_\alpha \partial\hat{\lambda}^\alpha\right), \tag{2.1}$$

where $(x^m, \theta^\alpha, \hat{\theta}^\alpha)$ are coordinates for the type IIB ten-dimensional superspace. So, the indices run as follows: $m = 0\ldots 9$, and $\alpha = 1,\ldots 16$. $(p_\alpha, \hat{p}_\alpha, \omega_\alpha, \hat{\omega}_\alpha)$ are the conjugate momenta to $(\theta^\alpha, \hat{\theta}^\alpha, \lambda^\alpha, \hat{\lambda}^\alpha)$, while λ^α and $\hat{\lambda}^\alpha$ satisfy the pure spinor condition in $d = 10$

$$\lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta = 0\,, \qquad \hat{\lambda}^\alpha(\gamma^m)_{\alpha\beta}\hat{\lambda}^\beta = 0, \tag{2.2}$$

where the matrices γ^m are generators of the Clifford algebra in $\mathbb{R}^{10}$. From now on, we will focus on the left moving variables in (2.1), keeping in mind that all the subsequent treatment is analogous for the right moving variables. From (2.1) we can find easily the OPE's

$$x^m(y)x^n(z) \to -\eta^{mn}\ln|y-z|^2, \qquad p_\alpha(y)\theta^\beta(z) \to \frac{\delta_\alpha^\beta}{(y-z)}. \tag{2.3}$$

Nevertheless, the pure spinor condition does not allow a direct computation of the OPE among λ^α and ω_α. As discussed in [1], the pure spinor spinor constraint must be solved, expressing λ^α in terms of 11 unconstrained $U(5)$ variables $(\lambda^+, \lambda_{ab}, \lambda^a)$, where $a = 1, \ldots 5$ and $\lambda_{ab} = -\lambda_{ba}$. Although those $U(5)$ fields are not manifestly Lorentz invariant, their OPE's are equivalent to Lorentz invariant OPE's involving λ^α, the pure spinor Lorentz current $N_{mn} = \frac{1}{2}\omega_\alpha(\gamma_{mn})^\alpha{}_\beta\lambda^\beta$ and the pure spinor ghost number current $J = \omega_\alpha\lambda^\alpha$. Furthermore, note that because of the pure spinor condition, there is a gauge invariance $\delta\omega_\alpha = \Lambda_m(\lambda^m\lambda)_\alpha$, so $(\lambda^\alpha, \omega_\alpha)$ must appear precisely in the gauge invariant combinations N_{mn} and J.The OPE's involving the pure spinor are

$$N_{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}\frac{(\gamma_{mn}\lambda)^\alpha(z)}{y-z}, \qquad J(y)\lambda^\alpha(z) \to \frac{\lambda^\alpha(z)}{y-z}, \tag{2.4}$$

$$N^{mn}(y)N^{pq}(z) \to -3\frac{\eta^{q[m}\eta^{n]p}}{(y-z)^2} + \frac{\eta^{p[n}N^{m]q}(z) - \eta^{q[n}N^{m]p}(z)}{y-z},$$

$$J(y)J(z) \to \frac{-4}{(y-z)^2}, \qquad J(y)N^{mn}(z) \to regular,$$

$$N_{mn}(y)T(z) \to \frac{N_{mn}(z)}{(y-z)^2}, \qquad J(y)T(z) \to \frac{-8}{(y-z)^3} + \frac{J(z)}{(y-z)^2},$$

where T is the energy momentum tensor

$$T = \frac{1}{2}\partial x^m \partial x_m + p_\alpha \partial\theta^\alpha - \omega_\alpha \partial\lambda^\alpha. \tag{2.5}$$

Note that the ghost number current and pure spinor Lorentz current have levels -4 and -3 respectively. Furthermore, the ghost number current has anomaly -8, which should be kept in mind for defining scattering amplitudes.

Besides the pure spinor λ^α, another key ingredient in this formalism is the BRST charge $Q = \oint dz \lambda^\alpha d_\alpha$, where

$$d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial x_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta) \tag{2.6}$$

is the supersymmetric Green-Schwarz constraint. Given the supersymmetric combination $\Pi^m = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta)$, d_α has the following OPEs

$$d_\alpha(y)d_\beta(z) \to \frac{\gamma^m_{\alpha\beta}\Pi_m}{y-z}, \qquad d_\alpha(y)\Pi^m(z) \to \frac{\gamma^m_{\alpha\beta}\partial\theta^\beta}{(y-z)}, \tag{2.7}$$

$$d_\alpha(y)f(x(z),\theta(z)) \to \frac{D_\alpha f(x(z),\theta(z))}{y-z}, \tag{2.8}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\theta\gamma^m)_\alpha\frac{\partial}{\partial x^m}$ is the supersymmetric derivative. From the first OPE in (2.7) it can easily be checked that $Q^2 = 0$ because of the pure spinor condition.

Vertex operators in the pure spinor formalism for the massless states are given by ghost number one and conformal weight zero objects. $V=\lambda^\alpha A_\alpha(x,\theta)$ is the most general object satisfying both conditions. Since V must be in the cohomology of Q, then

$$D_{(\alpha}A_{\beta)}=\gamma^m_{\alpha\beta}A_m, \tag{2.9}$$

where the indices on the left hand side are symmetrized and $A_m(x,\theta)$ is some superfield. The gauge invariance $\delta V=Q\Lambda$ implies that $\delta A_\alpha=D_\alpha\Lambda$ and $\delta A_m=\partial_m\Lambda$, which are the gauge invariance for super-Yang-Mills, while the equation (2.9) is the super-Yang-Mills equation of motion. In order to define scattering amplitudes, the integrated version of the vertex operators U is also needed. In the case of the massless vertex operator, through $\partial V=QU$ the explicit form of U is found

$$U=\partial\theta^\alpha A_\alpha(x,\theta)+\Pi^m A_m(x,\theta)+d_\alpha W_\alpha(x,\theta)+\frac{1}{2}N^{mn}F_{mn}(x,\theta), \tag{2.10}$$

where W^α and F_{mn} are the spinor and vector super-Yang-Mills superfield strengths respectively.

Since the pure spinor λ^α have 11 zero modes[5] in any Riemann surface Σ_g, it is necessary to absorb them when computing scattering amplitudes. The manner that they are absorbed is by introducing 11 PCO's

$$Y^I_C=C^I_\alpha\theta^\alpha\delta(C^I_\alpha\lambda^\alpha),\ \ I=1,\ldots 11, \tag{2.11}$$

inside the scattering amplitude [4], which for N-points at tree level is

$$\mathcal{A}=\langle V_1(z_1)V_2(z_2)V_3(z_3)\int dz_4U_4(z_4)\ldots\int dz_NU_N(z_N)Y^1_C(y_1)\ldots Y^{11}_C(y_{11})\rangle. \tag{2.12}$$

So, to perform this computation the OPE's (2.3), (2.4) and (2.7) are used to integrate over the non-zero modes, remaining an integral over the zero modes of the pure spinor and θ^α[6]. Note that $\delta(C_\alpha\lambda^\alpha)$ is a Dirac's delta function and C^I_α is a constant projective spinor, which can be thought as a point in the $\mathbb{C}P^{15}$ space. Although in [4] it was argued that the scattering amplitude was independent of the constant spinors C^I_α, it was later found in [17] that indeed the amplitude depends on the choice of C^I_α and also that Q exact states do not decouple. In the next section, we propose a new picture operators, which does not have that disadvantage.

3 The New Picture Changing Operators

In this section we introduce the new lowering picture changing operators. In particular, we will discuss why with this new proposal for the picture changing operators, the origin must be removed

[5]the spinors ω_α, p_α and θ^α have $11g$, $16g$ and 16 zero modes respectively for the Riemann surface of genus g

[6]The integration over the fields x is treated in detail in D'Hoker and Phong [20], we will not focus in those integrals.

from the pure spinor space. This will allow to write the tree-level scattering amplitude in terms of the projective pure spinor variables in the following section, and also, to find a relationship with the twistor space in section 7. In the end of the present section we give examples of choices for the constant spinors C^I's and discuss their implications.

3.1 The New Proposal for the PCO's

In this subsection we discuss some motivations which led us to define new lowering PCO's.

The bosonic spinor λ^α, constrained to satisfy the pure spinor condition $\lambda\gamma^m\lambda = 0$, constitutes an interesting and non-trivial complex space, which will be denoted through this paper as the pure spinor space PS or the pure spinor cone. Since the coordinates λ^α of such space are holomorphic, the integral

$$\int [\mathrm{d}\lambda]\delta(C\lambda)f(\lambda) \tag{3.1}$$

is only well defined if the domain of integration, i.e the cycles around which we integrate are known. Moreover, as shown by Nekrasov [14], the tip of the pure spinor cone $\lambda^\alpha = 0$ introduces anomalies. Then, by removing this point of the pure spinor space, the theory is anomaly free [7]. This is simple to see if one computes the de-Rham cohomology of the pure spinor minus the origin space

$$H^i\left(PS \setminus \{0\}\right) = \mathbb{R}, \quad \text{for } i = 0, 6, 15, 21, \tag{3.2}$$

so the first Chern class and second Chern character both vanish, $c_1(PS \setminus \{0\}) = ch_2(PS \setminus \{0\}) = 0$ and therefore the theory is anomaly free. This motivates us to make a new proposal for the PCO's, in such a way that the tip of the cone is naturally excluded. Furthermore, Skenderis and Hoogeveen [17] showed that the scattering amplitude, as formulated in [4], depends on the choice of the constant spinors C^I_α, having to integrate over them in order to obtain a manifestly Lorentz invariant prescription. Nevertheless, as we will show in section 6, the scattering amplitude will not depend on the constant spinors using the new PCO's.

Our proposal, which seems to be the most natural, is to define the PCO's as

$$Y^I_C = \frac{C^I_\alpha \theta^\alpha}{C^I_\alpha \lambda^\alpha}, \qquad I = 1, \ldots 11, \tag{3.3}$$

where C^I_α are again constant spinors. Just like the standard PCO's (2.11), this new PCO's are not manifestly Lorentz invariant. Also, since $QY^I_C = 1$, they are not BRST closed. Using these PCO's it will be necessary to modify the usual BRST charge of the minimal formalism in order to have a

[7] This is the unique singular point of the pure spinor space because it is a complex cone over the smooth manifold $SO(10)/U(5) \subset \mathbb{C}P^{15}$.

global description, as will be done in section 5. Then, we will be able to show in that section that the scattering amplitude is BRST, Lorentz and supersymmetric invariant.

Since we want to integrate over the pure spinor zero modes, basically as a multi-dimensional Cauchy's integral, we will start by considering the analogous of the poles. This role will be played by the denominators of the PCO's, so we start by defining the functions

$$f^I(\lambda) \equiv C^I_\alpha \lambda^\alpha, \tag{3.4}$$

which map the pure spinor space to the complex numbers for each value of $I = 1, ..., 11$, i.e $f^I : PS \rightarrow \mathbb{C}$. Given these functions, secondly we define the hypersurface "D_I" as the subspace $f^I = 0$

$$D_I = \{\lambda^\alpha \in PS : \, C^I_\alpha \lambda^\alpha = 0\}. \tag{3.5}$$

In order to have a well defined integration over the pure spinor space inside the scattering amplitude, it is necessary to impose the condition that the intersection between the D_I's satisfies $D_1 \cap D_2 ... \cap D_{11} = \{\text{finite number of points}\}$ in order to have a Cauchy like integral over PS. Just to be more explicit, using the $U(5)$ decomposition [1] for writing the pure spinor constraint, we require that the 16 equations

$$f^I = 0, \quad \text{and} \quad \chi^a = \lambda^+ \lambda^a - \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \quad \text{with} \quad I = 1, 2, ..., 11; \quad a, b, c, d, e = 1, 2, ..., 5, \tag{3.6}$$

intersect in a finite number of points. However, the five equations $\chi^a = 0$ must be taken carefully because with only this condition, there are more singular points besides $\lambda^\alpha = 0$. Therefore, a second set of equations $\zeta_a = \lambda^b \lambda_{ba} = 0$, must be taken into account. Both set of conditions $\chi^a = 0$ and $\zeta_a = 0$ come from the $U(5)$ decomposition of the pure spinor condition [1]. Although the first one implies in the second one when $\lambda^+ \neq 0$, as will be explained with one example in appendix A.1, disregarding the second one could lead to a not well defined tangent space at every point of PS. Therefore, both conditions will be considered when we construct an example for the C^I's in subsection 3.2.

To demand that the constant spinors C^I's are linearly independent in $\mathbb{C}^{16}$ is not enough to obtain an intersection in a finite number of points. However, clearly the origin $\{0\}$ is a common point in the intersection of all the hypersurfaces D_I. We claim that the only common point between the hypersurfaces D_I's is the origin because precisely, it is the unique anomalous or singular point of the theory. Therefore, the integration contours are those that go around the origin of the pure spinor space.

In the following subsection we give an example for the constant spinors C^I_α which allow for such a type of intersection.

3.2 Some Examples for the Constant Spinors C^I_α

In this subsection we will consider two examples. One where the C^I's are linearly independent, although do not allow for an intersection of the hypersurfaces D_I in a finite number of points. In the second example, we construct a set of C^I's which intersect just in the origin.

First Example We will make the same choice for the C^I's as in [17], so we consider this example basically to establish a comparison with this reference. Let the C^I_α's be in the $U(5)$ representation:

$$C^I_\alpha = (C^I_+, C^{I,ab}, C^I_a) \qquad a,b = 1,...,5,$$

where $C^{I,ab} = -C^{I,ba}$. Making the choice of [17]

$$C^1_\alpha = \delta^+_\alpha, \quad C^{2,ab} = \delta^{[a}_1 \delta^{b]}_2, \cdots, C^{11,ab} = \delta^{[a}_4 \delta^{b]}_5, \quad \text{all other} \quad C^I_\alpha = 0, \tag{3.7}$$

the functions f^I's are

$$f^1 = \lambda^+, \quad f^2 = \lambda_{12}, \quad f^3 = \lambda_{13}, \cdots, f^{11} = \lambda_{45}. \tag{3.8}$$

With the conditions $f^I = 0$ the pure spinor constraints are satisfied identically, but the parameters λ^a's are free, therefore the intersection is the space $\mathbb{C}^5$, in contrast with our requirement of intersecting just in the origin. With this choice we can "naively" compute the three point tree level amplitude only locally ($\lambda^+ \neq 0$), obtaining the same result as in [17] as we will review below. The answer will not be Lorentz invariant. For 3-points the computation is as follows:

$$\begin{aligned}
\mathcal{A} &= \langle \lambda^\alpha A_{1\alpha}(z_1) \lambda^\beta A_{2\beta}(z_2) \lambda^\gamma A_{3\gamma}(z_3) Y^1(z) \ldots Y^{11}(z) \rangle \\
&= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \frac{C^1\theta}{C^1\lambda} \cdots \frac{C^{11}\theta}{C^{11}\lambda} \\
&= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \frac{\theta^+}{\lambda^+} \frac{\theta_{12}}{\lambda_{12}} \cdots \frac{\theta_{45}}{\lambda_{45}} \\
&= \int_\Gamma \frac{\mathrm{d}\lambda^+ \wedge \mathrm{d}\lambda_{12} \wedge \ldots \wedge \mathrm{d}\lambda_{45}}{(\lambda^+)^3} \int \mathrm{d}^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \frac{\theta^+}{\lambda^+} \frac{\theta_{12}}{\lambda_{12}} \cdots \frac{\theta_{45}}{\lambda_{45}}.
\end{aligned} \tag{3.9}$$

where Γ is defined as $\Gamma = \{\lambda \in PS : |f^I| = \epsilon^I, I = 1,...,11, \epsilon^I \in \mathbb{R}^+\}$ and $[\mathrm{d}\lambda] = \mathrm{d}\lambda^+ \wedge \mathrm{d}\lambda_{12} \wedge \ldots \wedge \mathrm{d}\lambda_{45}/(\lambda^+)^3$ [4]. Note that naively $\lambda^+ = 0$ is a singularity, but we do not have access to it since we are on the patch $\lambda^+ \neq 0$. So, for this coordinate is possible that the cycle of integration is not well defined. Formally, we should choose a patch which allows to access the singularity. In this particular example, the singularity is $\mathbb{C}^5$, which is a non-compact and infinite space, that can not be contourned with a compact space defined by some cycle Γ. Therefore, in the Cauchy's sense this is a not well defined integral. That is what we meant with naively computing the integral. The only contribution to the integral above will come from $\alpha = \beta = \gamma = +$. In our case, in contrast with [17], there are no subtleties with the integrals coming from the other choices, which are of the form $\int_\Gamma \mathrm{d}\lambda_{ab} \frac{\lambda_{ab}}{\lambda_{ab}}$. For example,

$$\int_\Gamma [\mathrm{d}\lambda](\lambda^+)^2 \lambda_{cd} \frac{1}{\lambda^+} \frac{1}{\lambda_{12}} \cdots \frac{1}{\lambda_{45}} = \int_\Gamma \mathrm{d}\lambda^+ \mathrm{d}^{10}\lambda_{ab} \frac{\lambda_{cd}}{\lambda^+} \frac{1}{\lambda^+} \frac{1}{\lambda_{12}} \cdots \frac{1}{\lambda_{45}} \tag{3.10}$$

will give zero because there is a double pole in λ^+ and any choice of λ_{cd} will kill one of the poles λ_{ab}. Choosing $\alpha = \beta = \gamma = +$ we obtain

$$\mathcal{A} = \int \mathrm{d}^{16}\theta f_{+++}(\theta)\theta^+ \theta_{12} \ldots \theta_{45}, \tag{3.11}$$

which is exactly the same answer found by Skenderis and Hoogeveen in [17], as in their case, it is not Lorentz invariant. Now we give a geometrical explanation of why it is not Lorentz invariant. Remember that the intersection between the hypersurfaces is $\mathbb{C}^5$, $D_1 \cap \ldots \cap D_{11} = \mathbb{C}^5$, so the scattering amplitude is defined on the space

$$PS \setminus \mathbb{C}^5. \tag{3.12}$$

Since the $SO(10)$ group acts transitively up to scalings on the the pure spinor space PS, then it is always possible to have an element $g \in SO(10)$ such that if $\lambda \in (PS \setminus \mathbb{C}^5)$, then $(g\lambda) \notin (PS \setminus \mathbb{C}^5)$, i.e $(g\lambda) \in \mathbb{C}^5$. This argument implies that the scattering amplitude is not Lorentz invariant, since it is not invariant under $SO(10)$, and it is not globally defined on PS, because we can make a transformation from $(PS \setminus \mathbb{C}^5)$ to PS where the scattering is not defined. In the appendix A.4 we give further simple examples.

Note that the origin is the only fixed point under $SO(10)$ transformations acting on the pure spinor space[8], this means that the condition for the intersection of the D_I's in the origin, $D_1 \cap \ldots \cap D_{11} = \{0\}$, it is not just a sufficient condition, but actually it is necessary condition in order to get a well defined scattering amplitude, i.e that the scattering amplitude is invariant under the BRST, supersymmetry and Lorentz transformations, see section 5. Summarizing, we showed that this specific choice for the C^I's is not allowed, since it does not obey our requirement of the hypersurfaces intersecting at the origin.

Second Example Now, we show how to construct a set of C^I's which allow to satisfy $D_1 \cap \ldots \cap D_{11} = \{0\}$. This geometrical construction is as follows: Take eleven points satisfying the conditions $\chi^a = 0$ and $\zeta_a = 0$. Then, evaluate each one of the 10 gradient vectors V^a and A_a, corresponding to

[8] This is because the origin is the unique singular point in PS.

χ^a and ζ_a respectively, at each one of those eleven points (see the appendix A.1 for more details). With this vectors, we construct 11 planes through the origin, such that at each one of the 11 points in PS, the 10 gradients belong to the planes. We present the answer as an 11×16 matrix

$$C = \begin{pmatrix} 1 & 2 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & -1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & -2 & 4 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -2 & 3 & 4 & 3 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -3 & 0 & -3 & 0 & 0 & -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 2 & -1 & 1 & 0 & 1 \end{pmatrix} . \tag{3.13}$$

We computed $C^I_\alpha \lambda^\alpha$ and using Mathematica, we found the intersections of the 11 planes with the pure spinor condition

$$\chi^a = \lambda^+ \lambda^a - \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0.$$

The answer is 12 times the tip of the cone: $\lambda^\alpha = 0$. This number 12 is the *multiplicity* or number of times the hypersurfaces intersect. This will be further discussed in the next section. Nevertheless, there are 5 additional non-zero solutions[9]. This is not an issue, since this non-zero solutions are not in the remaining pure spinor equations $\zeta_a = \lambda^b \lambda_{ba} = 0$, therefore, we can discard them safely. Note that the 11 C^I's form a $\mathbb{C}^5$ space in $\mathbb{C}^{16}$, which is invariant by $U(5)$ group, so applying elements of $U(5)$ to the matrix C^I_α (3.13) we get an infinite numbers of C'^I's, for which the intersection with PS is the origin.

Instead of computing the scattering amplitude in this second example as we did in the first one, we will show in the next section how to find the answer without an explicit form for the C^I's. In conclusion, what we wanted to show with this example is that we can indeed find a set of constant spinors fulfilling our requirement of intersection of the planes and PS only at the origin.

[9]For these non-zero solution $\lambda^+ = 0$. Those are precisely the points for which the constrains $\chi_a = 0$ are not enough to describe the pure spinor space.

4 The Tree Level Scattering Amplitude and Čech-Dolbeault Equivalence in the Pure Spinor Formalism

In the present section we will compute the scattering amplitude in a covariant way. We start by defining the scattering amplitude and the integration contours. Then, we proceed to perform the scattering amplitude computation in the projective pure spinor space coordinates, where the singular point is explicitly removed. This scattering amplitude computation will become important in the rest of the paper. For instance, this computation will introduce the notion of degree of the projective pure spinor space, which will be useful to relate in a simple way the minimal and non-minimal formalisms. Actually, the framework in which we relate both formalisms is given by the Čech-Dolbeault language. This is not surprising because the PCO's are defined locally, so, the Čech language is a natural formalism to describe the scattering amplitude because it is a description in terms of patches. That is the reason why we include a subsection for reviewing the Čech-Dolbeault language.

At the end of this section we will argue that our picture changing operators are not related to any particular regulator.

4.1 Integration Contours

Before attempting to compute the tree level scattering amplitude, we must discuss which are the integration contours. This will allow to have a well defined amplitude.

The contours will be given by the homology cycles. In our case, they are naturally defined as

$$\Gamma = \{\lambda^\alpha \in PS : |f^I(\lambda)| = |C^I \lambda| = \varepsilon^I\}. \tag{4.1}$$

Clearly, Γ is an 11-cycle, i.e it has real dimension 11. Except for the integration contour, the tree level scattering amplitude corresponding to the zero modes has the same form as in the first example in the sub-section 3.2

$$\mathcal{A} = \int \mathrm{d}^{16}\theta \int_\Gamma W, \tag{4.2}$$

where W is given by

$$W = [\mathrm{d}\lambda] Y_C^1 ... Y_C^{11} \lambda^\alpha \lambda^\beta \lambda^\gamma \; f_{\alpha\beta\gamma}(\theta) \tag{4.3}$$

and Y_C^I's are the new PCO's (3.3). Since the integrand W satisfies $\mathrm{d}(W) = (\partial + \bar{\partial})(W) = 0$ then it belongs to the de-Rham cohomology group $H^{11}_{DR}(PS \setminus D)$, where $PS \setminus D$ is the space in which the W-form is defined, i.e D is the hypersurface on PS given by $D = D_1 \cup \ldots \cup D_{11}$. Then, the cycle Γ belongs to the homology group $H_{11}(PS \setminus D, \mathbb{Z})$. We will illustrate this with the following example. Consider for instance the integral $\int_\gamma \mathrm{d}z/z$, where γ is the circle $\gamma = \{z \in \mathbb{C} : |z| = \epsilon\}$. So any circle C around the origin is related to γ since $\gamma - C$ is the boundary of some annulus U, i.e $\partial(U) = \gamma - C$, therefore γ is an element of the homology group $H_1(\mathbb{C} \setminus \{0\}, \mathbb{Z})$ and by the Stokes theorem $\int_\gamma \mathrm{d}z/z = \int_C \mathrm{d}z/z$. In $\mathbb{C}^2$ we have an analogous situation, for example consider the integral $\int_\varphi \mathrm{d}z_1 \mathrm{d}z_2/(z_1 z_2)$. Here the torus φ, defined by $\varphi = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \epsilon_1, |z_2| = \epsilon_2\}$ is an element of the homology group $H_2(\mathbb{C}^2 \setminus \{(0, z_2) \,\text{and}\, (z_1, 0) : z_1, z_2 \in \mathbb{C}\}, \mathbb{Z}) = H_2((\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}), \mathbb{Z})$ and the integral depends only of the class of the torus φ. The same is true for the integral (4.2). Therefore, the integral (4.2) will depend only on the homology class cycle and the cohomology class cocycle. This is the principle that will allow us to show that the scattering amplitude is independent of the C^I's, which will be discussed in the section 6.

4.2 The Scattering Amplitude as an Integral over the Projective Pure Spinor Space

In the last subsection we have defined the integration contours in the pure spinor space in order to have a well defined tree level scattering amplitude. Now, in this subsection we will proceed to write the coordinates for the pure spinor space in terms of the projective pure spinor coordinates. Then, we will compute the tree-level scattering amplitude in this new coordinates.

As we will show in the next sub-subsection, in the projective coordinates we can make a simple analysis of the poles in the scattering amplitude integral. The cycle Γ previously defined will be used to obtain the integration contours in the projective pure spinor space.

4.2.1 Contours for the Amplitude in the Projective Pure Spinor Coordinates

We can write the pure spinor coordinates as $\lambda^\alpha = \gamma\,\tilde{\lambda}^\alpha$, where $\gamma \in \mathbb{C}$ and $\tilde{\lambda}^\alpha$ are global coordinates for the $SO(10)/U(5)$ space[10]. That is, $\tilde{\lambda}^\alpha$ satisfies the constraints $\tilde{\lambda}\gamma^m\tilde{\lambda} = 0$ and has the equivalence relation $\tilde{\lambda}^\alpha \sim c\tilde{\lambda}^\alpha$, where $c \in \mathbb{C}^*$. When $\gamma = 0$ then $\lambda^\alpha = 0$, but $\tilde{\lambda}^\alpha$ can take any value in the projective pure spinor space, i.e $SO(10)/U(5)$, also known as the twistor space [16]. In these coordinates the poles take the form

$$\begin{aligned} \gamma\tilde{f}^1 &\equiv \gamma\, C^1_\alpha \tilde{\lambda}^\alpha = 0, \\ \gamma\tilde{f}^2 &\equiv \gamma\, C^2_\alpha \tilde{\lambda}^\alpha = 0, \\ &\;\;\vdots \\ \gamma\tilde{f}^{11} &\equiv \gamma\, C^{11}_\alpha \tilde{\lambda}^\alpha = 0. \end{aligned} \tag{4.4}$$

[10] Actually γ is the fiber of the $\mathcal{O}(-1)$ line bundle over $SO(10)/U(5)$ [14].

When $\gamma \neq 0$, we have 11 constraints and 10 degrees of freedom for the projective pure spinor space, so, it is not possible to find a solution for the 11 constraints. On the other hand, when $\gamma = 0$, naively all the constraints behave as being zero. Nevertheless, we must consider this case inside the scattering amplitude. In the numerator of W there are 7 γ's coming from the integration measure plus 3 coming from the vertex operators, contributing in total γ^{10} in the numerator. Therefore, only one of the 11 γ's will remain in the denominator of W. This remaining γ kills one of the 11 functions $\tilde{f}^I$. Therefore, now the cycle Γ is given by

$$\Gamma = C \times \tilde{\Gamma}, \tag{4.5}$$

where C is the cycle $C = \{\gamma \in \mathbb{C} : |\gamma| = \epsilon\}$ and $\tilde{\Gamma}$ is a 10-cycle which we define in the following. After integrating around the contour $|\gamma| = \varepsilon$, which belongs to Γ and excludes the origin of the space, the denominator of W will have 11 $\tilde{f}^I$'s. However, remember that one of the $\tilde{f}^I$'s was killed by γ. Therefore, the cycle $\tilde{\Gamma}$ must be given by

$$\tilde{\Gamma} = \{|\tilde{f}^i| = \epsilon_i, \text{where the } i\text{'s are ten numbers between 1 to 11}\}. \tag{4.6}$$

After this simple analysis, now we proceed to compute the scattering amplitude.

4.2.2 The Tree Level Scattering Amplitude in the Projective Pure spinor Space

The tree-level scattering amplitude has the form

$$\mathcal{A} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta Y_C^1 ... Y_C^{11} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta). \tag{4.7}$$

As discussed in [4], the term $\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta)$ can always be written in the following form

$$\lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \propto (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)K, \tag{4.8}$$

up to BRST exact and global terms, which are decoupled as we will show later in section 5. K is the kinematic factor, which is a function of the polarizations and momenta[11]. Then the amplitude takes the form

$$\begin{aligned} \mathcal{A} &= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^1\theta}{C^1\lambda} ... \frac{C^{11}\theta}{C^{11}\lambda} (\lambda\gamma^m)_{\alpha_1} (\lambda\gamma^n)_{\alpha_2} (\lambda\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4\alpha_5} \theta^{\alpha_1}\theta^{\alpha_2}\theta^{\alpha_3}\theta^{\alpha_4}\theta^{\alpha_5} K \\ &= \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{\epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}} C^1_{\beta_1} ... C^{11}_{\beta_{11}}}{C^1\lambda ... C^{11}\lambda} (\lambda\gamma^m)_{\alpha_1} (\lambda\gamma^n)_{\alpha_2} (\lambda\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4\alpha_5} \theta^1 ... \theta^{16} K. \end{aligned} \tag{4.9}$$

In the coordinates $\lambda^\alpha = \gamma\tilde{\lambda}^\alpha$, we can choose the following parametrization for the projective pure spinor in the patch $\tilde{\lambda}^+ \neq 0$

$$\tilde{\lambda}^\alpha = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde} u_{bc} u_{de}). \tag{4.10}$$

[11]In general, when there are more than 3 vertex operators in the scattering amplitude, it must include integrals of the worldsheet coordinates $(z, \bar{z})$. However, we are not taking care of those terms.

So, as shown in [14], the integration measure becomes $[\mathrm{d}\lambda] = \gamma^7 \mathrm{d}\gamma \wedge \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}$ and the amplitude locally can be written as

$$\mathcal{A} = \int_\Gamma \frac{\mathrm{d}\gamma}{\gamma} \wedge \frac{\mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45} \epsilon^{\alpha_1 ... \alpha_5 \beta_1 ... \beta_{11}} C^1_{\beta_1} ... C^{11}_{\beta_{11}}}{C^1 \tilde{\lambda} ... C^{11} \tilde{\lambda}} (\tilde{\lambda}\gamma^m)_{\alpha_1} (\tilde{\lambda}\gamma^n)_{\alpha_2} (\tilde{\lambda}\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4 \alpha_5} K, \quad (4.11)$$

where the θ^α variables have been integrated. The integral around the contour $|\gamma| = \varepsilon$ is trivial, then

$$\mathcal{A} = (2\pi i) \int_{\tilde{\Gamma}} \frac{\mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45} \epsilon^{\alpha_1 ... \alpha_5 \beta_1 ... \beta_{11}} C^1_{\beta_1} ... C^{11}_{\beta_{11}}}{C^1 \tilde{\lambda} ... C^{11} \tilde{\lambda}} (\tilde{\lambda}\gamma^m)_{\alpha_1} (\tilde{\lambda}\gamma^n)_{\alpha_2} (\tilde{\lambda}\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4 \alpha_5} K, \quad (4.12)$$

where the contour $\tilde{\Gamma}$ was defined in (4.6). Note that up to a sign, the scattering amplitude is independent of the choice of the 10-cycle out of the 11 possibilities. To illustrate that we can consider the simplest and non trivial case of the projective pure spinor space, i.e the projective pure spinor space in $d = 4$, in this case the integral is (see also the appendix A.3)

$$\int_{\tilde{\gamma}} \frac{\epsilon_{ab} \tilde{\lambda}^a \mathrm{d}\tilde{\lambda}^b \epsilon^{cd} C^1_c C^2_d}{(C^1 \tilde{\lambda})(C^2 \tilde{\lambda})}, \quad (4.13)$$

where $\tilde{\lambda}^a = (\tilde{\lambda}^1, \tilde{\lambda}^2)$ are the homogeneous coordinates of $\mathbb{C}P^1$. In this case we have two choices. First we can take $\tilde{\gamma} = \{\tilde{\lambda} \in \mathbb{C}P^1 : |C^1 \tilde{\lambda}| = \epsilon\}$ and for simplicity we set $C^1 = (1, 0)$ and $C^2 = (0, 1)$. In the patch $\tilde{\lambda}^2 \neq 0$ we have the parametrization $\tilde{\lambda}^a = (u, 1)$, therefore, the contour $\tilde{\gamma}$ is well defined and the integral (4.13) is

$$\int_{|u|=\epsilon} \frac{\mathrm{d}u}{u}. \quad (4.14)$$

Note that in the patch $\tilde{\lambda}^1 \neq 0$ the cycle γ is not well defined. The second choice is $\tilde{\gamma} = \{\tilde{\lambda} \in \mathbb{C}P^1 : |C^2 \tilde{\lambda}| = \epsilon\}$. Here we must take the patch $\tilde{\lambda}^1 \neq 0$, where the parametrization is given by $\tilde{\lambda}^a = (1, v)$. Then, the integral (4.13) becomes

$$-\int_{|v|=\epsilon} \frac{\mathrm{d}v}{v}. \quad (4.15)$$

So, we have shown for the $d = 4$ projective pure spinor space, that different choices of the cycle $\tilde{\gamma}$ results just in changing the sign of (4.13). The same argument holds for the ten dimensional projective pure spinor space.

The integration measure in (4.12), $\mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}$, is the same found in a covariant manner by Berkovits and Cherkis in [16]. Therefore, we have the following identity.

Identity If $\tilde{\lambda}^\alpha$ is an element of the projective pure spinor space in 10 dimensions, i.e. if $\tilde{\lambda}^\alpha \in SO(10)/U(5)$, then the integration measure $[\mathrm{d}\tilde{\lambda}]$ defined by [16]

$$[\mathrm{d}\tilde{\lambda}] (\tilde{\lambda}\gamma^m)_{\alpha_1} (\tilde{\lambda}\gamma^n)_{\alpha_2} (\tilde{\lambda}\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4 \alpha_5} = \frac{2^3}{10!} \epsilon_{\alpha_1 ... \alpha_5 \beta_1 ... \beta_{11}} \mathrm{d}\tilde{\lambda}^{\beta_1} \wedge ... \wedge \mathrm{d}\tilde{\lambda}^{\beta_{10}} \tilde{\lambda}^{\beta_{11}}, \quad (4.16)$$

written in the parametrization $\tilde{\lambda}^\alpha = (\tilde{\lambda}^+, \tilde{\lambda}_{ab}, \tilde{\lambda}^a) = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de})$ is

$$[\mathrm{d}\tilde{\lambda}] = \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}. \tag{4.17}$$

This identity is proved in the appendix B

With this identity in mind, the amplitude (4.12) can be written in a covariant manner with respect to $SO(10)/U(5)$

$$\mathcal{A} \;=\; (2\pi i)\int_{\tilde{\Gamma}} \frac{[\mathrm{d}\tilde{\lambda}]\epsilon^{\alpha_1...\alpha_5\beta_1...\beta_{11}}C^1_{\beta_1}...C^{11}_{\beta_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}(\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}K. \tag{4.18}$$

This integral is the same as the 10 dimensional integral found in [16], so it is possible to have a twistor type version for the scattering amplitude at tree level. In [16] the integral is solved up to a proportionality factor. However, we will find a rigorous solution. Using (4.17) in (4.18) we get

$$\begin{aligned}\mathcal{A} &= (2\pi i)2^3\int_{\tilde{\Gamma}} \frac{1}{10!}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge ... \wedge \mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\frac{\epsilon^{\alpha_1...\alpha_5\gamma_1...\gamma_{11}}C^1_{\gamma_1}...C^{11}_{\gamma_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}K\\ &= (2\pi i)2^3\int_{\tilde{\Gamma}} \frac{5!}{10!}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge ... \wedge \mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}\delta^{\gamma_1}_{[\beta_1}\delta^{\gamma_2}_{\beta_2}...\delta^{\gamma_{11}}_{\beta_{11}]}\frac{C^1_{\gamma_1}...C^{11}_{\gamma_{11}}}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}K.\end{aligned} \tag{4.19}$$

Without loss of generality, we take $C^1\tilde{\lambda}, ..., C^{10}\tilde{\lambda}$ to define $\tilde{\Gamma}$, then (4.19) becomes

$$\begin{aligned}\mathcal{A} &= (2\pi i)2^3\int_{\tilde{\Gamma}} 5!\frac{(\mathrm{d}C^1\tilde{\lambda})\wedge ... \wedge(\mathrm{d}C^{10}\tilde{\lambda})(C^{11}\tilde{\lambda})}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}K\\ &= (2\pi i)2^3 5!\int_{\tilde{\Gamma}} \frac{(\mathrm{d}\tilde{f}^1)\wedge ... \wedge(\mathrm{d}\tilde{f}^{10})}{\tilde{f}^1...\tilde{f}^{10}}K\end{aligned} \tag{4.20}$$

where $\tilde{f}^I = C^I\tilde{\lambda}$. The others terms, like

$$\int_{\tilde{\Gamma}} \frac{(C^1\tilde{\lambda})(\mathrm{d}C^2\tilde{\lambda})\wedge ... \wedge(\mathrm{d}C^{10}\tilde{\lambda})\wedge(\mathrm{d}C^{11}\tilde{\lambda})}{C^1\tilde{\lambda}...C^{11}\tilde{\lambda}}$$

do not contribute since one of the poles $(C^1\tilde{\lambda}, ..., C^{10}\tilde{\lambda})$ is canceled, in this case, $(C^1\tilde{\lambda})$. Another choice of the C^I's just change the sign of (4.20).

Naively, it can be thought that the integral in (4.20) gives $(2\pi i)^{10}$. However, remember that $\tilde{f}^I$ are functions over the projective pure spinor space and $\tilde{\Gamma}$ is a 10-cycle in the projective pure spinor space. Therefore, this integral is non-trivial as in the flat space. Despite that the answer will just differ from this trivial case by a number, to know the formal answer will be extremely useful for relating the minimal and non-minimal pure spinor formalisms.

Before establishing the equivalence between the minimal and non-minimal pure spinor formalism for the tree level scattering amplitude, it is needed to give a short introduction to the Čech and Dolbeault language, which will be very useful to understand that correspondence.

4.3 Čech and Dolbeault Language

Due to the behavior $(1/\lambda)$ in the new lowering picture changing operators, they are defined locally in the pure spinor space. However, it will be interesting to have a global description, i.e patch independent, which can be achieved by introducing the Čech language. In this section we give a simple introduction to the Čech formalism and the Čech-Dolbeault isomorphism, which turns out to be useful for relating the minimal and non-minimal pure spinor formalism from the tree level scattering amplitude as we will show in subsection 4.4, and to check the BRST, Lorentz and SUSY symmetries in the section 5.

Given the new formulation for the PCO's

$$Y_C^I = \frac{C^I\theta}{C^I\lambda}, \qquad I = 1, ..., 11, \tag{4.21}$$

it is clear that Y_C^I is just defined in the patch $PS \setminus D_I$ where D_I is the hypersurface given by $f^I = C^I_\alpha \lambda^\alpha = 0$. Because 11 PCO's are needed in order to compute the tree level scattering amplitude, it is sufficient to have 11 patches to cover the pure spinor space at this order. Each patch is defined by the denominator of the picture operator, i.e we define the patch U_I as

$$U_I = PS \setminus D_I, \qquad D_I = \{\lambda \in PS : f^I \equiv C^I_\alpha \lambda^\alpha = 0\}. \tag{4.22}$$

The set $\underline{U} = \{U_I\}$ is a cover of the pure spinor space without the origin since we claimed that $D_1 \cap ... \cap D_{11} = \{0\}$. This means

$$PS \setminus \{0\} = U_1 \cup ... \cup U_{11} = \bigcup_{I=1}^{11} U_I. \tag{4.23}$$

This is as desired because the singular point is removed from the theory. Note that in the papers [9][21] the authors take the patches $\mathcal{U}_\alpha = PS \setminus \mathcal{D}_\alpha$ where $\mathcal{D}_\alpha = \{\lambda \in PS : \lambda^\alpha = 0\}$, $\alpha = 1, ..., 16$. Clearly these $\mathcal{D}_\alpha$'s satisfy $\mathcal{D}_1 \cap ... \cap \mathcal{D}_{16} = \{0\}$, therefore $PS \setminus \{0\} = \mathcal{U}_1 \cup ... \cup \mathcal{U}_{16}$ and we can define the PCO's as

$$Y_\alpha = \frac{\theta^\alpha}{\lambda^\alpha}, \qquad \alpha = 1, ..., 16. \tag{4.24}$$

Actually, for tree level scattering amplitudes this notation is not very convenient, as is explained in the appendix A.2.

Now, we introduce the Čech cochains because this terminology will be extensively used in this paper. The Čech k-cochain, denoted by $\psi_{I_1...I_{k+1}}$, is an holomorphic p-form in the intersection $U_{I_1...I_{k+1}} = U_{I_1} \cap U_{I_2} \cap ... \cap U_{I_{k+1}}$. I.e $\psi_{I_1...I_{k+1}} \in \Omega^p(U_{I_1...I_{k+1}})$ where $\Omega^p(U)$ is the abelian group of the holomorphic p-forms over U. We choose the abelian group of p-forms because it will be the group used in this paper. The Čech cochains must be antisymmetric in the Čech labels, for instance, $\psi_{I_1..I_i...I_j...I_{k+1}} = -\psi_{I_1..I_j...I_i...I_{k+1}}$. This is related to the orientation of the manifold, which in our

case is $PS \setminus \{0\}$.
We define the set of the 0-cochains on $PS \setminus \{0\}$ with values in the holomorphic p-forms as

$$C^0(\underline{U}, \Omega^p) = \bigoplus_{I=1}^{11} \Omega^p(U_I). \tag{4.25}$$

Similarly the 1-cochains are elements of the set

$$C^1(\underline{U}, \Omega^p) = \bigoplus_{I<J} \Omega^p(U_{IJ}) \tag{4.26}$$

and so on. We define the Čech operator as the map $\delta : C^k(\underline{U}, \Omega^p) \to C^{k+1}(\underline{U}, \Omega^p)$ given by

$$(\delta\psi)_{I_1....I_{k+2}} \equiv \psi_{I_2 I_3....I_{k+2}} - \psi_{I_1 I_3....I_{k+2}} + ... + (-1)^{k+1}\psi_{I_1 I_2....I_{k+1}}. \tag{4.27}$$

It is easy to show that δ is a nilpotent operator, $\delta^2 = 0$. If $(\delta\psi)_{I_1....I_{k+2}} = 0$ then $\psi_{I_1....I_{k+1}}$ is called a cocycle and the set of all cocycles in $C^k(\underline{U}, \Omega^p)$ is an abelian subgroup denoted by $Z^k(\underline{U}, \Omega^p)$. If $\psi_{I_1....I_{k+1}} = (\delta\rho)_{I_1....I_{k+1}}$ then $\psi_{I_1....I_{k+1}}$ is called a coboundary and the set of all coboundary in $C^k(\underline{U}, \Omega^p)$ is denoted by $B^k(\underline{U}, \Omega^p)$. Clearly every coboundary is a cocycle since $\delta^2 = 0$, then we can define the coset

$$H^k(PS \setminus \{0\}, \Omega^p) = \frac{Z^k(\underline{U}, \Omega^p)}{B^k(\underline{U}, \Omega^p)} \tag{4.28}$$

known as the k-Čech cohomology group with values in the Abelian group of holomorphic p-forms Ω^p on $PS \setminus \{0\}$. We refer the reader to [22][23] for more details about this topic.

Note that the PCO's are elements of $C^0(\underline{U}, \mathcal{O})$, where $\Omega^0 \equiv \mathcal{O}$ is a group of holomorphic functions, for instance

$$Y^I_C = \frac{C^I\theta}{C^I\lambda} \in \mathcal{O}(U_I) \tag{4.29}$$

is an holomorphic function on the patch U_I. It is easy to see that Y^I_C is not a cocycle

$$(\delta Y_C)^{IJ} = \left(\frac{C^J\theta}{C^J\lambda} - \frac{C^I\theta}{C^I\lambda}\right)\Bigg|_{U_{IJ}} = -\frac{C^{[I}\theta C^{J]}\lambda}{(C^I\lambda)(C^J\lambda)}\Bigg|_{U_{IJ}} \neq 0 \tag{4.30}$$

and therefore Y^I_C is not in the Čech cohomology. The PCO's have the particular property that the product of different PCO's is a Čech cochain, for example

$$\zeta^{I_1...I_k} \equiv Y^{I_1}_C...Y^{I_k}_C = \frac{C^{I_1}\theta...C^{I_k}\theta}{(C^{I_1}\lambda)...(C^{I_k}\lambda)} \in \mathcal{O}(U_{I_1...I_k}), \qquad k \leq 11 \tag{4.31}$$

is an element of $C^{k-1}(\underline{U}, \mathcal{O})$ because $\zeta^{I_1...I_k}$ is antisymmetry in its Čech labels. This happens because the variables θ^α are grassmann numbers, $\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$. When $k = 11$ we have

$$\zeta^{I_1...I_{11}} = \epsilon^{I_1...I_{11}} \frac{C^1\theta...C^{11}\theta}{(C^1\lambda)...(C^{11}\lambda)} \in \mathcal{O}(U_1 \cap ... \cap U_{11}). \tag{4.32}$$

This element is important because it is inside to the scattering amplitude. Since the cover $\underline{U}$ just has 11 patches and $(\delta\zeta)^{I_1...I_{11}I_{12}}$ is antisymmetric in all its Čech labels then

$$(\delta\zeta)^{I_1...I_{11}I_{12}} = 0 \tag{4.33}$$

so $\zeta^{I_1...I_{11}}$ belongs to Čech cohomology.

4.3.1 Čech-Dolbeault Isomorphism

Now we give a simple explanation about the Čech-Dolbeault isomorphism, which as we will show in section 4.4, is the base to obtain the relationship between the minimal and the non-minimal pure spinor formalisms. There is a simple way to relate the Čech and Dolbeault cocycles using the so called the partition of unity [22][23][9]. We can take the partition of unity as

$$\rho_I = \frac{f^I \bar{f}_I}{(|f^1|^2 + ... + |f^{11}|^2)}, \qquad I = 1, ..., 11 \tag{4.34}$$

where $f^I = C^I\lambda$, $\bar{f}_I$ is its complex conjugate: $\bar{f}_I = \bar{C}_I\bar{\lambda}$, and $\bar{\lambda}_\alpha = (\lambda^\alpha)^*$. It is clear that this partition of unity is subordinated to the cover $\underline{U}$, i.e, $\rho_I \neq 0$ only when $\lambda^\alpha \in U_I$, outside of the patch U_I the partition of unity is identically zero. Obviously this partition of unity satisfies the condition

$$\sum_{I=1}^{11} \rho_I = 1. \tag{4.35}$$

Let $\psi_{I_1...I_{k+1}}$ be a k-Čech cocycle ($\psi_{I_1...I_{k+1}} \in Z^k(\underline{U}, \Omega^p)$), then we define the corresponding η_ψ Dolbeault cocycle of type (p, k) as

$$\eta_\psi = \frac{1}{k!} \sum_{I_1...I_{k+1}=1}^{11} \psi_{I_1...I_{k+1}} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge ... \wedge \bar{\partial}\rho_{I_{k+1}}. \tag{4.36}$$

Note that η_ψ is a (p, k) form, which is p holomorphic and k antiholmorphic. As expected, $\bar{\partial}\eta_\psi = \mathrm{d}\bar{\lambda}_\alpha \wedge \frac{\partial}{\partial\bar{\lambda}_\alpha}\eta_\psi = 0$ because $\psi_{I_1...I_{k+1}}$ is a cocycle. Also $\psi_{I_1...I_{k+1}}$ is a coboundary, $\psi_{I_1...I_{k+1}} = (\delta\tau)_{I_1...I_{k+1}}$, then η_ψ is $\bar{\partial}$-exact, i.e $\eta_\psi = \bar{\partial}\eta_\tau$, where η_τ is the corresponding Dolbeault cochain to $\tau_{I_1...I_k}$

$$\eta_\tau = \frac{1}{(k-1)!} \sum_{I_1...I_k=1}^{11} \tau_{I_1...I_k} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge ... \wedge \bar{\partial}\rho_{I_k}, \tag{4.37}$$

i.e $\eta_\psi = \eta_{(\delta\tau)} = \bar{\partial}\eta_\tau$. Therefore we have a map between the Čech and Dolbeault cohomology groups $H^k(PS \setminus \{0\}, \Omega^p)$ and $H^{(p,k)}_{\bar{\partial}}(PS \setminus \{0\})$. Actually this map is an isomorphism but we do not show that statement here [22][23]. In particular we can consider the Čech cocycle

$$\beta_{I_1...I_{11}} = \epsilon_{I_1...I_{11}} \frac{\mathrm{d}(C^1\lambda) \wedge ... \wedge \mathrm{d}(C^{11}\lambda)}{(C^1\lambda)...(C^{11}\lambda)} \in \Omega^{11}(U_1 \cap ... \cap U_{11}) \tag{4.38}$$

which will appear in the section 4.4. Clearly $\beta_{I_1 \dots I_{11}}$ is an element of $H^{10}(PS \setminus \{0\}, \Omega^{11})$ so we can find its corresponding $\eta_\beta \in H^{(11,10)}_{\bar\partial}(PS \setminus \{0\})$. Applying the map (4.36) to $\beta_{I_1 \dots I_{11}}$ we get

$$\begin{aligned} \eta_\beta &= \frac{1}{10!} \sum_{I_1 \dots I_{11}=1}^{11} \beta_{I_1 \dots I_{11}} \rho_{I_1} \wedge \bar\partial \rho_{I_2} \wedge \dots \wedge \bar\partial \rho_{I_{11}} \\ &= (-1)^{i-1} \frac{\mathrm{d}(C^1\lambda) \wedge \dots \wedge \mathrm{d}(C^{11}\lambda) \wedge \bar\partial\rho_1 \wedge \dots \wedge \widehat{\bar\partial\rho_i} \wedge \dots \wedge \bar\partial\rho_{11}}{(C^1\lambda)\dots(C^{11}\lambda)} \end{aligned} \quad (4.39)$$

where $\widehat{\bar\partial\rho_i}$ means that it must be removed from (4.39). The C^I dependence is eliminated by a global transformation from the projective pure spinor space to itself, as will be done in the section 6. [12].

Since the pure spinor space without the origin ($PS \setminus \{0\}$) is contractible to $SO(10)/SU(5)$, i.e $PS \setminus \{0\}$ is deformed to $SO(10)/SU(5)$ [13], where one can think of $SO(10)/SU(5)$ as the boundary of the $PS \setminus \{0\}$ space, then the topological invariants of these two spaces are the same [24], in particular the following two groups are isomorphic

$$H^{(11,10)}_{\bar\partial}(PS \setminus \{0\}) \approx H^{21}_{DR}(SO(10)/SU(5)) \quad (4.40)$$

where DR means the de-Rham cohomology [23]. For the purposes of this paper it is enough to show that the map

$$i^* : H^{(11,10)}_{\bar\partial}(PS \setminus \{0\}) \longrightarrow H^{21}_{DR}(SO(10)/SU(5)) \quad (4.41)$$

is an injective homomorphism, i.e for any element $\eta \in H^{(11,10)}_{\bar\partial}(PS \setminus \{0\})$ there is just one element $i^*(\eta) \in H^{21}_{DR}(SO(10)/SU(5))$, where "$i$" is the map which embeds the $SO(10)/SU(5)$ space in the $PS \setminus \{0\}$ space and "i^*" is the pull back of the differential forms.

Proof

Let $\lambda = (\lambda^1, \dots, \lambda^{16}) = (\lambda^\alpha) \in \mathbb{C}^{16}$ be a point of the pure spinor space, $PS \setminus \{0\}$, i.e $\lambda\gamma^m\lambda = 0$ and $\lambda \neq 0$, then the $SO(10)/SU(5)$ space is embedded in $PS \setminus \{0\}$ by

$$SO(10)/SU(5) = \{(\lambda^\alpha) \in PS \setminus \{0\} : \lambda^\alpha \bar\lambda_\alpha = r^2\}, \qquad r \text{ is a positive constant, } r \in \mathbb{R}^+, \quad (4.42)$$

where $\bar\lambda_\alpha$ is the conjugate complex of λ^α [14]. Therefore (4.42) defines the injective map

$$i : SO(10)/SU(5) \longrightarrow PS \setminus \{0\}. \quad (4.43)$$

Now we must prove two statements in order to show that (4.41) is an injective homomorphism:

[12] We recommend to see the example in the appendix A.3 to get more information about this computation.

[13] For example the space $\mathbb{C} \setminus \{0\}$ can be deformed to S^1.

[14] Note that when $r \to \infty$ we can think the $SO(10)/SU(5)$ space like the boundary of the $PS \setminus \{0\}$.

1. First, we need to verify that the map (4.41) is well defined, in others words, if η is an (11,10)-form on $PS \setminus \{0\}$ which is $\bar{\partial}$ closed, i.e $\bar{\partial}\eta = 0$, then the 21-form on $SO(10)/SU(5)$ given by "$i^*\eta$" is "d" closed, i.e $\mathrm{d}(i^*\eta) = 0$.

   Since the exterior derivate operator d commutes with pull back, then we have

$$\mathrm{d}(i^*\eta) = i^*(\mathrm{d}\eta) = i^*[(\partial + \bar{\partial})\eta]. \tag{4.44}$$

   Remember that η is a (11,10)-form, this means $\partial\eta = 0$ because the complex dimension of the pure spinor space is 11, $\dim_{\mathbb{C}}(PS \setminus \{0\}) = 11$, so we have $\mathrm{d}(i^*\eta) = i^*(\bar{\partial}\eta)$. As $\bar{\partial}\eta = 0$ then we have shown $\mathrm{d}(i^*\eta) = 0$.

2. Finally, we must show that the homomorphism i^* is injective. To show this, it is sufficient to prove that i^* maps the zero to the zero. In others words, if η is a (11,10)-form on $PS \setminus \{0\}$ which is $\bar{\partial}$ exact, i.e $\eta = \bar{\partial}\tau$, where τ is a (11,9)-form on $PS \setminus \{0\}$, then the 21-form on $SO(10)/SU(5)$ given by "$i^*\eta$" is "d" exact, i.e $(i^*\eta) = \mathrm{d}(i^*\tau)$.

   Since τ is a (11,9)-form then $\eta = \bar{\partial}\tau = (\partial + \bar{\partial})\tau = \mathrm{d}\tau$, because $\partial\tau = 0$. So we have

$$i^*\eta = i^*(\mathrm{d}\tau) = \mathrm{d}(i^*\tau). \tag{4.45}$$

   Therefore we showed that the map (4.41) is an injective homomorphism.

To see more information about this topic we refer to [22][24].

This isomorphism will be very useful to obtain the equivalence between the minimal and non-minimal pure spinor formalism and to show that the scattering amplitude is invariant under BRST, Lorentz and supersymmetry transformations.

4.4 Equivalence Between the Minimal and Non-Minimal Formalism

In the previous subsection we gave the basic tools for writing the Dolbeault cocycle corresponding to the scattering amplitude. Using that, we will relate the minimal and non-minimal pure spinor formalisms. Specifically, we must find the Dolbeault cocycle corresponding to the scattering amplitude (4.9) using the isomorphism $H^{10}(PS \setminus \{0\}, \Omega^{11}) \approx H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$, which was explained in the previous subsection.
Since the elements of the group $H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$ are (11,10)-forms, they can not be evaluated in the whole space of the pure spinor minus the origin. However, $PS \setminus \{0\}$ can be contracted to the space $SO(10)/SU(5)$, which can be thought as the boundary in the infinite of the $PS \setminus \{0\}$ space. Then, by the isomorphism (4.41), the elements of $H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$ can be evaluated in the $SO(10)/SU(5)$ space. As will be explained in this section, this fact means that the picture lowering operators are not related to any particular regulator.

Now we show how to get the Dolbeault cocycle corresponding to (4.9). The scattering amplitude (4.9) can be written as

$$\begin{aligned}\mathcal{A} &= \int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{\alpha_1\dots\alpha_5\beta_1\dots\beta_{11}} C^1_{\beta_1}\dots C^{11}_{\beta_{11}}}{C^1\lambda\dots C^{11}\lambda} (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} K \\ &= \frac{1}{11!}\sum_{I_1\dots I_{11}} \epsilon_{I_1\dots I_{11}} \int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{\alpha_1\dots\alpha_5\beta_1\dots\beta_{11}} C^{I_1}_{\beta_1}\dots C^{I_{11}}_{\beta_{11}}}{C^{I_1}\lambda\dots C^{I_{11}}\lambda} (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} K \\ &\equiv \frac{1}{11!}\sum_{I_1\dots I_{11}} \epsilon_{I_1\dots I_{11}} \int_\Gamma \beta^{I_1\dots I_{11}} = \int_\Gamma \beta^{1,\dots,11}\end{aligned} \tag{4.46}$$

where the θ^α's have been integrated. Clearly $\beta^{I_1\dots I_{11}}$ is a Čech cochain[15]

$$\beta^{I_1\dots I_{11}} \in C^{10}(\underline{U}, \Omega^{11}) \tag{4.47}$$

where $\underline{U}$ is the cover of the $PS \setminus \{0\}$ space, which was defined in the subsection 4.3, i.e $\underline{U} = \{U_I\}$, $I = 1, ..., 11$, and the patches U_I's are given by $U_I = PS \setminus D_I$, where D_I is the hypersurface $D_I = \{\lambda^\alpha \in PS : C^I_\alpha \lambda^\alpha = 0\}$. Remember that $PS \setminus \{0\} = U_1 \cup ... \cup U_{11}$. Since there are 11 patches to cover $PS \setminus \{0\}$ then $\beta^{I_1\dots I_{11}}$ is in the Čech cohomology because $C^{11}(\underline{U}, \Omega^{11}) = \{0\}$ and $(\delta\beta)^{I_1\dots I_{12}} \in C^{11}(\underline{U}, \Omega^{11})$, so $(\delta\beta)^{I_1\dots I_{12}} = 0$, so we can write

$$\beta^{I_1\dots I_{11}} \in H^{10}(PS \setminus \{0\}, \Omega^{11}). \tag{4.48}$$

Now, using the partition of unity (4.34) we can find the Dolbeault cocycle, η_β, given by (4.36) and (4.39)

$$\eta_\beta = \frac{1}{10!} \sum_{I_1\dots I_{11}=1}^{11} \beta^{I_1\dots I_{11}} \rho_{I_1} \wedge \bar{\partial}\rho_{I_2} \wedge \dots \wedge \bar{\partial}\rho_{I_{11}}. \tag{4.49}$$

Note that, since $\beta^{I_1\dots I_{11}}$ is an element of $H^{10}(PS \setminus \{0\}, \Omega^{11})$, then $\eta_\beta \in H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$, as was explained in the sub-subsection 4.3.1. The Dolbeault cohomology group $H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\})$ was computed in [14], $H^{(11,10)}_{\bar{\partial}}(PS \setminus \{0\}) = \mathbb{C}$, so it only has one generator which is η_β. The computation (4.49) is not straightforward because it is needed to make a non-trivial global transformation from $SO(10)/U(5)$ to itself in order to find an expression for the Dolbeault cocycle η_β independent of the constants C^I's. See the simple example given in the appendix A.3. To avoid this difficulty, we will introduce the concept of the degree of the projective pure spinor space, in order to obtain η_β in a simpler way.

[15] In [14] was shown that the measure $[\mathrm{d}\lambda]$ is defined globally on $PS \setminus \{0\}$, so, the Čech indices come only from the PCO's.

4.4.1 The Projective Pure Spinor Degree

The last step (4.20) in the computation of the scattering amplitude with the projective pure spinor space variables was

$$\mathcal{A} \;=\; (2\pi i)2^3\int_{\tilde{\Gamma}} 5!\frac{(\mathrm{d}\tilde{f}^1)\wedge...\wedge(\mathrm{d}\tilde{f}^{10})}{\tilde{f}^1...\tilde{f}^{10}}K, \tag{4.50}$$

This integral is known [22][26] and its result is given by the intersection theory

$$\int_{\tilde{\Gamma}}\frac{(\mathrm{d}\tilde{f}^1)\wedge...\wedge(\mathrm{d}\tilde{f}^{10})}{\tilde{f}^1...\tilde{f}^{10}} = (2\pi i)^{10}\sum_{\nu}(\tilde{D}_1,...,\tilde{D}_{10})_{p_\nu} \tag{4.51}$$

where ν is the number of points p_ν where the hypersurfaces $\tilde{D}_I$ were defined by $\tilde{D}_I = \{\tilde{\lambda}^\alpha \in SO(10)/U(5)\ :]C^I\tilde{\lambda} = 0,\ I = 1,\ldots 10\}$ and $(\tilde{D}_1,...,\tilde{D}_{10})_{p_\nu} \equiv m_\nu$ is the multiplicity[16] in p_ν. Remember that the coordinates $\tilde{\lambda}^\alpha$, $\alpha = 1,...,16$ can be thought as coordinates of $\mathbb{C}^{16}\setminus\{0\}$ with the equivalence relation $\tilde{\lambda}^\alpha \sim c\tilde{\lambda}^\alpha$, $c \neq 0 \in \mathbb{C}$, satisfying the constraints $\tilde{\lambda}\gamma^m\tilde{\lambda} = 0$, so the projective pure spinor space $SO(10)/U(5)$ is embedded in $\mathbb{C}P^{15} = \mathbb{C}^{16}\setminus\{0\}/(\tilde{\lambda}^\alpha \sim c\tilde{\lambda}^\alpha)$, $c \in \mathbb{C}^*$. Therefore the hypersurface $\tilde{D}_I \subset SO(10)/U(5)$ is the intersection between the linear subspace $C^I_\alpha\tilde{\lambda}^\alpha = 0$ and $SO(10)/U(5)$, where now $\tilde{\lambda}^\alpha \in \mathbb{C}P^{15}$, i.e $\tilde{D}_I = \{\{C^I_\alpha\tilde{\lambda}^\alpha = 0\}\cap SO(10)/U(5),\ \text{where}\ \tilde{\lambda}^\alpha \in \mathbb{C}P^{15}\}$. Note that the intersection of the 10 linear subspaces $C^I\tilde{\lambda} = 0$, $I = 1,...,10$ in $\mathbb{C}P^{15}$ is the linear subspace $\mathbb{C}P^5$ embedded in $\mathbb{C}P^{15}$, therefore the intersection of the hypersurfaces $\tilde{D}_I$'s is just the intersection between $\mathbb{C}P^5$ and $SO(10)/U(5)$, this means

$$\tilde{D}_1\cap...\cap\tilde{D}_{10} = \mathbb{C}P^5\cap SO(10)/U(5)\Big|_{\mathbb{C}P^{15}}. \tag{4.52}$$

Since $SO(10)/U(5)$ is a smooth manifold on $\mathbb{C}P^{15}$, the multiplicity in each intersection point of (4.52) is one [22]. So, the sum of the multiplicity at each intersection point p_ν is the number of intersection points among $\mathbb{C}P^5$ and $SO(10)/U(5)$, denoted by ${}^{\#}(SO(10)/U(5)\cdot\mathbb{C}P^5)$

$$\sum_{\nu}(\tilde{D}_1,...,\tilde{D}_{10})_{p_\nu} = \ {}^{\#}(SO(10)/U(5)\cdot\mathbb{C}P^5), \tag{4.53}$$

This number is called the degree of the projective pure spinor space $\deg(SO(10)/U(5)) \equiv {}^{\#}(SO(10)/U(5)\cdot\mathbb{C}P^5)$. In [25] it was shown that the degree of this space is given by

$$\deg(SO(10)/U(5)) = \int_{SO(10)/U(5)}\frac{\omega^{10}}{(2\pi i)^{10}}, \tag{4.54}$$

where ω is

$$\omega = -\partial\overline{\partial}\ln(\tilde{\lambda}\bar{\tilde{\lambda}}), \tag{4.55}$$

[16] The multiplicity can be understood in the same way as in the solutions of a system of algebraic equations.

and $\tilde{\lambda}^\alpha$ is an holomorphic coordinate for $SO(10)/U(5)$. Therefore we have that

$$\begin{aligned} \mathcal{A} &= \int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{\alpha_1 \ldots \alpha_5 \beta_1 \ldots \beta_{11}} C^1_{\beta_1} \ldots C^{11}_{\beta_{11}}}{C^1\lambda \ldots C^{11}\lambda} (\lambda\gamma^m)_{\alpha_1} (\lambda\gamma^n)_{\alpha_2} (\lambda\gamma^p)_{\alpha_3} (\gamma_{mnp})_{\alpha_4\alpha_5} K \\ &= \int_{SO(10)/U(5)} (2\pi i)\, 2^3\; 5!\; \omega^{10}\, K \end{aligned} \tag{4.56}$$

Notice that using the pure spinor measure [25]

$$[\mathrm{d}\lambda](\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \frac{2^3}{11!}\epsilon_{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}} \mathrm{d}\lambda^{\beta_1} \wedge \ldots \wedge \mathrm{d}\lambda^{\beta_{11}}, \tag{4.57}$$

and replacing this measure in the amplitude (4.46) we obtain [17]

$$\mathcal{A} = 2^3 \int_\Gamma 5! \frac{(\mathrm{d}f^1) \wedge \ldots \wedge (\mathrm{d}f^{11})}{f^1 \ldots f^{11}} K \tag{4.58}$$

where $f^I = C^I\lambda$, $I = 1, .., 11$. In the same way as (4.50) this integral is given by the intersection theory [22][26]

$$\int_\Gamma \frac{(\mathrm{d}f^1) \wedge \ldots \wedge (\mathrm{d}f^{11})}{f^1 \ldots f^{11}} = (2\pi i)^{11} (D_1, \ldots, D_{11})_{\{0\}}, \tag{4.59}$$

where the origin is the unique point of intersection between the hypersurfaces D_I's given by (4.22), $D_I = \{\lambda^\alpha \in PS : f^I = 0\}, I = 1, \ldots, 11$, i.e $D_1 \cap \ldots \cap D_{11} = \{0\}$ as we claimed, and $(D_1, \ldots, D_{11})_{\{0\}}$ means the multiplicity of this intersection. So using (4.50), (4.51), (4.53) and (4.59) we can conclude

$$(D_1, \ldots, D_{11})_{\{0\}} = \deg(SO(10)/U(5)). \tag{4.60}$$

As computed in [25], the degree of the projective pure spinor space is 12. That explains why the multiplicity in the intersection point, i.e the origin, between the matrix (3.13) and PS is 12.

4.4.2 The Dolbeault Cocycle

Now, using the degree of the projective pure spinor space we can compute easily the Dolbeault cocycle corresponding to the form $\beta^{I_1 \ldots I_{11}}$. From (4.56) we have that the scattering amplitude is

$$\mathcal{A} = \int_\Gamma \beta^{1,\ldots,11} = (2\pi i)\; 2^3\; 5! \int_{SO(10)/U(5)} \omega^{10}\; K. \tag{4.61}$$

[17] The normalization factor 2^3 in the measure comes from the fact that $(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}(\gamma^q\bar{\lambda})^{\alpha_1}(\gamma^r\bar{\lambda})^{\alpha_2}(\gamma^s\bar{\lambda})^{\alpha_3}(\gamma_{qrs})^{\alpha_4\alpha_5} = 2^6 5!(\lambda\bar{\lambda})^3$.

Writing ω in coordinates as in the appendix B, we have

$$
\begin{aligned}
\int_\Gamma \beta^{1,\dots,11} &= (2\pi i)\, 2^3\, 5! \int_{SO(10)/U(5)} \omega^{10}\, K \\
&= (2\pi i)\ 2^3\ \ 5! \int_{\mathbb{C}^{20}} \frac{(10!)\ \ \bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1+\frac{1}{2}u_{ab}\bar{u}^{ab}+\frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8} K \\
&= \ \ 2^3\ \ 5! \int_{\mathbb{C}^{20}} \int_0^{2\pi} \frac{i(10!)\ \ \mathrm{d}\phi \bigwedge_{a<b,\, c<d} \mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1+\frac{1}{2}u_{ab}\bar{u}^{ab}+\frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8}\ K.
\end{aligned}
\tag{4.62}
$$

So (4.62) is a 21-form evaluated locally on the $SO(10)/SU(5)$ space given by (4.42). This can be seen in the following simple way: the variables u_{ab} parametrize the projective pure spinor space in the patch $\lambda^+ \neq 0$, i.e $\tilde{\lambda}^\alpha = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de})$, and ϕ parametrizes the circle $\gamma = e^{i\phi}$. So we have locally the space $SO(10)/U(5)\Big|_{\lambda^+\neq 0} \times U(1)$. Since $U(5) = U(1) \times SU(5)$ then we get the space $SO(10)/U(5)\Big|_{\lambda^+\neq 0} \times U(1) = SO(10)/SU(5)\Big|_{\lambda^+\neq 0}$. Note that we have done just a local analysis. Actually, it is impossible to write globally the space $SO(10)/SU(5)$ as the product between the projective pure spinor space and the circle, $SO(10)/SU(5) \neq SO(10)/U(5) \times U(1)$.

The expression (4.62) means that we found the Dolbeault cocycle η_β evaluated in the space $SO(10)/SU(5)$ locally, i.e we got $(i^*\eta_\beta)\Big|_{\lambda^+\neq 0}$, where i is the embedding $i : SO(10)/SU(5) \to PS \setminus \{0\}$, explained in the sub-subsection 4.3.1. In the following, we are going to obtain η_β in a covariant way in the $PS \setminus \{0\}$ space.

Remember that the holomorphic pure spinor measure $[\mathrm{d}\lambda]$ was given in (4.57). We define a new antiholomorphic 10-form in the $PS \setminus \{0\}$ space as

$$
\left[\mathrm{d}\bar{\lambda}\right]'(\bar{\lambda}\gamma^m)^{\alpha_1}(\bar{\lambda}\gamma^n)^{\alpha_2}(\bar{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon^{\alpha_1\dots\alpha_5\beta_1\dots\beta_{11}}\mathrm{d}\bar{\lambda}_{\beta_1}\wedge\dots\wedge\mathrm{d}\bar{\lambda}_{\beta_{10}}\bar{\lambda}_{\beta_{11}}, \tag{4.63}
$$

where $\bar{\lambda}_\alpha$ is a pure spinor, $\bar{\lambda}_\alpha(\gamma^m)^{\alpha\beta}\bar{\lambda}_\beta = 0$. Note that (4.63) has the same algebraic expression as in (B.3), with the difference that in this case $\bar{\lambda}_\alpha$ belongs to the $PS \setminus \{0\}$ space while the one in (B.3) it is a projective pure spinor. It is easy to see that in the parametrization on the patch $\lambda^+ \neq 0$

$$
\lambda^\alpha = \gamma(1, u_{ab}, \epsilon^{abcde}u_{bc}u_{de}/8), \qquad \bar{\lambda}_\alpha = \bar{\gamma}(1, \bar{u}^{ab}, \epsilon_{abcde}\bar{u}^{bc}\bar{u}^{de}/8), \tag{4.64}
$$

the (11,10)-form $[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]'$ becomes

$$
[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]' = \gamma^7\bar{\gamma}^8\mathrm{d}\gamma \wedge \mathrm{d}u_{12} \wedge \dots \wedge \mathrm{d}u_{45} \wedge \mathrm{d}\bar{u}^{12} \wedge \dots \wedge \mathrm{d}\bar{u}^{45}. \tag{4.65}
$$

The $SO(10)/SU(5)$ space given in (4.42) is parametrized on the patch $\lambda^+ \neq 0$ in the following way

$$
\lambda^\alpha = r\, e^{i\phi}(1, u_{ab}, \epsilon^{abcde}u_{bc}u_{de}/8), \qquad \text{where } r \text{ is positive constant.} \tag{4.66}
$$

So, we can write the 21-form of (4.62) as

$$\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\Bigg|_{SO(10)/SU(5)}\Bigg|_{\lambda^+\neq 0} = \frac{i\ \mathrm{d}\phi\bigwedge_{a<b,\,c<d}\mathrm{d}u_{ab}\mathrm{d}\bar{u}^{cd}}{(1+\frac{1}{2}u_{ab}\bar{u}^{ab}+\frac{1}{8^2}\epsilon^{abcde}\epsilon_{afghi}u_{bc}u_{de}\bar{u}^{fg}\bar{u}^{hi})^8}. \tag{4.67}$$

Using the pure spinor constraint it is not hard to verify that the (11,10)- form $[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'/(\lambda\bar{\lambda})^8$ is $\bar{\partial}$ closed on $PS\setminus\{0\}$:

$$\bar{\partial}\left(\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\right)=0. \tag{4.68}$$

Therefore, the (11,10)-form $[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'/(\lambda\bar{\lambda})^8$ belongs to cohomology group $H^{(11,10)}_{\bar{\partial}}(PS\setminus\{0\})$ and the pull back i^* is just the restriction

$$i^*\left(\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\right)=\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\Bigg|_{SO(10)/SU(5)}, \tag{4.69}$$

which is an element of the de-Rham cohomology group $H^{21}_{DR}(SO(10)/SU(5))$. Finally, we found the Dolbeault cocycle η_β corresponding to $\beta^{1,\ldots,11}$

$$\beta^{1,\ldots,11}\xrightarrow{\text{Čech-Dol}}\eta_\beta\equiv 2^3\,5!\,(10!)\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\,K, \tag{4.70}$$

and (4.62) in a covariant way is given by

$$\int_\Gamma\beta^{1,\ldots,11}=\int_{SO(10)/SU(5)}\eta_\beta\Big|_{SO(10)/SU(5)}\equiv 2^3\,5!\int_{SO(10)/SU(5)}(10!)\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\Bigg|_{SO(10)/SU(5)}K. \tag{4.71}$$

Using the Čech-Dolbeault isomorphism we have gone from a theory in an 11-cycle Γ to a theory in the whole $SO(10)/SU(5)$ space. Furthermore, notice that since the non-minimal pure spinor formalism is defined in the whole pure spinor space $PS\setminus\{0\}$, which is a non-compact space, then there are an infinite number of global functions on it such that the amplitude does not change. These functions are called regulators. This is in contrast with the $SO(10)/SU(5)$ space, which is a compact manifold whose unique generator is given by (4.69).

Note that integrating the non compact direction of the $PS\setminus\{0\}$ space we get the space $SO(10)/SU(5)$. This means that for any regulator in the non-minimal formalism after integrating the non compact direction of the $PS\setminus\{0\}$ space, one must get the expression (4.71). We will be more explicit by using coordinates in the following. If λ^α is a pure spinor, then it can be written as $\lambda^\alpha=\gamma\tilde{\lambda}^\alpha$, where $\gamma\in\mathbb{C}^*=U(1)\times\mathbb{R}^+$ and $\tilde{\lambda}^\alpha$ is a projective pure spinor. So, setting $\gamma=\rho\,e^{i\phi}$, where $e^{i\phi}\in U(1)$ and $\rho\in\mathbb{R}^+$ and integrating by ρ in the non-minimal formalism we must get (4.71) for any regulator. This implies that our picture changing operators does not correspond to any particular regulator and therefore we believe that the scattering amplitude prescription with the new picture operators is perhaps more fundamental than the prescription with regulators.

4.4.3 A Particular Regulator

In this sub-subsection we would like to illustrate what we said in the last paragraph with a particular regulator.

The most useful regulator in the non-minimal pure spinor formalism for computing the tree level scattering amplitude is

$$\mathcal{N} = \exp(-\bar{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha), \tag{4.72}$$

as given in [8], where r_α is a spinor such that $r_\alpha(\gamma^m)^{\alpha\beta}\bar{\lambda}_\beta = 0$. After integrating the variables r_α and θ^α we get [27]

$$\mathcal{A} = 2^3\, 5! \int_{PS} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-(\lambda\bar{\lambda})}\, (\lambda\bar{\lambda})^3\, K \tag{4.73}$$

where the measure $[\mathrm{d}r]$ was given in [25][8]

$$[\mathrm{d}r] = \frac{1}{2^3\, 5!\, 11!} (\bar{\lambda}\gamma^m)^{\alpha_1} (\bar{\lambda}\gamma^n)^{\alpha_2} (\bar{\lambda}\gamma^p)^{\alpha_3} (\gamma_{mnp})^{\alpha_4\alpha_5} \epsilon_{\alpha_1,\ldots,\alpha_5\beta_1\ldots\beta_{11}} \partial_r^{\beta_1} \ldots . \partial_r^{\beta_{11}}, \tag{4.74}$$

the factor 2^3 comes from a normalization explained in the footnote 17. We replaced the vertex operators in the amplitude by $(\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\, K$, where K is the kinematic factor. Using the coordinates $\lambda^\alpha = \gamma\tilde{\lambda}^\alpha = \rho\, e^{i\phi}\tilde{\lambda}^\alpha$, which were explained previously, the integration measure is [14]

$$\begin{aligned} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}] = (\gamma\bar{\gamma})^7\, \mathrm{d}\gamma \wedge \mathrm{d}\bar{\gamma} \wedge [\mathrm{d}\tilde{\lambda}] \wedge [\mathrm{d}\bar{\tilde{\lambda}}] \;&=\; -2\, i\, (\rho^2)^7\, \rho\, \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge [\mathrm{d}\tilde{\lambda}] \wedge [\mathrm{d}\bar{\tilde{\lambda}}] \\ &=\; -2\, i\, (\rho^2)^7\, \rho\, \mathrm{d}\rho \wedge [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]' \Big|_{SO(10)/SU(5)}\Big|_{r=1} \end{aligned} \tag{4.75}$$

where $SO(10)/SU(5)|_{r=1}$ means that the space $SO(10)/SU(5)$ has size $r = 1$ (see (4.42)). So, integrating the non-compact variable ρ from r_0 to r we get

$$\begin{aligned} & 2^3\, 5! \int_{PS} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-(\lambda\bar{\lambda})}\, (\lambda\bar{\lambda})^3\, K \\ = \;& 2^3\, 5! \int_{SO(10)/SU(5)} (10!) [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]' \left(\frac{e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{(\tilde{\lambda}\bar{\tilde{\lambda}})^8} + \frac{\rho^2 e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{(\tilde{\lambda}\bar{\tilde{\lambda}})^7} + \frac{\rho^4 e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{2(\tilde{\lambda}\bar{\tilde{\lambda}})^6} + \right. \\ & + \frac{\rho^6 e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{3!(\tilde{\lambda}\bar{\tilde{\lambda}})^5} + \frac{\rho^8 e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{4!(\tilde{\lambda}\bar{\tilde{\lambda}})^4} + \frac{\rho^{10} e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{5!(\tilde{\lambda}\bar{\tilde{\lambda}})^3} \frac{\rho^{12} e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{6!(\tilde{\lambda}\bar{\tilde{\lambda}})^2} + \frac{\rho^{14} e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{7!(\tilde{\lambda}\bar{\tilde{\lambda}})} + \\ & \left. + \frac{\rho^{16} e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{8!} + \frac{\rho^{18} (\tilde{\lambda}\bar{\tilde{\lambda}}) e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{9!} + \frac{\rho^{20} (\tilde{\lambda}\bar{\tilde{\lambda}})^2 e^{-\rho^2(\tilde{\lambda}\bar{\tilde{\lambda}})}}{10!} \right) \Bigg|_{SO(10)/SU(5)|_{\rho=r_0}}^{SO(10)/SU(5)|_{\rho=r}} K, \end{aligned} \tag{4.76}$$

Note that $SO(10)/SU(5)|_{\rho=r} - SO(10)/SU(5)|_{\rho=r_0}$ is the boundary of the finite pure spinor space, i.e

$$PS_{r_0,r} \equiv \{\lambda^\alpha \in \mathbb{C}^{16} : \lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta = 0 \text{ and } r_0^2 \leq \lambda^\alpha\bar{\lambda}_\alpha \leq r^2\}, \tag{4.77}$$

where r_0, r are positive constants. In order to obtain the whole pure spinor space, $PS \setminus \{0\}$, we must take the limits $r_0 \to 0$ and $r \to \infty$, so we get the equivalence

$$2^3\,5! \int_{PS} [\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]\, e^{-(\lambda\bar{\lambda})}\, (\lambda\bar{\lambda})^3 K = 2^3\,5! \int_{SO(10)/SU(5)} (10!) \frac{[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\Bigg|_{SO(10)/SU(5)} K = \int_{\Gamma} \beta^{1,\ldots,11}. \tag{4.78}$$

This is the reason why we say that $SO(10)/SU(5)$ is the "boundary" of the $PS \setminus \{0\}$ space, although it is a non-compact space. The equivalence (4.78) holds for any regulator because the Čech-Dolbeault isomorphism relates the minimal formalism with a formalism in $SO(10)/SU(5)$, which only has one cohomology generator given by $([\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]'/(\lambda\bar{\lambda})^8)|_{SO(10)/SU(5)}$.

Although the equivalence between the minimal and non-minimal formalisms is somewhat premature because in tree level we can absorb any number in the coupling constant $e^{-2\mu}$ [27], the previous result is beautiful and it will be very import for computing loop amplitudes [28].

5 Symmetries of the Scattering Amplitude

In this section we analyze the symmetries of the scattering amplitude with the new PCO's. Namely, we will show that the scattering amplitude is invariant under BRST, Lorentz and supersymmetry transformations. Here we will often use the Čech language and the Čech-Dolbeault isomorphism presented in the subsection 4.3.

5.1 BRST Invariance

We will show that the tree level scattering amplitude is BRST invariant and that the Q exact states are decoupled.

As we discussed in the subsection 4.3, the PCO's are defined locally because they behave like $1/\lambda$: $Y_C^I = \frac{C^I\theta}{C^I\lambda}$ and they are well defined only in $U_I = PS \setminus D_I$. Therefore, as proposed in [9] one must add to the old BRST charge

$$Q = \oint \mathrm{d}z\, \lambda^\alpha d_\alpha, \tag{5.1}$$

where d_α are the constraints (2.6), the Čech operator δ given in (4.27). The δ operator play an important role in the construction of the b-ghost, as we will discuss in the section 8. So the total BRST charge is

$$Q_T = \oint \lambda^\alpha d_\alpha + \delta \equiv Q + \delta. \tag{5.2}$$

By definition, if the tree level scattering amplitude $\mathcal{A}$ is physical then it must be Q_T closed, i.e $Q_T\,\mathcal{A} = 0$.

In the following we will show that the amplitude is Q_T closed. First of all, remember that in the tree level scattering amplitude the vertex operators can always be written as a global function in $PS\setminus\{0\}$ given by $\lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)$ [4], where the k_i's are the momenta and the e_i's are the polarizations of the vertex operators. Since the tree level scattering amplitude is given by

$$\mathcal{A} = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{I=1}^{11} Y_C^I\, \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i). \tag{5.3}$$

and the measure $[\mathrm{d}\lambda]$ is globally defined on $PS\setminus\{0\}$ [14], then the scattering amplitude is δ closed. This was explained carefully in the subsection 4.4 (see the explanation after (4.46)). Now it remains to show $Q\,\mathcal{A}=0$. Because

$$Q\, Y_C^I = 1, \tag{5.4}$$

therefore we have

$$\begin{aligned} Q(\mathcal{A}) &= Q\left(\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{I=1}^{11} Y_C^I\, \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)\right) \\ &= \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1,\ldots,I_{11}} Q\left(\int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^{I_1}\theta \ldots C^{I_{11}}\theta}{C^{I_1}\lambda \ldots C^{I_{11}}\lambda}\, \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)\right) \\ &= \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1,\ldots,I_{11}} \int_\Gamma (\delta\tau)^{I_1,\ldots,I_{11}} \end{aligned} \tag{5.5}$$

where $\tau^{I_1,\ldots,I_{10}}$ is the holomorphic 11-form

$$\tau^{I_1,\ldots,I_{10}} = [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^{I_1}\theta \ldots C^{I_{10}}\theta}{C^{I_1}\lambda \ldots C^{I_{10}}\lambda}\, \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \in C^9(\underline{U}, \Omega^{11}) \tag{5.6}$$

where $\underline{U}$ is the cover of the $PS\setminus\{0\}$ space given in the subsection 4.3. Clearly $(\delta\tau)^{I_1,\ldots,I_{11}}$ is a trivial element of the Čech cohomology group $H^{11}(PS\setminus\{0\},\Omega^{11})$, so its corresponding Dolbeault cocycle

$$\eta_{(\delta\tau)} = \frac{1}{10!}\sum_{I_1,\ldots,I_{11}=1}^{11} (\delta\tau)^{I_1,\ldots,I_{11}} \rho_{I_1} \wedge \bar\partial\rho_{I_2} \wedge \ldots \wedge \bar\partial\rho_{I_{11}}, \tag{5.7}$$

where ρ_I is partition of unity (4.34), is a trivial element of the Dolbeault cohomology group $H_{\bar\partial}^{(11,10)}(PS\setminus\{0\})$, i.e

$$\eta_{(\delta\tau)} = \bar\partial(\eta_\tau) \qquad \text{(11,10)-form on } PS\setminus\{0\}, \tag{5.8}$$

where η_τ is the (11,9)-form given by

$$\eta_\tau = \frac{1}{9!}\sum_{I_1,\ldots,I_{10}=1}^{11} \tau^{I_1,\ldots,I_{10}} \rho_{I_1} \wedge \bar\partial\rho_{I_2} \wedge \ldots \wedge \bar\partial\rho_{I_{10}}, \tag{5.9}$$

as explained in the sub-subsection 4.3.1. So we can write (5.5) as

$$\int_\Gamma (\delta\tau)^{1,\dots,11} = \int_{SO(10)/SU(5)} i^*(\bar\partial(\eta_\tau)) = \int_{SO(10)/SU(5)} \mathrm{d}(i^*(\eta_\tau)), \tag{5.10}$$

where "i" is the map $i: SO(10)/SU(5) \to PS \setminus \{0\}$ given in the sub-subsection 4.3.1. Finally, applying the Stokes theorem

$$\int_\Gamma (\delta\tau)^{1,\dots,11} = \int_{SO(10)/SU(5)} \mathrm{d}(i^*(\eta_\tau)) = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\tau) \tag{5.11}$$

and since the $SO(10)/SU(5)$ space is a compact manifold without boundary, $\partial(SO(10)/SU(5)) = \varnothing$, then we can conclude

$$Q(\mathcal{A}) = 0. \tag{5.12}$$

Thus, we have shown that the tree level scattering amplitude is Q_T closed.

Now we will show that the global (i.e $(\delta\Omega)^{IJ} = \Omega^J - \Omega^I = 0$) and Q exact (i.e $\langle\, Q(\Omega)\,\rangle$) functions are decoupled, that is, they are $Q_T = Q + \delta$ exact. A Q exact function inside to the scattering amplitude is given by

$$\langle\, Q(\Omega)\,\rangle = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta Y_C^1 ... Y_C^{11} Q(\Omega(\lambda,\theta,k)). \tag{5.13}$$

Only the terms with 5 θ's and 3 λ's in $Q(\Omega)$ will contribute, because there are 11 θ's coming from the 11 PCO's and the scattering amplitude must have ghost number zero. So, we focus on the global term

$$\Omega(\lambda,\theta,k,e) = \lambda^\alpha \lambda^\beta \theta^{\gamma_1} ... \theta^{\gamma_6} f_{\alpha\beta\gamma_1 ... \gamma_6}(k_i, e_i), \tag{5.14}$$

where k_i are the momenta and e_i are the polarizations. We can write (5.13) as

$$\begin{aligned} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta\, Y_C^1 ... Y_C^{11} Q(\Omega(\lambda,\theta,k)) \;&=\; - \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta\, Q(Y_C^1 ... Y_C^{11}\Omega(\lambda,\theta,k)) \\ &\quad + \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta\, Q(Y_C^1 ... Y_C^{11})\Omega(\lambda,\theta,k). \end{aligned} \tag{5.15}$$

The term $Y_C^1 ... Y_C^{11}\Omega(\lambda,\theta,k)$ is identically zero because there are 17 θ's. So

$$\begin{aligned} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta\, Y_C^1 ... Y_C^{11} Q(\Omega(\lambda,\theta,k)) \;&=\; \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta\, Q(Y_C^1 ... Y_C^{11})\Omega(\lambda,\theta,k) \\ &= \frac{1}{11!} \sum_{I_1,\dots,I_{11}=1}^{11} \epsilon_{I_1 ... I_{11}} \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta\, Q(Y_C^{I_1} ... Y_C^{I_{11}})\Omega(\lambda,\theta,k) \\ &= \frac{1}{11!} \sum_{I_1,\dots,I_{11}=1}^{11} \epsilon_{I_1 ... I_{11}} \int_\Gamma (\delta\kappa)^{I_1 ... I_{11}}, \end{aligned} \tag{5.16}$$

where $\kappa^{I_1 ... I_{10}}$ is the holomorphic 11-form

$$\kappa^{I_1 ... I_{10}} = [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \frac{C^{I_1}\theta ... C^{I_{10}}\theta}{C^{I_1}\lambda ... C^{I_{10}}\lambda} \lambda^\alpha \lambda^\beta \theta^{\gamma_1} ... \theta^{\gamma_6} f_{\alpha\beta\gamma_1 ... \gamma_6}(k_i, e_i) \in C^9(\underline{U}, \Omega^{11}). \tag{5.17}$$

Note that $\delta \langle Q(\Omega) \rangle = 0$, since $(\delta(\delta\kappa))^{I_1 \ldots I_{12}} = 0$. Using the same procedure that allowed as go from (5.5) and to conclude in (5.12) we have

$$\langle Q(\Omega) \rangle = \frac{1}{11!} \sum_{I_1,\ldots,I_{11}=1}^{11} \epsilon_{I_1 \ldots I_{11}} \int_\Gamma (\delta\kappa)^{I_1 \ldots I_{11}} = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\kappa) = 0. \tag{5.18}$$

Therefore we have shown that every global and exact function inside to the scattering amplitude is decoupled.

For a general case we must show that the scattering amplitude decouple the states which are Q_T exact, i.e

$$\langle (Q+\delta)(\Omega) \rangle = 0 \tag{5.19}$$

for any Ω.

First of all, we know that the BRST operator is nilpotent $(Q+\delta)^2 = 0$ and we want to show that the BRST exact terms are decoupled from the scattering amplitude. The Q_T exact terms are written as

$$\langle (Q+\delta)(\Omega) \rangle = \int_\Gamma [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{I=1}^{11} Y_C^I (Q+\delta)(\Omega). \tag{5.20}$$

However, as the 11-form $[\mathrm{d}\lambda] \prod_{I=1}^{11} Y_C^I$ is a 10-Čech cochain, $C^{10}(\underline{U}, \Omega^{11})$, then it is possible that the product of the cochains $[\mathrm{d}\lambda] \prod_{I=1}^{11} Y_C^I$ and $(Q+\delta)(\Omega)$ is not well defined, because the product of two cochains is not always a cochain. For instance, let us consider the following 2 cochains

$$Y_C^I = \frac{C^I\theta}{C^I\lambda} \in C^0(\underline{U}, \mathcal{O}), \qquad \Omega^J = \frac{\Lambda_{mn}(C^J\gamma^{mn}\theta)}{(C^J\lambda)} \in C^0(\underline{U}, \mathcal{O}). \tag{5.21}$$

Clearly

$$\Psi^{IJ} \equiv Y_C^I \, \Omega^J = \frac{(C^I\theta)\,(C^J\gamma^{mn}\theta)\Lambda_{mn}}{(C^I\lambda)(C^J\lambda)} \neq -\frac{(C^J\theta)\,(C^I\gamma^{mn}\theta)\Lambda_{mn}}{(C^I\lambda)(C^J\lambda)} \notin C^1(\underline{U}, \mathcal{O}), \tag{5.22}$$

In the particular case when Ω is a global holomorphic function in $PS \setminus \{0\}$ the product with any Čech cochain is well defined, for example the vertex operators in (5.3), or as in the computation (5.13). Note also that the Čech operator is not a derivate operator, i.e it does not satisfy the Leibniz rule. So it is not well defined acting on the product (5.22)

$$(\delta\Psi)^{IJK} \neq (\delta Y)^{IJ}\Omega^K \pm Y^I (\delta\Omega)^{JK}. \tag{5.23}$$

Therefore the expressions $(Q+\delta) \langle (Q+\delta)(\Omega) \rangle$ and $\langle (Q+\delta)(Q+\delta)(\Omega) \rangle$ are not equal i.e $(Q+\delta) \langle (Q+\delta)(\Omega) \rangle \neq \langle (Q+\delta)(Q+\delta)(\Omega) \rangle$, and in most cases the left hand side is not defined when Ω has Čech labels. Therefore the expression (5.19) does not make sense unless that Ω will be a global holomorphic function, like we assumed in (5.13).

From the analysis above we can conclude that for the tree level scattering amplitudes, the naive existence of the homotopy operator [8][9] given by

$$\xi = \left.\frac{C^I\theta}{C^I\lambda}\right|_{U_I} + \left.\frac{C^I\theta C^J\theta}{C^I\lambda C^J\lambda}\right|_{U_I\cap U_J} + ... + \left.\frac{C^1\theta C^2\theta ... C^{11}\theta}{C^1\lambda C^2\lambda ... C^{11}\lambda}\right|_{U_1\cap U_2\cap ...\cap U_{11}}, \tag{5.24}$$

which by definition satisfies

$$(Q+\delta)(\xi V_1V_2V_3U_1...U_{N-3}) = V_1V_2V_3U_1...U_{N-3} \tag{5.25}$$

for $V_1V_2V_3$ unintegrated vertex operators and $U_1...U_{N-3}$ integrated vertex operators, is not allowed because

$$\xi V_1V_2V_3U_1...U_{N-3} \tag{5.26}$$

is not a global function on $PS \setminus \{0\}$. Therefore at tree level it is sufficient to decouple the global and Q exact functions, see (5.13).

5.2 Lorentz and Supersymmetry Invariance

Now we show that although the new lowering picture operators are neither Lorentz nor supersymmetry invariant, the scattering amplitude is invariant under both transformations.

5.2.1 Lorentz Invariance

It is easy to show that the action of the Lorentz generators $M^{mn} = (1/2)\int \mathrm{d}z[(\omega\gamma^{mn}\lambda) + (p\gamma^{mn}\theta)]$ on the PCO's is Q exact:

$$M^{mn}Y_C^I = -\frac{1}{2}Q\left[\frac{(C^I\gamma^{mn}\theta)(C^I\theta)}{(C^I\lambda)^2}\right], \tag{5.27}$$

then, replacing this in the scattering amplitude we get

$$\begin{aligned} & M^{mn}(\mathcal{A}) \qquad\qquad (5.28) \\ = \; & \frac{1}{2\ 11!}\sum_{I_1,...,I_{11}=1}^{11}\epsilon_{I_1,...,I_{11}}\int_\Gamma[\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta\sum_{i=1}^{11}(-1)^iQ\left[\frac{(C^{I_i}\gamma^{mn}\theta)(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}\right]\frac{C^{I_1}\theta...\widehat{C^{I_i}\theta}...C^{I_{11}}\theta}{C^{I_1}\lambda...\widehat{C^{I_i}\lambda}...C^{I_{11}}\lambda} \\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \\ = \; & \frac{1}{2\ 11!}\sum_{I_1,...,I_{11}=1}^{11}\epsilon_{I_1,...,I_{11}}\int_\Gamma[\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta\sum_{i=1}^{11}(-1)^{i-1}\frac{(C^{I_i}\gamma^{mn}\theta)(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}Q\left(\frac{C^{I_1}\theta...\widehat{C^{I_i}\theta}...C^{I_{11}}\theta}{C^{I_1}\lambda...\widehat{C^{I_i}\lambda}...C^{I_{11}}\lambda}\right) \\ & \lambda^\alpha\lambda^\beta\lambda^\gamma\ f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \end{aligned}$$

where the term $\frac{\widehat{C^{I_i}\theta}}{\widehat{C^{I_i}\lambda}}$ means it must be removed from the expression. Making an algebraic manipulation we find

$$\sum_{i=1}^{11}(-1)^{i-1}\frac{(C^{I_i}\gamma^{mn}\theta)(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}Q\left(\frac{C^{I_1}\theta...\widehat{C^{I_i}\theta}...C^{I_{11}}\theta}{C^{I_1}\lambda...\widehat{C^{I_i}\lambda}...C^{I_{11}}\lambda}\right) \equiv \sum_{i=1}^{11}(-1)^{i-1}\pi^{I_i}Q\left(Y_C^{I_1}...\widehat{Y_C^{I_i}}...Y_C^{I_{11}}\right) = (\delta\psi)^{I_1...I_{11}} \tag{5.29}$$

where we define

$$\pi^I \equiv \frac{(C^I\gamma^{mn}\theta)(C^I\theta)}{(C^I\lambda)^2} \tag{5.30}$$

and $\psi^{I_1...I_{10}}$ is given by

$$\begin{aligned}\psi^{I_1...I_{10}} &= -\frac{1}{9!}\pi^{[I_1}Y_C^{I_2}...Y_C^{I_{10}]} \qquad (5.31)\\ &\equiv -\frac{1}{9!}\left(\pi^{I_1}Y_C^{I_2}Y_C^{I_3}...Y_C^{I_{10}} - \pi^{I_2}Y_C^{I_1}Y_C^{I_3}...Y_C^{I_{10}} + \text{all permutation}\right) \in C^9(\underline{U},\mathcal{O}).\end{aligned}$$

We define the holomorphic 11-form

$$\Psi^{I_1...I_{10}} = [\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta\,\psi^{I_1...I_{10}}\lambda^\alpha\lambda^\beta\lambda^\gamma\; f_{\alpha\beta\gamma}(\theta,\,k_i,\,e_i) \in C^9(\underline{U},\Omega^{11}). \tag{5.32}$$

So we can write (5.28) in the following way

$$M^{mn}(\mathcal{A}) = \frac{1}{2\;11!}\sum_{I_1,...,I_{11}=1}^{11}\epsilon_{I_1,...,I_{11}}\int_\Gamma(\delta\Psi)^{I_1...I_{11}}. \tag{5.33}$$

With the same procedure used to show the BRST invariance of the amplitude, i.e following the steps from (5.5) to (5.12) we obtain

$$\sum_{I_1,...,I_{11}=1}^{11}\epsilon_{I_1,...,I_{11}}\int_\Gamma(\delta\Psi)^{I_1...I_{11}} = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\Psi) = 0 \tag{5.34}$$

since $\partial(SO(10)/SU(5)) = \varnothing$. Finally, we conclude the tree level scattering amplitude is Lorentz invariant

$$M^{mn}(\mathcal{A}) = 0. \tag{5.35}$$

5.2.2 Invariance under Supersymmetry

Now we show that the tree level scattering amplitude is invariant under supersymmetry transformations. We call the supersymmetry generator "q", which is given by

$$q = \varepsilon^\alpha q_\alpha \tag{5.36}$$

where ε^α is a Grassmann constant spinor spinor and

$$q_\alpha = \int \mathrm{d}z \, (p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta \partial x_m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta).$$

It is easy to see that the action of q on the PCO's is

$$q(Y^I_C) = \varepsilon^\alpha q_\alpha(Y^I_C) = Q\left[\frac{(\varepsilon C^I)(C^I\theta)}{(C^I\lambda)^2}\right]. \tag{5.37}$$

Therefore in the tree level scattering amplitude we have

$$\begin{aligned} & q(\mathcal{A}) \qquad (5.38)\\ = \; & \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11}\epsilon_{I_1,\ldots,I_{11}}\int_\Gamma [\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta \sum_{i=1}^{11}(-1)^{i-1} Q\left[\frac{(\varepsilon C^{I_i})(C^{I_i}\theta)}{(C^{I_i}\lambda)^2}\right]\frac{C^{I_1}\theta \ldots \widehat{C^{I_i}\theta}\ldots C^{I_{11}}\theta}{C^{I_1}\lambda\ldots\widehat{C^{I_i}\lambda}\ldots C^{I_{11}}\lambda}\\ & \lambda^\alpha\lambda^\beta\lambda^\gamma \; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)\\ = \; & \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11}\epsilon_{I_1,\ldots,I_{11}}\int_\Gamma [\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta \sum_{i=1}^{11}(-1)^{i}\frac{(\varepsilon C^{I_i})(C^{I_i}\theta)}{(C^{I_i}\lambda)^2} Q\left(\frac{C^{I_1}\theta \ldots \widehat{C^{I_i}\theta}\ldots C^{I_{11}}\theta}{C^{I_1}\lambda\ldots\widehat{C^{I_i}\lambda}\ldots C^{I_{11}}\lambda}\right)\\ & \lambda^\alpha\lambda^\beta\lambda^\gamma \; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i). \end{aligned}$$

Making a similar algebraic manipulation like in (5.29) we get

$$\begin{aligned} & [\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta \sum_{i=1}^{11}(-1)^{i}\frac{(\varepsilon C^{I_i})(C^{I_i}\theta)}{(C^{I_i}\lambda)^2} Q\left(\frac{C^{I_1}\theta \ldots \widehat{C^{I_i}\theta}\ldots C^{I_{11}}\theta}{C^{I_1}\lambda\ldots\widehat{C^{I_i}\lambda}\ldots C^{I_{11}}\lambda}\right)\\ & \lambda^\alpha\lambda^\beta\lambda^\gamma \; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i)\\ = \; & (\delta\Phi)^{I_1\ldots I_{11}}, \end{aligned} \tag{5.39}$$

where we have the following definitions

$$\Phi^{I_1\ldots I_{10}} = [\mathrm{d}\lambda]\int \mathrm{d}^{16}\theta \, \phi^{I_1\ldots I_{10}} \, \lambda^\alpha\lambda^\beta\lambda^\gamma \; f_{\alpha\beta\gamma}(\theta,\, k_i,\, e_i) \in C^9(\underline{U}, \Omega^{11}), \tag{5.40}$$

$$\phi^{I_1\ldots I_{10}} \;\; = \;\; \frac{1}{9!}\varphi^{[I_1}\, Y^{I_2}_C \ldots Y^{I_{10}]}_C, \tag{5.41}$$

$$\varphi^I = \frac{(\varepsilon C^I)(C^I\theta)}{(C^I\lambda)^2}. \tag{5.42}$$

Using the same argument as in (5.34) it is clear that

$$q(\mathcal{A}) = \frac{1}{11!}\sum_{I_1,\ldots,I_{11}=1}^{11}\epsilon_{I_1,\ldots,I_{11}}\int_\Gamma (\delta\Phi)^{I_1\ldots I_{11}} = \int_{\partial(SO(10)/SU(5))} i^*(\eta_\Phi) = 0. \tag{5.43}$$

So the tree level scattering amplitude is invariant under supersymmetry transformations.

In conclusion, we succeeded in proving the invariance under the total BRST, Lorentz and supersymmetry transformations, where the Čech-Dolbeault language played a central role.

6 Independence of the Constant Spinors C^I_α's

In this section, our goal is to show that the scattering amplitude is independent of the choice of the constant spinors C^I's. This implies that they do not need to be integrated, in contrast with the analysis presented in [4][17], where it did was necessary.

We will present an example of pure spinors in four dimensions, where the conditions of linear independence for the C^I's and the intersection of the hyperplanes D_I's in the origin are equivalent. However, in ten dimensions it is not sufficient that the C^I's are linearly independent, so, based on the assumption that the hypersurfaces $D_I = \{C^I\lambda = 0\}, I = 1, ..., 11$, meet just in the origin, we will show that the scattering amplitude is independent of the C^I's choice.

6.1 Pure Spinors in $d = 4$: A Simple Example

Before we show the independence of the C^I's for pure spinors in ten dimensions, we give a simple example in four dimensions in order to understand how this can be achieved.

Consider the pure spinor space in $d = 4$ dimensions, i.e the flat space $\mathbb{C}^2$. In this case the integral corresponding to (4.46) is given by

$$\int_\Gamma \vartheta = \int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{cd} C^1_c\, C^2_d}{(C^1\lambda)(C^2\lambda)}, \qquad c, d = 1, 2 \tag{6.1}$$

where $\lambda^a = (\lambda^1, \lambda^2)$ are the coordinates of $\mathbb{C}^2$ and the measure is simply $[\mathrm{d}\lambda] = 2^{-1}\epsilon_{ab}\, \mathrm{d}\lambda^a \wedge \mathrm{d}\lambda^b = \mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2$. Clearly, the vectors $C^j, j = 1, 2$ must be linearly independent in order to obtain an integral different from zero, i.e the determinant $\det(C^j_a) \neq 0$. This implies that the intersection of the hyperplanes $C^j\lambda = 0$, $j = 1, 2$ is just the origin. To compute (6.1), firstly we consider the simple case when $C^j_a = \delta^j_a$. Then the integral is

$$\int_\Gamma \vartheta = \int_\Gamma \frac{\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2}{\lambda^1\lambda^2}. \tag{6.2}$$

Secondly we define in a natural way the 2-cycle Γ as the torus $\Gamma = \{\lambda^a \in \mathbb{C}^2 : |\lambda^1| = \epsilon^1 \text{ and } |\lambda^2| = \epsilon^2\}$, where ϵ^1, ϵ^2 are positive arbitrary constants. So (6.2) is a trivial integral and its answer is

$$\int_\Gamma \vartheta = (2\pi i)^2. \tag{6.3}$$

Once the answer is known, we must know what happens if we choose two arbitrary vectors C^i_a but keep the same 2-cycle $\Gamma = \{\lambda^a \in \mathbb{C}^2 : |\lambda^1| = \epsilon^1 \text{ and } |\lambda^2| = \epsilon^2\}$. In other words, we want to know the answer to the question: is (6.1) independent of the constants C^j's?. We will show that the answer is affirmative and its result is the same as in (6.3). From (6.1) we have

$$\int_{|\lambda^2|=\epsilon^2} \int_{|\lambda^1|=\epsilon^1} \frac{(a_1b_2 - a_2b_1)\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2}{(a_1\lambda^1 + a_2\lambda^2)(b_1\lambda^1 + b_2\lambda^2)}, \tag{6.4}$$

where $C^1 = (a_1, a_2)$ and $C^2 = (b_1, b_2)$. Without loss of generality, we can set $a_2, b_1 \neq 0$. To solve (6.4), first note that since ϵ^1 is an arbitrary constant, it can be set to a very large value such that the pole $-(b_2/b_1)\lambda^2$ is inside of the cycle $|\lambda^1| = \epsilon^1$, so integrating λ^1 we have

$$\int_{|\lambda^2|=\epsilon^2} \int_{|\lambda^1|=\epsilon^1} \frac{(a_1b_2 - a_2b_1)\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2}{(a_1\lambda^1 + a_2\lambda^2)(b_1\lambda^1 + b_2\lambda^2)} = (2\pi i) \int_{|\lambda^2|=\epsilon^2} \frac{(a_1b_2 - a_2b_1)\mathrm{d}\lambda^2}{(a_1b_2 - a_2b_1)\lambda^2}, \tag{6.5}$$

getting the same answer as in (6.3). However, since the integral depends on a very large value of ϵ^1, this is not a satisfactory way for computing, so we must look for a better analysis.
As $\det(C^j_a) = (a_1b_2 - a_2b_1) \neq 0$, then we can make the following change of variables

$$\begin{aligned} \lambda^1 &= M^{-1}(b_2z^1 - a_2z^2) \\ \lambda^2 &= M^{-1}(-b_1z^1 + a_1z^2) \end{aligned} \tag{6.6}$$

where $M = (a_1b_2 - a_2b_1)$. Using these new coordinates (6.5) becomes

$$\int_\Gamma \frac{\mathrm{d}z^1 \wedge \mathrm{d}z^2}{z^1z^2}. \tag{6.7}$$

where Γ is the 2-cycle given by $|b_2z^1 - a_2z^2| = \epsilon^1|M|$ and $|a_1z^2 - b_1z^1| = \epsilon^2|M|$. Since ϵ^1 and ϵ^2 are positive arbitrary constants then $|z_1| > 0$, $|z_2| > 0$ and applying the triangle inequality we get

$$\begin{aligned} 0 < |z^1| &\leq (\epsilon^1|a_1| + \epsilon^2|a_2|) \\ 0 < |z^2| &\leq (\epsilon^1|b_1| + \epsilon^2|b_2|) \end{aligned} \tag{6.8}$$

where without loss of generality, we set $a_2, b_1 \neq 0$. Therefore the torus $\Gamma = \{(z^1, z^2) \in \mathbb{C}^2 : |b_2z^1 - a_2z^2| = \epsilon^1|M|, |a_1z^2 - b_1z^1| = \epsilon^2|M|\}$ can be deformed to the torus $\Gamma' = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1| = (\epsilon^1|a_1| + \epsilon^2|a_2|)/2, |z^2| = (\epsilon^1|b_1| + \epsilon^2|b_2|)/2\}$. So, we have shown that the integral (6.1) is independent of the constants C^j's when we fix the integration cycle, because it can always be deformed to a cycle of the type $|C^i\lambda| = \varepsilon^j$, $j = 1, 2$, for some ε^j, $j = 1, 2$. Formally we are saying the following: remember that the integral (6.1) just depends of the classes of the homology cycle $[\Gamma]$ and the cocycle of cohomology $[\vartheta]$ (see subsection 4.1). So, if $\det(C^j_a) \neq 0$ then all the holomorphic 2-forms ϑ are in the same cohomology class $[\vartheta]$ and all the 2-cycle $\Gamma = \{(\lambda^1, \lambda^2) \in \mathbb{C}^2 : |C^j\lambda| = \epsilon^j, j = 1, 2\}$ are in the same homology class $[\Gamma]$. Now we must show the same for pure spinors in $d = 10$.

6.2 Pure Spinors in $d = 10$

In the previous example the conditions $\det(C^j_a) \neq 0$ and $\{C^1\lambda = 0\} \cap \{C^2\lambda = 0\} = \{0\}$ were equivalent. However, in the pure spinor space in $d = 10$ the condition $\det(C^I_\alpha) \neq 0$ does not make sense because $I = 1, .., 11$ and $\alpha = 1, ..., 16$, but remember that we have always claimed that $D_1 \cap ... \cap D_{11} = \{0\}$, where $D_I = \{\lambda^\alpha \in PS : C^I_\alpha\lambda^\alpha = 0\}$. In this case is not easy to follow the same analysis of the previous example because PS is not a flat space. Therefore we will make use of the ideas presented previously in this paper, like the Čech-Dolbeault isomorphism, to prove that the scattering amplitude is independent of the C^I's.

From (4.46) we have that the amplitude is given by

$$\mathcal{A} = \int_\Gamma \beta^{1,\dots,11} \tag{6.9}$$

where the 11-cycle Γ was defined as $\Gamma = \{\lambda^\alpha \in PS : |C^I\lambda| = \epsilon^I,\ I = 1, ..., 11\}$, $\epsilon^I \in \mathbb{R}^+$. In the sub-subsection 4.4.2 we found the Dolbeault cocycle

$$\eta_\beta = 2^3\, 5!\, \frac{[\mathrm{d}\lambda] \wedge [\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^8}\bigg|_{SO(10)/SU(5)} K \tag{6.10}$$

corresponding to $\beta^{1,\dots,11}$, thanks to the isomorphism from the group $H^{10}(PS \setminus \{0\}, \Omega^{11})$ to $H^{21}_{DR}(SO(10)/SU(5))$ (see sub-subsection 4.3.1). As the Dolbeault cocycle η_β is independent of the constants C^I's, then, choosing another set of constant spinors C'^I, $I = 1, .., 11$, such that they satisfy the same condition $D'_1 \cap ... \cap D'_{11} = \{0\}$, its Čech cocycle $\beta'^{1,\dots,11}$ is in the same cohomology class as $\beta^{1,\dots,11}$, because $\beta'^{1,\dots,11}$ and $\beta^{1,\dots,11}$ have the same corresponding Dolbeault cocycle η_β and the groups $H^{10}(PS \setminus \{0\}, \Omega^{11})$ and $H^{21}_{DR}(SO(10)/SU(5))$ are isomorphic. It means that

$$\int_\Gamma \beta^{1,\dots,11} = \int_\Gamma \beta'^{1,\dots,11}, \tag{6.11}$$

because the cohomology classes $[\beta^{1,\dots,11}]$ and $[\beta'^{1,\dots,11}]$ are the same.
So we have shown that the tree level scattering amplitude is independent of the constant spinors C^I's and therefore it is not needed to integrate over them.

7 Relation with Twistor Space

In the sub-subsection 4 we studied the tree-level scattering amplitude in the projective pure spinor space. In this section we will show that the result found there, given by the integral (4.20), is the same found by Berkovits and Cherkis in [16]. In that reference, the projective pure spinor space allowed to get the Green's function for a massless scalar field in $d = 10$ dimensions.

The Green's function for a massless scalar field in $d = 10$ dimensions is given by the integral

$$\Phi(x) = \int_{\tilde{\Gamma}} [\mathrm{d}\tilde{\lambda}]\ F(\tilde{\lambda}, \omega)|_{\omega = x\tilde{\lambda}}, \tag{7.1}$$

which is written covariantly [16]. In this integral, $\tilde{\lambda}^\alpha$ is a projective pure spinor, $[\mathrm{d}\tilde{\lambda}]$ is the measure of the projective pure spinor space given by (B.1), while $F(\tilde{\lambda}, \omega)|_{\omega = x\tilde{\lambda}}$ is given by

$$F(\tilde{\lambda}, \omega) = \frac{\epsilon_{\alpha_1 \dots \alpha_{11}\beta_1 \dots \beta_5} A_1^{\alpha_1} \dots A_{11}^{\alpha_{11}} (\gamma^m \omega)^{\beta_1} (\gamma^n \omega)^{\beta_2} (\gamma^p \omega)^{\beta_3} (\gamma_{mnp})^{\beta_4 \beta_5}}{\prod_{r=1}^{11} (A_r^\alpha \omega_\alpha)}, \quad \omega_\alpha = (x \cdot \gamma \tilde{\lambda})_\alpha. \tag{7.2}$$

A_I^α's are constant spinors and the cycle $\tilde{\Gamma}$ is given by ten out of the eleven poles of $F(\tilde{\lambda},\omega)$. The measure (B.1) is not suitable for obtaining the relationship between twistors and scattering amplitude, so we will modify it as follows. First note that for $|x|\neq 0$ then $(x\cdot\gamma\tilde{\lambda})_\alpha$ is a pure spinor if and only if $\tilde{\lambda}^\alpha$ is also a pure spinor. This is very easy to show:

1. In the backward direction: If $\tilde{\lambda}^\alpha$ is a pure spinor in ten dimensions, then $(x\cdot\gamma\tilde{\lambda})_\alpha$ is also a pure spinor. So, we must prove that $(x\cdot\gamma\tilde{\lambda})_\alpha(\gamma^m)^{\alpha\beta}(x\cdot\gamma\tilde{\lambda})_\beta = 0$ using the condition $\tilde{\lambda}\gamma\tilde{\lambda}=0$. Then,

$$
\begin{aligned}
(x\cdot\gamma\tilde{\lambda})_\alpha(\gamma^m)^{\alpha\beta}(x\cdot\gamma\tilde{\lambda})_\beta &= x^n x^p\{\tilde{\lambda}^\delta(\gamma_n)_{\delta\alpha}(\gamma^m)^{\alpha\beta}(\gamma_p)_{\beta\rho}\tilde{\lambda}^\rho\} \\
&= 2\,x^m x^p(\tilde{\lambda}\gamma_p\tilde{\lambda}) - x^n x^p(\tilde{\lambda}\gamma^m\gamma_n\gamma_p\tilde{\lambda}) \\
&= -\frac{1}{2}x^n x^p(\tilde{\lambda}\gamma^m\{\gamma_n,\gamma_p\}\tilde{\lambda}) \\
&= -\frac{1}{2}x\cdot x(\tilde{\lambda}\gamma^m\tilde{\lambda}) = 0.
\end{aligned}
\tag{7.3}
$$

2. Let's now make the prove in the forward direction, i.e, assuming that $(x\cdot\gamma\tilde{\lambda})_\alpha$ is a pure spinor, then $\tilde{\lambda}^\alpha$ is also a pure spinor. We start defining the pure spinor ρ_α: $\rho_\alpha \equiv (x\cdot\gamma\tilde{\lambda})_\alpha$. Then, writing $\tilde{\lambda}^\alpha$ in terms of ρ_α we find

$$
\tilde{\lambda}^\alpha = \frac{1}{x\cdot x}(\rho\gamma\cdot x)^\alpha. \tag{7.4}
$$

   Since ρ_α is a pure spinor, then performing a similar computation as in the proof in the backward direction, it is trivial to show that $\tilde{\lambda}^\alpha$ is a pure spinor: $\tilde{\lambda}\gamma^m\tilde{\lambda}=0$.

Using the previous property we redefine the measure $[\mathrm{d}\tilde{\lambda}]$ given in (B.1) by

$$
\begin{aligned}
&[\mathrm{d}\tilde{\lambda}]'(\tilde{\lambda}\gamma\cdot x\gamma^m)^{\alpha_1}(\tilde{\lambda}\gamma\cdot x\gamma^n)^{\alpha_2}(\tilde{\lambda}\gamma\cdot x\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \\
&\frac{2^3}{10!|x|^8}\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathrm{d}(\tilde{\lambda}\gamma\cdot x)_{\beta_1}\wedge\ldots\wedge\mathrm{d}(\tilde{\lambda}\gamma\cdot x)_{\beta_{10}}(\tilde{\lambda}\gamma\cdot x)_{\beta_{11}}.
\end{aligned}
\tag{7.5}
$$

Performing a simple computation as in (B.5) we can show that $[\mathrm{d}\tilde{\lambda}] = e^{i\phi}[\mathrm{d}\tilde{\lambda}]'$, where $\phi\in\mathbb{R}$ is a constant. Then, up to a phase factor, we write

$$
\Phi(x) = \int_{\tilde{\Gamma}}[\mathrm{d}\tilde{\lambda}]'\;F(\tilde{\lambda},\omega)|_{\omega=x\tilde{\lambda}}. \tag{7.6}
$$

Replacing $F(\tilde{\lambda},\omega)|_{\omega=x\tilde{\lambda}}$ and using the measure (7.5) we get

$$
\Phi(x) = \frac{2^3 5!}{|x|^8}\int_{\tilde{\Gamma}}\frac{\mathrm{d}(A_1 x\cdot\gamma\tilde{\lambda})\wedge\ldots\wedge\mathrm{d}(A_{10}x\cdot\gamma\tilde{\lambda})}{(A_1 x\cdot\gamma\tilde{\lambda})\ldots(A_{10}x\cdot\gamma\tilde{\lambda})}, \tag{7.7}
$$

where without loss of generality, we chose $\tilde{\Gamma} = \{\lambda^\alpha \in SO(10)/U(5) : |(A_I x \cdot \gamma\tilde{\lambda})| = \epsilon_I,\ I = 1, ..., 10\}$. From (4.20) and (7.7) the relationship between the tree level scattering amplitude and the Green's function for the massless scalar field is clear

$$\begin{aligned} C^I_\alpha &\rightarrow (A_I \gamma \cdot x)_\alpha, \\ K &\rightarrow \frac{1}{(2\pi i)|x|^8}. \end{aligned} \tag{7.8}$$

This result was not known using the old PCO's. Although the construction for the scattering amplitudes at the genus g is in progress we think that it is likely to have a relationship between loops scattering amplitudes and massless solutions for higher-spin [16].

8 Comments About the Loop-Level

Now we give a glance about the scattering amplitude at the loop-level. The loop level in the minimal pure spinor formalism has two fundamental ingredients: the picture raising operators and the b-ghost. The picture raising operators are needed to absorb the zero modes of the field ω_α and some of the zero modes of the field p_α, given in the action (2.1).
Because at the loop level the complex structure of the Riemann surfaces have deformations, known as the moduli space, in order to fix these deformations it is necessary to introduce the b-ghost. In the pure spinor formalism the b-ghost is not a fundamental field [4][8] and therefore its construction in terms of the others fields is such that satisfies

$$\{Q_T, b(z)\} = T(z), \tag{8.1}$$

where Q_T is the BRST charge and $T(z)$ is the stress-energy tensor.
In [9] was given the b-ghost for the minimal pure spinor formalism. Nevertheless it has not been used to compute scattering amplitudes. One reason for not using it is the difficulty for dealing with the Čech indices inside the scattering amplitude. In this section we will give some directions for computing scattering amplitudes with b-ghost in the minimal pure spinor formalism.

8.1 Product of Čech Cochains

In this sub-subsection we define a product between the Čech cochains such that the result will be also a Čech cochain. The aim is to obtain a well defined scattering amplitude, i.e since the loop level scattering amplitude includes the b-ghost and the lowering and raising picture changing operators, which are mathematical objects define locally, then it is necessary that the product of all these objects will be a Čech cochain, such that the BRST operator $Q_T = Q + \delta$ is also well defined allowing in this manner to establish a relationship between the minimal and non-minimal formalism.

As we have shown in the example (5.22), the product of two Čech cochains is not in general a Čech cochain. This implies that the Čech operator δ is not defined acting on this product, because it does not satisfy the Leibniz rule, i.e it is not a derivative operator, see the example (5.23). So we are going to define a product between the Čech cochains and show how the Čech operator acts on them. This is a small step towards the definition of loop-level scattering amplitudes.

As we showed previously with the example (5.22), considering two general cochains ψ^I and τ^I in the Abelian group of holomorphic function, the product in most cases is not a Čech cochain

$$\psi^I \tau^J \notin C^1(\underline{U}, \mathcal{O}) \tag{8.2}$$

because $\psi^I \tau^J \neq -\psi^J \tau^I$. So, we define the following antisymmetric product

$$\psi^I * \tau^J \equiv \frac{1}{2}\left(\psi^I \tau^J - \psi^J \tau^I\right)\Big|_{U_I \cap U_J} \equiv \frac{1}{2}\psi^{[I}\tau^{J]} = \pi^{IJ} \in C^1(\underline{U}, \mathcal{O}), \tag{8.3}$$

which looks like an exterior product. Obviously this product is antisymmetric in the index I, J, i.e $\psi^I * \tau^J = -\psi^J * \tau^I$, however the exchange of the ψ^I and τ^J depends if they are grassmann or bosonic variables, this means

$$\psi^I * \tau^J = (-)^{(deg(\psi^I)\cdot deg(\tau^J))}\tau^J * \psi^I, \tag{8.4}$$

where $deg(\psi^I) = 0$ or 1 if ψ^I is a bosonic or grassmann variable respectively. Note that if the product of the Čech cochains is well defined, as in the case of the product of the picture lowering operators $Y_C^I Y_C^J$, then it is in agreement with (8.3):

$$Y_C^I * Y_C^J = \frac{1}{2}Y_C^{[I}Y_C^{J]} = Y_C^I Y_C^J. \tag{8.5}$$

Now, acting with the Čech operator on (8.3) we get

$$\begin{aligned}(\delta\pi)^{IJK} &= [\delta(\psi * \tau)]^{IJK} = \frac{1}{2^2}(\delta\psi)^{[IJ}\tau^{K]} = -\frac{1}{2^2}\psi^{[I}(\delta\tau)^{JK]} = \frac{3}{4}\frac{1}{3!}\left((\delta\psi)^{[IJ}\tau^{K]} - \psi^{[I}(\delta\tau)^{JK]}\right) \\ &= \frac{3}{4}\left((\delta\psi)^{IJ} * \tau^K - \psi^I * (\delta\tau)^{JK}\right) \in C^2(\underline{U}, \mathcal{O}). \end{aligned} \tag{8.6}$$

Note that δ acts like the exterior derivative over elements with star product, but with a coefficient in the front. This is a beautiful property. If we have three Čech cochains $\psi^I, \tau^J, \rho^K \in C^0(\underline{U}, \mathcal{O})$ then we define

$$\psi^I * \tau^J * \rho^K = \frac{1}{3!}\psi^{[I}\tau^J\rho^{K]} = \chi^{IJK} \in C^2(\underline{U}, \mathcal{O}). \tag{8.7}$$

It is simple to see that this product is associative. Acting with the Čech operator we have

$$
\begin{aligned}
(\delta\chi)^{IJKL} &= \frac{1}{3!\,2}(\delta\psi)^{[IJ}\tau^K\rho^{L]} = -\frac{1}{3!\,2}\psi^{[I}(\delta\tau)^{JK}\rho^{L]} = \frac{1}{3!\,2}\psi^{[I}\tau^J(\delta\rho)^{KL]} \\
&= \frac{4}{3!}\frac{1}{4!}\left((\delta\psi)^{[IJ}\tau^K\rho^{L]} - \psi^{[I}(\delta\tau)^{JK}\rho^{L]} + \psi^{[I}\tau^J(\delta\rho)^{KL]}\right) \\
&= \frac{4}{3!}((\delta\psi)^{IJ} * \tau^K * \rho^L - \psi^I * (\delta\tau)^{JK} * \rho^L + \psi^I * \tau^J * (\delta\rho)^{KL}) \in C^3(\underline{U}, \mathcal{O}).
\end{aligned} \tag{8.8}
$$

Again, the Čech operator acts like the exterior derivative operator over elements with the star product, nevertheless it has a coefficient in the front. It is straightforward to generalize this procedure for higher cochains with values on any Abelian group. Notice that the expressions (5.31) and (5.41) are just the $*$ product.

If we use this product between the homotopy operator (5.24) and the PCO's inside the tree level scattering amplitude the result vanishes because there are 11 patches to cover the pure spinor space.

8.2 The b-ghost

Unlike the tree-level scattering amplitude, in higher orders of the genus expansion the cover $\underline{U} = \{U_I\}$ given by the eleven patches $U_I = PS \setminus D_I$, $I = 1, .., 11$ (see 4.22) is not enough. The explanation is simple, since the b-ghost is a linear combination of 0,1,2 and 3-Čech cochains in the pure spinor space [9] and the product of the 11 picture lowering operators $\prod_{I=1}^{11} Y_C^I$ is a 10-Čech cochain then, with the antisymmetric $*$ product it is clear that the scattering amplitude will vanish if the number of patches is less than $11 + 4(3g-3)$, $g > 1$, where g is the genus of the Riemann surface. In the particular case when $g = 1$ this number is $11+4$. One can think to add more patches to the tree level scattering amplitude (see the appendix A.2) and so to apply the $*$ product with the naive homotopy operator, however this product needs to be better understood, since actually the operator δ is not a derivate operator strictly speaking because of those coefficients in the front of (8.6) and (8.8) [18]. Note also that the tree level scattering amplitude must be δ closed in contrast to the genus-g, as we discuss later.

So, in this approach the b-ghost is given by

$$
b = b_{(0)} + b_{(1)} + b_{(2)} + b_{(3)} \tag{8.9}
$$

with

$$
b^{\mu}_{(0)} = \frac{A^{\mu}_{\alpha}G^{\alpha}}{(A^{\mu}\lambda)}, \quad b^{\mu\nu}_{(1)} = \frac{A^{\mu}_{\alpha}A^{\nu}_{\beta}H^{[\alpha\beta]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)}, \quad b^{\mu\nu\rho}_{(2)} = \frac{A^{\mu}_{\alpha}A^{\nu}_{\beta}A^{\rho}_{\gamma}K^{[\alpha\beta\gamma]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)(A^{\rho}\lambda)}, \quad b^{\mu\nu\rho\kappa}_{(3)} = \frac{A^{\mu}_{\alpha}A^{\nu}_{\beta}A^{\rho}_{\gamma}A^{\kappa}_{\delta}L^{[\alpha\beta\gamma\delta]}}{(A^{\mu}\lambda)(A^{\nu}\lambda)(A^{\rho}\lambda)(A^{\kappa}\lambda)},
$$

where the specific form of the numerators (G, H, K, L) above can be found in [8] and the A^{μ}_{α}'s belong to a bigger set of constant spinors $A^{\mu} \in \{C^I, V^i\} \equiv \{C^1, ..., C^{11}, V^1, ..., V^{4(3g-3)}\}$, i.e $\mu \in \{I, i\}$,

[18] Actually, we wish that the operator $Q_T = Q + \delta$ acts like the exterior derivate, however we do not succeed yet.

where the V^i's are linearly independent vectors in $\mathbb{C}^{16}$ such that the hypersurfaces $P^i \equiv \{\lambda^\alpha \in PS : V^i_\alpha \lambda^\alpha = 0\}$ satisfy $D_1 \cap .. \cap D_{11} \cap P^1 \cap ... \cap P^{4(3g-3)} = \{0\}$. We define the patches $U_I = PS \setminus D_I$, $U^i = PS \setminus P^i$ and get the cover $\mathfrak{U} = \{U_I, U^i\}$ where $PS \setminus \{0\} = \bigcup_{I=1}^{11} U_I \bigcup_{i=1}^{4(3g-3)} U^i$. Note that the C^I's are the same as in the tree level case, so the cycle Γ given in (4.1) is a good definition to compute the scattering amplitude. Using the commutators and anticommutators given in [9]

$$\{Q, G^\alpha(z)\} = \lambda^\alpha T(z), \quad \left[Q, H^{[\alpha\beta]}\right] = \lambda^{[\alpha} G^{\beta]}, \quad \{Q, K^{[\alpha\beta\gamma]}\} = \lambda^{[\alpha} H^{\beta\gamma]}, \tag{8.10}$$

$$\left[Q, L^{[\alpha\beta\gamma\delta]}\right] = \lambda^{[\alpha} K^{\beta\gamma\delta]}, \quad \lambda^{[\eta} L^{\alpha\beta\gamma\delta]} = 0,$$

where $T(z)$ is the stress-energy tensor given in the section 2, it is easy to verify that the b-ghost (8.9) satisfies

$$\{Q + \delta, b(z)\} = T(z), \tag{8.11}$$

where, for instance

$$(\delta\, b_{(0)})^{\mu\nu} = \frac{A^\nu_\alpha G^\alpha}{(A^\nu \lambda)} - \frac{A^\mu_\alpha G^\alpha}{(A^\mu \lambda)} = \frac{A^\mu_\alpha A^\nu_\beta \lambda^{[\alpha} G^{\beta]}}{(A^\mu \lambda)(A^\nu \lambda)} = \frac{A^\mu_\alpha A^\nu_\beta \left[Q, H^{[\alpha\beta]}\right]}{(A^\mu \lambda)(A^\nu \lambda)}.$$

8.2.1 The Ghost Number Bidegree

As in bosonic string theory, the b-ghost must have ghost number -1 and the BRST charge must increase the ghost number by one unit. Note that $b_{(0)}$ has ghost number -1 but $b_{(1)}, b_{(2)}, b_{(3)}$ have ghost number $-2, -3, -4$ respectively (the numerators have ghost number zero, see [9]), where the ghost current is given by $J_\lambda = \lambda^\alpha \omega_\alpha$. However, since the total BRST charge is $Q_T = Q + \delta$ and the Čech operator increases the number of patches in one, then the number of patches is also a ghost number. So we define the total ghost number by

$$J_T = \int \mathrm{d}z J_\lambda + J_\delta,$$

$$J_\delta \equiv (\sum_\mu \mu \partial_\mu - 1),$$

where the operator J_δ acts on the Čech cochains in the Čech labels, for example

$$(\sum_\eta \eta \partial_\eta - 1)\; b^{\mu\nu\rho\kappa}_{(3)} = 3\, b^{\mu\nu\rho\kappa}_{(3)}.$$

Therefore the b-ghost (8.9) has J_T ghost number -1, as expected. In the tree level scattering amplitude the J_δ ghost number is not relevant because this amplitude is δ closed, see the subsection 5.1. Nevertheless, at loop level this ghost number become very important since the relation (8.11) means that the scattering amplitude is $Q_T = Q + \delta$ closed up to boundary terms in the moduli space [20], i.e if we consider a loop level scattering amplitude where the $*$ product is used to insert the b-ghost, then we expect to get

$$(Q+\delta)\left\langle \,....\int \mathrm{d}z\mu_{\bar{z}}^{z}(z)b(z)\right\rangle = \left\langle \,....\int \mathrm{d}z\mu_{\bar{z}}^{z}(z)(Q+\delta)(b(z))\right\rangle \;=\; \left\langle \,....\int \mathrm{d}z\mu_{\bar{z}}^{z}(z)T(z)\right\rangle$$
$$=\; \int_{\mathcal{M}} \frac{\partial}{\partial \tau^{i}}\left\langle ....\right\rangle , \qquad (8.12)$$

where .... means the global insertions, $\mu_{\bar{z}}^{z}(z)$ is the Beltrami differential, τ^{i}'s are the Teichmüller parameters and $\mathcal{M}$ is the Moduli space. Now, with the aim to see the importance of the J_δ ghost number at the loop level we can regard the 1-loop scattering amplitude. In this amplitude we have 11 zero modes of the pure spinor λ^α and 11 zero modes of the spinor ω_α [4], so at 1 loop the zero modes of λ^α and ω_α form the pure spinor phase space. Integrating somehow the zero modes of the fields ω_α, d_α and θ^α then the scattering amplitude shall behave as

$$\int_\Gamma [\mathrm{d}\lambda] \frac{A^{I_1}...A^{I_{11}}}{(A^{I_1}\lambda)...(A^{I_{11}}\lambda)} \frac{A^{I_{12}}A^{I_{13}}\lambda^4}{(A^{I_{12}}\lambda)(A^{I_{13}}\lambda)}, \qquad (8.13)$$

where we are not being careful with the spinorial indices. Note that this amplitude is a 12-Čech cochain and have J_λ ghost number -1. However this ghost number must always be zero, therefore it should be compensated with one J_δ ghost number[19] and so we will get an 11-Čech cochain. As the tree level scattering amplitude is a 10-Čech cochain and it is related to the Green's function for the massless scalar field then the 1 loop amplitude, which should be a 11-Čech cochain, suggests that it should be related to the Green's function for the massless higher-spin field [16]. This was only a simple and crude analysis about the 1-loop scattering amplitude. Actually the full analysis must be over the whole pure spinor phase space. This is because, for instance, at two loops we should get a Čech cochain in the pure spinor space bigger than 11, so its corresponding Dolbeault cochain will be identically zero and the amplitude will vanish. Therefore it is necessary to regard the whole pure spinor phase space, i.e the space of the λ^α's and ω_α's.

One could think that the scattering amplitude must have $J_T = J_\lambda + J_\delta$ ghost number zero, but that is not true. For example, in the tree level amplitude it is impossible to construct the picture lowering operator such that the amplitude has J_T ghost number zero and the origin is removed from the pure spinor space. So the conditions that the scattering amplitude has J_λ ghost number zero is necessary in order to get a physical amplitude, i.e a $(Q+\delta)$ closed scattering amplitude.

This was just a glance about the loop level scattering amplitudes, which is a work in progress [28].

[19] Perhaps because (8.13) is δ exact.

9 Conclusions

We proposed a new "picture lowering" operator and computed the scattering amplitude at tree level in such a way that we eliminated the singular point of the pure spinor space, getting in this way a theory free of anomalies [14]. Since the new picture operators are defined just on each patch of the pure spinor space, it is necessary to introduce the Čech operator as part of the BRST charge in order to have a well defined formalism. Therefore, we have introduced the Čech formalism for the scattering amplitudes computation, which seems to be the correct formulation [9][21]. Given the Čech-Dolbeault isomorphism, we found the corresponding Dolbeault cocycle for the scattering amplitude. What is interesting here is that the Dolbeault cocycle must not be evaluated in whole pure spinor space, but in the $SO(10)/SU(5)$, which can be thought like a sphere in the pure spinor space. This confirms that the singular point was removed from the pure spinor space. Moreover, since the de-Rham cohomology group of this manifold has just one generator given by

$$\left.\frac{[\mathrm{d}\lambda]\wedge[\mathrm{d}\bar{\lambda}]'}{(\lambda\bar{\lambda})^{8}}\right|_{SO(10)/SU(5)},$$

i.e $H^{21}_{DR}(SO(10)/SU(5)) = \mathbb{C}$ [14], then these picture operates do not correspond to any particular regulator of the non-minimal formalism. This may suggest that the minimal pure spinor formalism is, in this sense, more fundamental than the non-minimal formalism since it directly involves cohomology generators. Note that in this paper the tree level scattering amplitude was always computed using three unintegrated vertex operators and the rest were integrated vertex operators. In contrast with the minimal formalism, in the non-minimal formulation it is possible to compute tree level amplitude with all the vertex operators unintegrated [8]. The difficulty in the minimal formalism is the b-ghost. Although we gave a glance about how to treat this issue, in order to continue with the loop-level this subject must be further developed [28].

Using the Čech-Dolbeault isomorphism, we also showed in an elegant manner that the tree-level scattering amplitude is BRST, Lorentz and supersymmetric invariant.

In contrast with the PCO's proposed in [4], with the new PCO's proposed in this paper the tree level scattering amplitude is independent of the choice of the constants spinors C^I's. That is because the cohomology class of the scattering amplitude is the same when the constants C^I's satisfy the constraint $\{C^1\lambda = 0\} \cap \{C^2\lambda = 0\} \cap ... \cap \{C^{11}\lambda = 0\} = \{0\}$, for λ^α satisfying the pure spinor condition.

Finally, we obtained a relationship between the tree-level scattering amplitude in the pure spinor formalism and the Green's function for the massless scalar field in the twistor formalism [16]. We believe that perhaps there is a relationship between the loop-level scattering amplitudes in the pure spinor formalism and the Green's function for the higher-spin massless fields [16], which we would like to explore in the future.

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A Some Simple Examples

A.1 The Pure Spinor Condition in the $U(5)$ decomposition

We will give an example of a point $p \in PS$, for which in the $U(5)$ decomposition, is necessary to consider both conditions $\chi^a = 0$ and $\zeta_a = 0$ in order to have a well defined tangent space at p.

Consider for instance the point $p = (\lambda^+ = 0, \lambda_{ab} = 0, \lambda^a = \delta^{1a})$ in the pure spinor space. Then, the gradient vectors $V^a = (\lambda^a, -\frac{1}{4}\epsilon^{abcde}\lambda_{bc}, \lambda^+\delta^{ab})$, which generate the holomorphic tangent space to the cone given by

$$\chi^a \equiv \lambda^+\lambda^a - \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de} = 0, \quad a,b,c,d,e = 1,...,5,$$

do not generate a tangent space of complex dimension 11 at the point p. This is because $V^i = (0,...,0)$ for $i = 2,..,5$, i.e only V^1 is different from zero at p, which means that p is a singular point[20] of the space $\chi^a = 0$, $a = 1,...,5$. So $\chi^a = 0$ does not describe completely the pure spinor space since actually PS only has one singular point: $\lambda^\alpha = 0$. For that reason, we must consider the rest of the pure spinor equations

$$\zeta_a = \lambda^b\lambda_{ba} = 0, \qquad a,b = 1,...,5. \tag{A.1}$$

Note that p is actually a point in the pure spinor space since it satisfies both $\chi^a = 0$ and $\zeta_a = 0$. In contrast, there exists points which do not satisfy simultaneously both set of equations. To the

[20] We say that p is a singular point of PS (or any space) if and only if it is not possible to define an unique tangent space in p with the same dimension of PS.

five equations $\zeta_a = 0$ corresponds five gradient vectors $A_a = (0, \lambda^b \delta_a^c - \lambda^c \delta_a^b, \lambda_{ba}) = (0, \lambda^{[b} \delta_a^{c]}, \lambda_{ba})$. Therefore at the point p we have in addition to V^1, four linearly independent vectors

$$A_i = (0, \lambda^{[1} \delta_i^{j]}, 0), \qquad i, j = 2, 3, .., 5, \tag{A.2}$$

where $[1j]$ are the components $([12], [13], ..., [15])$ of such vectors and we have a well defined tangent space at p. Summarizing, with this particular example for the point p, we argued the necessity of considering the conditions $\zeta_a = 0$ and now we have at every point of the pure spinor space, except for the origin, a tangent space of complex dimension 11. In other words, the pure spinor space without the origin is a smooth manifold embedded in $\mathbb{C}^{16}$. Note however that for $\lambda^+ \neq 0$, the five vectors V^a's are linearly independent and the solutions of the equations $\chi^a = 0$ satisfy trivially the equations $\zeta_a = 0$. Therefore, when $\lambda^+ \neq 0$ the five equations $\chi^a = 0$ are enough to describe the pure spinor space.

A.2 Another Cover For The Pure Spinor Space

We argued that the constant spinors C^I's given in (3.7) are not a good choice because the intersection of the hypersurfaces $\{C^I \lambda = 0\}, I = 1, ..., 11$ is the non compact space $\mathbb{C}^5$. This means that the union of the patches $U_I = PS \setminus \{C^I \lambda = 0\}$, where the scattering amplitude is supported, is not the whole pure spinor space, i.e

$$U_1 \cup ... \cup U_{11} = PS \setminus \mathbb{C}^5. \tag{A.3}$$

Then one question arises: Is it possible to complete the patches U_I in such a way that they form a cover of the $PS \setminus \{0\}$ space? Obviously the answer is positive. Here we show what is the difficulty for completing the patches for the tree level scattering amplitude.

In [9] it was proposed the cover for the pure spinor space $\mathcal{U} = \{U_\alpha\}$, $\alpha = 1, ..., 16$, with the patches U_α's given by

$$U_\alpha = PS \setminus \mathcal{D}_\alpha, \qquad \mathcal{D}_\alpha \equiv \{\lambda^\alpha \in PS : \lambda^\alpha = 0\}. \tag{A.4}$$

Clearly $\mathcal{U}$ is a cover of the pure spinor space without the origin

$$\bigcup_{\alpha=1}^{16} U_\alpha = U_1 \cup ... \cup U_{16} = PS \setminus \{0\}. \tag{A.5}$$

So, we can define the following picture operators

$$Y^\alpha = \frac{\theta^\alpha}{\lambda^\alpha}. \tag{A.6}$$

The first difficulty here is that there are 16 PCO's instead of 11, however this is not really a problem. Note that in the $U(5)$ decomposition, i.e $\lambda^\alpha = (\lambda^+, \lambda_{ab}, \lambda^a)$, $a, b = 1, ..., 5$ and $\lambda_{ab} = -\lambda_{ba}$, and choosing the picture operators

$$Y^{+} = \frac{\theta^{+}}{\lambda^{+}} \quad \text{and} \quad Y_{ab} = \frac{\theta_{ab}}{\lambda_{ab}}, \tag{A.7}$$

we fall in the first example of the subsection 3.2, with the difference that now we have a cover for $PS \setminus \{0\}$. So the tree level scattering amplitude is given by

$$\mathcal{A} = \int_{\Gamma} [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{i=1}^{11} \frac{\theta^{i}}{\lambda^{i}} \lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} f_{\alpha\beta\gamma}(\theta) \tag{A.8}$$

where

$$\frac{\theta^{1}}{\lambda^{1}} \equiv \frac{\theta^{+}}{\lambda^{+}}, \quad \frac{\theta^{2}}{\lambda^{2}} \equiv \frac{\theta_{12}}{\lambda_{12}}, \cdots, \frac{\theta^{11}}{\lambda^{11}} \equiv \frac{\theta_{45}}{\lambda_{45}} \tag{A.9}$$

and the cycle Γ is given by $\Gamma = \{\lambda^{\alpha} \in PS : |\lambda^{i}| = \varepsilon^{i}, i = 1, ..., 11\}$, $\varepsilon^{i} \in \mathbb{R}^{+}$. Now we must verify if (A.8) is a physical amplitude, i.e if it is $Q_T = Q + \delta$ closed.
In the same way as in (5.5) it is not hard to show that (A.8) is Q closed. Then now we must show that the amplitude (A.8) is δ closed. Since in this case there are 16 patches then the analysis can not be similar to the one presented in the subsection 4.4. Acting with the δ operator in (A.8) we get

$$(\delta\mathcal{A})^{1,\dots,11,j} = \int_{\Gamma} [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \left(\prod_{i=2}^{11} \frac{\theta^{i}}{\lambda^{i}} \frac{\theta^{j}}{\lambda^{j}} - \prod_{i=3}^{11} \frac{\theta^{1}}{\lambda^{1}} \frac{\theta^{j}}{\lambda^{j}} \frac{\theta^{i}}{\lambda^{i}} + ... + \prod_{i=1}^{11} \frac{\theta^{i}}{\lambda^{i}} \right) \lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} f_{\alpha\beta\gamma}(\theta), \tag{A.10}$$

where j is any number from 12 to 16. Naively (A.10) can be written as

$$(\delta\mathcal{A})^{1,\dots,11,j} = \int_{\Gamma} [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \, Q \left(\prod_{i=1}^{11} \frac{\theta^{i}}{\lambda^{i}} \frac{\theta^{j}}{\lambda^{j}} \right) \lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} f_{\alpha\beta\gamma}(\theta), \tag{A.11}$$

nevertheless that is not true. In the subsection 4.1 we said that the scattering amplitude also depends of the homology class of the cycle Γ. Since the computation (A.11) has 12 Čech labels and Γ is a 11-cycle we need to be careful. From (A.10) we can see that just the term

$$\int_{\Gamma} [\mathrm{d}\lambda] \int \mathrm{d}^{16}\theta \prod_{i=1}^{11} \frac{\theta^{i}}{\lambda^{i}} \lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} f_{\alpha\beta\gamma}(\theta) \tag{A.12}$$

contributes, since the other terms vanish because the cycle $|\lambda^{j}| = \varepsilon^{j}$ is not in Γ. Therefore the scattering amplitude (A.8) is not physical.

Actually, we have shown that the cycle Γ is a trivial element of the homology group $H_{11}(PS \setminus \mathcal{D})$, where $\mathcal{D} = \mathcal{D}_1 \cup ... \cup \mathcal{D}_{16}$. This is because the intersection $\mathcal{D}_1 \cap ... \cap \mathcal{D}_{11}$ is $\mathbb{C}^5$, so the difficulty of using the cover $\mathfrak{U}$ and the PCO's (A.6) is to get a well defined cycle Γ such that we can write $(\delta\mathcal{A})^{1,..,11,j}$ like in (A.11), i.e a non trivial element of the homology group $H_{11}(PS \setminus \mathcal{D})$. Note that if we add to the cover $\underline{U} = \{U_I\}, I = 1,..,11$, where the patches U_I's are given in (4.22), more patches, then there is not problem. The reason is simple, since the condition $D_1 \cap ... \cap D_{11} = \{0\}$, for the D_I's given in (4.22), then the cycle Γ (4.1) will always be a non trivial element of the homology class, so applying the δ operator to the amplitude we get something equal to (A.11).

In conclusion, for tree level scattering amplitude with 3 unintegrated vertex operators and the remaining integrated, it is sufficient to work with 11 patches such that they cover the pure spinor space without the origin $PS \setminus \{0\}$.

A.3 The Čech-Dolbeault Correspondence for Pure Spinor in $d = 4$

Our next simple example is the pure spinor space in $d = 4$ dimensions, i.e $PS = \mathbb{C}^2$. We choose the coordinates $\lambda^a = (\lambda^1, \lambda^2)$ and consider the integral

$$I = \int_\Gamma \frac{\mathrm{d}(C^1\lambda) \wedge \mathrm{d}(C^2\lambda)}{(C^1\lambda)(C^2\lambda)} = \int_\Gamma \psi, \tag{A.13}$$

where $C^i\lambda = C^i_a\lambda^a$, $\det(C^i_a) \neq 0$ and Γ is given by $\Gamma = \{\lambda^a \in \mathbb{C}^2 : |C^i\lambda| = \varepsilon^i\}$, $\varepsilon^i \in \mathbb{R}^+$. We can write (A.13) as

$$I = \int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{ab}C^1_a C^2_b}{(C^1\lambda)(C^2\lambda)}, \tag{A.14}$$

where $[\mathrm{d}\lambda] = (1/2)\epsilon_{ab}\mathrm{d}\lambda^a \wedge \mathrm{d}\lambda^b = \mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2$.
Note that $\mathbb{C}^2$ can be seen as the total space of the universal line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^1$, i.e $\lambda^a = \gamma\tilde{\lambda}^a$ where γ is the fiber and $\tilde{\lambda}^a$ are the coordinates of $\mathbb{C}P^1$. So, without loss of generality we choose $\Gamma = \{\lambda^a \in \mathcal{O}(-1) : |\gamma| = \varepsilon, |C^1\tilde{\lambda}| = \varepsilon^1, \text{ where } \tilde{\lambda}^a \in \mathbb{C}P^1\}\, \varepsilon, \varepsilon^1 \in \mathbb{R}^+$ (as in the sub-subsection 4.2.1) and the measure $[\mathrm{d}\lambda]$ is given like in reference [14] by $[\mathrm{d}\lambda] = \gamma\mathrm{d}\gamma \wedge [\mathrm{d}\tilde{\lambda}]$, where $[\mathrm{d}\tilde{\lambda}] = \epsilon_{ab}\mathrm{d}\tilde{\lambda}^a\tilde{\lambda}^b$ is the measure for the twistor space in $d = 4$ [16]. Then, integrating γ we get

$$\int_\Gamma [\mathrm{d}\lambda] \frac{\epsilon^{ab}C^1_a C^2_b}{(C^1\lambda)(C^2\lambda)} = \int_\Gamma \frac{\mathrm{d}\gamma}{\gamma} \wedge [\mathrm{d}\tilde{\lambda}] \frac{\epsilon^{ab}C^1_a C^2_b}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = (2\pi i) \int_{|C^1\tilde{\lambda}|=\varepsilon_1} [\mathrm{d}\tilde{\lambda}] \frac{\epsilon^{ab}C^1_a C^2_b}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})}, \tag{A.15}$$

where the right hand side has the same form as the Green's function for the massless scalar field in $d = 4$ using the twistor language [16]. This result in $d = 4$ is analogous to the one obtained in $d = 10$, see (4.20).

Now, using the partition of unity

$$\rho_i = \frac{|C^i\lambda|^2}{(|C^1\lambda|^2 + |C^2\lambda|^2)}, \qquad i = 1, 2 \tag{A.16}$$

subordinated to the cover $\mathcal{U} = \{U_1, U_2\}$, where

$$U_i = \mathbb{C}^2 \setminus \{C^i\lambda = 0\}, \qquad i = 1, 2,$$

we find the Dolbeault cocycle corresponding to ψ. Note that from the condition $\det(C^i_a) \neq 0$ then $\{C^1\lambda = 0\} \cap \{C^2\lambda = 0\} = \{0\}$, so we get $\mathbb{C}^2 \setminus \{0\} = U_1 \cup U_2$.
Since ψ is a (2,0) holomorphic form over $U_1 \cap U_2$ then ψ is 1-Čech cochain, i.e $\psi \in C^1(\mathfrak{U}, \Omega^2)$, where $\Omega^2(\mathbb{C}^2 \setminus \{0\})$ is the abelian group of the (2,0) holomorphic forms over $\mathbb{C}^2 \setminus \{0\}$. So, using (4.36), the Dolbeault cocycle corresponding to $\psi \equiv \psi_{12} = -\psi_{21}$ is given by

$$\eta_\psi = \sum_{\alpha,\beta=1}^{2} \psi_{\alpha\beta}\rho_\alpha \wedge \bar{\partial}\rho_\beta = \psi_{12}\rho_1 \wedge \bar{\partial}\rho_2 + \psi_{21}\rho_2 \wedge \bar{\partial}\rho_1 = \psi_{12} \wedge \bar{\partial}\rho_2 \tag{A.17}$$

where $1, 2$ are the Čech labels. Replacing ψ_{12} and ρ_2 in η_ψ we get

$$\eta_\psi = \frac{\mathrm{d}(C^1\lambda) \wedge \mathrm{d}(C^2\lambda) \wedge \left[(\bar{C}^1\bar{\lambda})\mathrm{d}(\bar{C}^2\bar{\lambda}) - (\bar{C}^2\bar{\lambda})\mathrm{d}(\bar{C}^1\bar{\lambda})\right]}{(|C^1\lambda|^2 + |C^2\lambda|^2)^2}. \tag{A.18}$$

Note that this (2,1)-form is global on $\mathbb{C}^2 \setminus \{0\}$.
Therefore from the Čech-Dolbeault correspondence we have

$$\int_\Gamma \psi_{12} = \int_{S^3} \eta_\psi|_{S^3}, \tag{A.19}$$

where S^3 is the sphere $|\lambda^1|^2 + |\lambda^2|^2 = r^2$, $r \in \mathbb{R}^+$. Since S^3 is a $U(1)$-line bundle over $\mathbb{C}P^1$ space then we can write η_ψ in the S^3 coordinates

$$\lambda^a = r\, e^{i\theta}(1, u),$$

where $e^{i\theta}$ parametrizes the fiber $U(1)$, u parametrizes the $\mathbb{C}P^1$ space and r is the size of S^3. So,

$$\eta_\psi|_{S^3} = i\frac{|\epsilon^{ab}C^1_a C^2_b|^2}{(|C^1_1 + C^1_2\,u|^2 + |C^2_1 + C^2_2\,u|^2)^2}\mathrm{d}\theta \wedge \mathrm{d}u \wedge \mathrm{d}\bar{u}. \tag{A.20}$$

Note that the constant r does not appear and the $U(1)$ part is decoupled. Therefore we can perform a global transformation from $\mathbb{C}P^1 \to \mathbb{C}P^1$ to eliminate the C^I's constants. This transformation is known as the Möbius transformation

$$v = \frac{C^1_1 + C^1_2\,u}{C^2_1 + C^2_2\,u}, \quad \text{where} \quad \begin{pmatrix} C^1_1 & C^1_2 \\ C^2_1 & C^2_2 \end{pmatrix} \in GL(2, \mathbb{C}). \tag{A.21}$$

With this transformation we obtain

$$\eta_\psi|_{S^3} = i\frac{1}{(1 + v\bar{v})^2}\mathrm{d}\theta \wedge \mathrm{d}v \wedge \mathrm{d}\bar{v}. \tag{A.22}$$

(A.22) is the $d = 4$ equivalent to (4.62) for pure spinors in $d = 10$ and $\eta_\psi|_{S^3}$ is a generator of the de-Rham cohomology group $H^3_{DR}(S^3) = \mathbb{C}$ in coordinates. Integrating by $\mathrm{d}\theta$ we have the following equality

$$\int_{|C^1\tilde{\lambda}|=\varepsilon_1} [\mathrm{d}\tilde{\lambda}]\frac{\epsilon^{ab}C^1_a C^2_b}{(C^1\tilde{\lambda})(C^2\tilde{\lambda})} = \int_{\mathbb{C}^2} \frac{1}{(1 + v\bar{v})^2}\mathrm{d}v \wedge \mathrm{d}\bar{v} = (2\pi i)\int_{\mathbb{C}P^1} H, \tag{A.23}$$

where the hyperplane class H is written locally as $H = (1/(2\pi i))(1 + v\bar{v})^{-2}\mathrm{d}v \wedge \mathrm{d}\bar{v}$ [25]. So (A.23) is just $(2\pi i)$ times the degree of the projective complex space $\mathbb{C}P^1$, which is one.

A.4 Global Integrals

Now we want to give a simple example with the aim to explore the global definition of the degree of a hypersurface. Let us consider the following cone in $\mathbb{C}^4$

$$\chi \equiv z_1 z_2 - z_3 z_4 = 0 \tag{A.24}$$

and the integral

$$I = \int_\Gamma \frac{\mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3}{f^1 f^2 f^3}, \tag{A.25}$$

where $f^i = C^i Z = C_1^i z_1 + C_2^i z_2 + C_3^i z_3 + C_4^i z_4$ and $\Gamma = \{Z \in \mathbb{C}^4 : \chi = 0 \text{ and } ||f^i|| = \varepsilon_i\}$, $\varepsilon_i \in \mathbb{R}^+$. We choose the C^i's in a similar way to (3.7), i.e, $f^1 = z_1,\ f^2 = z_2,\ f^3 = z_3$.

Note that the intersection

$$\{\tilde{f}^1 = 0\} \cap \{\tilde{f}^2 = 0\} \cap \{\tilde{\chi} = 0\}\Big|_{\mathbb{C}P^3} = \{[0,0,1,0],[0,0,0,1]\}$$

where $\{\tilde{f}^i = 0\} \equiv \{f^i = 0\}/\sim$ and the equivalence relation is given by $Z \sim cZ$, $c \in \mathbb{C}^*$. The same is true for $\tilde{\chi}$. This means that the degree of the smooth manifold $\tilde{\chi} = 0$ embedded in $\mathbb{C}P^3$ is deg($\tilde{\chi} = 0$)=2. So we would expect that (A.25) will be $(2\pi i)^3 2$ from the discussion of the sub-subsection 4.4.1.

Now, it is important to note that the intersection

$$\{f^1 = 0\} \cap \{f^2 = 0\} \cap \{f^3 = 0\} \cap \{\chi = 0\}\Big|_{\mathbb{C}^4} = \mathbb{C},$$

which, as we will explain, implies that the integral (A.25) is not well defined. Replacing the f^i's in (A.25) we have an integral like in $\mathbb{C}^3$

$$\int_{|z_i|=\varepsilon_i} \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{z_1 z_2 z_3} = (2\pi i)^3, \tag{A.26}$$

where we have lost all the information about the cone, in fact we are in one chart. If we want to obtain global information, we must write the integral in the following way

$$I = \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3 \wedge \mathrm{d}\chi}{f^1 f^2 f^3 \chi} = \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}(z_1 z_2 - z_3 z_4)}{z_1 z_2 z_3 (z_1 z_2 - z_3 z_4)}, \tag{A.27}$$

where R is given by $R = \{Z \in \mathbb{C}^4 : ||f^i|| = \varepsilon_i, |\chi| = \varepsilon\}$. Integrating first z_1 and then z_2, z_3 and z_4, we would obtain as a result $(2\pi i)^3$. Nevertheless, note that the pole f^3 is eliminated and it should be recovered from χ. This implies that the cycles $|f^3| = \varepsilon_3$ and $|\chi| = \varepsilon$ were mixed. To understand this better, let us first integrate over the cycle $|f^3| = \varepsilon_3$ in (A.27). We will obtain

$$\begin{aligned} \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}(z_1 z_2 - z_3 z_4)}{z_1 z_2 z_3 (z_1 z_2 - z_3 z_4)} &= \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_3 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge (-z_3)\mathrm{d}z_4}{z_1 z_2 z_3 (z_1 z_2 - z_3 z_4)} \\ &= \frac{-1}{(2\pi i)} \int_R \frac{\mathrm{d}z_3 \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_4}{z_1 z_2 z_3 (\frac{z_1 z_2}{z_3} - z_4)}, \end{aligned} \tag{A.28}$$

so we get an infinite in the denominator and the integral is zero. Therefore we have a contraction and (A.25) is not well defined for $f^i = z_i$, $i = 1, 2, 3$. This contraction comes from the fact that the integral is not well defined globally for those f^i's, i.e changing the order in which we compute the integral (A.27) is equivalent to a change of chart in the cone.
In the pure spinor formalism the f^I's, given by the constant spinors C^I's (3.7), have the same problem. Although we can not do the same trick as (A.27), because the constraints (3.6) $\chi_a = 0$ do not describe the whole pure spinor space, it can be useful to understand this more complicated problem. For the constrains $\chi_a = 0$ we have

$$I = \int_R \frac{(\mathrm{d}f^1) \wedge ... \wedge (\mathrm{d}f^{11}) \wedge \mathrm{d}(\chi_1) \wedge ... \wedge \mathrm{d}(\chi_5)}{f^1 ... f^{11} (\chi_1) ... (\chi_5)} \tag{A.29}$$

where R goes around every pole. Integrating first by the cycle $|\lambda^+| = \varepsilon$ we get an infinite in the denominator, just as in the previous example.

Now, changing $f^3 = z_3$ by $f^3 = z_3 - z_4$ in the example of the cone $\chi = z_1 z_2 - z_3 z_4 = 0$, we get the intersection

$$\{f^1 = 0\} \cap \{f^2 = 0\} \cap \{f^3 = 0\} \cap \{\chi = 0\}\Big|_{\mathbb{C}^4} = \{0\}$$

with multiplicity $m_{\{0\}} = 2$, which comes from the equation $z_4^2 = 0$. So, we have the integral

$$I = \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}f^1 \wedge \mathrm{d}f^2 \wedge \mathrm{d}f^3 \wedge \mathrm{d}\chi}{f^1 f^2 f^3 \chi} = \frac{1}{(2\pi i)} \int_R \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}(z_3 - z_4) \wedge \mathrm{d}(z_1 z_2 - z_3 z_4)}{z_1 z_2 (z_3 - z_4)(z_1 z_2 - z_3 z_4)}. \tag{A.30}$$

This integral does not have any problems and its result is the expected $(2\pi i)^3 2$ (which was explained in the sub-subsection 4.4.1 and matches with the Bezout theorem [22]).

B Proof of the Identity $[\mathrm{d}\tilde{\lambda}] = \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}$.

Let us give again the statement that we want to proof.
If $\tilde{\lambda}^\alpha$ is an element of the projective pure spinors space in 10 dimensions, i.e. if $\tilde{\lambda}^\alpha \in SO(10)/U(5)$, then the integration measure $[\mathrm{d}\tilde{\lambda}]$ defined by [16]

$$[\mathrm{d}\tilde{\lambda}](\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon_{\alpha_1...\alpha_5\beta_1...\beta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1} \wedge ... \wedge \mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}, \tag{B.1}$$

written in the parametrization $\tilde{\lambda}^\alpha = (\tilde{\lambda}^+, \tilde{\lambda}_{ab}, \tilde{\lambda}^a) = (1, u_{ab}, \frac{1}{8}\epsilon^{abcde}u_{bc}u_{de})$ is

$$[\mathrm{d}\tilde{\lambda}] = \mathrm{d}u_{12} \wedge ... \wedge \mathrm{d}u_{45}. \tag{B.2}$$

Proof

Since $SO(10)/U(5)$ is a complex manifold we can write an anti-holomorphic measure as [25]

$$[\mathrm{d}\bar{\tilde{\lambda}}](\bar{\tilde{\lambda}}\gamma^m)^{\alpha_1}(\bar{\tilde{\lambda}}\gamma^n)^{\alpha_2}(\bar{\tilde{\lambda}}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5} = \frac{2^3}{10!}\epsilon^{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\mathrm{d}\bar{\tilde{\lambda}}_{\beta_1}\wedge\ldots\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_{10}}\bar{\tilde{\lambda}}_{\beta_{11}}, \tag{B.3}$$

or in a more appropriate way as

$$[\mathrm{d}\bar{\tilde{\lambda}}] = \frac{1}{2^3 5! 10!}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}(\tilde{\lambda}\gamma^m)_{\alpha_1}(\tilde{\lambda}\gamma^n)_{\alpha_2}(\tilde{\lambda}\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\mathrm{d}\bar{\tilde{\lambda}}_{\delta_1}\wedge\ldots\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_{10}}\bar{\tilde{\lambda}}_{\delta_{11}}, \tag{B.4}$$

where $\bar{\tilde{\lambda}}_\alpha = (\bar{\tilde{\lambda}}_+, \bar{\tilde{\lambda}}^{ab}, \bar{\tilde{\lambda}}_a) = (1, \bar{u}^{ab}, \frac{1}{8}\epsilon_{abcde}\bar{u}^{bc}\bar{u}^{de})$. From (B.1) and (B.4) it is simple to see that

$$\begin{aligned}
[\mathrm{d}\tilde{\lambda}]\wedge[\mathrm{d}\bar{\tilde{\lambda}}] &= \frac{1}{5!(10!)^2}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\epsilon_{\alpha_1\ldots\alpha_5\beta_1\ldots\beta_{11}}\epsilon^{\alpha_1\ldots\alpha_5\delta_1\ldots\delta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge\ldots\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\tilde{\lambda}^{\beta_{11}}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_1}\wedge\ldots\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_{10}}\bar{\tilde{\lambda}}_{\delta_{11}} \\
&= \frac{1}{(10!)^2}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\delta^{\delta_1}_{[\beta_1}\delta^{\delta_2}_{\beta_2}\ldots\delta^{\delta_{11}}_{\beta_{11}]}\tilde{\lambda}^{\beta_{11}}\bar{\tilde{\lambda}}_{\delta_{11}}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge\ldots\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_1}\wedge\ldots\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\delta_{10}} \\
&= \frac{1}{10!}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^2}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge\ldots\wedge\mathrm{d}\tilde{\lambda}^{\beta_{10}}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_1}\wedge\ldots\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_{10}} \\
&\quad -\frac{10}{10!}\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\mathrm{d}\tilde{\lambda}^{\beta_1}\wedge\ldots\wedge\mathrm{d}\tilde{\lambda}^{\beta_9}\wedge\bar{\tilde{\lambda}}_{\alpha_1}\mathrm{d}\tilde{\lambda}^{\alpha_1}\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_1}\wedge\ldots\wedge\mathrm{d}\bar{\tilde{\lambda}}_{\beta_9}\wedge\tilde{\lambda}^{\alpha_2}\mathrm{d}\bar{\tilde{\lambda}}_{\alpha_2} \\
&= -\frac{1}{10!}\Bigg(\frac{1}{(\tilde{\lambda}\bar{\tilde{\lambda}})^2}\partial\overline{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge\ldots\wedge\partial\overline{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}}) \\
&\quad -\frac{10}{(\tilde{\lambda}\bar{\tilde{\lambda}})^3}\partial(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge\overline{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge\partial\overline{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\wedge\ldots\wedge\partial\overline{\partial}(\tilde{\lambda}\bar{\tilde{\lambda}})\Bigg) \\
&= \frac{1}{10!}\left(i(\tilde{\lambda}\bar{\tilde{\lambda}})^{8/10}\partial\overline{\partial}\ln(\tilde{\lambda}\bar{\tilde{\lambda}})\right)^{10}.
\end{aligned} \tag{B.5}$$

In [25] it was shown that

$$\frac{\omega^{10}}{10!} = \frac{\mathrm{d}u_{12}\wedge\ldots\wedge\mathrm{d}u_{45}\wedge\mathrm{d}\bar{u}^{12}\wedge\ldots\wedge\mathrm{d}\bar{u}^{45}}{(\tilde{\lambda}\bar{\tilde{\lambda}})^8}, \tag{B.6}$$

where

$$\omega = -\partial\overline{\partial}\ln(\tilde{\lambda}\bar{\tilde{\lambda}}) \tag{B.7}$$

and

$$(\tilde{\lambda}\bar{\tilde{\lambda}}) = (1 + \frac{1}{2}u_{ab}\bar{u}^{ab} + \frac{1}{8^2}\epsilon^{a_1b_1c_1d_1e_1}\epsilon_{a_1b_2c_2d_2e_2}u_{b_1c_1}u_{d_1e_1}\bar{u}^{b_2c_2}\bar{u}^{d_2e_2}).$$

So, we have shown that

$$[\mathrm{d}\tilde{\lambda}] = \exp(i\phi)\mathrm{d}u_{12}\wedge\ldots\wedge\mathrm{d}u_{45} \tag{B.8}$$

where $\phi\in\mathbb{R}$ is a constant. Since this phase factor does not affect the amplitude we can set $\phi = 0$ and thus the identity was proven ■

References

[1] N. Berkovits, "Super-Poincaré Covariant Quantization of the Superstring," JHEP 04 (2000) 018, hep-th/0001035

[2] E. Cartan, "Lecons sur la Theorie des Spineurs", Hermann, Paris, 1937

[3] P.S. Howe, "Pure spinor lines in superspace and ten-dimensional supersymmetric theories," Phys. Lett. B258: 141 (1991).

[4] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **0409**, 047 (2004) [arXiv:hep-th/0406055].

[5] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **0601**, 005 (2006) [arXiv:hep-th/0503197].

[6] N. Berkovits and C.R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalism", Phys. Rev. Lett. 96:011602 (2006), [arXiv:hep-th/0509234]

[7] C.R. Mafra, "Four-point one-loop amplitude computation in the pure spinor formalism ", JHEP 0601:075 (2006), [arXiv:hep-th/0512052]

[8] N. Berkovits, "Pure spinor formalism as an N = 2 topological string," JHEP **0510**, 089 (2005) [arXiv:hep-th/0509120].

[9] N. Berkovits and N. Nekrasov, "Multiloop superstring amplitudes from non-minimal pure spinor formalism," JHEP **0612**, 029 (2006) [arXiv:hep-th/0609012].

[10] N. Berkovits and C. Mafra, "Some Superstring Amplitude Computations with the Non-Minimal Pure Spinor Formalism," JHEP **0611** 079 (2006), [arXiv:hep-th/0607187]

[11] C. Mafra and C. Stahn, "The One-loop Open Superstring Massless Five-point Amplitude with the Non-Minimal Pure Spinor Formalism", JHEP **0903** 126 (2009), [arXiv:0902.1539 [hep-th]]

[12] P.A. Grassi and P. Vanhove, "Higher-loop Amplitudes in the non-minimal Pure Spinor Formalism", JHEP **0905** 089 (2009), [arXiv:0903.3903 [hep-th]]

[13] Y. Aisaka and N. Berkovits, "Pure Spinor Vertex Operators in Siegel Gauge and Loop Amplitude Regularization", JHEP **0907** 062 (2009), [arXiv:0903.3443 [hep-th]]

[14] N. A. Nekrasov, "Lectures on curved beta-gamma systems, pure spinors, and anomalies," arXiv:hep-th/0511008.

[15] L.P. Hughston, "Applications of $SO(8)$ Spinors", pp.253-287 in Gravitation and Geometry: a volume in honor of Ivor Robinson (eds. W. Rindler and A. Trautman), Bibliopolis, Naples (1987).

[16] N. Berkovits and S. A. Cherkis, "Pure spinors are higher-dimensional twistors," JHEP **0412**, 049 (2004) [arXiv:hep-th/0409243].

[17] J. Hoogeveen and K. Skenderis, "Decoupling of unphysical states in the minimal pure spinor formalism I," JHEP **1001**, 041 (2010) [arXiv:0906.3368 [hep-th]].

[18] N. Berkovits, J. Hoogeveen and K. Skenderis, "Decoupling of unphysical states in the minimal pure spinor formalism II," JHEP **0909**, 035 (209) [arXiv:0906.3371 [hep-th]].

[19] J. Hoogeveen and K. Skenderis, "BRST Quantization of the Pure Spinor Superstring", JHEP **0711** 081 (2007), [arXiv:0710.2598 [hep-th]].

[20] E. D'Hoker and D. H. Phong, "The Geometry of String Perturbation Theory," Rev. Mod. Phys. **60**, 917 (1988).

[21] Y. Aisaka, E. A. Arroyo, N. Berkovits and N. Nekrasov, "Pure Spinor Partition Function and the Massive Superstring Spectrum," JHEP **0808**, 050 (2008) [arXiv:0806.0584 [hep-th]].

[22] Griffiths and Harris, "Principles of Algebraic Geometry", [Wiley Classics Library Edition Published 1994]

[23] Raoul Bott and Loring W.Tu "Differential Forms in Algebraic Topology", [Springer-Verlag published 1982]

[24] W.S. Massey, "A Basic Course in Algebraic Topology", [Springer 1991]

[25] H. Gomez, "One-loop Superstring Amplitude From Integrals on Pure Spinors Space," JHEP **0912**, 034 (2009) [arXiv:0910.3405 [hep-th]].

[26] William Fulton, "Intersection theory", [Springer 1998]

[27] H. Gomez and C. R. Mafra, "The Overall Coefficient of the Two-loop Superstring Amplitude Using Pure Spinors," JHEP **1005**, 017 (2010) [arXiv:1003.0678 [hep-th]].

[28] Work in progress