HEUN'S EQUATION, GENERALIZED HYPERGEOMETRIC FUNCTION AND EXCEPTIONAL JACOBI POLYNOMIAL

KOUICHI TAKEMURA

Abstract. We study Heun's differential equation in the case that one of the singularities is apparent. In particular we conjecture a relationship with generalized hypergeometric differential equation and establish it in some cases. We apply our results to exceptional Jacobi polynomials.

1. Introduction

The hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2}+(\gamma-(\alpha+\beta+1)z)\frac{dy}{dz}-\alpha\beta y=0, \tag{1.1}$$

is one of the most important differential equation in mathematics and physics. Several properties of the hypergeometric differential equation, i.e. integral representation of solutions, explicit description of monodromy, algebraic solutions, orthogonal polynomials, etc. are studied very well, and they are applied to various problems in mathematics and physics. The hypergeometric differential equation has three singularities $\{0,1,\infty\}$ and it is a canonical form of Fuchsian differential equations of second order with three singularities.

Several generalizations of the hypergeometric differential equation have been studied so far.

A generalization is given by adding regular singularities. Heun's differential equation is a canonical form of a second-order Fuchsian equation with four singularities, which is given by

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0, \tag{1.2}$$

with the condition

$$\gamma+\delta+\epsilon=\alpha+\beta+1, \quad t\neq 0,1. \tag{1.3}$$

It has been applied to several problems in physics (e.g. see ([10, 12])). The parameter q is independent from the local exponents and is called an accessory parameter. Although it is much more difficult to study global structure of Heun's differential equation than that of the hypergeometric differential equation, several special solutions of Heun's differential equation have been investigated. In the case that one of the regular singularities $\{0,1,t,\infty\}$ is apparent, the solutions of Heun's differential equation have integral representations ([14]). On the other hand, if $\alpha\in\mathbb{Z}_{\leq 0}$ and

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q is special, then Heun's differential equation has polynomial solutions. We will investigate the case that Heun's differential equation has polynomial solutions and the regular singularity $z = t$ is apparent. Propositions on the structure between two conditions will be given in section 5.

Another generalization of the hypergeometric equation is given by increasing the degree of the differential. Let ${}_pF_q\,(a_1,\dots,a_p;b_1,\dots,b_q;z)$ be the generalized hypergeometric function defined by

(1.4)
$$ {}_pF_q\,(a_1,\dots,a_p;b_1,\dots,b_q;z) = {}_pF_q\left(\begin{array}{c} a_1,a_2,\dots,a_p \\ b_1,b_2,\dots,b_q \end{array};z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\dots(a_p)_n}{(b_1)_n(b_2)_n\dots(b_q)_n n!}z^n, $$

where $(a)_n = a(a+1)\dots(a+n-1)$. Then it satisfies the generalized hypergeometric differential equation

(1.5)
$$ \left\{\frac{d}{dz}\left(z\frac{d}{dz}+b_1-1\right)\dots\left(z\frac{d}{dz}+b_q-1\right)-\left(z\frac{d}{dz}+a_1\right)\dots\left(z\frac{d}{dz}+a_p\right)\right\}y=0. $$

In the case $p = 2$ and $q = 1$, the function ${}_2F_1\,(\alpha,\beta;\gamma;z)$ is called Gauss hypergeometric function, and Eq.(1.5) is just the hypergeometric differential equation. If $p = q+1$, then the differential equation (1.5) is Fuchsian with singularities $z = 0, 1, \infty$, and it is known to be rigid ([4]), i.e. there is no accessory parameter in Eq.(1.5). Consequently we have integral representations of solutions of the generalized hypergeometric differential equation.

In this paper we study some cases that the generalized hypergeometric differential equation is factorized and Heun's differential equation appears as a factorized component. Let $L_{a_1,\dots,a_{q+1};b_1\dots b_q}$ be the monic differential operator of order $q+1$ such that $L_{a_1,\dots,a_{q+1};b_1\dots b_q}y = 0$ is equivalent to Eq.(1.5). For example

$$ \begin{aligned} L_{a_1,a_2,a_3;b_1,b_2} &= \frac{d^3}{dz^3}+\frac{(a_1+a_2+a_3+3)z-(b_1+b_2+1)}{z(z-1)}\frac{d^2}{dz^2} \\ &+\frac{(a_1a_2+a_1a_3+a_2a_3+a_1+a_2+a_3+1)z-b_1b_2}{z^2(z-1)}\frac{d}{dz}+\frac{a_1a_2a_3}{z^2(z-1)}. \end{aligned} \tag{1.6} $$

Letessier, Valent and Wimp ([7]) studied generalized hypergeometric differential equations in reducible cases. They proved that the function

$$ {}_{p+r}F_{q+r}\left(\begin{array}{c} a_1,\dots,a_p,e_1+1,\dots,e_r+1 \\ b_1,b_2,\dots,b_q,e_1,\dots,e_r \end{array};z\right) \tag{1.7} $$

satisfies a linear differential equation of order $\max(p,q+1)$ whose coefficients are polynomials. We now explain it in the case $p = 2$, $q = 1$, $r = 1$. Let $f_0(z) = {}_2F_1\,(\alpha,\beta;\gamma;z)$. Then the function $f_1(z) = zf_0'(z)/e_1 + f_0(z)$ is equal to the function ${}_3F_2\left(\begin{array}{c}\alpha,\beta,e_1+1\\ \gamma,e_1\end{array};z\right)$ and it satisfies $L_{\alpha,\beta,e_1+1;\gamma,e_1}f_1(z) = 0$. On the other hand, it also satisfies

$$ \tilde{L}_{\alpha,\beta;\gamma;e_1}f_1(z) = 0, \tag{1.8} $$

where

$$\tilde{L}_{\alpha,\beta;\gamma;e_1} = \frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\alpha+\beta-\gamma+2}{z-1} - \frac{1}{z-t}\right)\frac{d}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}, \tag{1.9}$$
$$t = \frac{e_1(e_1+1-\gamma)}{(e_1-\alpha)(e_1-\beta)}, \quad q = \frac{\alpha\beta(e_1+1)(\gamma-e_1-1)}{(e_1-\alpha)(e_1-\beta)},$$

and we have the factorization

$$L_{\alpha,\beta,e_1+1;\gamma,e_1} = \left(\frac{d}{dz} + \frac{e_1+1}{z} + \frac{1}{z-1} + \frac{1}{z-t}\right)\tilde{L}_{\alpha,\beta;\gamma;e_1}. \tag{1.10}$$

Since the point $z = t$ is not singular with respect to the differential equation $L_{\alpha,\beta,e_1+1;\gamma,e_1} y = 0$, it is an apparent singularity with respect to $\tilde{L}_{\alpha,\beta;\gamma;e_1} y = 0$. Maier ([8]) observed the fact conversely and he established that Heun's equation with the apparent singularity $z = t$ whose exponents are 0, 2 appears as a right factor of the generalized hypergeometric equation $L_{\alpha,\beta,e_1+1;\gamma,e_1} y = 0$ with a suitable value e_1 (see Proposition 4.1). In this paper we generalize Maier's result and propose a conjecture that solutions of the Fuchsian differential equation with singularities $z = 0, 1, \infty, t_1, \dots, t_M$ such that the singularities $z = t_1, \dots, t_M$ are apparent also satisfy a generalized hypergeometric equation (see Conjecture 1).

Gomez-Ullate, Kamran and Milson ([3]) introduced X_1-Jacobi polynomials as an orthogonal system within the Sturm-Liouville theory. They are remarkable and are stuck out the classical framework because the sequence of polynomials starts from a polynomial of degree one. Sasaki et al. ([9, 11]) extended it to two types of X_ℓ-Jacobi polynomials ($\ell = 1, 2, \dots$) and studied properties of them. It is known that X_1-Jacobi polynomials satisfy Heun's differential equation. We apply results in this paper to X_1-Jacobi polynomials. Then we may understand a position of X_1-Jacobi polynomials in the theory of Heun's differential equation. Moreover we establish that X_1-Jacobi polynomials are also expressed by generalized hypergeometric functions.

This paper is organized as follows. In section 2, we review definitions and properties of Heun's differential equation, apparent singularity and Heun polynomial. In section 3, we recall an integral transformation of Heun's differential equation and its application to the case that singularity $z = t$ is apparent. In section 4, we give a conjecture on Heun's differential equation with an apparent singularity and reducible generalized hypergeometric equation, and verify it for some cases. In section 5, we explain propositions in the case that Heun's differential equation has polynomial solutions and the regular singularity $z = t$ is apparent. In section 6, we give applications to X_1-Jacobi polynomials.

2. Heun's differential equation, apparent singularity and Heun polynomial

2.1. **Local solution.** Let us consider local solutions of Heun's differential equation

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)} y = 0, \tag{2.1}$$

$(\gamma+\delta+\epsilon=\alpha+\beta+1)$ about $z=t$. The exponents about $z=t$ are 0 and $1-\epsilon$. If $\epsilon\notin\mathbb{Z}$, then we have a basis of local solutions about $z=t$ as follows;

$$f(z)=\sum_{j=0}^{\infty}c_j(z-t)^j,\ (c_0\neq 0),\quad g(z)=(z-t)^{1-\epsilon}\sum_{j=0}^{\infty}\tilde{c}_j(z-t)^j,\ (\tilde{c}_0\neq 0). \tag{2.2}$$

The coefficients c_i are recursively determined by $\epsilon t(t-1)c_1+(\alpha\beta t-q)c_0=0$ and

$$\begin{aligned} &i(i+\epsilon-1)t(t-1)c_i+(i+\alpha-2)(i+\beta-2)c_{i-2}\\ &+[(i-1)(i-2)(2t-1)+(i-1)\{(\gamma+\delta+2\epsilon)t-\gamma-\epsilon\}+\alpha\beta t-q]c_{i-1}=0, \end{aligned} \tag{2.3}$$

for $i\geq 2$. Hence c_i is a polynomial of the variable q of order i.

2.2. **Apparent singularity.** We now define an apparent singularity in the case $\epsilon\in\mathbb{Z}$. If $\epsilon\in\mathbb{Z}_{\leq 0}$, then we have a basis of local solutions as follows;

$$f(z)=\sum_{j=0}^{\infty}c_j(z-t)^j+A\,g(z)\log(z-t),\ g(z)=(z-t)^{1-\epsilon}\sum_{j=0}^{\infty}\tilde{c}_j(z-a)^j, \tag{2.4}$$

If the logarithmic term in Eq.(2.4) disappears, i.e. $A=0$, then the singularity $z=t$ is called apparent. Note that the apparency of a regular singularity is equivalent to that the monodromy about $z=t$ is trivial i.e. the monodromy matrix is the unit.

Now we describe an explicit condition that the regular singularity $z=t$ of Heun's differential equation is apparent in the case $\epsilon\in\mathbb{Z}_{\leq 1}$. It is written as

$$\begin{aligned} &(\alpha-\epsilon-1)(\beta-\epsilon-1)c_{-\epsilon-1}\\ &+[\epsilon(\epsilon+1)(2t-1)-\epsilon\{(\gamma+\delta+2\epsilon)t-\gamma-\epsilon\}+\alpha\beta t-q]c_{-\epsilon}=0, \end{aligned} \tag{2.5}$$

where $c_1,\dots,c_{-\epsilon}$ are determined recursively by Eq.(2.3). Note that the equation is obtained by Eq.(2.3) in the case $i=1-\epsilon$. By setting $n=1-\epsilon$, we have $\delta=\alpha+\beta-\gamma+n$ and the condition that the singularity $z=t$ is apparent is written as $P^{\mathsf{app}}(q)=0$, where $P^{\mathsf{app}}(q)$ is a polynomial of the variable q of order n, which is also a polynomial of t,α,β,γ.

Example 1. *(i) If $\epsilon=0$ ($n=1$), then the condition that the regular singularity $z=t$ is apparent is written as $P^{\mathsf{app}}(q)=q-\alpha\beta t=0$ and it follows that the singularity $z=t$ disappears.*
(ii) If $\epsilon=-1$ ($n=2$), then the condition that the regular singularity $z=t$ is apparent is written as

$$P^{\mathsf{app}}(q)=q^2-\{(2\alpha\beta+\alpha+\beta)t-\gamma+1\}q+\alpha\beta t\{(\alpha+1)(\beta+1)t-\gamma\}=0. \tag{2.6}$$

(iii) If $\epsilon=-2$ ($n=3$), then the condition that the regular singularity $z=t$ is apparent is written as

$$\begin{aligned} P^{\mathsf{app}}(q)=&\,q^3+\{(-3\alpha\beta-3\alpha-3\beta-1)t+(3\gamma-4)\}q^2\\ &+[\{3\alpha^2\beta^2+6\alpha\beta(\alpha+\beta)+10\alpha\beta+2(\alpha^2+\beta^2)+2\alpha+2\beta\}t^2\\ &\quad+\{(-6\alpha\beta-4\alpha-4\beta)\gamma+4\alpha\beta+4\alpha+4\beta\}t+2(\gamma-1)(\gamma-2)]q\\ &-\alpha\beta t\{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)t^2-\gamma(3\alpha\beta+4\alpha+4\beta+4)t+2\gamma(\gamma-1)\}=0. \end{aligned} \tag{2.7}$$

(iv) If $\epsilon = -3$ $(n = 4)$, then the condition that the regular singularity $z = t$ is apparent is written as

$$
\begin{aligned}
&P^{\mathsf{app}}(q) = q^4 - 2\{(2\alpha\beta + 3\alpha + 3\beta + 2)t - 3\gamma + 5\}q^3 + [\{6\alpha^2\beta^2 + 18\alpha\beta(\alpha+\beta) \\
&+ 44\alpha\beta + 11\alpha^2 + 11\beta^2 + 18\alpha + 18\beta + 4\}t^2 + \{-(18\alpha\beta + 22\alpha + 22\beta + 8)\gamma + 20\alpha\beta \\
&+ 30\alpha + 30\beta + 12\}t + 11\gamma^2 - 40\gamma + 33]q^2 + [-\{4\alpha^3\beta^3 + 18\alpha^2\beta^2(\alpha+\beta) + 76\alpha^2\beta^2 \\
&+ 22\alpha\beta(\alpha^2+\beta^2) + 84\alpha\beta(\alpha+\beta) + 80\alpha\beta + 6\alpha^3 + 6\beta^3 + 18\alpha^2 + 18\beta^2 + 12\alpha + 12\beta\}t^3 \\
&+ \{(18\alpha^2\beta^2 + 44\alpha\beta(\alpha+\beta) + 82\alpha\beta + 18\alpha^2 + 18\beta^2 + 18\alpha + 18\beta)\gamma \\
&- (10\alpha^2\beta^2 + 30\alpha\beta(\alpha+\beta) + 66\alpha\beta + 18\alpha^2 + 18\beta^2 + 18\alpha + 18\beta)\}t^2 \\
&+ \{(-22\alpha\beta - 18\alpha - 18\beta)\gamma^2 + (50\alpha\beta + 54\alpha + 54\beta)\gamma - 24\alpha\beta - 36\alpha - 36\beta\}t \\
&+ 6(\gamma-1)(\gamma-2)(\gamma-3)]q + \alpha\beta t[(\alpha+1)(\alpha+2)(\alpha+3)(\beta+1)(\beta+2)(\beta+3)t^3 \\
&- 2\gamma\{3\alpha^2\beta^2 + 11\alpha\beta(\alpha+\beta) + 37\alpha\beta + 9\alpha^2 + 9\beta^2 + 27\alpha + 27\beta + 18\}t^2 \\
&+ \gamma\{(10\alpha\beta + 18\alpha + 18\beta + 18)(\gamma-1) + \alpha\beta\gamma\}t - 6\gamma(\gamma-1)(\gamma-2)] = 0.
\end{aligned} \tag{2.8}
$$

2.3. **Heun polynomial.** We determine a condition that Heun's differential equation has a non-zero polynomial solutions of degree $N-1$. Then the solution has an asymptotic $(1/z)^{1-N}$ as $z \to \infty$ and we have $1 - N = \alpha$ or $1 - N = \beta$, because the exponents about $z = \infty$ are α and β. We now assume that $1 - \alpha = N \in \mathbb{Z}_{\geq 0}$. If the accessory parameter q satisfies $c_N = 0$ where c_N is determined by Eq.(2.3), then it follows from Eq.(2.3) in the case $i = N+1$ that $c_{N+1} = 0$. Thus we have $c_i = 0$ for $i \geq N+2$ and we obtain a polynomial solution of degree $N-1$. The polynomial is called Heun polynomial. Note that the condition $c_N = 0$ is written as $P^{\mathsf{pol}}(q) = 0$ by multiplying a suitable constant, where $P^{\mathsf{pol}}(q)$ is a polynomial of the variable q of order N, which is also a polynomial of $t, \beta, \gamma, \epsilon$. $(\delta = \beta - N - \gamma - \epsilon + 2)$

Example 2. *(i) If $\alpha = 0$ $(N = 1)$ and $\beta \notin \mathbb{Z}$, then the condition for existence of non-zero polynomial solution of Heun's equation is written as $P^{\mathsf{pol}}(q) = q = 0$ and a polynomial solution is $y = 1$.*
(ii) If $\alpha = -1$ $(N = 2)$ and $\beta \notin \mathbb{Z}$, then the condition for existence of non-zero polynomial solution of Heun's equation is written as

$$P^{\mathsf{pol}}(q) = q^2 + ((\beta - \epsilon)t + \gamma + \epsilon)q + \beta\gamma t = 0, \tag{2.9}$$

and a polynomial solution is

$$y = t(t-1)\epsilon + (q + \beta t)(z - t). \tag{2.10}$$

(iii) If $\alpha = -2$ $(N = 3)$ and $\beta \notin \mathbb{Z}$, then the condition for existence of non-zero polynomial solution of Heun's equation is written as

$$
\begin{aligned}
P^{\mathsf{pol}}(q) = q^3 &+ \{(3\beta - 3\epsilon - 1)t + 3\gamma + 3\epsilon + 2\}q^2 \\
&+ \{2(\beta-\epsilon)(\beta-\epsilon-1)t^2 - 4(\epsilon^2 + (\gamma - \beta + 2)\epsilon - (2\gamma+1)\beta)t \\
&\quad + 2(\gamma+\epsilon)(\gamma+\epsilon+1)\}q + 4\beta\gamma t((\beta-\epsilon)t + \gamma + \epsilon + 1) = 0,
\end{aligned} \tag{2.11}
$$

and a polynomial solution is

$$y = 2t^2(t-1)^2\epsilon(\epsilon+1) + 2t(t-1)(\epsilon+1)(q+2\beta t)(z-t) + \{q^2 + ((3\beta-\epsilon+1)t+\gamma+\epsilon)q + 2\beta t((\beta+1)t+\gamma)\}(z-t)^2. \tag{2.12}$$

Polynomial-type solutions of Heun's differential equation are written as

$$y = z^{\sigma_0}(z-1)^{\sigma_1}(z-t)^{\sigma_t}p(z), \tag{2.13}$$

where $p(z)$ is a polynomial, $\sigma_0 \in \{0, 1-\gamma\}$, $\sigma_1 \in \{0, 1-\delta\}$, $\sigma_t \in \{0, 1-\epsilon\}$. We described above the condition for existing a non-zero polynomial-type solution in the case $\sigma_0 = \sigma_1 = \sigma_t = 0$. The condition for existing a polynomial-type solution in the case $\sigma_0 = 1-\gamma$, $\sigma_1 = 1-\delta$ and $\sigma_t = 0$ is described as $\epsilon - \alpha \in \mathbb{Z}_{\leq -1}$ (or $\epsilon - \beta \in \mathbb{Z}_{\leq -1}$) and $\tilde{P}(q) = 0$, where $\tilde{P}(q)$ is a polynomial of the variable q of order $-\epsilon+\alpha$ (or $-\epsilon+\beta$). Then the order of $p(z)$ is $-\epsilon+\alpha-1$ (or $-\epsilon+\beta-1$).

3. Integral transformation and its application

Let p be an element of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and γ_p be a cycle on the Riemann sphere with variable w which starts from $w = o$, goes around $w = p$ in a counter-clockwise direction and ends at $w = o$. Let $[\gamma_z, \gamma_p] = \gamma_z \gamma_p \gamma_z^{-1} \gamma_p^{-1}$ be the Pochhammer contour. Kazakov and Slavyanov ([5]) established that Heun's differential equation admits integral transformations.

Proposition 3.1. ([5, 13]) *Set*

$$\begin{aligned} &(\eta-\alpha)(\eta-\beta) = 0,\ \gamma' = \gamma-\eta+1,\ \delta' = \delta-\eta+1,\ \epsilon' = \epsilon-\eta+1, \\ &\{\alpha', \beta'\} = \{2-\eta, \alpha+\beta-2\eta+1\}, \\ &q' = q + (1-\eta)(\epsilon+\delta t+(\gamma-\eta)(t+1)). \end{aligned} \tag{3.1}$$

Let $v(w)$ be a solution of

$$\frac{d^2v}{dw^2} + \left(\frac{\gamma'}{w} + \frac{\delta'}{w-1} + \frac{\epsilon'}{w-t}\right)\frac{dv}{dw} + \frac{\alpha'\beta' w - q'}{w(w-1)(w-t)}v = 0. \tag{3.2}$$

Then the function

$$y(z) = \int_{[\gamma_z, \gamma_p]} v(w)(z-w)^{-\eta}dw \tag{3.3}$$

is a solution of

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}y = 0, \tag{3.4}$$

for $p \in \{0, 1, t, \infty\}$.

It was obtained in ([14]) that polynomial-type solutions of Heun's equation correspond to solutions which have an apparent singularity by the integral transformation. In particular we have the following proposition by setting $\eta = \beta$ in Proposition 3.1.

Proposition 3.2. ([14]) *If $\epsilon \in \mathbb{Z}_{\leq 0}$, $\alpha, \beta, \beta - \gamma, \beta - \delta \notin \mathbb{Z}$ and the singularity $z = t$ of Eq.(3.4) is apparent, then there exists a non-zero solution of Eq.(3.2) which can be written as $v(w) = w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)$ where $h(w)$ is a polynomial of degree $-\epsilon$ and the functions*

$$\int_{[\gamma_z, \gamma_p]} w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)(z-w)^{-\beta}dw, \tag{3.5}$$

$(p = 0, 1)$ *are non-zero solutions of Eq.(3.4).*

We may drop the condition $\alpha, \beta, \beta - \gamma, \beta - \delta \notin \mathbb{Z}$ in Proposition 3.2 by replacing to that $h(w)$ is a polynomial of degree no more than $-\epsilon$ and the solutions in Eq.(3.5) for $p = 0, 1$ may be zero.

Corollary 3.3. *If $\epsilon \in \mathbb{Z}_{\leq 0}$, $\alpha, \beta, \beta - \gamma, \beta - \delta \notin \mathbb{Z}$ and the singularity $z = t$ of Eq.(3.4) is apparent, then any solutions of Eq.(3.4) can be expressed by a finite sum of hypergeometric functions.*

Proof. It follow from Proposition 3.2 in the case $\epsilon = 0$ and $q = \alpha\beta t$ that the functions

$$F_p(z) = \int_{[\gamma_z, \gamma_p]} w^{\beta-\gamma}(w-1)^{\gamma-\alpha-1}(z-w)^{-\beta}dw, \tag{3.6}$$

$(p = 0, 1)$ are non-zero solutions of hypergeometric differential equation, if $\alpha, \beta, \beta - \gamma, \beta - \delta \notin \mathbb{Z}$. Since $F_0(z) \sim z^{1-\gamma}$ $(z \to 0)$ and $F_1(z) \sim (z-1)^{\gamma-\alpha-\beta}$ $(z \to 1)$, we have

$$\begin{aligned} F_0(z) &= d_{\alpha,\beta,\gamma} z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z), \\ F_1(z) &= \tilde{d}_{\alpha,\beta,\gamma}(1-z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - z), \end{aligned} \tag{3.7}$$

where $d_{\alpha,\beta,\gamma}, \tilde{d}_{\alpha,\beta,\gamma}$ are constants. By expanding $h(w) = \sum_{i=0}^{-\epsilon} c'_i w^i$ (resp. $h(w) = \sum_{i=0}^{-\epsilon} \tilde{c}'_i (1-w)^i$) and applying the formula, we have the corollary. □

We describe Proposition 3.2 and Corollary 3.3 in the case $\epsilon = -2$ explicitly.

Proposition 3.4. *Set $\epsilon = -2$. The condition that the singularity $z = t$ of Eq.(3.4) is apparent is written as Eq.(2.7). Then there exists a non-zero solution of Eq.(3.2) written as $v(w) = w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)$ where*

$$\begin{aligned} h(w) = {} & 2\alpha(\alpha+1)w^2 + 2(\alpha+1)\{q - \alpha(\beta+2)t\}w \\ & + q^2 - \{2\alpha\beta + 3\alpha + \beta + 1)t - \gamma + 2\}q + \alpha t\{t(\alpha+1)(\beta+1)(\beta+2) - \beta\gamma\}, \end{aligned} \tag{3.8}$$

and the functions

$$\int_{[\gamma_z, \gamma_p]} w^{\beta-\gamma}(w-1)^{\beta-\delta}h(w)(z-w)^{-\beta}dw, \tag{3.9}$$

$(p = 0, 1)$ *are non-zero solutions of Eq.(3.4).*

4. Generalized hypergeometric equation and Heun's differential equation with an apparent singularity

We propose a conjecture on Fuchsian differential equations which have apparent singularities and generalized hypergeometric equations.

Conjecture 1. *Set*

$$\tilde{L}=\frac{d^2}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}-\sum_{k=1}^{M}\frac{m_k}{z-t_k}\right)\frac{d}{dz}+\frac{s_M z^M+\dots+s_0}{z(z-1)(z-t_1)\dots(z-t_M)},$$

and assume that $0,1,t_1,\dots,t_M$ are distinct mutually, $m_1,\dots m_M\in\mathbb{Z}_{\geq 1}$ and the singularities $z=t_k$ of $\tilde{L}y=0$ are apparent for $k=1,\dots,M$. Then there exists a generalized hypergeometric differential operator $L_{\alpha,\beta,e_1+1,\dots,e_N+1;\gamma,e_1,\dots,e_N}$ $(N\leq\sum_{k=1}^{M}m_k)$ which admits the factorization

$$L_{\alpha,\beta,e_1+1,\dots,e_N+1;\gamma,e_1,\dots,e_N}=\tilde{D}\tilde{L}, \tag{4.1}$$

where $\tilde{D}$ is a differential operator of order N whose coefficients are rational functions.

Here we verify the conjecture for the cases $n=1$, $m_1=1,2,3$. Set

$$H_{[\epsilon=-n]}=\frac{d^2}{dz^2}+\left(\frac{\gamma}{z}+\frac{n+\alpha+\beta-\gamma+1}{z-1}-\frac{n}{z-t}\right)\frac{d}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}. \tag{4.2}$$

The case $n=1$ and $m_1=1$ is essentially due to Maier ([8]).

Proposition 4.1. ([8]) *If the singularity $z=t$ of Heun's differential equation $H_{[\epsilon=-1]}y=0$ is apparent (see Eq.(2.6)), then the generalized hypergeometric differential operator $L_{\alpha,\beta,e_1+1;\gamma,e_1}$ admits the factorization*

$$L_{\alpha,\beta,e_1+1;\gamma,e_1}=\left(\frac{d}{dz}+\frac{e_1+1}{z}+\frac{1}{z-1}+\frac{1}{z-t}\right)H_{[\epsilon=-1]},\qquad e_1=\frac{\alpha\beta t}{q-\alpha\beta t}. \tag{4.3}$$

Remark that if $q-\alpha\beta t=0$ in Proposition 4.1, then we have $\alpha\beta t(1-t)=0$. If $\alpha=0$ (resp. $\beta=0$) and $q=0$, then $e_1=(\beta t+1-\gamma)/(t-1)$ (resp. $e_1=(\alpha t+1-\gamma)/(t-1)$).

The following theorems are verified by straightforward calculations.

Theorem 4.2. *If the singularity $z=t$ of Heun's differential equation written as $H_{[\epsilon=-2]}y=0$ is apparent (see Eq.(2.7)), then there exists a generalized hypergeometric differential operator $L_{\alpha,\beta,e_1+1,e_2+1;\gamma,e_1,e_2}$ which admits the factorization*

$$L_{\alpha,\beta,e_1+1,e_2+1;\gamma,e_1,e_2}=\left(\frac{d^2}{dz^2}+\left(\frac{e_1+e_2+3}{z}+\frac{2}{z-1}+\frac{2}{z-t}\right)\frac{d}{dz}+v(z)\right)H_{[\epsilon=-2]}, \tag{4.4}$$

such that

$$\begin{aligned} v(z)=[&(e_1+3)(e_2+3)z^2+\{q-((e_1+1)(e_2+1)+(\alpha+2)(\beta+2))t \\ &-(e_1+3)(e_2+3)+2(\gamma+1)\}z+t(e_1+1)(e_2+1)]/\{z^2(z-1)(z-t)\}, \end{aligned} \tag{4.5}$$

$$e_1+e_2=-3+\frac{q-(\alpha+2)(\beta+2)t+2\gamma}{(1-t)},$$

$$e_1e_2=\frac{\alpha\beta t(q-(\alpha\beta+2\alpha+2\beta+2)t+2(\gamma-1))}{(q-\alpha\beta t)(1-t)}.$$

If $q=\alpha\beta t$ and $\alpha\beta\neq 0$, then we have $(\alpha+\beta+1)t=\gamma-1$, $e_1+e_2=-1$ and $e_1e_2=-(\alpha+1)(\beta+1)(\gamma-1)/(\alpha+\beta-\gamma+2)$. Note that the condition $\alpha+\beta-\gamma+2=0$ implies $t=1$ or $\gamma=1=-\alpha-\beta$. If $\gamma=1=-\alpha-\beta$, then we have $e_1+e_2=-1$ and $e_1e_2=$

$\alpha\beta t/(t-1)$. If $q=\alpha\beta t$ and $\alpha=0$, then $q=0$, $e_1+e_2=\{(2\beta+1)t+3-2\gamma\}/(t-1)$ and $e_1e_2=\{\beta(\beta+1)t^2-2\beta(\gamma-1)t+(\gamma-1)(\gamma-2)\}/(t-1)^2$.

Theorem 4.3. *If the singularity $z=t$ of Heun's differential equation written as $H_{[\epsilon=-3]}y=0$ is apparent (see Eq.(2.8)), then there exists a hypergeometric differential operator $L_{\alpha,\beta,e_1+1,e_2+1,e_3+1;\gamma,e_1,e_2,e_3}$ which admits the factorization*

$$(4.6)\quad L_{\alpha,\beta,e_1+1,e_2+1,e_3+1;\gamma,e_1,e_2,e_3}= \left(\frac{d^3}{dz^3}+\left(\frac{e_1+e_2+e_3+6}{z}+\frac{3}{z-1}+\frac{3}{z-t}\right)\frac{d^2}{dz^2}+v(z)\frac{d}{dz}+w(z)\right)H_{[\epsilon=-2]},$$

such that

(4.7)

$$e_1+e_2+e_3=\frac{q-(\alpha\beta+3\alpha+3\beta+3)t+3(\gamma-2)}{1-t},$$
$$e_1e_2+e_1e_3+e_2e_3=\frac{1}{2(t-1)^2}[q^2-\{(2\alpha\beta+5\beta+5\alpha+4)t-5\gamma+9\}q$$
$$+\{(\alpha\beta+5\alpha+5\beta+19)\alpha\beta+6\alpha^2+6\beta^2+12\alpha+12\beta+4\}t^2$$
$$+\{-(5\alpha\beta+12\alpha+12\beta+6)\gamma+2(3\alpha\beta+9\alpha+9\beta+5)\}t+2(3\gamma^2-12\gamma+11)],$$
$$e_1e_2e_3=\frac{\alpha\beta t}{(2(t-1)^2(q-\alpha\beta t))}[q^2-\{(2\alpha\beta+5\alpha+5\beta+6)t-5\gamma+7\}q$$
$$+\{(\alpha\beta+5\alpha+5\beta+21)\alpha\beta+6\alpha^2+6\beta^2+18\alpha+18\beta+12\}t^2$$
$$+\{-(5\alpha\beta+12\alpha+12\beta+12)\gamma+4(\alpha\beta+3\alpha+3\beta+3)\}t+6(\gamma-1)(\gamma-2)].$$

If $q=\alpha\beta t$ then we have

(4.8)

$$(2\alpha+\beta+2)(\alpha+2\beta+2)t^2-\{4(\alpha+\beta+1)(\gamma-1)+\alpha\beta\}t+2(\gamma-1)(\gamma-2)=0,$$
$$e_1+e_2+e_3=-\frac{3\{(2\alpha+\beta+2)(\alpha+2\beta+2)t-2(\gamma-2)(\alpha+\beta+1)\}}{2(\alpha+\beta-\gamma+2)},$$
$$e_1e_2+e_1e_3+e_2e_3=\frac{3(2\alpha+\beta+2)(\alpha+2\beta+2)t-\gamma(6\alpha+6\beta+4)+10(\alpha+\beta)+8}{2(\alpha+\beta-\gamma+2)},$$
$$e_1e_2e_3=-\frac{(\alpha+1)(\beta+1)\{(2(\alpha+\beta+2)(\gamma-1)+\alpha\beta)t-2(\gamma-1)(\gamma-2)\}}{2(\alpha+\beta-\gamma+2)(t-1)}.$$

If $\alpha+\beta-\gamma+2=0$ and $q=\alpha\beta t$, then it follows from $t\neq 0,1$ that $t=2(\alpha+\beta)(\alpha+\beta+1)/\{(2\alpha+\beta+2)(\alpha+2\beta+2)\}$, $\alpha\beta+4\alpha+4\beta+4\neq 0$, $e_1+e_2+e_3=3(\alpha+\beta)(\alpha\beta+2\alpha+2\beta+2)/(\alpha\beta+4\alpha+4\beta+4)$, $e_1e_2+e_1e_3+e_2e_3=-(3\alpha\beta(\alpha+\beta)+13\alpha\beta+6\alpha^2+6\beta^2+10\alpha+10\beta+4)/(\alpha\beta+4\alpha+4\beta+4)$ and $e_1e_2e_3=-4(\alpha+\beta)(\alpha+\beta+1)^2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)/(\alpha\beta+4\alpha+4\beta+4)^2$.

5. Polynomial-type solutions with an apparent singularity

If $\epsilon\in\mathbb{Z}_{\leq 0}$, then the condition that $z=t$ is apparent is written as $P^{\mathsf{app}}(q)=0$, where $P^{\mathsf{app}}(q)$ is monic polynomial of q with degree $1-\epsilon$. On the other hand, if $\alpha\in\mathbb{Z}_{\leq 0}$ and $\beta\notin\mathbb{Z}_{\leq 0}$, then the condition that Eq.(3.4) has a polynomial solution is written as $P^{\mathsf{pol}}(q)=0$, where $P^{\mathsf{pol}}(q)$ is a monic polynomial of q with degree $1-\alpha$, and the degree of the polynomial solution of Eq.(3.4) is $-\alpha$. In this section we investigate a relationship of equations $P^{\mathsf{app}}(q)=0$ and $P^{\mathsf{pol}}(q)=0$ in the case $\epsilon\in\mathbb{Z}_{\leq 0}$ and $\alpha\in\mathbb{Z}_{\leq 0}$

Lemma 5.1. *Assume that $\alpha\in\mathbb{Z}$, $\epsilon\in\mathbb{Z}$ and the singularity $z=t$ is apparent. Then the monodromy representation of solutions of Eq.(3.4) is reducible.*

Proof. Let $y_1(z)$, $y_2(z)$ be a basis of solutions of Eq.(3.4). Since the singularity $z=t$ is apparent, the monodromy matrix around $z=t$ is a unit matrix. Let $M^{(p)}$ $(p=0,1,\infty)$ be the monodromy matrix on the cycle around the singularity $w=p$ anti-clockwise with respect to the basis $y_1(z)$, $y_2(z)$. For the moment we assume that $\gamma,\delta\notin\mathbb{Z}$. Then $M^{(0)}$ (resp. $M^{(1)}$) is conjugate to the diagonal matrix with eigenvalues 1 and $e^{2\pi\sqrt{-1}\gamma}$ (resp. 1 and $e^{2\pi\sqrt{-1}\delta}$). Since the exponents about $z=\infty$ are α, β and we have the relation $M^{(0)}M^{(1)}=(M^{(\infty)})^{-1}$, the matrix $M^{(0)}M^{(1)}$ has an eigenvalue 1. Then the matrices $M^{(0)}$ and $M^{(1)}$ have an common invariant one-dimensional subspace, because if we set

$$M^{(0)}=\begin{pmatrix}1&0\\0&g\end{pmatrix},\ M^{(1)}=P\begin{pmatrix}1&0\\0&d\end{pmatrix}P^{-1},\ P=\begin{pmatrix}p&q\\r&s\end{pmatrix},\tag{5.1}$$

the condition that $M^{(0)}M^{(1)}$ has an eigenvalue 1 is written as $0=1-\mathrm{tr}(M^{(0)}M^{(1)})+\det(M^{(0)}M^{(1)})=qr(d-1)(g-1)/(qr-ps)$, and we have an common one-dimensional eigenspace for the case $q=0$, $r=0$, $d=1$ or $g=1$ respectively. In the case $\gamma\in\mathbb{Z}$ (resp. $\delta\in\mathbb{Z}$), the matrix $M^{(0)}$ (resp. $M^{(1)}$) has the multiple eigenvalue 1, and we can also show that the matrices $M^{(0)}$ and $M^{(1)}$ have an common invariant one-dimensional subspace by expressing the matrices in the form of Jordan normal forms. Hence the monodromy representation of solutions of Eq.(3.4) is reducible. $\square$

Remark that Lemma 5.1 is also a consequence of the multiplicative Deligne-Simpson problem for a special case ([4]).

Proposition 5.2. *Assume that $\alpha\in\mathbb{Z}$, $\epsilon\in\mathbb{Z}_{\leq 0}$ and the singularity $z=t$ is apparent. Set $n=-\epsilon(\in\mathbb{Z}_{\geq 0})$.*
(i) If $\alpha>0$, then there exists a non-zero solution $y(z)$ such that $y(z)=z^{1-\gamma}(z-1)^{1-\delta}h(z)$ and $h(z)$ is a polynomial of degree no more than $\alpha+n-1$.
(ii) If $\alpha<1-n$, then there exists a non-zero solution $y(z)$ such that $y(z)$ is a polynomial of degree no more than $-\alpha$.
(iii) If $1-n\leq\alpha\leq 0$, then there exists a non-zero solution $y(z)$ such that $y(z)$ is a polynomial of degree $-\alpha$ or there exists a non-zero solution $y(z)$ such that $y(z)=z^{1-\gamma}(z-1)^{1-\delta}h(z)$ and $h(z)$ is a polynomial of degree no more than $\alpha+n-1$.

Proof. Assume that $\beta\notin\mathbb{Z}$, $\gamma\notin\mathbb{Z}$ and $\beta-\gamma\notin\mathbb{Z}$ for the moment. It follows from reducibility of monodromy that there exists a non-zero solution $y(z)$ of Eq.(3.4) such that $y(z)=z^{\theta_0}(z-1)^{\theta_1}(z-t)^{\theta_t}h(z)$ such that $h(z)$ is a polynomial, $h(0)h(1)h(t)\neq 0$, $\theta_0\in\{0,1-\gamma\}$, $\theta_1\in\{0,1-\delta\}$, $\theta_t\in\{0,1+n\}$, and $\alpha=-\deg h(z)-\theta_0-\theta_1-\theta_t$ or $\beta=-\deg h(z)-\theta_0-\theta_1-\theta_t$ (see [14, Proposition 3.1]). Since $n\in\mathbb{Z}_{\geq 0}$ and $(z-t)^n$ is a polynomial in z, we have a non-zero solution $y(z)$ of Eq.(3.4) such that $y(z)=z^{\theta_0}(z-1)^{\theta_1}h(z)$ such that $h(z)$ is a polynomial, $h(0)h(1)\neq 0$, $\theta_0\in\{0,1-\gamma\}$,

$\theta_1 \in \{0, 1-\delta\}$, and $\alpha = -\deg h(z) - \theta_0 - \theta_1$ or $\beta = -\deg h(z) - \theta_0 - \theta_1$. Because $\deg h(z)$ is a non-negative integer, the possible cases under the consition $\alpha \in \mathbb{Z}$, $\beta \notin \mathbb{Z}$, $\gamma \notin \mathbb{Z}$ and $\beta - \gamma = \delta - n - 1 - \alpha \notin \mathbb{Z}$ are the cases $\deg h(z) = -\alpha \in \mathbb{Z}_{\geq 0}$ ($\alpha \leq 0$, $(\theta_0, \theta_1) = (0,0)$) and $\deg h(z) = \alpha + n - 1 \in \mathbb{Z}_{\geq 0}$ ($\alpha \geq 1-n$, $(\theta_0, \theta_1) = (1-\gamma, 1-\delta)$). Hence we have the proposition under the condition $\alpha \in \mathbb{Z}$, $\beta \notin \mathbb{Z}$, $\gamma \notin \mathbb{Z}$ and $\beta - \gamma = \delta - n - 1 - \alpha \notin \mathbb{Z}$.

Since the monic characteristic polynomial in q for existence of polynomial-type solutions $y(z) = z^{\theta_0}(z-1)^{\theta_1} h(z)$ ($h(z)$: a polynomial, $(\theta_0, \theta_1) = (0,0)$, $(1-\gamma, 1-\delta)$) is continuos with respect to the parameters β and γ, we obtain the proposition for all β and γ by continuity argument. □

Theorem 5.3. *Assume that $\epsilon \in \mathbb{Z}_{\leq 0}$, $\alpha \in \mathbb{Z}_{\leq 0}$ and $\beta \notin \mathbb{Z}_{\leq 0}$.*
(i) If $-\alpha \leq -\epsilon$ and Heun's differential equation (Eq.(3.4)) has a polynomial solution (i.e. the accessory parameter q satisfies $P^{\mathsf{pol}}(q) = 0$), then the singularity $z = t$ is apparent (i.e. $P^{\mathsf{app}}(q) = 0$).
(ii) If $-\epsilon \leq -\alpha$ and the singularity $z = t$ is apparent (i.e. $P^{\mathsf{app}}(q) = 0$), then Eq.(3.4) has a polynomial solution (i.e. $P^{\mathsf{pol}}(q) = 0$).

Proof. (ii) follows from Proposition 5.2 (ii).

We show (i). If $\epsilon \in \mathbb{Z}_{\leq 0}$, then a basis of local solutions about $z = t$ is written as

$$f(z) = (z-t)^{1-\epsilon} \sum_{j=0}^{\infty} c_j (z-t)^j, \quad g(z) = \sum_{j=0}^{\infty} \tilde{c}_j (z-t)^j + A f(z) \log(z-t). \tag{5.2}$$

Apparency of the singularity $z = t$ is described as the condition $A = 0$. If there exists a polynomial solution $y = p(z)$ of Eq.(3.4), then $\deg_z p(z) = -\alpha \leq -\epsilon$. Since the expansion of $f(z)$ starts from $1-\epsilon$, the solution $p(z)$ is proportional to $g(z)$. Hence $A = 0$ and we obtain (i). □

Proposition 5.2 and Theorem 5.3 are also valid for the case the singularity $z = 0$ or $z = 1$ is apparent.

6. X_1 Jacobi polynomial

We now review a definition of X_1-Jacobi polynomials and their properties ([3, 9, 11]). Let $P_k(\eta)$ be the Jacobi polynomial parametrized as

$$P_k(\eta) = \frac{(g+\frac{1}{2})_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j (k+g+h+2)_j}{j!(g+\frac{1}{2})_j} \left(\frac{1-\eta}{2}\right)^j. \tag{6.1}$$

The X_1-Jacobi polynomials $\hat{P}_k(\eta)$ ($k = 0, 1, 2, \dots$) are defined in the case $g, h \notin \{-1/2, -3/2, -5/2, \dots\}$ by

$$\hat{P}_k(\eta) = \frac{1}{k+h+\frac{1}{2}} \left((h+\tfrac{1}{2}) \tilde{\xi}(\eta) P_k(\eta) + (1+\eta)\xi(\eta) \frac{d}{d\eta} P_k(\eta) \right), \tag{6.2}$$
$$\xi(\eta) = \frac{g-h}{2}\eta + \frac{g+h+1}{2}, \quad \tilde{\xi}(\eta) = \frac{g-h}{2}\eta + \frac{g+h+3}{2}.$$

Hence $\deg_\eta \hat{P}_k(\eta) = k+1$. The X_1-Jacobi polynomials in the case $g, h > -1/2$ are orthogonal with respect to the following inner product;

$$\int_{-1}^{1} \hat{P}_k(\eta)\hat{P}_{k'}(\eta)\mathcal{W}(\eta)d\eta = C_k\delta_{k,k'}, \quad \mathcal{W}(\eta) = \frac{(1-\eta)^{g+\frac{1}{2}}(1+\eta)^{h+\frac{1}{2}}}{2^{g+h+2}\xi(\eta)^2}, \tag{6.3}$$

where C_k is a non-zero constant. The X_1-Jacobi polynomial $\hat{P}_k(\eta)$ satisfies the following differential equation;

$$\begin{aligned}(1-\eta^2)\frac{d^2}{d\eta^2}\hat{P}_k(\eta) + \left(h - g - (g+h+3)\eta - 2\frac{(1-\eta^2)\xi'(\eta)}{\xi(\eta)}\right)\frac{d}{d\eta}\hat{P}_k(\eta) \\ + \left(-\frac{2(h+\frac{1}{2})(1-\eta)\tilde{\xi}'(\eta)}{\xi(\eta)} + k(k+g+h+2) + g - h\right)\hat{P}_k(\eta) = 0.\end{aligned} \tag{6.4}$$

By setting $\eta = 1-2z$ and $y = \hat{P}_k(\eta)$, we obtain a specific case of Heun's differential equation whose parameters are given by

(6.5)

$$\begin{aligned}&\alpha = -k-1, \; \beta = k+g+h+1, \; \gamma = g+3/2, \; \delta = \alpha+\beta-\gamma+3 = h+3/2, \\ &\epsilon = -2, \; t = \frac{1-\gamma}{\alpha+\beta-2\gamma+3} = \frac{g+1/2}{g-h}, \\ &q = \frac{(1-\gamma)(\alpha\beta+2\alpha+2\beta-2\gamma+4)}{\alpha+\beta-2\gamma+3} = \frac{(g+1/2)}{h-g}\{k^2+(g+h+2)k+g-h\}.\end{aligned}$$

The condition that the singularity $z=t$ of the differential equation

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\alpha+\beta-\gamma+3}{z-1} - \frac{2}{z-t}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}y = 0, \tag{6.6}$$

is apparent is written as Eq.(2.7). By substituting $t = (1-\gamma)/(\alpha+\beta-2\gamma+3)$ into Eq.(2.7), we have the factorization

$$\begin{aligned}&\left(q + \frac{(\gamma-1)(\alpha\beta+2\alpha+2\beta-2\gamma+4)}{\alpha+\beta-2\gamma+3}\right)\cdot \\ &\left(q^2 - \frac{4\gamma^2-(2\alpha\beta+4\alpha+4\beta+12)\gamma+(2\alpha\beta+5\alpha+5\beta+9)}{\alpha+\beta-2\gamma+3}q \right.\\ &\left. \quad - \frac{\alpha\beta(\gamma-1)(4\gamma^2-(\alpha\beta+4\alpha+4\beta+8)\gamma+(\alpha+1)(\beta+1)}{(\alpha+\beta-2\gamma+3)^2}\right) = 0.\end{aligned} \tag{6.7}$$

Hence the singularity $z = t = (1-\gamma)/(\alpha+\beta-2\gamma+3)$ is apparent with respect to the second order differential equation which X_1-Jacobi polynomial $\hat{P}_k(1-2z)$ satisfies.

Next we investigate the condition that Eq.(6.6) has a non-zero polynomial solution under the assumption that the singularity $z=t$ is apparent (see Eq.(2.7)). If $\alpha = -k-1$, $\beta \neq 0$ and $k \in \mathbb{Z}_{\geq 1}$, then it follows from Theorem 5.3 that the differential equation has a non-zero polynomial solution. If $\alpha = -1$ ($k=0$), then the condition that the differential equation has a non-zero polynomial solution is written as Eq.(2.9) and we have

$$(q-1+\gamma)\left(q-\frac{\beta\gamma}{\beta-2\gamma+2}\right)=0. \tag{6.8}$$

by substituting $t=(1-\gamma)/(-1+\beta-2\gamma+3)$ and $\epsilon=-2$. On the other hand we have $q=1-\gamma$ by Eq.(6.5) in the case $k=0$ ($\alpha=-1$). Hence we confirm that there exists a non-zero polynomial which corresponds to the X_1-Jacobi polynomial in the case $k=0$. If $\alpha=0$, then the condition that the differential equation has a non-zero polynomial solution is written as $q=0$ and a solution is constant, and it does not agree with Eq.(6.5), i.e. $q=2(1-\gamma)(\beta-\gamma+2)/(\beta-2\gamma+3)$. Hence the constant does not belong to parameters of Heun's differential equation concerning to X_1-Jacobi polynomials. It follows from apparency of the singularity $z=t=(1-\gamma)/(\beta-2\gamma+3)$ and Proposition 5.2 that there exists a non-zero solution $y(z)$ of Heun's differential equation with the parameters in Eq.(6.5) such that $y(z)=z^{1-\gamma}(z-1)^{1-\delta}h(z)$, $\deg h(z)=1$ and the polynomial $h(z)$ is calculated as $h(z)=(\beta-2\gamma+3)z+\gamma-2$.

It follows from Theorem 4.2 and apparency of the singularity $z=t$ that the polynomial $\hat{P}_k(1-2z)$ also satisfies the generalized hypergeometric differential equation $L_{-k-1,k+g+h+1,e_1+1,e_2+1;g+3/2,e_1,e_2}y=0$, where

$$\begin{aligned} &e_1+e_2=2\gamma-3=2g, \\ &e_1e_2=\frac{\alpha\beta(\gamma-1)}{-\gamma+2+\alpha+\beta}=\frac{-(k+1)(k+g+h+1)(2g+1)}{2h+1}. \end{aligned} \tag{6.9}$$

Thus we have the following proposition;

Theorem 6.1. *The X_1-Jacobi polynomials are expressed in terms of generalized hypergeometric functions,*

$$\hat{P}_k(\eta)=D_k\cdot {}_4F_3\left(\begin{matrix}-k-1,k+g+h+1,e_1+1,e_2+1\\ g+3/2,e_1,e_2\end{matrix};\frac{1-\eta}{2}\right), \tag{6.10}$$

$$e_1+e_2=2g,\quad e_1e_2=\frac{-(k+1)(k+g+h+1)(2g+1)}{2h+1}, \tag{6.11}$$

where D_k is a non-zero constant.

Proof. Let $\hat{Q}_k(\eta)$ be the generalized hypergeometric function defined by the right hand side of Eq.(6.10). Then the functions $\hat{Q}_k(1-2z)$ and $\hat{P}_k(1-2z)$ are holomorphic solutions of the generalized hypergeometric differential equation

$$L_{-k-1,k+g+h+1,e_1+1,e_2+1;g+3/2,e_1,e_2}y=0 \tag{6.12}$$

about $z=0$, where e_1 and e_2 are given by Eq.(6.11). The exponents of the differential equation about $z=0$ are 0, $-g-1/2$, $1-e_1$ and $1-e_2$. If $g+1/2,e_1,e_2\notin\mathbb{Z}$, then the dimension of holomorphic solutions of the differential equation is one and the function $\hat{P}_k(1-2z)$ is proportional to $\hat{Q}_k(1-2z)$. By continuity argument, the function $\hat{P}_k(1-2z)$ is proportional to $\hat{Q}_k(1-2z)$ in the case $g,h\notin\{-1/2,-3/2,-5/2,\dots\}$, the case that the functions are well-defined. □

7. Concluding remarks

It is known that two types of X_ℓ-Jacobi polynomials ($\ell = 1, 2, \dots$) satisfies a second-order Fuchsian differential equation which satisfies the assumption of Conjecture 1 by setting $\eta = 1 - 2z$ (see ([9, 11]) etc.). Thus relationships between X_ℓ-Jacobi polynomials and generalized hypergeometric polynomials should be studied further. On the other hand, several researchers including the authors in ([1, 2, 6]) studied generalized Jacobi polynomials. It would be interesting to consider relationship among those polynomials.

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Department of Mathematics, Faculty of Science and Technology, Chuo University, 1-13-27 Kasuga, Bunkyo-ku Tokyo 112-8551, Japan.

E-mail address: `takemura@math.chuo-u.ac.jp`