Exact Poisson pencils, τ-structures and topological hierarchies

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To Boris Dubrovin on the occasion of his 60th birthday, with friendship and admiration.

Abstract: We discuss, in the framework of Dubrovin-Zhang's perturbative approach to integrable evolutionary PDEs in $1+1$ dimensions, the role of a special class of Poisson pencils, called exact Poisson pencils. In particular we show that, in the semisimple case, exactness of the pencil is equivalent to the constancy of the so-called "central invariants" of the theory that were introduced by Dubrovin, Liu and Zhang.

1 Introduction

Integrable hierarchies of evolutionary PDEs of the form

$$q^i_t = V^i_j(q)q^j_x + \sum_{k=1}^{\infty} \epsilon^k F^i_k(q, q_x, q_{xx}, \dots, q_{(n)}, \dots) \tag{1.1}$$

have been extensively studied in the last years (see, e.g., [14, 27, 15, 16, 5, 28]).

In particular, great attention to the so-called *topological hierarchies* also beacuase of their relation to the theory of Gromov-Witten invariants, the theory of singularities, and other seemengly unrelated topics of Mathematics and Theoretical Physics. These hierarchies possess some additional structures: they are bi-Hamiltonian, they admit a tau-structure and satisfy Virasoro constraints [14]. The notion of τ-structure (or τ-function) is perhaps among the oldest ones in the theory of evolutionary equations in $1+1$ dimensions, having been introduced by Hirota as the major character in the bilinear formulation of integrable PDEs. Its properties were further exploited by the

Japanese school (see, e.g., [24, 10, 35]). In the present approach, the existence of a τ-structure for an integrable hierarchy of $1+1$ evolutionary PDEs will be understood as the possibility of defining special densities h_i^* for the mutually conserved quantities of the PDEs that satisfy the symmetry requirement

$$\frac{\partial h_i^*}{\partial t_j} = \frac{\partial h_j^*}{\partial t_i},$$

where $\frac{\partial}{\partial t_k}$ is some suitable one-sequence ordering of the various times of the hierarchy.

Virasoro symmetries are also well known objects of the theory; in particular here we refer to the Virasoro-type algebras of additional (explicitly time(s)-dependent) symmetries of the classes of PDEs we are concerned with. In particular, they gained much attention in the light of the celebrated results by Kontsevich and Witten [25, 36] that identified a particular τ-function of the KdV hierarchy with the partition function of 2D Quantum gravity,

As it is well known, the existence of a bi-Hamiltonian structure means that the equations of the hierarchy can be written in Hamiltonian form with respect to two compatible Poisson bivectors P_1 and P_2 and that the Poisson pencil $P_2 - \lambda P_1$ is a Poisson bivector for any λ [29]. A remarkable result established in [14], and subsequently refined in [5] is that, if the pencil P_λ is semisimple, (in a sense to be precised later) and admits a τ-function the above requirements fix uniquely the hierarchy once the dispersionless limit

$$q^i_t = V^i_j(q) q^j_x \tag{1.2}$$

and its bi-Hamiltonian structure (ω_1, ω_2) are given.The semisimplicity of the pencil is related to the existence of a special set of coordinates $(u^1, \ldots, u^n)$ called *canonical coordinates*. If one relaxes the hypothesis of existence of a *tau*-structure, the deformations are parametrized by certain functional parameters called *central invariants* that are constants in the case of topological hierarchies. In turn, further results in [15], suggest that the constancy of these central invariants be related with the existence of the τ-function of the hierarchy.

In this paper we will show that the Poisson pencil

$$\Pi_\lambda = P_2 - \lambda P_1$$

of a topological hierarchy is exact, in the sense that there exist a vector field Z (to be called *Liouville* vector field of the pencil) such that

$$\mathrm{Lie}_Z P_2 = P_1, \quad \text{and} \quad \mathrm{Lie}_Z P_1 = 0. \tag{1.3}$$

Moreover, we show that there exist a Miura transformation reducing simultaneously Z to its dispersionless limit:

$$Z \to e = \sum_{i=1}^{n} \frac{\partial}{\partial u^i}$$

and the pencil Π_λ to the form

$$\omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)}.$$

The hint for our works stems from the observation(s) (to be briefly recalled in Section 2) that the geometry of exact bihamitonian manifolds provides *somehow for free* the needed "toolkit" requested for the existence of a τ-function for the hierarchy. Indeed, on general grounds, on the one hand the bihamiltonian hierarchies defined on exact bihamitonian manifolds exhibit additional symmetries of Virasoro type [37]. On the other hand, the action of the Liouville field on the Hamiltonian of the hierarchy naturally provides new densities for the conserved quantities.

Actually, we are not going to tackle these problems directly and abstractly as a problem in the general theory of Poisson manifolds; rather, we use these "nice" properties of exact Poisson pencils as suggestions for their realization within the perturbative approach developed in recent years by Boris Dubrovin and his collaborators for the classification problem of $1+1$ evolutionary integrable PDEs of KdV-type. In particular, we borrow from them methods as well as a number of explicit results, with the aim of showing that the geometric notion of exactness of a Poisson pencil can be fruitfully used in this field.

The paper is organized as follows:in Section 2 we collect some (more or less known) results about exact Poisson pencils; then in Section 3 we study exact semisimple Poisson pencil of hydrodynamic type and we show that for such pencils the vector field Z coincides with the unity vector field e of the underlying Frobenius manifold. In Sections 4 and 5 we recall (following [15] and [27]) some definitions and results about central invariants and bi-Hamiltonian cohomology necessary for the subsequent Section 6 which is devoted to the proof of the main result of the paper. Section 7 contains a brief summary of the paper and some indications of further possible steps to generalize the results herewith presented.

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2 Geometry of exact Bihamiltonian manifolds

In this section we collect some results on the geometry of *exact* bihamiltonian manifolds, and their relations with the hierarchies therein supported. It is fair to say that, in one form or the other, these results are known in the literature. However, we deem useful to collect them together here, as they somehow provide the guiding principle for the arguments contained in the core of the paper.

Definition 1. *Let $P_\lambda := P_2 - \lambda P_1$ a pencil of Poisson bivectors, defined on a manifold $\mathcal{M}$. We say that P_λ is an* exact *Poisson pencil if there exists a vector field $Z \in \mathcal{X}(M)$ such that*

$$P_1 = \mathrm{Lie}_Z P_2; \quad \mathrm{Lie}_Z P_1 (= \mathrm{Lie}^2_Z P_2) = 0. \tag{2.1}$$

The vector field Z will be referred to as the Liouville *field of the exact Poisson pencil.*

We remark that, on general grounds, the Liouville vector field Z is not uniquely defined. For instance adding a bi-Hamiltonian vector field to a Liouville vector field one obtains a new Liouville vector field. In the known examples (e.g. in the case of the A_n-Drinfel'd-Sokolov hierarchies), there are some natural choices for it. Indeed, in the paper, we shall see that this is the case.

2.1 Exact BH manifolds and the second Hamiltonian function(s)

Let us consider an exact bihamiltonian manifold, whose Lenard Magri chains be "anchored" according to the Gel'fand-Zakharevich definition [22], that is all chain originate from a Casimir function of P_1. Let $\mathcal{H}(\lambda) := \mathcal{H}_0 + \frac{\mathcal{H}_1}{\lambda} + \frac{\mathcal{H}_2}{\lambda^2} + \cdots$ a Casimir of the pencil, that is a formal Laurent series in λ satisfying

$$P_\lambda(d\mathcal{H}(\lambda) = 0 (\Rightarrow P_1 d\mathcal{H}_0 = 0), \tag{2.2}$$

and consider the pencil of bihamitonian vector fields X_λ of the hierarchy, to be represented as

$$X_\lambda = P_1 d\mathcal{H}(\lambda). \tag{2.3}$$

Proposition 2. *Let $\mathcal{H}^*(\lambda) := -\mathrm{Lie}_Z \mathcal{H}(\lambda)$; then the one parameter family of vector fields X_λ can be represented as*

$$X_\lambda = P_\lambda d\mathcal{H}^*(\lambda), \tag{2.4}$$

that is, the deformed Hamiltonians $\mathcal{H}^*_i = \mathrm{Lie}_Z \mathcal{H}_i$ define the same GZ foliation of the phase space $\mathcal{M}$.

Proof. It follows from the straightforward chain of equality

$$\begin{aligned}0 =&\mathrm{Lie}_Z\left(P_\lambda d\mathcal{H}(\lambda)\right) = \mathrm{Lie}_Z(P_\lambda)d\mathcal{H}(\lambda) + P_\lambda \mathrm{Lie}_Z(d\mathcal{H}(\lambda))\\ &= P_1 d\mathcal{H}(\lambda) + P_\lambda(d\mathrm{Lie}_Z(\mathcal{H}(\lambda)) = X_\lambda - P_\lambda d\mathcal{H}^*(\lambda).\end{aligned}$$

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Remark: Exact bihamiltonian pencils, besides having "historically" provided the first instances of such structures, naturally enter the so-called *method of argument translation* related with Lie-Poisson pencils on Lie algebras (see [30]). In this field can be found in [14], §3; we notice however that in our case, the Gel'fand-Zakharevich sequences start form Casimir of the "deformed" tensor P_1, rather than with Casimirs of the (Lie Poisson) tensor P_2. In the case of PDEs, this might be a non-trivial difference.

2.2 Exact BH manifolds and the Virasoro algebra

Master symmetries are a very classical topics in the theory of integrable PDEs [21, 33]. In [37] it was observed that the Galileian symmetry of the KdV equation could be used as a generator of a whole (albeit formal) family of such symmetries, and that such a family is isomorphic to the nilpotent upper subalgebra of the Virasoro algebra, that is, the subalgebra generated by the elements ℓ_k with $k \geq 0$. Here we shall show (see also [1, 32]) that this is a common feature of all exact bihamiltonian manifolds, and, in particular, that the Liouville vector field can be added as the Virasoro generator ℓ_{-1}.

Definition 3. *[37] Let P_λ be a Poisson pencil of GZ type, and let $N := P_2 \cdot {P_1}^{-1}$ its formal recursion operator. A vector field Y is called a conformal symmetry of the pencil if it holds*

$$\mathrm{Lie}_Y N = N\,. \tag{2.5}$$

Proposition 4. *Let (P_λ, Z) be an exact bihamiltonian pencil. Then the field $Y_0 := N\,Z$ is a conformal symmetry of P_λ.*

Proof: Since $N = P_2\,{P_1}^{-1}$ we have $\mathrm{Lie}_Z(N) = \mathbf{1}$. Now, let us define $Y_0 := N\,(Z)$; obviously

$$[Y_0, NX] = \mathrm{Lie}_{Y_0}(NX) = \mathrm{Lie}_{Y_0}(N)\,X + N\,\mathrm{Lie}_{Y_0}(X) = \mathrm{Lie}_{Y_0}(N)\,X + N\,[Y_0, X]. \tag{2.6}$$

The vanishing of the Nijenhuis torsion of N (which, as it is well known to experts in the theory of Poisson pencil, is implied by the compatibility of P_2 and P_1) reads, for every pair of vector fields W, X

$$[NW, NX] = N\,[NW, X] + N\,[W, NX] - N^2\,[W, X]. \tag{2.7}$$

Substituting $Y_0 = N\,Z$ in (2.6) and using the vanishing of the torsion of N we get

$$\begin{aligned}
\mathrm{Lie}_{N\,Z}(N)\,X + N\,[N\,Z, X] = N\,[NZ, X] + N\,[Z, NX] - N^2\,[Z, X] = \\
N\,[NZ, X] + N\,(\mathrm{Lie}_Z(N)\,X) + N^2\,[Z, X] - N^2\,[Z, X] \\
\text{which yields } \mathrm{Lie}_{N\,Z}(N)\,X = N\,X\,\forall\,X, \quad \text{since } \mathrm{Lie}_Z\,N = \mathbf{1}.
\end{aligned} \tag{2.8}$$

■

As a corollary, we have the following result (see [37] for the full proof, which holds obviously also for the slight generalization herewith presented). It is based on the properties

$$\mathrm{Lie}_Z N^j = j\,N^{j-1} \tag{2.9}$$

Proposition 5. *Let*

$$Y_j := N^{j+1} Z(\quad \textit{so that } Y_{-1} \equiv Z)$$

be the family of vector fields obtained formally by the action of the recursion operator on the Liouville vector field Z. Then the commutation relations of the Virasoro algebra

$$[Y_j, Y_k] = (k - j) Y_{k+j}$$

hold.

2.3 The exact GD pencil and its τ-function

The nowadays standard formulation of the n-th GD hierarchy is based on the its Lax representation; namely, the phase space is identified with the affine space of differential operators of the form

$$L = \partial^{n+1} + U_n \partial^{n-1} + U_{n-1}\partial^{n-2} + \cdots + U_1,$$

and its bihamiltonian structure can be represented by means of the Hamilton operators

$$\begin{aligned}
\dot{L} &= P_1(X) = [L, X]_+ \\
\dot{L} &= P_2(X) = (LX)_+ L - L(XL)_+ - \frac{1}{n+1}[L, (\partial^{-1}[X, L]_{-1})]
\end{aligned} \tag{2.10}$$

where X represents a one-form on the phase space, that is, a purely non-local pseudodifferential operator. As it is customary, the subscript $(\cdot)_+$ refers to the purely differential part of the operator and $(\cdot)_{-1}$ is the residue.

It is well known – and easily ascertained from (2.10) – that the Poisson pencil $P_2 - \lambda P_1$ is exact, and admits as a Liouville vector field is the field

$$Z := \dot{U}_1 = 1.$$

It is also well known that the densities of conserved quantities of the GD n-th hierarchy can be collected in a generating function $h([U],z)$ of the form

$$h([U],z) = z + \sum_{i=-1}^{\infty} \frac{h_i([U])}{z^i}$$

where $z^{n+1} = \lambda$, and the symbol $[U]$ stands for "differential polynomial in the dependent fields $U_i(x)$"; $h([U])$ is related with the Baker Akhiezer function ψ of the theory by $\psi = \mathrm{e}^{(\int^x h([U],z)dx)}\,\mathrm{e}^{\sum_i t_i z^i)}$, and is the unique solution of the above form of the Riccati-type equation

$$h^{(n+1)} + \sum_{j=0}^{n-1} U_{j+1} h^{(j)} = z^{n+1},$$

where $h^{(0)} \equiv 1$ and, by definition, $h^{(k+1)} = \partial_x h^{(k)} + h([U],z)h^{(k)}$.

In [19] the following representation for the nGD (and KP) flows was highlighted: The GD flows imply the local conservation laws

$$\frac{\partial}{\partial t_j} h([U],z) = \partial_x H^{(j)}([U],z), \tag{2.11}$$

where $H^{(j)}([U],z)$ are formal series of the form $H^{(j)}([U],z) = z^j + \sum_{k=1}^{\infty} \frac{H_k^j([U])}{z^k}$ and z is related with the parameter λ of the Poisson pencil by $\lambda = z^{n+1}$Along the GD flows these "currents" obey the equations

$$\frac{\partial}{\partial t_j} H^{(k)}([U],z) = H^{(j+k)} - H^{(j)}H^{(k)} + \sum_{l=1}^{k} H_l^j H^{(k-l)} + \sum_{l=1}^{j} H_l^k H^{(j-l)}. \tag{2.12}$$

Let us consider the generating function of the densities of the second (or dual) hamiltonian $h^*([U],z)$. According to Proposition (2) it must satisfy as well suitable conservation laws, to be written as

$$\frac{\partial}{\partial t_j} h^*([U],z) = \partial_x H^*_{(j)}([U],z), \tag{2.13}$$

in terms of "dual" currents $H^*_{(j)}([U],z)$ that have the form

$$H^*_{(j)}([U],z) = j z^{j-1} + \sum_{k=1}^{\infty} \frac{H^*_{jk}([U])}{z^{k+1}}. \tag{2.14}$$

It turns out[1] that, if we denote by $H_{(l)} = z^{l-1} - \sum_{k\geq 1} H_l^k z^{-(k+1)}$, the dual currents are given by $H^*_{(j)} = \sum_{l=1}^{j} H_{(l)} H^{(j-l)}$. By using this representation, and working a bit on the component-wise form of (2.12), and in particular on the formula

$$isdevoted h^*([U],z) = \frac{\partial}{\partial z} h([U],z) - \frac{1}{z^{n+1}} \frac{\partial}{\partial t^n} h([U],z), \tag{2.15}$$

[1] See [6] (where computations are done in the KP case) for more details.

one can show that the coefficients H^*_{jk} are symmetric in j,k, i.e., $H^*_{jk}=H^*_{kj}$, and along the flows their evolution satisfies $\dfrac{\partial H^*_{jk}}{\partial t_l}=\dfrac{\partial H^*_{lk}}{\partial t_j}$. Therefore, there exists a function $\tau(t_1,t_2,\dots)$ (independent of the spectral parameter z) such that

$$H^*_{jk}=\frac{\partial^2}{\partial t_j\partial t_k}\log\tau. \tag{2.16}$$

This function is the Hirota τ–function of the GD hierarchy; the outcome that we want to herewith remark is that, in this picture, the τ-function appears as the (logarithmic) potential for the densities of conservation laws associated with (2.13) the second Gel'fand-Zakharevich Hamiltonian naturally defined on the exact bi-Hamiltonian phase space of the KdV equation.

3 The dispersionless case

Let us consider the an integrable system of the form (1.1), i.e.

$$q^i_t=V^i_j(q)q^j_x+\sum_{k=1}^{\infty}\epsilon^k F^i_k(q,q_x,q_{xx},\dots,q_{(n)},\dots), \tag{3.1}$$

and consider its dispersionless (or hydrodynamical) limit. The equations of the dispersionless hierarchy have the form

$$q^i_t=V^i_j(q)q^j_x \tag{3.2}$$

For such systems, the class of Hamiltonian structures to be considered were introduced by Dubrovin and Novikov. Let us briefly outline the key points in their construction. Consider functionals

$$\mathcal{F}[q]:=\int_{S^1}f(q^1(x),\dots,q^n(x))\,dx,\qquad\text{and}\qquad G[q]:=\int_{S^1}g(q^1(x),\dots,q^n(x))\,dx$$

and define a bracket between them as follows:

$$\{F,G\}[q]:=\iint_{S^1\times S^1}\frac{\delta F}{\delta q^i(x)}\,\omega^{ij}(x,y)\,\frac{\delta G}{\delta q^j(y)}\,dxdy=\iint_{S^1\times S^1}\frac{\partial f}{\partial q^i(x)}\,\omega^{ij}\,\frac{\partial g}{\partial q^j(y)}\,dxdy, \tag{3.3}$$

where $\frac{\delta}{\delta q^i}$ denotes the variational derivative with respect to q^i. The bivector $\omega^{ij}(x,y)$ has the following (local, hydrodynamical) form

$$\omega^{ij}=g^{ij}\delta'(x-y)+\Gamma^{ij}_k q^k_x\delta(x-y). \tag{3.4}$$

A deep result geometrically characterizes the conditions for a bracket (3.3) be Poisson:

Theorem 6. *[12] If* $\det(g^{ij}) \neq 0$*, then the bracket* (3.3) *is Poisson if and only if the metric* g^{ij} *is flat and the functions* Γ^{ij}_k *are related to the Christoffel symbols of* g_{ij} *(the inverse of* g^{ij}*) by the formula* $\Gamma^{ij}_k = -g^{il}\Gamma^j_{lk}$.

Let us now consider a pair of Poisson bivectors of hydrodynamic type ω_1^{ij}, ω_2^{ij}, associated with a pair of flat metrics g_1 and g_2. As shown by Dubrovin in [13] the flat metrics define a *bi-Hamiltonian structure of hydrodynamic type* iff

1. the Riemann tensor R_λ of the pencil $g_\lambda := g_2^{ij} - \lambda g_1^{ij}$ vanishes for any value of λ;
2. the Christoffel symbols $(\Gamma_\lambda)^{ij}_k$ of the pencil are given by $\Gamma^{ij}_{(2)k} - \lambda\Gamma^{ij}_{(1)k}$.

In this paper we will consider Poisson pencils of hydrodynamic type satisfying two additional assumptions that can be expressed on the pencil g_λ as follows:

Assumption I: The roots $u^1(q), \dots, u^n(q)$ of the characteristic equation

$$\det g_\lambda = \det(g_2 - \lambda g_1) = 0$$

are functionally independent.

Assumption II: The Poisson pencil associated to the flat pencil of metrics g_λ according to the Dubrovin-Novikov recipe is an *exact* Poisson pencil ω_λ. By definition this means that $\mathrm{Lie}_Z\omega_2 = \omega_1$ and $\mathrm{Lie}_Z\omega_1 = 0$ for a suitable vector field Z.

The pencil g_λ satisfying Assumption I is called *semisimple* and the functions $u^i(q)$ are called *canonical coordinates*. It can be shown that, in canonical coordinates both metrics are diagonal [20]:

$$g_1^{ij} = f^i\delta_{ij}, \qquad g_2^{ij} = u^i f^i \delta_{ij}$$

and the the Poisson pencil ω_λ becomes

$$\omega_\lambda = g_2^{ij}(u)\delta'(x-y) + \Gamma^{ij}_{(2)k}u^k_x\delta(x-y) - \lambda\left(g_1^{ij}(u)\delta'(x-y) + \Gamma^{ij}_{(1)k}u^k_x\delta(x-y)\right)$$

where the Christoffel symbols vanish if all the indices are different and

$$\Gamma^{ii}_{(1)j} = \frac{1}{2}\frac{\partial f^i}{\partial u^j},\ \Gamma^{ij}_{(1)i} = -\frac{1}{2}\frac{f^j}{f^i}\frac{\partial f^i}{\partial u^j},\ \Gamma^{ij}_{(1)j} = \frac{1}{2}\frac{f^i}{f^j}\frac{\partial f^j}{\partial u^i},\ (i \neq j) \qquad \Gamma^{ii}_{(1)i} = \frac{1}{2}\frac{\partial f^i}{\partial u^i}$$
$$\Gamma^{ii}_{(2)j} = u^i\Gamma^{ii}_{(1)j},\ \Gamma^{ij}_{(2)i} = u^j\Gamma^{ij}_{(1)i},\ \Gamma^{ij}_{(2)j} = u^i\Gamma^{ij}_{(1)j},\ (i \neq j) \qquad \Gamma^{ii}_{(2)i} = \frac{1}{2}f^i + u^i\Gamma^{ii}_{(1)i}.$$

Remark 7. *In canonical coordinates also the equations of the dispersionless hierarchy become diagonal.*

The following property will be crucial in the computations we shall perform in the core of the paper

Theorem 8. *A semisimple bi-Hamiltonian structure of hydrodynamic type is exact if and only if the condition*

$$\sum_{k=1}^{n}\frac{\partial f^i}{\partial u^k}=0. \tag{3.5}$$

is satisfied.

Moreover, in canonical coordinates all the components of the vector field Z are equal to 1.

Proof. By means of a straightforward computation, using formula [27]

$$\begin{aligned}&\mathrm{Lie}_Z P^{ij}= \qquad\qquad (3.6)\\ &\sum_{k,s}\left(\partial_x^s Z^k(u(x),\dots)\frac{\partial P^{ij}}{\partial u^k_{(s)}(x)}-\frac{\partial Z^i(u(x),\dots)}{\partial u^k_{(s)}(x)}\partial_x^s P^{kj}-\frac{\partial Z^j(u(y),\dots)}{\partial u^k_{(s)}(y)}\partial_y^s P^{ik}\right),\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{Lie}_Z P_2 \;=\;& \left(Z^k\frac{\partial g^{ij}_{(2)}}{\partial u^k}-\frac{\partial Z^i}{\partial u^k}g^{kj}_{(2)}-\frac{\partial Z^j}{\partial u^k}g^{ik}_{(2)}\right)\delta'(x-y)+\\ &\left(Z^k\frac{\partial \Gamma^{ij}_{(2)l}}{\partial u^k}-\frac{\partial Z^i}{\partial u^k}\Gamma^{kj}_{(2)l}-\frac{\partial Z^j}{\partial u^k}\Gamma^{ik}_{(2)l}-g^{ik}_{(2)}\frac{\partial^2 Z^j}{\partial u^k\partial u^l}\right)u^l_x\,\delta(x-y)=P_1\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\mathrm{Lie}_Z P_1 \;=\;& \left(Z^k\frac{\partial g^{ij}_{(1)}}{\partial u^k}-\frac{\partial Z^i}{\partial u^k}g^{kj}_{(1)}-\frac{\partial Z^j}{\partial u^k}g^{ik}_{(1)}\right)\delta'(x-y)+\\ &\left(Z^k\frac{\partial \Gamma^{ij}_{(1)l}}{\partial u^k}-\frac{\partial Z^i}{\partial u^k}\Gamma^{kj}_{(1)l}-\frac{\partial Z^j}{\partial u^k}\Gamma^{ik}_{(1)l}-g^{ik}_{(1)}\frac{\partial^2 Z^j}{\partial u^k\partial u^l}\right)u^l_x\,\delta(x-y)=0\end{aligned}$$

The vanishing of the coefficients of $\delta'(x-y)$ implies

$$\mathrm{Lie}_Z g_2=g_1,\qquad \mathrm{Lie}_Z g_1=0,$$

or, more explicitly

$$\begin{aligned}(\mathrm{Lie}_Z g_1)^{ii}&=Z^k\frac{\partial f^i}{\partial u^k}-2f^i\frac{\partial Z^i}{\partial u^i}=0\\ (\mathrm{Lie}_Z g_2)^{ii}&=Z^k u^i\frac{\partial f^i}{\partial u^k}+Z^i f^i-2u^i f^i\frac{\partial Z^i}{\partial u^i}=f^i.\end{aligned}$$

Taking into account the first equation, the second equation implies

$$Z^i=1,\; i=1,\dots,n, \tag{3.7}$$

and, as a consequence, the first equation reduces to (3.5). It remains to verify

$$Z^k \frac{\partial \Gamma^{ij}_{(1)l}}{\partial u^k} - \frac{\partial Z^i}{\partial u^k}\Gamma^{kj}_{(1)l} - \frac{\partial Z^j}{\partial u^k}\Gamma^{ik}_{(1)l} - g^{ik}_{(1)}\frac{\partial^2 Z^j}{\partial u^k \partial u^l} = Z^k \frac{\partial \Gamma^{ij}_{(1)l}}{\partial u^k} = 0$$

and

$$Z^k \frac{\partial \Gamma^{ij}_{(2)l}}{\partial u^k} - \frac{\partial Z^i}{\partial u^k}\Gamma^{kj}_{(2)l} - \frac{\partial Z^j}{\partial u^k}\Gamma^{ik}_{(2)l} - g^{ik}_{(1)}\frac{\partial^2 Z^j}{\partial u^k \partial u^l} = Z^k \frac{\partial \Gamma^{ij}_{(2)l}}{\partial u^k} = \Gamma^{ij}_{(1)l}.$$

It is easy to check that both follow from (3.5).

■

Remark 1. *In the above computations we have used the same letter (Z) to denote a vector field on the manifold M and the corresponding vector field on the loop space $\mathcal{L}(M)$.*

Remark 2. *The semisimple Poisson pencil of hydrodynamic type associated with a semisimple Frobenius manifold is always exact [13]. In this case the vector filed Z, usually denoted by e is the unity vector field.*

3.1 The n–GD example

Let us consider the dispersionless limit of the A_n Drinfel'd-Sokolov bi-Hamiltonian structure. In this case we have the following generating functions for the contravariant components of the metrics of the pencil [34, 16]

$$\begin{aligned} g_1(q,p) &= \sum_{i,j=1}^{n} g_1^{ij} p^{i-1} q^{j-1} = \frac{\lambda'(p) - \lambda'(q)}{p-q} \\ g_2(q,p) &= \sum_{i,j=1}^{n} g_2^{ij} p^{i-1} q^{j-1} = \frac{\lambda'(p)\lambda(q) - \lambda'(q)\lambda(p)}{p-q} + \frac{\lambda'(p)\lambda'(q)}{n+1} \end{aligned}$$

where

$$\lambda(p) = p^{n+1} + U^n p^{n-1} + \cdots + U^2 p + U^1.$$

Clearly, since λ' does not depend on U_1 and $\frac{\partial \lambda}{\partial U_1} = 1$, we have

$$\mathrm{Lie}_Z g_2 = g_1, \qquad \mathrm{Lie}_Z g_1 = 0,$$

with $Z = \frac{\partial}{\partial U^1}$, that is the Poisson pencil associated with g_1 and g_2 is exact. Moreover it is also semisimple. The canonical coordinates $(u^1, \ldots, u^n)$ are the critical values of λ. If we denote by $v_1, \ldots, v_n$ the critical points of λ (by definition they do not depend on U^1):

$$\lambda'(p) = (n+1)p^n + (n-1)U_n p^{n-2} + \cdots + U_2 = (n+1)\prod_{k=1}^{n}(p - v_k) = 0,$$

the canonical coordinates are

$$u^i = v_i^{n+1} + U_n v_i^{n-1} + \cdots + U_2 v_i + U_1.$$

As expected, in canonical coordinates, the vector field Z reads

$$Z = \sum_{i=1}^{n} \frac{\partial u^i}{\partial U^1}\frac{\partial}{\partial u^i} = \sum_{i=1}^{n} \frac{\partial}{\partial u^i}.$$

4 Central invariants

The main problem in the approach of the Dubrovin's school to the theory of integrable systems is the classification of Poisson pencil of the form (see for instance [14, 27, 15, 16, 28, 4])

$$\Pi^{ij}_\lambda = \omega_2^{ij} + \sum_{k\geq 1}\epsilon^k \sum_{l=0}^{k+1} A^{ij}_{(2)k,l}(q, q_x, \ldots, q_{(l)})\delta^{(k-l+1)}(x-y)$$
$$-\lambda\left(\omega_1^{ij} + \sum_{k\geq 1}\epsilon^k \sum_{l=0}^{k+1} A^{ij}_{(1)k,l}(q, q_x, \ldots, q_{(l)})\delta^{(k-l+1)}(x-y)\right)$$

where ω_1 and ω_2 are semisimple Poisson bivectors of hydrodynamic type and $A^{ij}_{k,l}$ are differential polynomial of degree l.

Two pencils Π_λ and $\tilde{\Pi}_\lambda$ are considered equivalent if they are related by a Miura transformation

$$\tilde{q}^i = F_0^i(q) + \sum_{k\geq 1}\epsilon^k F_k(q, q_x, \ldots, q_{(k)}), \qquad \det\frac{\partial F_0^i}{\partial q^j} \neq 0,\ \deg F_k^i = k.$$

In the semisimple case [27] (that is if ω_λ is semisimple) equivalence classes of equivalent Poisson pencil are labelled by n functional parameters called *central invariants.* More precisely two pencils having the same leading order are Miura equivalent if and only if they have the same central invariants. The problem of costructing a pencil for a given choice of the leading term ω_λ and of the central invariants has been solved only in certain cases. In general, even to prove the existence of the pencil is a non trivial problem.

Let us recall the definition of the central invariants of a Poisson pencil.

At each order in ϵ the coefficient of the term containing the highest derivative of the delta function is a tensor field of type $(2,0)$, symmetric for odd derivatives and skewsymmetric for even derivatives. Consider the formal series

$$\pi^{ij}(p, \lambda, q^1, \ldots, q^n) = g_2^{ij}p + \sum_{k\geq 1} A^{ij}_{(2)k,0}p^{k+1} - \lambda\left(g_1^{ij}p + \sum_{k\geq 1} A^{ij}_{(1)k,0}p^{k+1}\right)$$

and denote by $\lambda^i(q,p)$ the roots of the equation

$$\det \pi^{ij}(p,\lambda,q^1,\dots,q^n)=0.$$

Expanding $\lambda^i(q,p)$ at $p=0$ we obtain

$$\lambda^i = u^i + \lambda_2^i p^2 + \mathcal{O}(p^4)$$

Following [16] we can define the central invariant c_i as

$$c_i = \frac{1}{3}\frac{\lambda_2^i(q)}{f^i(q)} \tag{4.1}$$

It turns out [27, 15] that the central invariants c_i depend only on the canonical coordinates u^i and are given by the following expression:

$$c_i(u^i) = \frac{1}{3(f^i)^2}\left(Q_2^{ii} - u^i\, Q_1^{ii} + \sum_{k\neq i}\frac{(P_2^{ki}-u^i\, P_1^{ki})^2}{f^k(u^k-u^i)}\right), \quad i=1,\dots,n. \tag{4.2}$$

where P_1^{ij}, P_2^{ij}, Q_1^{ij}, Q_2^{ij} are the components of the tensor fields $A^{(1)ij}_{2,0}$, $A^{(2)ij}_{2,0}$, $A^{(1)ij}_{3,0}$, $A^{(2)ij}_{3,0}$ in canonical coordinates. This means that, in such coordinates, the pencil has the following expansion in ϵ:

$$\begin{aligned}\Pi^{ij}_\lambda &= \omega_2 + \epsilon\left(P_2^{ij}\delta''(x-y)+\cdots\right)+\epsilon^2\left(Q_2^{ij}\delta'''(x-y)+\cdots\right)++\mathcal{O}(\epsilon^3)\\ &\quad -\lambda\left[\omega_1+\epsilon\left(P_1^{ij}\delta''(x-y)+\cdots\right)+\epsilon^2\left(Q_1^{ij}\delta'''(x-y)+\cdots\right)+\mathcal{O}(\epsilon^3)\right]\end{aligned}$$

As a remark, we notice that we can define central invariants in an alternative way, as

$$c_i = -\frac{1}{3\,f^i}\,\mathrm{Res}_{\lambda=u^i}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda \tag{4.3}$$

where the tensor A^{ij} is defined by

$$A^{ij}_\lambda = Q^{ij}_\lambda + (g_\lambda^{-1})_{lk}P^{li}_\lambda P^{kj}_\lambda.$$

with

$$Q^{ij}_\lambda = Q_2^{ij} - \lambda Q_1^{ij}, \qquad P^{ij}_\lambda = P_2^{ij} - \lambda P_1^{ij}.$$

To prove this identity we notice that the identity (4.2) can be written in terms of the tensor A^{ij}_λ as

$$3c_i(u^i)(f^i)^2 = \left\{A^{ii}_\lambda\right\}_{\lambda=u^i} = \mathrm{Res}_{\lambda=u^i}\sum_{k=1}^n \frac{A^{kk}_\lambda}{\lambda-u^k}. \tag{4.4}$$

and therefore, dividing both sides by f^i and using the properties of residues, we obtain

$$\begin{aligned}
3c_i(u^i)\,f^i &= \mathrm{Res}_{\lambda=u^i}\sum_{k=1}^n \frac{A_\lambda^{kk}}{f^i(\lambda-u^k)} = \\
&\mathrm{Res}_{\lambda=u^i}\sum_{k=1}^n \frac{A_\lambda^{kk}}{f^k(\lambda-u^k)} = \\
&-\mathrm{Res}_{\lambda=u^i}\sum_{k=1}^n (g_\lambda^{-1})_{kl}A_\lambda^{lk} = -\mathrm{Res}_{\lambda=u^i}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda
\end{aligned}$$

The formula (4.3) clearly holds true in any coordinate system and, due to the results of Liu and Zhang, it remains valid also after a Miura transformation.

Let us show how this alternative definition works in a couple of example:

AKNS Let us consider the Poisson pencil $\omega_2+\epsilon P_2^{(1)}-\lambda\omega_1$ with

$$\omega_2+\epsilon P_2^{(1)}-\lambda\omega_1 = \begin{pmatrix} (2u\partial_x+u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} + \epsilon\begin{pmatrix} 0 & -\delta'' \\ \delta'' & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix} \tag{4.5}$$

where, to compactify the formulas, we write δ instead of $\delta(x-y)$. This is the Poisson pencil of the so-called AKNS (or two-boson) hierarchy.

In this case

$$g_\lambda = \begin{pmatrix} 2u & v-\lambda \\ v-\lambda & -2 \end{pmatrix}.$$

After some computations we get $A_\lambda = \frac{g_\lambda}{\det g_\lambda}$ and therefore, taking into account that

$$u^1 = v+\sqrt{-4u},\; u^2 = v-\sqrt{-4u}. \qquad f^1 = \frac{8}{u_2-u_1},\; f^2 = \frac{8}{u_1-u_2},$$

using formula (4.3) we obtain

$$\begin{aligned}
c_1 &= -\frac{1}{3f^1}\mathrm{Res}_{\lambda=u^1}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda = -\frac{1}{3f^1}\mathrm{Res}_{\lambda=u^1}\frac{2}{\det g_\lambda} = -\frac{1}{12} \\
c_2 &= -\frac{1}{3f^2}\mathrm{Res}_{\lambda=u^2}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda = -\frac{1}{3f^2}\mathrm{Res}_{\lambda=u^2}\frac{2}{\det g_\lambda} = -\frac{1}{12}
\end{aligned}$$

Two component CH Moving $P_2^{(1)}$ from P_2 to P_1 in the Poisson pencil of the AKNS hierarchy one obtains the following Poisson pencil [18, 27]

$$P_\lambda = \begin{pmatrix} (2u\partial_x+u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} - \lambda\begin{pmatrix} 0 & \delta'-\epsilon\delta'' \\ \delta'+\epsilon\delta'' & 0 \end{pmatrix} \tag{4.6}$$

which is the Poisson pencil defining the so called CH_2 hierarchy. The pencil g_λ and the canonical coordinates are the same of the previous example, while $A_\lambda = \frac{\lambda^2 g_\lambda}{\det g_\lambda}$. Using formula (4.3) we obtain

$$\begin{aligned} c_1 &= -\frac{1}{3f^1}\mathrm{Res}_{\lambda=u^1}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda = -\frac{1}{3f^1}\mathrm{Res}_{\lambda=u^1}\frac{2\lambda^2}{\det g_\lambda} = -\frac{(u^1)^2}{12} \\ c_2 &= -\frac{1}{3f^2}\mathrm{Res}_{\lambda=u^2}\mathrm{Tr}\, g_\lambda^{-1}A_\lambda = -\frac{1}{3f^2}\mathrm{Res}_{\lambda=u^2}\frac{2\lambda^2}{\det g_\lambda} = -\frac{(u^2)^2}{12}. \end{aligned}$$

Remark 3. *Notice that in both examples the matrix $g_\lambda^{-1}A_\lambda$ is the identity matrix times a scalar function. In the first case this function is $\frac{1}{\det g_\lambda}$ while in the second case it is $\frac{\lambda^2}{\det g_\lambda}$.*

5 Bi-Hamiltonian cohomology

In this section we collect, for the reader's convenience, some definitions and results about (Bi)-Hamiltonian cohomologies and the Dubrovin-Zhang complex (see [14] for full details and proofs). Let g be a flat metric on a manifold M and ω be the associated Poisson bivector of hydrodynamic type. In analogy with the case of finite dimensional Poisson manifolds [26] one defines Poisson cohomology groups in the following way:

$$H^j(\mathcal{L}(M),\omega) := \frac{\ker\{d_\omega : \Lambda^j_{\mathrm{loc}} \to \Lambda^{j+1}_{\mathrm{loc}}\}}{\mathrm{im}\{d_\omega : \Lambda^{j-1}_{\mathrm{loc}} \to \Lambda^j_{\mathrm{loc}}\}} \tag{5.1}$$

where $d_\omega := [\omega, \cdot]$ (the square brackets denote the Schouten brackets) and Λ^j_{loc} is the space of local j-multivectors on the loop space of the manifold M (see [14] for more details on the definition of this complex). Since Λ^j_{loc} has a natural decomposition in homogeneous components which is preserved by d_ω, we have

$$H^j(\mathcal{L}(M),\omega) = \oplus_k H^j_k(\mathcal{L}(M),\omega). \tag{5.2}$$

For Poisson structures of hydrodynamic type like (3.4), it has been proved in [23] (see also [11] for an independent proof of the cases $n = 1, 2$) that $H^k(\mathcal{L}(M),\omega) = 0$ for $k = 1, 2, \dots$. The vanishing of these cohomology groups implies that any deformation of a Poisson bivector of a hydrodynamic type

$$P^\epsilon = \omega + \sum_{n=1}^{\infty} \epsilon^n P_n, \tag{5.3}$$

where $P_k \in \Lambda^2_{k+2,\mathrm{loc}}$ can be obtained from ω by performing a Miura transformation.

In order to study deformations of Poisson pencil of hydrodynamic type it is necessary to introduce bi-Hamiltonian cohomology groups [22, 14, 27]. For $i \geq 2$ they are defined as

$$H^i_k(\mathcal{L}(M),\omega_1,\omega_2)=\frac{\mathrm{Ker}\left(d_{\omega_1}d_{\omega_2}\,|_{\Lambda^{i-1}_{k,\mathrm{loc}}}\right)}{\mathrm{Im}\left(d_{\omega_1}|_{\Lambda^{i-2}_{k-2,\mathrm{loc}}}\right)\oplus\mathrm{Im}\left(d_{\omega_1}|_{\Lambda^{i-2}_{k-2,\mathrm{loc}}}\right)}.$$

Liu and Zhang showed that, in the semisimple case,

$$H^2_k(\mathcal{L}(M),\omega_1,\omega_2)=0\quad\forall k\neq 2,$$

and that the elements of

$$H^2_2(\mathcal{L}(M),\omega_1,\omega_2)$$

have the form

$$d_2\left(\sum_{i=1}^n\int c^i(u^i)u^i_x\mathrm{log}u^i_x\,dx\right)-d_1\left(\sum_{i=1}^n\int u^ic^i(u^i)u^i_x\mathrm{log}u^i_x\,dx\right)\tag{5.4}$$

where $c^i(u^i)$ are the central invariants introduced in the previous section. More explicitly, the components of these vector fields, in canonical coordinates, are given by

$$X^i=\sum_{j=1}^n\left[\left(\frac{1}{2}\delta_{ij}\partial_xf^i+A^{ij}\right)c^ju^j_x+(2\delta_{ij}f^i-L^{ij})\partial_x(c^ju^j_x)\right],\ i=1,\dots,n.\tag{5.5}$$

with

$$A^{ij}=\frac{1}{2}\left(\frac{f^i}{f^j}\frac{\partial f^j}{\partial u^i}u^j_x-\frac{f^j}{f^i}\frac{\partial f^i}{\partial u^j}u^i_x\right)\tag{5.6}$$

$$L^{ij}=\frac{1}{2}\delta_{ij}f^i+\frac{(u^i-u^j)f^i}{2f^j}\frac{\partial f^j}{\partial u^i}.\tag{5.7}$$

We will use these facts later.

6 Constant central invariants and exactness

This section is devoted to the proof of the main result of the paper.

Theorem 9. *Let*

$$\Pi_\lambda=P_2-\lambda P_1=\omega_2+\sum_{k=1}^\infty\epsilon^{2k}P_2^{(2k)}-\lambda\left(\omega_1+\sum_{k=1}^\infty\epsilon^{2k}P_1^{(2k)}\right).\tag{6.1}$$

be a Poisson pencil whose dispersionless limit $\omega_2-\lambda\omega_1$ is semisimple and exact. Then its central invariants are constant if and only if it is, in the sense of formal series of Poisson pencils, exact.

In particular, we recall that Theorem 8 states that a Poisson pencil of hydrodynamic type is exact if and only if the quantities f^j satisfy

$$\sum_{k=1}^{n} \frac{\partial f^j}{\partial u^k} = 0, \quad j = 1, \ldots, n.$$

We split the proof of the main theorem into the proof of some Lemmas.

Lemma 10. *There exists a Miura transformation reducing the pencil* (6.1) *to the form*

$$\Pi_\lambda = \omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)}. \tag{6.2}$$

*P*roof. The lemma is a consequence of the vanishing of the second Poisson cohomology group [23, 11, 14] associated to Poisson structure of hydrodynamic type and of the triviality of the odd order deformations [27, 15].

Let us restrict our attention to exact Poisson pencil of the form (6.2). This means that there exists a vector field $Z = \sum_{k=0}^{\infty} \epsilon^{2k} Z_{2k}$ ($\deg Z_{2k} = 2k$) such that

$$\mathrm{Lie}_Z(\omega_1) = 0, \tag{6.3}$$

$$\mathrm{Lie}_Z(\omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)}) = \omega_1. \tag{6.4}$$

From (6.3) and (6.4) it follows that

$$\mathrm{Lie}_{Z_0}\omega_1 = 0 \tag{6.5}$$

$$\mathrm{Lie}_{Z_0}\omega_2 = \omega_1. \tag{6.6}$$

We have seen (see Theorem 8) that this implies (3.5) and that, in canonical coordinates $Z_0^i = 1$, that is $Z_0 = e$.

Lemma 11. *There exists a Miura transformation preserving ω_1 that reduces Z to e.*

Proof. From (6.3) it follows that

$$\mathrm{Lie}_{Z_{2k}}(\omega_1) = 0, \qquad k = 1, 2, \ldots. \tag{6.7}$$

This means, in particular, that $Z_2 = d_{\omega_1} H_2$ for a suitable functional H_2. The Miura transformation generated by the vector field $d_1 \tilde{H}_2$ with

$$\mathrm{Lie}_e \tilde{H}_2 = H_2 \tag{6.8}$$

mantains the form of the pencil: $\Pi_\lambda \to \tilde{\Pi}_\lambda = \omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} \tilde{P}_2^{(2k)}$ and reduces Z to the form

$$\begin{aligned} Z &= e + \epsilon^2(\mathrm{Lie}_{d_{\omega_1}\tilde{H}_2} e + d_{\omega_1} H_2) + \mathcal{O}(\epsilon^4) = \\ &\quad e + \epsilon^2(d_{\omega_1}(-\mathrm{Lie}_e \tilde{H}_2) + d_{\omega_1} H_2) + \mathcal{O}(\epsilon^4) = \\ &\quad e + \mathcal{O}(\epsilon^4). \end{aligned}$$

We can apply the same arguments to higher order deformations and construct a Miura transformation that maps Z into e. ■

Remark 12. *For completeness, let us further discuss the solvability of (6.8), that is, of an equation of the form*

$$\mathrm{Lie}_e \tilde{K} = K \tag{6.9}$$

for the unknown functional $\tilde{K} = \int_{S^1} \tilde{k}\, dx$. *In canonical coordinates equation* (6.9) *reads*

$$\int_{S^1} \sum_{i=1}^{n} \frac{\partial \tilde{k}}{\partial u^i}\, dx = \int_{S^1} k\, dx.$$

Indeed taking into account the periodic boundary conditions the l.h.s. of (6.9) *is equal to*

$$\sum_{i=1}^{n} \int_{S^1} e^i \frac{\delta \tilde{K}}{\delta u^i}\, dx = \sum_{i=1}^{n} \int_{S^1} \left[\frac{\partial \tilde{k}}{\partial u^i} + \partial_x \sum_{k=1}^{\infty} (-1)^k \partial_x^{k-1} \left(\frac{\partial \tilde{k}}{\partial u^i_{(k)}} \right) \right] dx = \int_{S^1} \sum_{i=1}^{n} \frac{\partial \tilde{k}}{\partial u^i}\, dx,$$

where $u^i_{(k)}$ *is the* $k-th$ *derivative with respect to* x *of* u^i. *A solution can be found solving the equation*

$$\sum_{i=1}^{n} \frac{\partial \tilde{k}}{\partial u^i} = k$$

for the density of the functional $\tilde{K}$. *It is equivalent to the system of equations*

$$\sum_{i=1}^{n} \frac{\partial \tilde{A}_j}{\partial u^i} = A_j, \qquad \sum_{i=1}^{n} \frac{\partial \tilde{B}_{jm}}{\partial u^i} = B_{jm}, \qquad \ldots$$

for the coefficients $\tilde{A}_i, \tilde{B}_{ij}, \ldots$ *of the homogenous differential polynomial*

$$\tilde{k} = \tilde{A}_i u^i_{(N)} + \tilde{B}_{ij} u^i_x u^j_{(N-1)} + \ldots$$

With a linear change of coordinates $(u^1, \ldots, u^n) \to (w^1, \ldots, w^n)$ *we can reduce* $\sum_{k=1}^{n} \frac{\partial}{\partial u^i}$ *to* $\frac{\partial}{\partial w^1}$. *In such coordinates the solution is obtained integrating the coefficients of* k *along* w^1. *Clearly the solution is not unique and in the coordinates* $(w^1, \ldots, w^n)$ *is defined up to functions of* $(w^2, \ldots, w^n)$.

The next lemma shows that the constancy of the central invariants is related to the exactness at the second order of the pencil.

Lemma 13. *Still in the hypotheses of Theorem 8 (namely, if the condition* (3.5) *is satisfied), the central invariants of* Π_λ *are constant if and only if the second order condition*

$$\mathrm{Lie}_e P_2^{(2)} = 0. \tag{6.10}$$

is satisfied.

Proof. Suppose that $\mathrm{Lie}_e P_2^{(2)} = 0$. According to the results of [27], in canonical coordinates $P_2^{(2)}$ is given by the formula

$$P_2^{(2)} = \mathrm{Lie}_X \left(g^{ij}_{(1)}(u)\delta'(x-y) + \Gamma^{ij}_{(1)k} u^k_x \delta(x-y) \right).$$

where X is given by formula (5.5). By straightforward computation, using formula (3.6), we obtain

$$\begin{gathered}\left(P_2^{(2)}\right)^{ij} = -\left(\frac{\partial X^i}{\partial u^j_{xx}} f^j + \frac{\partial X^j}{\partial u^i_{xx}} f^i \right) \delta'''(x-y) + \dots = \\ \left[-3c_i (f^i)^2 \delta^i_j + \frac{u^i - u^j}{2} \left(\frac{\partial f^j}{\partial u^i} c^j - \frac{\partial f^i}{\partial u^j} c^i \right) \right] \delta'''(x-y) + \dots\end{gathered}$$

where the dots are the additional terms containing derivatives of the delta function of order lower than 3. Taking into account condition (3.5), condition

$$\left(\mathrm{Lie}_e P_2^{(2)}\right)^{ij} = \sum_k \frac{\partial (P_2^{(2)})^{ij}}{\partial u^k} = 0$$

for $i = j$ implies

$$-3 \sum_k \frac{\partial c_i}{\partial u^k} (f^i)^2 = -3 \frac{\partial c_i}{\partial u^i} (f^i)^2 = 0, \qquad i = 1, \dots, n.$$

Suppose now that the central invariants are constant. In this case one can easily check that the vector fields X and e commute. Indeed

$$[e, X]^i = \sum_k \frac{\partial X^i}{\partial u^k}$$

and the result follows from (3.5). Using this fact we have

$$\mathrm{Lie}_e P_2^{(2)} = \mathrm{Lie}_e \mathrm{Lie}_X \omega_1 = \mathrm{Lie}_X \mathrm{Lie}_e \omega_1 = 0$$

■

Lemma 13 relates the condition (6.10) to the constancy of the central invariants but does not give us any information about the higher order conditions entering the definition of exactness. In order to push our analysis further up in the ϵ expansion, we need the results about bi-Hamiltonian cohomology we recalled in the previous section.

Lemma 14. *If the condition* (3.5) *is satisfied, and the pencil*

$$\Pi_\lambda = \omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)}.$$

satisfies

$$\mathrm{Lie}_e P_2^{(2)} = 0$$

then there exist a Miura transformation such that

$$\Pi_\lambda \to \tilde{\Pi}_\lambda = \omega_\lambda + \sum_{k=1}^{\infty} \epsilon^{2k} \tilde{P}_2^{(2k)}.$$

with

$$\mathrm{Lie}_e \tilde{P}_2^{(2k)} = 0, \qquad k = 1, 2, \ldots$$

Proof. We construct the Miura transformation by induction. Suppose that the pencil Π_λ satisfies

$$\mathrm{Lie}_e P_2^{(2k)} = 0, \ldots, N$$

but at the subsequent order,

$$\mathrm{Lie}_e P_2^{(2N+2)} \neq 0.$$

We show that it is possible to define a Miura transformation such that the transformed pencil $\tilde{\Pi}_\lambda$ satisfies the above condition, that is, is exact up to order $2N+2$, with Liouville vector field still given by $Z = e$. To construct such a transformation we will use the following strategy:

- First we will show that

  $$\mathrm{Lie}_e P_2^{(2N+2)} = \mathrm{Lie}_{X_2^{(2N+2)}} \omega_1$$

  and that the vector field $X_2^{(2N+2)}$ belongs to $H^2_{2N+2}(\mathcal{L}(M), \omega_1, \omega_2)$. Due to the triviality of this cohomology group for $N > 0$ this implies that

  $$X_2^{(2N+2)} = d_{\omega_1} H_2^{(2N+2)} + d_{\omega_2} K_2^{(2N+2)}$$

  for two suitable local functionals $H_2^{(2N+2)}$ and $K_2^{(2N+2)}$ having densities which are differential polynomials of degree $2N+2$.

- Second we will show that the pencil $\tilde{\Pi}_\lambda$ related to Π_λ by the Miura transformation generated by the vector field $d_{\omega_1} \tilde{K}_2^{(2N+2)}$, with

  $$\mathrm{Lie}_e \tilde{K}_2^{(2N+2)} = K_2^{(2N+2)}, \tag{6.11}$$

  has the required property.

Concerning the first point we have to show that

$$d_{\omega_1}\left(\mathrm{Lie}_e P_2^{(2N+2)}\right) = 0 \tag{6.12}$$

$$d_{\omega_2}\left(\mathrm{Lie}_e P_2^{(2N+2)}\right) = 0. \tag{6.13}$$

This can be easily proved using the following consequences of graded Jacobi identity:

$$\mathrm{Lie}_e\, d_{\omega_1} - d_{\omega_1}\mathrm{Lie}_e = 0 \tag{6.14}$$

$$\mathrm{Lie}_e\, d_{\omega_2} - d_{\omega_2}\mathrm{Lie}_e = d_{\omega_1}. \tag{6.15}$$

Indeed, (6.12) follows immediately from (6.14) and $d_{\omega_1}P_2^{(2N+2)} = 0$. To ascertain the validity of (6.13) we first observe that from $[P_2, P_2] = 0$ it follows

$$d_{\omega_2}P_2^{(2N+2)} = -\frac{1}{2}\sum_{k=1}^{N}[P_2^{(2k)}, P_2^{(2N+2-2k)}];$$

then using (6.15) and graded Jacoby we obtain

$$d_{\omega_2}\,\mathrm{Lie}_e P_2^{(2N+2)} = \mathrm{Lie}_e d_{\omega_2}P_2^{(2N+2)} - d_{\omega_1}P_2^{(2N+2)} = -\frac{1}{2}\sum_{k=1}^{N}\mathrm{Lie}_e[P_2^{(2k)}, P_2^{(2N+2-2k)}] = 0$$

Concerning the second point (that is, Equation (6.11)), we observe that the Miura transformation generated by the vector field $\epsilon^{2N+2}d_{\omega_1}\tilde{K}_2^{(2N+2)}$ reduces the pencil to the form

$$\begin{aligned}\tilde{\Pi}_\lambda &= \omega_\lambda + \epsilon^2 P_2^{(2)} + \cdots + \epsilon^{2N+2}\tilde{P}_2^{(2N+2)} + \mathcal{O}(\epsilon^{2N+4}) = \\ &= \omega_\lambda + \epsilon^2 P_2^{(2)} + \cdots + \epsilon^{2N+2}\left(P_2^{(2N+2)} + \mathrm{Lie}_{d_{\omega_1}\tilde{K}_2^{(2N+2)}}\omega_2\right) + \mathcal{O}(\epsilon^{2N+4})\end{aligned}$$

and

$$\begin{aligned}\mathrm{Lie}_e\tilde{P}_2^{(2N+2)} &= \mathrm{Lie}_e P_2^{(2N+2)} + \mathrm{Lie}_e d_{\omega_2}d_{\omega_1}\tilde{K}_2^{(2N+2)} = \\ & d_{\omega_1}d_{\omega_2}K_2^{(2N+2)} + d_{\omega_2}d_{\omega_1}\mathrm{Lie}_e\tilde{K}_2^{(2N+2)} = \\ & d_{\omega_1}d_{\omega_2}K_2^{(2N+2)} + d_{\omega_2}d_{\omega_1}K_2^{(2N+2)} = 0\end{aligned}$$

■

Remark 4. *The identity* (6.14) *is the counterpart at the level of the double complex defined by* $(d_{\omega_1}, d_{\omega_2})$ *of the exactness of the pencil* $\omega_2 - \lambda\omega_1$.

Collecting the results of all the previous Lemmas we can finally prove the main theorem.

Proof of the main theorem. Due to lemma 10, without loss generality we can assume the pencil of the form (6.2). Suppose that the pencil (6.2) is exact, i.e. it satisfies (6.3) and (6.4). Due to lemma 11, performing a Miura transformation preserving ω_1, we can reduce Z to e. The exactness of the pencil implies (6.10) and consequently, due to lemma 13, the constancy of the central invariants.
Suppose now that the central invariants of the pencil (6.2) are constant. Due again to lemma 13 the pencil satisfies the condition (6.10). In order to prove that (6.2) is exact it is enough to prove that it is Miura equivalent to an exact Poisson pencil. But this follows from lemma 14. ■

We close this section discussing how the above procedure works for the case of the AKNS hierarchy. Let us consider the Poisson pencil (4.5). We have already shown that it has constant central invariants. According to theorem 9 it is an exact Poisson pencil. The Liouville vector field is $Z = e = \frac{\partial}{\partial v}$.

Notice that

$$\begin{pmatrix} 0 & -\delta'' \\ \delta'' & 0 \end{pmatrix} = -\mathrm{Lie}_X \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix}$$

where

$$X = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \xi} \\ \frac{\delta H}{\delta \eta} \end{pmatrix}, \qquad H = -\int_{S^1} \frac{\eta(x)^2}{4}\, dx.$$

This means that the Miura transformation generated by the vector field X (up to terms of order $\mathcal{O}(\epsilon^3)$) reduces the pencil (4.5) to the form $P'_\lambda =$

$$\begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix} +$$
$$\frac{\epsilon^2}{2}\mathrm{Lie}_X^2 \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} + \frac{\epsilon^3}{6}\mathrm{Lie}_X^3 \begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} + \cdots =$$
$$\begin{pmatrix} (2u\partial_x + u_x)\delta & v\delta' \\ \partial_x(v\delta) & -2\delta' \end{pmatrix} - \lambda \begin{pmatrix} 0 & \delta' \\ \delta' & 0 \end{pmatrix} + \frac{\epsilon^2}{2}\begin{pmatrix} 0 & 0 \\ 0 & \delta''' \end{pmatrix} + \frac{\epsilon^3}{6}\begin{pmatrix} 0 & -\delta'''' \\ \delta'''' & 0 \end{pmatrix} + \dots$$

Notice also that the vector field $Z = e = \frac{\partial}{\partial \eta}$ is left invariant by the Miura transformation generated by X (indeed Z and X commute). Moreover according to lemma 13 $\mathrm{Lie}_e P_2^{'(2)} = 0$.

7 Conclusions and outlook

In this paper we elaborated on the circle of ideas connecting exact bihamiltonian pencils, tau structures, and the central invariants of hierarchies admitting hydrodynamical limit, as defined by Dubrovin and collaborators. We have provided the characterization of a semisimple exact pencil of hydrodynamical type in canonical coordinates. If this is related to a Frobenius manifold, then the Liouville vector field must coincide with the unity vector field. We have shown that the exactness of the pencil is equivalent to the constancy of the central invariants defined by the dispersive expansion of the Poisson pencil of the hierarchy, and, in particular, that exactness at order 2 in the ε expansion is sufficient to ensure exactness at all orders. We believe that this property is intimately related with the properties of the vector field e that although not belonging to the Dubrovin-Zhang complex, defines an outer derivation of the complex, and satisfies (6.14).

Still, many important examples of bi-Hamiltonian hierarchies of PDEs do not have constant central invariants (and are believed not to admit τ-structures, at least in the strong sense herewith understood. Among them the Camassa-Holm equation and its multicomponent generalizations [7, 27, 8, 18], and other examples belonging to the so called r-KdV-CH-hierarchy [31, 2, 3, 9]. In particular in [27] it has been shown that the CH equation possesses linear central invariants, while, e.g., the CH_2 equation has *quadratic* central invariants. A natural question would be whether the point of view exposed in the present paper can be applied to characterize these hierarchies. Work in this direction is in progress, to be detailed elsewhere; in particular, according to some preliminary results, this method can be applied almost *verbatim* to the case of linear central invariants. It corresponds to the geometric relation, well known in the CH case,

$$\mathrm{Lie}_Z{}^2(P_2) = 0, \quad \text{but} \quad \mathrm{Lie}_Z P_2 \neq P_1.$$

On the other hand, in the higher degree case, the iteration procedure seems to require further condition on the pencil, whose meaning is currently being investigated.

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