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Ward-Takahashi identities for the colored Boulatov model

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Ward-Takahashi identities of the colored Boulatov model are derived using a generic unitary field transformation. In a specific instance, this generic transformation turns out to be a symmetry of the interaction so that particular classes of reduced Ward-Takahashi identities for that symmetry are consequently identified.

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I. INTRODUCTION

Group field theories (GFTs) are generalization of matrix models as higher rank tensor quantum field theories over a group manifold [1]. They provide a relevant framework for the quantization of gravity [2–6] as well as possess, as recently shown for the particular class of colored GFT models [7–9], an equivalent formulation of a large $1/N$ limit expansion [10–12] and exhibit a critical behavior [13, 14].

It should be emphasized that the presence of colored fields in GFT plays an increasing role in the search of symmetry of these theories. Indeed, some investigations pertaining to the symmetry aspects of such colored GFTs have been led recently [15–18] and one proved that, to mention a few, the colored theories are endowed with a genuine quantum group symmetry [15] which encodes a notion of diffeomorphism symmetry in GFTs [17].

Let us remind that the partition function of GFT models in the sense of Boulatov-Ooguri [1] are defined through an interaction and a Gaussian measure with a degenerate covariance. This covariance is indeed made of a group averaging and product of delta functions which in fact projects onto the gauge invariant sector of the space of square integrable functions $L^2(G^D,\mathbb{C})$, for a D dimensional GFT. Hence the kinetic term in the Lagrange formulation of the action, can be seen as trivial (this term is of the mass kind when restricting field on the gauge invariant sector) or even inexistent from the quantum field theory point of view. This peculiar feature, in returns, renders unclear the ordinary definition of classical symmetry and the corresponding notion of Noether theorem (the notion of Noether currents may only reduce to the Lagrangian density itself for translations for instance). At the current stage of investigations on symmetries of GFTs, there are three ways to address this issue: either to deal with a quantum group symmetry and making use of Hopf algebra techniques [15], or to introduce a nontrivial kinetic term (motivated indeed by renormalization requirements in [25] [4]), or, finally, to state directly the Noether theorem for a given symmetry at the quantum level. The latter is well-known to be related with the identification of Ward-Takahashi (WT) identities associated with a particular field symmetry. To shed more light on the last aspect is the purpose of the present work.

In this paper, we study the WT identities for the colored Boulatov GFT model using a generic unitary field transformation. This general unitary field transformation turns out to be a symmetry for the Boulatov action provided a specific way that one chooses to act on the field arguments. Associated with that symmetry, reduced WT equations satisfied by the correlations functions are determined. These various kind of identities should be useful for in-depth perturbative and nonperturbative renormalization programs of GFTs [18–25]. For example, in quantum electrodynamics the WT identities relate the full three-point function with the two-point function hence the wave function renormalization with the vertex renormalization. In the context of noncommutative quantum field theory, they play a crucial role in the proof of asymptotic safety at all orders of perturbation theory [26, 27].

The paper is organized as follows: Section 2 reviews the basics of the Boulatov model. Section 3 introduces the unitary field transformation that will be used in order to define the variation of the different parts entering in the definition of the partition function. Mainly, one can define a general unitary transformation and one more specific unitary with the property that it preserves the gauge invariance of fields and is a symmetry of the interaction term.

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For the general and the more specific symmetry the study of WT identities are discussed in Section 4. A summary of the results is given in Section 5 and, finally, a detailed appendix provides basics facts on the theory, on unitary field transformations and other precisions on derivations used in the text.

II. THE COLORED BOULATOV MODEL

Let G be some compact multiplicative Lie group, and denote h its elements, e its unit, and $\int dh$ the integral with respect to the Haar measure. Let $\bar{\varphi}^i, \varphi^i$, $i = 0, 1, 2, 3$ be four couples of complex scalar fields over three copies of G, $\varphi^i : G \times G \times G \to \mathbb{C}$. We denote $\delta^N(h)$ the delta function over G with some cutoff such that $\delta^N(e)$ is finite, but diverges (polynomially) when N goes to infinity. For $G = SU(2)$ (denoting $\chi^j(h)$ the character of h in the representation j) respectively $G = U(1)$ we can choose

$$\delta^N(h)\Big|_{G=SU(2)} = \sum_{j=0}^{N} d_j\, \chi^j(h)\;, \qquad \delta^N(\varphi)\Big|_{G=U(1)} = \sum_{p=-N}^{N} e^{\imath p \varphi}\;, \tag{1}$$

where $d_j = 2j + 1$.

The partition function of the colored Boulatov model [7] over G is the path integral

$$Z(\lambda, \bar{\lambda}) = e^{-F(\lambda,\bar{\lambda})} = \int \prod_{i=0}^{4} d\mu_C(\bar{\varphi}^i, \varphi^i)\; e^{-S^{\rm int}(\bar{\varphi}^i,\varphi^i)}\;, \tag{2}$$

with normalized Gaussian measure of covariance C (denoting $\varphi_{hpq} := \varphi(h, p, q)$ and the Haar measure over a group variable h as $\int_h := \int dh$):

$$C^{kj}_{h_0h_1h_2;\, h'_0h'_1h'_2} = \int d\mu_C(\bar{\varphi}^i, \varphi^i)\; \bar{\varphi}^k_{h_0h_1h_2} \varphi^j_{h'_0h'_1h'_2} = \delta^{kj} \int_h \delta^N\big(h_0 h (h'_0)^{-1}\big) \delta^N\big(h_1 h (h'_1)^{-1}\big) \delta^N\big(h_2 h (h'_2)^{-1}\big)\;, \tag{3}$$

and interaction wisely chosen

$$\begin{aligned} S^{\rm int} &= \frac{\lambda}{\sqrt{\delta^N(e)}} \int_{h_{ij}} \varphi^0_{h_{03}h_{02}h_{01}} \bar{\varphi}^1_{h_{10}h_{13}h_{12}} \varphi^2_{h_{21}h_{20}h_{23}} \bar{\varphi}^3_{h_{32}h_{31}h_{30}} \\ &+ \frac{\bar{\lambda}}{\sqrt{\delta^N(e)}} \int_{h_{ij}} \bar{\varphi}^0_{h_{03}h_{02}h_{01}} \varphi^1_{h_{10}h_{13}h_{12}} \bar{\varphi}^2_{h_{21}h_{20}h_{23}} \varphi^3_{h_{32}h_{31}h_{30}}\;, \end{aligned} \tag{4}$$

where one should identify $h_{ij} = h_{ji}$ and the symbol $\int_{h_{ij}}$ means that the integral is performed over all variables labelled by h_{ij} (here above six copies of G). The resulting pairing of field arguments of this nonlocal interaction can be dually and graphically associated with a tetrahedron (each argument for each edge) and Feyman graphs, in this theory, are noting but a collection of tetrahedra (simplicial complexes) glued along one of their faces (triangle) following the covariance rule. More precisions on the GFT diagrammatics can be found in [3]. For a colored theory, there is an additional gluing constraint enforcing that only colors of the same index can be glued together (hence the presence of δ^{jk} in the covariance). One also notes that, in the ordinary colored GFT [7], the interaction with coupling constant λ is of the form $\varphi^0\varphi^1\varphi^2\varphi^3$ and the one with coupling $\bar{\lambda}$ can be chosen as its complex conjugate. Here, we have just renamed $\varphi^{1,3}$ as $\bar{\varphi}^{1,3}$ (and vice-versa) in order to have a correct notion of field transformation with respect to colors. Hence, in the following, the formalism that we will develop holds without loss of generality in the ordinary colored GFT, with another field transformation.

As an operator, C can act also onto functions. We will use indifferently the compact notations, when no confusion may occur

$$\begin{aligned} [C\varphi]^i_{[g]} &= C^{ij}_{[g][\tilde{g}]}\, \varphi^j_{[\tilde{g}]} = \int_{[\tilde{g}]} C^{ij}_{[g][\tilde{g}]}\, \varphi^j_{[\tilde{g}]} := \int_{\tilde{g}_i} C^{ij}_{g_1g_2g_3;\, \tilde{g}_1\tilde{g}_2\tilde{g}_3} \varphi^j_{\tilde{g}_1\tilde{g}_2\tilde{g}_3}\;, \\ \bar{\varphi}C\varphi &= \bar{\varphi}^j_{[g]} [C\varphi]^j_{[g]} = \int_{[g]} \bar{\varphi}^j_{[g]} [C\varphi]^j_{[g]} := \int_{g_i\tilde{g}_i} \bar{\varphi}^j_{g_1g_2g_3} C^{ji}_{g_1g_2g_3;\, \tilde{g}_1\tilde{g}_2\tilde{g}_3} \varphi^i_{\tilde{g}_1\tilde{g}_2\tilde{g}_3}\;. \end{aligned} \tag{5}$$

III. UNITARY TRANSFORMATIONS

A. General unitary transformations

From now on, we will restrict to the situation where $G = SU(2)$ though most of the ensuing developments may found an extension for an arbitrary Lie group.

Consider the partition function Z including source terms and the associated free energy W for the Boulatov model, namely

$$e^{-W(\bar\eta,\eta)} = Z(\bar\eta,\eta) = \int d\mu_C(\bar\varphi,\varphi) e^{-S^{\rm int}\,(\varphi,\bar\varphi)+\bar\varphi\eta+\varphi\bar\eta} \;, \tag{6}$$

where notations remain the same as earlier and the source term incorporates color indices:

$$\bar\varphi\eta+\varphi\bar\eta := \sum_i \int_{g_i} \left(\bar\varphi^i_{g_1g_2g_3}\eta^i_{g_1g_2g_3} + \bar\eta^i_{g_1g_2g_3}\varphi^i_{g_1g_2g_3}\right) . \tag{7}$$

In order to avoid confusion, we will use different notation for a single index a and a triple index $[a]=(a_1a_2,a_3)$. Hence the composition of the covariance and a field will be written henceforth $C_{[a][b]}\varphi_{[b]}$ and an operator T acting onto a single group element of fields will be denoted as $T_{ab}\varphi_b$.

Let us consider now a general unitary operator U satisfying a composition law $U^{ij}_{[a][b]}\bar U^{ij'}_{[c][b]} = \delta^{jj'}\delta_{[a][c]}$, with $\delta_{[a][c]}$ the kernel of the unit operator, such that the fields transform under U as

$$^U\varphi^i_{[a]} = U^{ij}_{[a][b]}\varphi^j_{[b]} \;, \qquad ^U\bar\varphi^i_{[a]} = \bar\varphi^j_{[b]}\bar U^{ij}_{[a][b]} \;, \qquad \frac{\delta}{\delta\, ^U\varphi^i_{[a]}} = \frac{\delta}{\delta\varphi^j_{[b]}}(U^{-1})^{ji}_{[b][a]} \;, \qquad \frac{\delta}{\delta\, ^U\bar\varphi^i_{[a]}} = U^{ij}_{[a][b]}\frac{\delta}{\delta\bar\varphi^j_{[b]}} \tag{8}$$

(one can prove that such operators exist, see Appendices B and B 2). Let us keep at the moment these as formal expressions.

Under U, $S^{\rm int}$ transforms as

$$\begin{aligned} ^U S^{\rm int} &= \frac{\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}} {}^U\varphi^0_{h_{03}h_{02}h_{01}}\; {}^U\bar\varphi^1_{h_{10}h_{13}h_{12}}\; {}^U\varphi^2_{h_{21}h_{20}h_{23}}\; {}^U\bar\varphi^3_{h_{32}h_{31}h_{30}} \\ &\quad + \frac{\bar\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}} {}^U\bar\varphi^0_{h^{03}h^{02}h^{01}}\; {}^U\varphi^1_{h^{10}h^{13}h^{12}}\; {}^U\bar\varphi^2_{h^{21}h^{20}h^{23}}\; {}^U\varphi^3_{h^{32}h^{31}h^{30}} \;, \\ &= \frac{\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\int_{[a][b][c][d]} U^{0i_0}_{h_{03}h_{02}h_{01}\,[a]}\varphi^{i_0}_{[a]}\;\bar\varphi^{i_1}_{[b]}\bar U^{1i_1}_{h_{10}h_{13}h_{12}\;[b]}\;U^{2i_2}_{h_{21}h_{20}h_{23}\,[c]}\;\varphi^{i_2}_{[c]}\;\bar\varphi^{i_3}_{[d]}\bar U^{3i_3}_{h_{32}h_{31}h_{30}\,[d]} \\ &\quad + \frac{\bar\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\int_{[a][b][c][d]} \bar\varphi^{i_0}_{[a]}\bar U^{0i_0}_{h^{03}h^{02}h^{01}[a]}\varphi^{i_1}_{[b]}U^{1i_1}_{h^{10}h^{13}h^{12}[b]}\;\bar\varphi^{i_2}_{[c]}\bar U^{2i_2}_{h^{21}h^{20}h^{23}\,[c]}\;U^{3i_3}_{h^{32}h^{31}h^{30}\,[d]}\varphi^{i_3}_{[d]} \;. \end{aligned} \tag{9}$$

The partition function under (8) can be mapped onto (see Eq.(C.6) of Appendix C 1)

$$Z(\bar\eta,\eta) = \int d\mu_{UCU^{-1}}(\bar\varphi,\varphi)\; e^{-S^{\rm int}\,(U\varphi,\bar\varphi U^{-1})+\bar\varphi U^{-1}\eta+\bar\eta U\varphi} \;. \tag{10}$$

By Lemma 3 (see Appendix A), we write this modified partition function as

$$Z(\bar\eta,\eta) = \int d\mu_C(\bar\varphi,\varphi)\; e^{\delta_\varphi(UCU^{-1}-C)\delta_{\bar\varphi}}\; e^{-S^{\rm int}\,(U\varphi,\bar\varphi U^{-1})+\bar\varphi U^{-1}\eta+\bar\eta U\varphi} \;. \tag{11}$$

Close to the identity, we can set $U^{ij}_{[a][b]} = \delta^{ij}\delta_{[a][b]} + \imath B^{ij}_{[a][b]}$, where B is an Hermitian kernel in the sense that $\bar B^{ji}_{[b][a]} = B^{ij}_{[a][b]}$. At first order in B, the variation of the interaction part is of the form (see Appendix C 1 for precisions concerning the notations and derivations of the following infinitesimal variations)

$$\begin{aligned} &\delta_B S^{\rm int} := \imath\lambda\Big[[B\varphi]^0\bar\varphi^1\varphi^2\bar\varphi^3 - \varphi^0[\bar\varphi B]^1\varphi^2\bar\varphi^3 + \varphi^0\bar\varphi^1[B\varphi]^2\bar\varphi^3 - \varphi^0\bar\varphi^1\varphi^2[\bar\varphi B]^3\Big] \;+\; \imath\bar\lambda\{\bar\varphi\} \;, \\ &\bar\lambda\{\bar\varphi\} = -[\bar\varphi^0 B]\varphi^1\bar\varphi^2\varphi^3 + \bar\varphi^0[B\varphi]^1\bar\varphi^2\varphi^3 - \bar\varphi^0\varphi^1[\bar\varphi B]^2\varphi^3 + \bar\varphi^0\varphi^1\bar\varphi^2[B\varphi]^3 \;. \end{aligned} \tag{12}$$

Meanwhile, the source terms have the infinitesimally transformations

$$\delta_B(\bar\eta\varphi+\bar\varphi\eta) = \imath\sum_i\int_{[g][a]}\Big(-\bar\varphi^i_{[a]}B^{ij}_{[a][g]}\eta^j_{[g]} + \bar\eta^i_{[g]}B^{ij}_{[g][a]}\varphi^j_{[a]}\Big) =: \imath(-\bar\varphi B\eta + \bar\eta B\varphi) \;, \tag{13}$$

and varying the covariance, we get

$$[UCU^{-1}]^{ij}_{[a][b]} - C^{ij}_{[a][b]} = \imath[B^{li}_{[c][a]}C^{lj}_{[c][b]} - C^{il'}_{[a][c']}B^{jl'}_{[b][c']}] =: \imath[BC-CB]^{ij}_{[a][b]} \;. \tag{14}$$

B. Right invariant unitary transformations

Among unitary operators, there exists a particular class that we propose also to study. This class includes unitary operators called *right invariant unitaries* for having the particular property to preserve the gauge invariance of fields[1] and let $S^{\rm int}$ invariant. A detailed discussion about these right invariant unitaries can be found in Appendix B 1 .

Working with a right invariant unitary U in the sense of the 1-action of Appendix B 2 Eq. (B.19), we get a change of field variables such that

$$
\begin{aligned}
&{}^U\varphi^0_{a.} = U_{ab}\varphi^0_{b.} \ , \qquad {}^U\bar\varphi^0_{a.} = \bar\varphi^0_{b.}(U^{-1})_{ba} = \bar U_{ab}\bar\varphi^0_{b.} \ , \qquad \frac{\delta}{\delta\, {}^U\varphi^0_{a.}} = \frac{\delta}{\delta\varphi^0_{b.}}(U^{-1})_{ba} \ , \qquad \frac{\delta}{\delta\, {}^U\bar\varphi^0_{a.}} = U_{ab}\frac{\delta}{\delta\bar\varphi^0_{b.}} \ , \\
&{}^U\varphi^1_{.a} = U_{ab}\varphi^1_{.b} \ , \qquad {}^U\bar\varphi^1_{.a} = \bar\varphi^1_{.b}(U^{-1})_{ba} = \bar U_{ab}\bar\varphi^1_{.b} \ , \qquad \frac{\delta}{\delta\, {}^U\varphi^1_{.a}} = \frac{\delta}{\delta\varphi^1_{.b}}(U^{-1})_{ba} \ , \qquad \frac{\delta}{\delta\, {}^U\bar\varphi^1_{.a}} = U_{ab}\frac{\delta}{\delta\bar\varphi^1_{.b}} \ , \qquad (15)
\end{aligned}
$$

and colors 2 and 3 transform like 0 and 1, respectively. The subscript a or b refers to a unique group element and the dot notifies the position of the remaining arguments. Thus 0 and 2 are transformed with respect to their first argument whereas 1 and 3 to their last argument.

$S^{\rm int}$ remains invariant under this transformation (see Appendix C 2 for details of the identities in the remaining of this section)

$$
{}^U S^{\rm int} = S^{\rm int} \ . \qquad (16)
$$

In fact, for the colored Boulatov model there are six such right invariant unitaries, namely one for each couple of arguments in the interaction, leaving the colored GFT interaction invariant. More generally, a D dimensional colored GFT will be invariant under $D(D+1)/2$ of such basic transformations that one can think as *minimal* symmetries. In the sequel, we will use one of these minimal symmetry in order to simplify some general WT identities and, consequently, to prove that the WT identities derived in this work have a non-trivial content.

Under (15), the partition function takes the form

$$
Z(\bar\eta,\eta) = \int d\mu_{U^{-1}CU}(\bar\varphi,\varphi)e^{-S^{\rm int}\,(U\varphi,\bar\varphi U^{-1})+\bar\varphi U^{-1}\eta+\bar\eta U\varphi} \ , \qquad (17)
$$

where the action of U on the covariance is defined by

$$
\begin{aligned}
&\bar U_{h'_0c'}C^{ii=0,2}_{h_0h_1h_2;\,h'_0h'_1h'_2}U_{h_0c} = \int_{abh_0h'_0} U_{h_0c}\bar U_{h_0a}\bar U_{h'_0c'}U_{h'_0b}\,\bar\varphi^{i=0,2}_{ah_1h_2}\,\varphi^{i=0,2}_{bh'_1h'_2} := [UCU^{-1}]^{ii=0,2}_{ch_1h_2;\,c'h'_1h'_2} \ , \\
&\bar U_{h'_2c'}C^{ii=1,3}_{h_0h_1h_2;\,h'_0h'_1h'_2}U_{h_2c} = \int_{abh_2h'_2} U_{h_2c}\bar U_{h_2a}\bar U_{h'_2c'}U_{h'_2b}\bar\varphi^{i=1,3}_{h_0h_1a}\varphi^{i=1,3}_{h'_0h'_1b} := [UCU^{-1}]^{ii=1,3}_{h_0h_1c;\,h'_0h'_1c'} \ . \qquad (18)
\end{aligned}
$$

Turning the discussion to infinitesimal transformations, we have $\delta_B S^{\rm int} = 0$,

$$
\begin{aligned}
&[UCU^{-1}]^{ii=0,2}_{[a][b]} - C^{ii=0,2}_{[a][b]} = \imath[-C^{ii=0,2}_{[a];\,c'b_1b_2}B_{b_0c'} + B_{ca_0}C^{ii=0,2}_{ca_1a_2;\,[b]}] =: \imath[BC-CB]^{ii=0,2}_{[a][b]} \ , \\
&[UCU^{-1}]^{ii=1,3}_{[a][b]} - C_{[a][b]} = \imath[-C^{ii=0,2}_{[a];\,b_0b_1c'}B_{b_2c'} + B_{ca_2}C^{ii=0,2}_{a_0a_1c;\,[b]}] =: \imath[BC-CB]^{ii=1,3}_{[a][b]} \ , \qquad (19)
\end{aligned}
$$

whereas varying source terms yields

$$
\begin{aligned}
\delta_B(\bar\eta\varphi+\bar\varphi\eta) &= \sum_i\int_{[g]}\Big({}^U\bar\varphi^i_{[g]}\eta^i_{[g]} + \bar\eta^i_{[g]}\ {}^U\varphi^i_{[g]}\Big) - (\bar\eta\varphi+\bar\varphi\eta) \\
&= \imath\int_{g_ia}\Big\{\sum_{i=0,2}\big(-\bar\varphi^i_{a.}B_{ag_0}\eta^i_{g_0.} + \bar\eta^i_{g_0.}B_{g_0a}\varphi^i_{a.}\big) + \sum_{i=1,3}\big(-\bar\varphi^i_{.a}B_{ag_2}\eta^i_{.g_2} + \bar\eta^i_{.g_2}B_{g_2a}\varphi^i_{.a}\big)\Big\} \\
&=: \imath(-\bar\varphi B\eta + \bar\eta B\varphi) \ . \qquad (20)
\end{aligned}
$$

Having collected all infinitesimal terms, we are in position to study the WT identities of the model and that will be the focus of the rest of this work.

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[1] GFTs can be indeed defined with gauge invariant fields. Hence, the mentioned transformation will preserve this property of fields.

IV. WARD-TAKAHASHI IDENTITIES FOR THE COLORED BOULATOV MODEL

A. General unitary transformation

Our starting point is the partition function (11). Considering the infinitesimal transformations (12), (13) and (14), Z may be written

$$
\begin{aligned}
Z(\bar\eta,\eta) &= \int d\mu_C(\bar\varphi,\varphi)\big(1+\imath\delta_\varphi[BC-CB]\delta_{\bar\varphi}\big)e^{-S^{\rm int}(\varphi,\bar\varphi)-\delta_B S^{\rm int}+\bar\varphi\eta+\bar\eta\varphi+\delta_B(\bar\varphi\eta+\bar\eta\varphi)}\\
&= \int d\mu_C(\bar\varphi,\varphi)\big(1+\imath\delta_\varphi[BC-CB]\delta_{\bar\varphi}\big)e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi)}\times\\
&\Big\{1-\imath\bar\varphi B\eta+\bar\eta B\varphi-\imath\lambda\Big[[B\varphi]^0\bar\varphi^1\varphi^2\bar\varphi^3-\varphi^0[\bar\varphi B]^1\varphi^2\bar\varphi^3+\varphi^0\bar\varphi^1[B\varphi]^2\bar\varphi^3-\varphi^0\bar\varphi^1\varphi^2[\bar\varphi B]^3\Big]\;-\;\imath\bar\lambda\{\bar\varphi\}\Big\}\,. \qquad (21)
\end{aligned}
$$

Being interesting only on connected functions, we now derivate the free energy with respect to the infinitesimal parameter $B^{ij}_{[\mu][\nu]}$:

$$
\begin{aligned}
\frac{\delta\ln Z(\eta,\bar\eta)}{\imath\delta B^{ij}_{[\mu][\nu]}} &= 0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{\delta_{\varphi^j_{[\nu]}}C^{il}_{[\mu][\alpha]}\delta_{\bar\varphi^l_{[\alpha]}}-\delta_{\varphi^l_{[\alpha]}}C^{lj}_{[\alpha][\nu]}\delta_{\bar\varphi^i_{[\mu]}}-\bar\varphi^i_{[\mu]}\eta^j_{[\nu]}+\bar\eta^i_{[\mu]}\varphi^j_{[\nu]} \qquad (22)\\
&-\lambda\Big[\delta^{i0}\varphi^j_{[\nu]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}-\delta^{j1}\bar\varphi^i_{[\mu]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}+\delta^{i2}\varphi^j_{[\nu]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}-\delta^{j3}\bar\varphi^i_{[\mu]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\\
&-\bar\lambda\{\bar\varphi\}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi)}\,,
\end{aligned}
$$

where the new notations mean

$$
\begin{aligned}
[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]} &:= \int_{h_{ij}}\bar\varphi^1_{\mu_{10}h_{13}h_{12}}\varphi^2_{h_{21}\mu_{20}h_{23}}\bar\varphi^3_{h_{32}h_{31}\mu_{30}}\,,\\
[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]} &:= \int_{h_{ij}}\bar\varphi^1_{h_{03}h_{02}\nu_{01}}\varphi^2_{\nu_{21}h_{20}h_{23}}\bar\varphi^3_{h_{32}\nu_{31}h_{30}}\,,\\
[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]} &:= \int_{h_{ij}}\varphi^0_{h_{03}\mu_{02}h_{01}}\bar\varphi^1_{h_{10}h_{13}\mu_{12}}\bar\varphi^3_{\mu_{32}h_{31}h_{30}}\,,\\
[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]} &:= \int_{h_{ij}}\varphi^0_{\nu_{03}h_{02}h_{01}}\bar\varphi^1_{h_{10}\nu_{13}h_{12}}\varphi^2_{h_{21}h_{20}\nu_{23}}\,. \qquad (23)
\end{aligned}
$$

After some algebra (the details of which are collected in Appendix D 1), the variation of the free energy function (22) can be recast as

$$
\begin{aligned}
\frac{\delta\ln Z(\eta,\bar\eta)}{\imath\delta B^{ij}_{[\mu][\nu]}} = 0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{&-[C\bar\varphi]^i_{[b]}\,[\eta C^\dagger]^j_{[a]}+[C\bar\eta]^i_{[b]}\,[\varphi C^\dagger]^j_{[a]}\\
&-\lambda\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}-\bar C^{j1}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\\
&+C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}-\bar C^{j3}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\;-\;\bar\lambda\{\bar\varphi\}\\
&+C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^j_{[a]}-\bar C^{jj'}_{[a][\nu]}\bar\varphi^{j'}_{[\nu]}\varphi^i_{[b]}\Big\}\;e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi)}\,. \qquad (24)
\end{aligned}
$$

WT identity for two-point functions - The next stage is to differentiate the expression (24) using the operator[2] $\partial_{\eta^p}\partial_{\bar\eta^k}(\cdot)|_{\eta^p=\bar\eta^k=0}$ for getting the connected components of the correlation functions (we shall denote $J=\bar\varphi\eta+\bar\eta\varphi$ $=\sum_i\bar\varphi^i\eta^i+\bar\eta^i\varphi^i$)

$$
\begin{aligned}
0 = \langle\partial_\eta\partial_{\bar\eta}\Big[&-[C\bar\varphi]^i_{[b]}\,[\eta C^\dagger]^j_{[a]}+[C\bar\eta]^i_{[b]}\,[\varphi C^\dagger]^j_{[a]}\\
&-\lambda\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}-\bar C^{j1}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}
\end{aligned}
$$

[2] The indices p,k are fixed in $\partial_{\eta^p}\partial_{\bar\eta^k}(\cdot)|_{\eta^p=\bar\eta^k=0}$. Moreover, omitting for a moment these indices, $\partial_{\eta^p}\partial_{\bar\eta^k}(\cdot)|_{\eta^p=\bar\eta^k=0}$ will be denoted by $\partial_\eta\partial_{\bar\eta}$ in the following.

$$+C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]} - \bar C^{j3}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big] \;-\; \bar\lambda\{\bar\varphi\}$$
$$+C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^j_{[a]} - \bar C^{jj'}_{[a][\nu]}\bar\varphi^{j'}_{[\nu]}\varphi^i_{[b]}\Big]e^J|_{\eta=\bar\eta=0}\rangle_c\,. \tag{25}$$

A direct computation yields at first

$$\begin{aligned}
&\langle\Big[C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^j_{[a]} - [C^\dagger]^{j'j}_{[\nu][a]}\bar\varphi^{j'}_{[\nu]}\varphi^i_{[b]}\Big]\partial_\eta(\bar\varphi\eta)\partial_{\bar\eta}(\bar\eta\varphi)|_{\eta=\bar\eta=0}\rangle_c\\
&-\lambda\,\langle\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}\Big]\partial_\eta(\bar\varphi\eta)\partial_{\bar\eta}(\bar\eta\varphi)|_{\eta=\bar\eta=0}\rangle_c + \lambda\,\langle\Big[[C^\dagger]^{1j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\Big]\partial_\eta(\bar\varphi\eta)\partial_{\bar\eta}(\bar\eta\varphi)|_{\eta=\bar\eta=0}\rangle_c\\
&-\lambda\,\langle\Big[C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}\Big]\partial_\eta(\bar\varphi\eta)\partial_{\bar\eta}(\bar\eta\varphi)|_{\eta=\bar\eta=0}\rangle_c + \lambda\,\langle\Big[[C^\dagger]^{3j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\partial_\eta(\bar\varphi\eta)\partial_{\bar\eta}(\bar\eta\varphi)|_{\eta=\bar\eta=0}\rangle_c\\
&-\bar\lambda\{\bar\varphi\} = \langle\Big[\partial_\eta[C\bar\varphi]^i_{[b]}\,[\eta C^\dagger]^j_{[a]}\partial_{\bar\eta}(\bar\eta\varphi) - \partial_{\bar\eta}[C\bar\eta]^i_{[b]}\,[\varphi C^\dagger]^j_{[a]}\partial_\eta(\bar\varphi\eta)\Big]|_{\eta=\bar\eta=0}\rangle_c\,.
\end{aligned}\tag{26}$$

Performing the explicit differentiation with respect to $\eta^p_{[m]}$ and $\bar\eta^k_{[n]}$, one gets

$$\begin{aligned}
&\langle\Big[[C\bar\varphi]^i_{[b]}[C^\dagger]^{pj}_{[m][a]}\varphi^k_{[n]} - [\varphi C^\dagger]^j_{[a]}C^{ik}_{[b][n]}\bar\varphi^p_{[m]}\Big]\rangle_c =\\
&\langle\Big[C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^j_{[a]} - [C^\dagger]^{j'j}_{[\nu][a]}\bar\varphi^{j'}_{[\nu]}\varphi^i_{[b]}\Big]\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c\\
&-\lambda\,\langle\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}\Big]\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c + \lambda\,\langle\Big[[C^\dagger]^{1j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\Big]\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c\\
&-\lambda\,\langle\Big[C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}\Big]\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c + \lambda\,\langle\Big[[C^\dagger]^{3j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c - \bar\lambda\{\bar\varphi\}\,,
\end{aligned}\tag{27}$$

so that summing over $[\mu]$ and $[\nu]$, the following statement holds:

Theorem 1. *Two-point functions of the colored Boulatov under a generic unitary field transformation satisfy the relation, for $i, j, p, k = 0, 1, 2, 3$,*

$$\begin{aligned}
&\langle[C\bar\varphi]^i_{[b]}[C^\dagger]^{pj}_{[m][a]}\varphi^k_{[n]}\rangle_c - \langle[\varphi C^\dagger]^j_{[a]}C^{ik}_{[b][n]}\bar\varphi^p_{[m]}\rangle_c =\\
&\langle\bar\varphi^j_{[a]}[C\varphi]^i_{[b]}\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c - \langle[\bar\varphi C^\dagger]^j_{[a]}\varphi^i_{[b]}\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c\\
&-\lambda\,\langle[\varphi C^\dagger]^j_{[a]}\;{}^{i0}[C\cdot\bar\varphi^1\varphi^2\bar\varphi^3]_{[b]}\,\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c + \lambda\,\langle[C\bar\varphi]^i_{[b]}\;{}^{1j}[\varphi^0\varphi^2\bar\varphi^3\cdot C^\dagger]_{[a]}\,\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c\\
&-\lambda\,\langle[\varphi C^\dagger]^j_{[a]}\;{}^{i2}[C\cdot\varphi^0\bar\varphi^1\bar\varphi^3]_{[b]}\,\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c + \lambda\,\langle[C\bar\varphi]^i_{[b]}\;{}^{3j}[\varphi^0\bar\varphi^1\varphi^2\cdot C^\dagger]_{[a]}\,\bar\varphi^p_{[m]}\varphi^k_{[n]}\rangle_c - \bar\lambda\{\bar\varphi\}\,,
\end{aligned}\tag{28}$$

where we introduced the notations

$$\begin{aligned}
{}^{i0}[C\cdot\bar\varphi^1\varphi^2\bar\varphi^3]_{[a]} &:= C^{i0}_{[a][\mu]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}\,, \qquad {}^{1j}[\varphi^0\varphi^2\bar\varphi^3\cdot C^\dagger]_{[a]} := [C^\dagger]^{1j}_{[\nu][a]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\,,\\
{}^{i2}[C\cdot\varphi^0\bar\varphi^1\bar\varphi^3]_{[a]} &:= C^{i2}_{[a][\mu]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}\,, \qquad {}^{3j}[\varphi^0\bar\varphi^1\varphi^2\cdot C^\dagger]_{[a]} := [C^\dagger]^{3j}_{[\nu][a]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\,,
\end{aligned}\tag{29}$$

for which repeated indices are summed.

The case of four external colors - Let us consider that the external color labels i, j, p, k are pairwise distinct. For definiteness, let us assume that $i = 0$, $j = 1$, $p = 2$ and $k = 3$, then the WT identity becomes

$$\begin{aligned}
0 &= \langle\bar\varphi^1_{[a]}[C\varphi]^0_{[b]}\bar\varphi^2_{[m]}\varphi^3_{[n]}\rangle_c - \langle[\bar\varphi C^\dagger]^1_{[a]}\varphi^0_{[b]}\bar\varphi^2_{[m]}\varphi^3_{[n]}\rangle_c\\
&-\lambda\,\langle[\varphi C^\dagger]^1_{[a]}\;{}^{00}[C\cdot\bar\varphi^1\varphi^2\bar\varphi^3]_{[b]}\,\bar\varphi^2_{[m]}\varphi^3_{[n]}\rangle_c + \lambda\,\langle[C\bar\varphi]^0_{[b]}\;{}^{11}[\varphi^0\varphi^2\bar\varphi^3\cdot C^\dagger]_{[a]}\,\bar\varphi^2_{[m]}\varphi^3_{[n]}\rangle_c - \bar\lambda\{\bar\varphi\}\,.
\end{aligned}\tag{30}$$

This is an identity for a four-point function with four external color which might be useful in the study of the coupling constant renormalization.

The case of two external colors - Let us assume now that $i = k = 0$ and $j = p = 1$

$$\begin{aligned}
&\langle[C\bar\varphi]^0_{[b]}[C^\dagger]^{11}_{[m][a]}\varphi^0_{[n]}\rangle_c - \langle[\varphi C^\dagger]^1_{[a]}C^{00}_{[b][n]}\bar\varphi^1_{[m]}\rangle_c =\\
&\langle\bar\varphi^1_{[a]}[C\varphi]^0_{[b]}\bar\varphi^1_{[m]}\varphi^0_{[n]}\rangle_c - \langle[\bar\varphi C^\dagger]^1_{[a]}\varphi^0_{[b]}\bar\varphi^1_{[m]}\varphi^0_{[n]}\rangle_c\\
&-\lambda\,\langle[\varphi C^\dagger]^1_{[a]}\;{}^{00}[C\cdot\bar\varphi^1\varphi^2\bar\varphi^3]_{[b]}\,\bar\varphi^1_{[m]}\varphi^0_{[n]}\rangle_c + \lambda\,\langle[C\bar\varphi]^0_{[b]}\;{}^{11}[\varphi^0\varphi^2\bar\varphi^3\cdot C^\dagger]_{[a]}\,\bar\varphi^1_{[m]}\varphi^0_{[n]}\rangle_c - \bar\lambda\{\bar\varphi\}\,.
\end{aligned}\tag{31}$$

WT identities for four-point functions - To obtain higher order point functions, we derivate again the free energy. Derivating twice $\ln Z$ for computing the four-point function identities, we have

$$
\begin{aligned}
&\langle \Big[C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^{j}_{[a]} - [C^\dagger]^{j'j}_{[\nu][a]}\bar\varphi^{j'}_{[\nu]}\varphi^{i}_{[b]}\Big]\partial_{\eta_2}J\partial_{\bar\eta_2}J\partial_{\eta_1}J\partial_{\bar\eta_1}J|_{\eta=\bar\eta=0}\rangle_c \\
&-\lambda\,\langle\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}\Big]\partial_{\eta_2}J\partial_{\bar\eta_2}J\partial_{\eta_1}J\partial_{\bar\eta_1}J|_{\eta=\bar\eta=0}\rangle_c \\
&+\lambda\,\langle\Big[[C^\dagger]^{1j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\Big]\partial_{\eta_2}J\partial_{\bar\eta_2}J\partial_{\eta_1}J\partial_{\bar\eta_1}J|_{\eta=\bar\eta=0}\rangle_c \\
&-\lambda\,\langle\Big[C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}\Big]\partial_{\eta_2}J\partial_{\bar\eta_2}J\partial_{\eta_1}J\partial_{\bar\eta_1}J|_{\eta=\bar\eta=0}\rangle_c \\
&+\lambda\,\langle\Big[[C^\dagger]^{3j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\partial_{\eta_2}J\partial_{\bar\eta_2}J\partial_{\eta_1}J\partial_{\bar\eta_1}J|_{\eta=\bar\eta=0}\rangle_c - \bar\lambda\{\bar\varphi\} \\
&= \langle\Big[[C\bar\varphi]^i_{[b]}\partial_{\eta_1}[\eta C^\dagger]^j_{[a]}\partial_{\bar\eta_1}J - \partial_{\bar\eta_1}[C\bar\eta]^i_{[b]}[\varphi C^\dagger]^j_{[a]}\partial_{\eta_1}J\Big]\partial_{\bar\eta_2}J\partial_{\eta_2}J|_{\eta=\bar\eta=0}\rangle_c + (1\leftrightarrow 2)\,. \qquad (32)
\end{aligned}
$$

Fixing the indices of $(\eta_1)^k_{[m]}$,$(\bar\eta_1)^l_{[n]}$ $(\eta_2)^t_{[p]}$ and $(\bar\eta_2)^s_{[q]}$, the differentiations yield

$$
\begin{aligned}
&\langle\Big[C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^{j}_{[a]} - [C^\dagger]^{j'j}_{[\nu][a]}\bar\varphi^{j'}_{[\nu]}\varphi^{i}_{[b]}\Big]\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c \\
&-\lambda\,\langle\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}\Big]\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c \\
&+\lambda\,\langle\Big[[C^\dagger]^{1j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\Big]\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c \\
&-\lambda\,\langle\Big[C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}\Big]\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c \\
&+\lambda\,\langle\Big[[C^\dagger]^{3j}_{[\nu][a]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c - \bar\lambda\{\bar\varphi\} \\
&= \langle\Big[[C\bar\varphi]^i_{[b]}[C^\dagger]^{kj}_{[m][a]}\varphi^l_{[n]} - C^{il}_{[b][n]}[\varphi C^\dagger]^j_{[a]}\bar\varphi^k_{[m]}\Big]\bar\varphi^t_{[p]}\varphi^s_{[q]}\rangle_c + [(k,l)\;\leftrightarrow\;(t,s)]\,. \qquad (33)
\end{aligned}
$$

Summing over $[\mu]$ and $[\nu]$, on this last expression rests our

Theorem 2. *Four-point functions of the colored Boulatov model under a generic unitary field transformation satisfy the relation, for* $i,j,k,l,s,t = 0,1,2,3,$

$$
\begin{aligned}
&\langle[C\bar\varphi]^i_{[b]}\varphi^l_{[n]}[C^\dagger]^{kj}_{[m][a]}\bar\varphi^t_{[p]}\varphi^s_{[q]}\rangle_c - \langle[\varphi C^\dagger]^j_{[a]}C^{il}_{[b][n]}\bar\varphi^k_{[m]}\bar\varphi^t_{[p]}\varphi^s_{[q]}\rangle_c + [(k,l)\;\leftrightarrow\;(t,s)] = \\
&\langle\bar\varphi^j_{[a]}[C\varphi]^i_{[b]}\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c - \langle[\bar\varphi C^\dagger]^j_{[a]}\varphi^i_{[b]}\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c \\
&-\lambda\,\langle[\varphi C^\dagger]^j_{[a]}\;{}^{i0}[C\cdot\bar\varphi^1\varphi^2\bar\varphi^3]_{[b]}\,\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c + \lambda\,\langle[C\bar\varphi]^i_{[b]}\;{}^{1j}[\varphi^0\varphi^2\bar\varphi^3\cdot C^\dagger]_{[a]}\,\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c \\
&-\lambda\,\langle[\varphi C^\dagger]^j_{[a]}\;{}^{i2}[C\cdot\varphi^0\bar\varphi^1\bar\varphi^3]_{[b]}\,\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c + \lambda\,\langle[C\bar\varphi]^i_{[b]}\;{}^{3j}[\,\varphi^0\bar\varphi^1\varphi^2\cdot C^\dagger]_{[a]}\,\bar\varphi^t_{[p]}\varphi^s_{[q]}\bar\varphi^k_{[m]}\varphi^l_{[n]}\rangle_c - \bar\lambda\{\bar\varphi\}\,. \qquad (34)
\end{aligned}
$$

WT identities for even-pt functions - The WT identities for $[n = 2p \geq 2]$-point functions can be deduced by simple recursion from the aforementioned equations. We first need to introduce some notations

$$
\begin{aligned}
&\mathcal{F} = [C\bar\varphi]^i_{[b]}\,[\eta C^\dagger]^j_{[a]} - [C\bar\eta]^i_{[b]}\,[\varphi C^\dagger]^j_{[a]}\,, \qquad \mathcal{H} = \mathcal{F}e^J\,, \\
&\partial_{\bar\eta_k}\partial_{\eta_k}\mathcal{H} = (\mathcal{F}_k + (\partial_{\bar\eta_k}J)(\partial_{\eta_k}J)\,\mathcal{F})\,e^J\,, \qquad \mathcal{F}_k = [C\bar\varphi]^i_{[b]}\partial_{\eta_k}[\eta C^\dagger]^j_{[a]}(\partial_{\bar\eta_k}J) - \partial_{\bar\eta_k}[C\bar\eta]^i_{[b]}[\varphi C^\dagger]^j_{[a]}\,(\partial_{\eta_k}J)\,. \qquad (35)
\end{aligned}
$$

It is by simple recursion that one proves that

$$
\Big[\prod_{l=1}^n \partial_{\eta_l}\partial_{\bar\eta_l}\Big]\mathcal{H} = \Big[\sum_{k=1}^n \mathcal{F}_k\prod_{l\neq k}^n[\partial_{\eta_l}J\partial_{\bar\eta_l}J] + \prod_{l=1}^n[\partial_{\eta_l}J\partial_{\bar\eta_l}J]\,\mathcal{F}\Big]\,e^J\,, \qquad (36)
$$

$$
\Big[\prod_{l}^n \partial_{\eta_l}\partial_{\bar\eta_l}\Big]\mathcal{H}|_{\eta=\bar\eta=0} = \sum_{k=1}^n \mathcal{F}_k\prod_{l\neq k}^n[\partial_{\eta_l}J\partial_{\bar\eta_l}J]\,|_{\eta=\bar\eta=0}\,. \qquad (37)
$$

From the last line (37), we explicitly obtain by fixing the derivative with respect to the indices such that $(\eta_l)^{\alpha_l}_{[a_l]}$ and $(\bar\eta_l)^{\beta_l}_{[b_l]}$:

$$
\Big[\prod_{l=1}^n \partial_{\eta_l}\partial_{\bar\eta_l}\Big]\mathcal{H}|_{\eta=\bar\eta=0} = \sum_{k=1}^n\Big([C\bar\varphi]^i_{[b]}[C^\dagger]^{\alpha_k j}_{[a_k][a]}\varphi^{\beta_k}_{[b_k]} - C^{i\beta_k}_{[b][b_k]}[\varphi C^\dagger]^j_{[a]}\bar\varphi^{\alpha_k}_{[a_k]}\Big)\prod_{l\neq k}^n\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\,. \qquad (38)
$$

The generalized WT identity for an even-point function can be written as

$$\Big\langle \Big[C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^{j}_{[a]} - [C^\dagger]^{j'j}_{[\nu][a]}\bar\varphi^{j'}_{[\nu]}\varphi^{i}_{[b]}\Big]\prod_{l=1}^{n}\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\Big\rangle_c$$
$$-\lambda\Big\langle \Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^{j}_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]}\Big]\prod_{l=1}^{n}\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\Big\rangle_c$$
$$+\lambda\Big\langle \Big[[C^\dagger]^{1j}_{[\nu][a]}[C\bar\varphi]^{i}_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}\Big]\prod_{l=1}^{n}\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\Big\rangle_c$$
$$-\lambda\Big\langle \Big[C^{i2}_{[b][\mu]}[\varphi C^\dagger]^{j}_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]}\Big]\prod_{l=1}^{n}\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\Big\rangle_c$$
$$+\lambda\Big\langle \Big[[C^\dagger]^{3j}_{[\nu][a]}[C\bar\varphi]^{i}_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big]\prod_{l=1}^{n}\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\Big\rangle_c - \bar\lambda\{\bar\varphi\}$$
$$=\sum_{k=1}^{n}\Big\langle\Big[[C\bar\varphi]^{i}_{[b]}[C^\dagger]^{\alpha_k j}_{[a_k][a]}\varphi^{\beta_k}_{[b_k]} - C^{i\beta_k}_{[b][b_k]}[\varphi C^\dagger]^{j}_{[a]}\bar\varphi^{\alpha_k}_{[a_k]}\Big]\prod_{l\neq k}^{n}\Big[\bar\varphi^{\alpha_l}_{[a_l]}\varphi^{\beta_l}_{[b_l]}\Big]\Big\rangle_c . \tag{39}$$

Summing over the remaining running indices $[\mu]$ and $[\nu]$, we have in the same anterior notations:

Theorem 3. *Even n-point functions of the colored Boulatov model under a generic unitary field transformation satisfy the relation*

$$\sum_{k=1}^{n}\Big\{\Big\langle[C\bar\varphi]^{i}_{[b]}[C^\dagger]^{\alpha_k j}_{[a_k][a]}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c - \Big\langle C^{i\beta_k}_{[b][b_k]}[\varphi C^\dagger]^{j}_{[a]}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\Big\}$$
$$=\Big\langle\bar\varphi^{j}_{[a]}[C\varphi]^{i}_{[b]}\prod_{\ell=1}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c - \Big\langle[\bar\varphi C^\dagger]^{j}_{[a]}\varphi^{i}_{[b]}\prod_{\ell=1}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$-\lambda\Big\langle[\varphi C^\dagger]^{j}_{[a]}\;{}^{i0}[C\cdot\bar\varphi^1\varphi^2\bar\varphi^3]_{[b]}\prod_{\ell=1}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c + \lambda\Big\langle[C\bar\varphi]^{i}_{[b]}\;{}^{1j}[\varphi^0\varphi^2\bar\varphi^3\cdot C^\dagger]_{[a]}\prod_{\ell=1}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$-\lambda\Big\langle[\varphi C^\dagger]^{j}_{[a]}\;{}^{i2}[C\cdot\varphi^0\bar\varphi^1\bar\varphi^3]_{[b]}\prod_{\ell=1}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c + \lambda\Big\langle[C\bar\varphi]^{i}_{[b]}\;{}^{3j}[\varphi^0\bar\varphi^1\varphi^2\cdot C^\dagger]_{[a]}\prod_{\ell=1}^{n}\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c - \bar\lambda\{\bar\varphi\} . \tag{40}$$

The WT identity (40) is valid in full generality regarding a generic unitary field transformation without requiring that transformation to be a symmetry of the action. We can make the following striking observation: the equation (24) (and hence the general WT identity (40)) generates without ambiguity the WT identities associated with independent non identically distributed matrix models with invertible covariances as the models studied in [26, 27]. Indeed, forgetting the color index and restricting all tensors to matrices, one has just to use the facts that, on one hand, these models are covariant under a unitary symmetry in order to cancel the interaction terms in $\lambda, \bar\lambda$ and, on the other hand, to invert two extra covariances such that (24) and (40) generate the corresponding WT equations for these more simple cases. Hence, it is often useful to specify which kind of WT identities could be inferred from the same reasoning with now a well defined symmetry of the model. It is the main purpose of the remaining of this paper.

B. Ward-Takahashi identities for the 1-action symmetry

We begin with the partition function (11) now under infinitesimal transformations given by $\delta_B S^{\rm int} = 0$, (19) and (20) and that we symbolically write

$$Z(\bar\eta,\eta) = \int d\mu_C(\bar\varphi,\varphi)\Big(1 - \imath\bar\varphi B\eta + \bar\eta B\varphi + \imath\delta_\varphi[BC - CB]\delta_{\bar\varphi}\Big)e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} . \tag{41}$$

Derivating the free energy with respect to $\imath B_{\mu\nu}$ yields

$$\frac{\ln Z(\eta,\bar\eta)}{\imath\delta B_{\mu\nu}} = 0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\int_{g_\alpha h_\beta}\Big(-\bar\varphi^{0,2}_{\mu.}\eta^{0,2}_{\nu.} + \bar\eta^{0,2}_{\mu.}\varphi^{0,2}_{\nu.} - \bar\varphi^{1,3}_{.\mu}\eta^{1,3}_{.\nu} + \bar\eta^{1,3}_{.\mu}\varphi^{1,3}_{.\nu}$$
$$+\frac{\delta}{\delta\varphi^{0,2}_{\nu.}}C^{0,2}_{\mu.;\,h_1h_2h_3}\frac{\delta}{\delta\bar\varphi^{0,2}_{h_0h_1h_2}} - \frac{\delta}{\delta\varphi^{0,2}_{h_0h_1h_2}}C^{0,2}_{h_0h_1h_2;\,\nu.}\frac{\delta}{\delta\bar\varphi^{0,2}_{\mu.}}$$

$$+\frac{\delta}{\delta\varphi^{1,3}_{.\nu}}C^{1,3}_{.\mu;\,h_1h_2h_3}\frac{\delta}{\delta\bar\varphi^{1,3}_{h_0h_1h_2}}-\frac{\delta}{\delta\varphi^{1,3}_{h_0h_1h_2}}C^{1,3}_{h_0h_1h_2;\,.\nu}\frac{\delta}{\delta\bar\varphi^{1,3}_{.\mu}}\Big)e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\,, \tag{42}$$

where repeated indices and $(\cdot)$ arguments (called henceforth dot arguments) are summed. The latter expression can be computed to (see Appendix D 2 for derivations and notations)

$$0=\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big[-\bar\varphi^{0,2}_{\mu.}\eta^{0,2}_{\nu.}+\bar\eta^{0,2}_{\mu.}\varphi^{0,2}_{\nu.}-\bar\varphi^{1,3}_{.\mu}\eta^{1,3}_{.\nu}+\bar\eta^{1,3}_{.\mu}\varphi^{1,3}_{.\nu}+\varphi^{0,2}_{\mu.}\bar\eta^{0,2}_{\nu.}-\bar\varphi^{0,2}_{\nu.}\eta^{0,2}_{\mu.}+\varphi^{1,3}_{.\mu}\bar\eta^{1,3}_{.\nu}-\bar\varphi^{1,3}_{.\nu}\eta^{1,3}_{.\mu}$$
$$-\lambda\,\varphi^{0,2}_{\mu.}\cdot[\bar\varphi^1\varphi^{2,0}\bar\varphi^3]_{\nu.}-\bar\lambda\,\varphi^{1,3}_{.\mu}\cdot[\bar\varphi^0\bar\varphi^2\varphi^{3,1}]_{.\nu}+\lambda\,\bar\varphi^{0,2}_{\nu.}\cdot[\varphi^1\bar\varphi^{2,0}\varphi^3]_{\mu.}+\bar\lambda\,\bar\varphi^{1,3}_{.\nu}\cdot[\varphi^0\varphi^2\bar\varphi^{3,1}]_{.\mu}\Big]e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \tag{43}$$

and therefore, differentiating by an even product of $\eta^{\alpha_\ell}_{[a_\ell]}\bar\eta^{\beta_\ell}_{[b_\ell]}$, we can readily identify the WT identities as given by the equation

$$\sum_{k=1}^n\Big\{\Big\langle\bar\varphi^{0,2}_{\mu.}\,[\delta]^{\alpha_k,0,2}_{[a_k]\nu.}\,\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c-\Big\langle[\delta]^{\beta_k,0,2}_{[b_k]\mu.}\,\varphi^{0,2}_{\nu.}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$-\Big\langle\varphi^{0,2}_{\mu.}\,[\delta]^{\beta_k,0,2}_{[b_k]\nu.}\,\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\Big\langle[\delta]^{\alpha_k,0,2}_{[a_k]\mu.}\,\bar\varphi^{0,2}_{\nu.}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$+\Big\langle\bar\varphi^{1,3}_{.\mu}\,[\delta]^{\alpha_k,1,3}_{[a_k].\nu}\,\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c-\Big\langle[\delta]^{\beta_k,1,3}_{.\mu[b_k]}\,\varphi^{1,3}_{.\nu}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$-\Big\langle\varphi^{1,3}_{.\mu}\,[\delta]^{\beta_k,1,3}_{[b_k].\nu}\,\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\Big\langle[\delta]^{\alpha_k,1,3}_{.\mu[a_k]}\,\bar\varphi^{1,3}_{.\nu}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\Big\}$$
$$=-\lambda\Big\langle\varphi^{0,2}_{\mu.}\cdot[\bar\varphi^1\varphi^{2,0}\bar\varphi^3]_{\nu.}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c-\bar\lambda\Big\langle\varphi^{1,3}_{.\mu}\cdot[\bar\varphi^0\bar\varphi^2\varphi^{3,1}]_{.\nu}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$+\lambda\Big\langle\bar\varphi^{0,2}_{\nu.}\cdot[\varphi^1\bar\varphi^{2,0}\varphi^3]_{\mu.}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\bar\lambda\Big\langle\bar\varphi^{1,3}_{.\nu}\cdot[\varphi^0\varphi^2\bar\varphi^{3,1}]_{.\mu}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\,, \tag{44}$$

where $[\delta]^{\alpha,i,j}_{[a][b]}=(\delta^{\alpha,i}+\delta^{\alpha,j})\delta_{[a][b]}$. This leads us to the following statement

Theorem 4. *Even n-point functions of the colored Boulatov model under a right invariant unitary transformation satisfy the relation*

$$\sum_{k=1}^n\Big\{\sum_{\alpha=0,2}\Big[\delta^{\alpha_k\alpha}\delta_{a_{k1}\nu}\Big\langle\bar\varphi^{\alpha}_{\mu a_{k2}a_{k3}}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c-\delta^{\alpha\beta_k}\delta_{\mu b_{k1}}\Big\langle\varphi^{\alpha}_{\nu b_{k2}b_{k3}}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$-\delta^{\alpha\beta_k}\delta_{b_{k1}\nu}\Big\langle\varphi^{\alpha}_{\mu b_{k2}b_{k3}}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\delta^{\alpha\alpha_k}\delta_{\mu a_{k1}}\Big\langle\bar\varphi^{\alpha}_{\nu a_{k2}a_{k3}}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\Big]$$
$$+\sum_{\alpha=1,3}\Big[\delta^{\alpha_k\alpha}\delta_{a_{k3}\nu}\Big\langle\bar\varphi^{\alpha}_{a_{k1}a_{k2}\mu}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c-\delta^{\alpha\beta_k}\delta_{\mu b_{k3}}\Big\langle\varphi^{\alpha}_{b_{k1}b_{k2}\nu}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c$$
$$-\delta^{\alpha\beta_k}\delta_{b_{k3}\nu}\Big\langle\varphi^{\alpha}_{b_{k1}b_{k2}\mu}\bar\varphi^{\alpha_k}_{[a_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\delta^{\alpha\alpha_k}\delta_{\mu a_{k3}}\Big\langle\bar\varphi^{\alpha}_{a_{k1}a_{k2}\nu}\varphi^{\beta_k}_{[b_k]}\prod_{\ell\neq k}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\Big]\Big\}$$
$$=\lambda\sum_{\alpha=0,2}\Big[-\Big\langle\varphi^{\alpha}_{\mu.}\cdot[\bar\varphi^1\varphi^{\bar\alpha}\bar\varphi^3]_{\nu.}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\Big\langle\bar\varphi^{\alpha}_{\nu.}\cdot[\varphi^1\bar\varphi^{\bar\alpha}\varphi^3]_{\mu.}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\Big]$$
$$+\bar\lambda\sum_{\alpha=1,3}\Big[-\Big\langle\varphi^{\alpha}_{.\mu}\cdot[\bar\varphi^0\bar\varphi^2\varphi^{\bar\alpha}]_{.\nu}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c+\Big\langle\bar\varphi^{\alpha}_{.\nu}\cdot[\varphi^0\varphi^2\bar\varphi^{\bar\alpha}]_{.\mu}\prod_{\ell=1}^n\Big[\bar\varphi^{\alpha_\ell}_{[a_\ell]}\varphi^{\beta_\ell}_{[b_\ell]}\Big]\Big\rangle_c\Big], \tag{45}$$

where $\bar\alpha=2,0$ if $\alpha=0,2$, respectively, and $\bar\alpha=3,1$ if $\alpha=1,3$, respectively, and in the left hand side of the equality, the notations explicitly mean

$$\varphi^0_{\mu.}\cdot[\bar\varphi^1\varphi^2\bar\varphi^3]_{.\nu}:=\int_{hgh_{ij}}\varphi^0_{\mu hg}\bar\varphi^1_{gh_{13}h_{12}}\varphi^2_{h_{21}hh_{23}}\bar\varphi^3_{h_{32}h_{31}\nu}\,,$$
$$\varphi^2_{\mu.}\cdot[\bar\varphi^1\varphi^0\bar\varphi^3]_{.\nu}:=\int_{hgh_{ij}}\varphi^0_{h_{03}hh_{01}}\bar\varphi^1_{h_{10}h_{13}\nu}\varphi^2_{\mu hg}\bar\varphi^3_{gh_{31}h_{30}}\,,$$

$$\varphi^1_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^3]_{.\nu} := \int_{hgh_{ij}} \bar\varphi^0_{h_{03}h_{02}h} \varphi^1_{hg\mu} \bar\varphi^2_{\nu h_{20}h_{23}} \varphi^3_{h_{32}gh_{30}} \,,$$
$$\varphi^3_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^1]_{.\nu} := \int_{hgh_{ij}} \bar\varphi^0_{\nu h_{02}h_{01}} \varphi^1_{h_{10}gh_{12}} \bar\varphi^2_{h_{21}h_{20}h} \varphi^3_{hg\mu} \,, \tag{46}$$

and the analogous for $\bar\varphi^{0,2}_{\nu.} \cdot [\varphi^1 \bar\varphi^{2,0} \varphi^3]_{\mu.}$ *and* $\bar\varphi^{1,3}_{.\nu} \cdot [\varphi^0 \varphi^2 \bar\varphi^{3,1}]_{.\mu}$.

At the first sight, one may wonder why Theorem 3 looks simpler than Theorem 4. This is really an illusion because, in the second case, the symmetry constrains much more the equality and fewer terms will survive. From the general WT identity (45), we can derive some more specific relations characterizing particular graphs. Let us for instance discuss the case of two-point graphs. For this category of graphs, (45) simplifies to

$$\begin{aligned}
&\sum_{\alpha=0,2} \Big[\delta^{\alpha_0\alpha} \delta_{a_1\nu} \Big\langle \bar\varphi^\alpha_{\mu a_2 a_3} \varphi^{\beta_0}_{[b]} \Big\rangle_c - \delta^{\alpha\beta_0} \delta_{\mu b_1} \Big\langle \varphi^\alpha_{\nu b_2 b_3} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c \\
&- \delta^{\alpha\beta_0} \delta_{b_1\nu} \Big\langle \varphi^\alpha_{\mu b_2 b_3} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c + \delta^{\alpha\alpha_0} \delta_{\mu a_1} \Big\langle \bar\varphi^\alpha_{\nu a_2 a_3} \varphi^{\beta_0}_{[b]} \Big\rangle_c \Big] \\
&+ \sum_{\alpha=1,3} \Big[\delta^{\alpha_0\alpha} \delta_{a_3\nu} \Big\langle \bar\varphi^\alpha_{a_1 a_2 \mu} \varphi^{\beta_0}_{[b]} \Big\rangle_c - \delta^{\alpha\beta_0} \delta_{\mu b_3} \Big\langle \varphi^\alpha_{b_1 b_2 \nu} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c \\
&- \delta^{\alpha\beta_0} \delta_{b_3\nu} \Big\langle \varphi^\alpha_{b_1 b_2 \mu} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c + \delta^{\alpha\alpha_0} \delta_{\mu a_3} \Big\langle \bar\varphi^\alpha_{a_1 a_2 \nu} \varphi^{\beta_0}_{[b]} \Big\rangle_c \Big] \\
&= -\lambda \Big\langle \varphi^{0,2}_{\mu.} \cdot [\bar\varphi^1 \varphi^{2,0} \bar\varphi^3]_{\nu.} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c - \bar\lambda \Big\langle \varphi^{1,3}_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^{3,1}]_{.\nu} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c \\
&+ \lambda \Big\langle \bar\varphi^{0,2}_{\nu.} \cdot [\varphi^1 \bar\varphi^{2,0} \varphi^3]_{\mu.} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c + \bar\lambda \Big\langle \bar\varphi^{1,3}_{.\nu} \cdot [\varphi^0 \varphi^2 \bar\varphi^{3,1}]_{.\mu} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c .
\end{aligned} \tag{47}$$

where a sum is performed repeated color indices and on dot arguments whereas $[a]$ and $[b]$ are kept fixed. Using a minimal symmetry, $\varphi^0 \to {}^U\varphi^0$ and $\varphi^3 \to {}^U\varphi^3$, which makes again $S^{\rm int} = {}^U S^{\rm int}$, the whole analysis gets simplified further. We then assume that only remains terms involving $\alpha = 0$ and 3, then the above WT identity can be recast in the following way

$$\begin{aligned}
&\delta^{\alpha_0 0} \delta_{a_1\nu} \Big\langle \bar\varphi^0_{\mu a_2 a_3} \varphi^{\beta_0}_{[b]} \Big\rangle_c - \delta^{0\beta_0} \delta_{\mu b_1} \Big\langle \varphi^0_{\nu b_2 b_3} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c - \delta^{0\beta_0} \delta_{b_1\nu} \Big\langle \varphi^0_{\mu b_2 b_3} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c + \delta^{0\alpha_0} \delta_{\mu a_1} \Big\langle \bar\varphi^0_{\nu a_2 a_3} \varphi^{\beta_0}_{[b]} \Big\rangle_c \\
&+ \delta^{\alpha_0 3} \delta_{a_3\nu} \Big\langle \bar\varphi^3_{a_1 a_2 \mu} \varphi^{\beta_0}_{[b]} \Big\rangle_c - \delta^{3\beta_0} \delta_{\mu b_3} \Big\langle \varphi^3_{b_1 b_2 \nu} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c - \delta^{3\beta_0} \delta_{b_3\nu} \Big\langle \varphi^3_{b_1 b_2 \mu} \bar\varphi^{\alpha_0}_{[a]} \Big\rangle_c + \delta^{3\alpha_0} \delta_{\mu a_3} \Big\langle \bar\varphi^3_{a_1 a_2 \nu} \varphi^{\beta_0}_{[b]} \Big\rangle_c \\
&= -\lambda \Big\langle \varphi^0_{\mu.} \cdot [\bar\varphi^1 \varphi^2 \bar\varphi^3]_{\nu.} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c - \bar\lambda \Big\langle \varphi^3_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^1]_{.\nu} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c \\
&+ \lambda \Big\langle \bar\varphi^0_{\nu.} \cdot [\varphi^1 \bar\varphi^2 \varphi^3]_{\mu.} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c + \bar\lambda \Big\langle \bar\varphi^3_{.\nu} \cdot [\varphi^0 \varphi^2 \bar\varphi^1]_{.\mu} \, \bar\varphi^{\alpha_0}_{[a]} \varphi^{\beta_0}_{[b]} \Big\rangle_c .
\end{aligned} \tag{48}$$

From the fact that (see [21] Lemma 2.1) in a color model, an even point function with a color missing on the external legs has external colors appearing always in pairs, therefore, assuming further that two-point functions do not vanish, we can require that $\alpha_0 = \beta_0 = 0$.[3] Taking into account these assumptions, the relation (48) becomes

$$\begin{aligned}
&\delta_{a_1\nu} \Big\langle \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{[b]} \Big\rangle_c - \delta_{\mu b_1} \Big\langle \bar\varphi^0_{[a]} \varphi^0_{\nu b_2 b_3} \Big\rangle_c - \delta_{b_1\nu} \Big\langle \bar\varphi^0_{[a]} \varphi^0_{\mu b_2 b_3} \Big\rangle_c + \delta_{\mu a_1} \Big\langle \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{[b]} \Big\rangle_c \\
&= -\lambda \Big\langle \varphi^0_{\mu.} \cdot [\bar\varphi^1 \varphi^2 \bar\varphi^3]_{\nu.} \bar\varphi^0_{[a]} \varphi^0_{[b]} \Big\rangle_c - \bar\lambda \Big\langle \varphi^3_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^1]_{.\nu} \bar\varphi^0_{[a]} \varphi^0_{[b]} \Big\rangle_c \\
&+ \lambda \Big\langle \bar\varphi^0_{\nu.} \cdot [\varphi^1 \bar\varphi^2 \varphi^3]_{\mu.} \bar\varphi^0_{[a]} \varphi^0_{[b]} \Big\rangle_c + \bar\lambda \Big\langle \bar\varphi^3_{.\nu} \cdot [\varphi^0 \varphi^2 \bar\varphi^1]_{.\mu} \, \bar\varphi^0_{[a]} \varphi^0_{[b]} \Big\rangle_c .
\end{aligned} \tag{49}$$

The following cases could be studied
Case 1: $\mu = \nu = a_1 = b_1$ yielding a trivial relation

$$\begin{aligned}
0 &= -\lambda \Big\langle \varphi^0_{\mu.} \cdot [\bar\varphi^1 \varphi^2 \bar\varphi^3]_{\mu.} \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\mu b_2 b_3} \Big\rangle_c - \bar\lambda \Big\langle \varphi^3_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^1]_{.\mu} \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\mu b_2 b_3} \Big\rangle_c \\
&+ \lambda \Big\langle \bar\varphi^0_{\mu.} \cdot [\varphi^1 \bar\varphi^2 \varphi^3]_{\mu.} \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\mu b_2 b_3} \Big\rangle_c + \bar\lambda \Big\langle \bar\varphi^3_{.\mu} \cdot [\varphi^0 \varphi^2 \bar\varphi^1]_{.\mu} \, \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\mu b_2 b_3} \Big\rangle_c .
\end{aligned} \tag{50}$$

Case 2: $\mu \neq \nu$, and $a_1 = \nu$ and $b_1 = \mu$. These assumptions leads to

$$\Big\langle \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\mu b_1 b_2} \Big\rangle_c - \Big\langle \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\nu b_2 b_3} \Big\rangle_c = 0$$

---

[3] This is without loss of generality since the case $\alpha_0 = \beta_0 = 3$ can be inferred by symmetry and will lead to the similar conclusion.

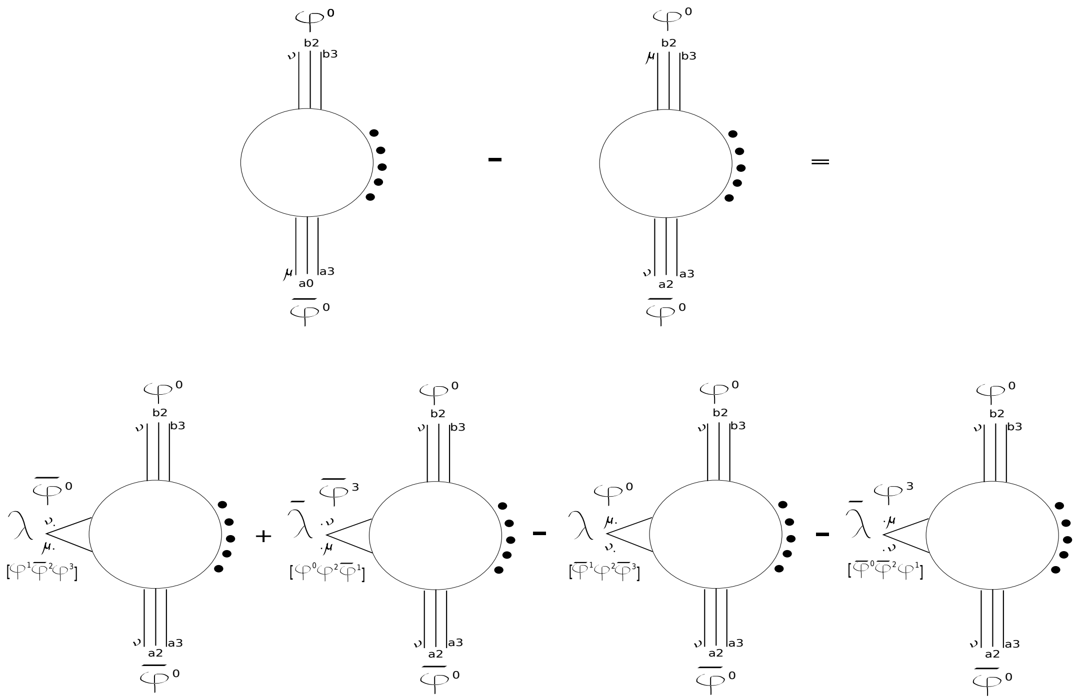

FIG. 1. Ward-Takahashi identity for the 1-action for colors 0 and 3

$$
\begin{aligned}
0 = &-\lambda \left\langle \varphi^0_{\mu.} \cdot [\bar\varphi^1 \varphi^2 \bar\varphi^3]_{\nu.} \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\mu b_1 b_2} \right\rangle_c - \bar\lambda \left\langle \, \varphi^3_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^1]_{.\nu} \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\mu b_1 b_2} \right\rangle_c \\
&+ \lambda \left\langle \bar\varphi^0_{\nu.} \cdot [\varphi^1 \bar\varphi^2 \varphi^3]_{\mu.} \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\mu b_1 b_2} \right\rangle_c + \bar\lambda \left\langle \bar\varphi^3_{.\nu} \cdot [\varphi^0 \varphi^2 \bar\varphi^1]_{.\mu} \, \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\mu b_1 b_2} \right\rangle_c ,
\end{aligned} \tag{51}
$$

where one should use the fact that any correlation including a beginning and an end-point with the same index (for instance, $\left\langle \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\mu b_1 b_2} \right\rangle_c$ involving a open strand with amplitude $\delta(\mu(\prod h)\mu^{-1}))$ does not depend on that point (say μ). Again, this WT identity is trivial.
Case 3: $\mu \neq \nu$ and assume that $a_1 = b_1 = \nu$, these restrictions give

$$
\begin{aligned}
&\left\langle \bar\varphi^0_{\mu a_2 a_3} \varphi^0_{\nu b_2 b_3} \right\rangle_c - \left\langle \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\mu b_2 b_3} \right\rangle_c \\
&= -\lambda \left\langle \varphi^0_{\mu.} \cdot [\bar\varphi^1 \varphi^2 \bar\varphi^3]_{\nu.} \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\nu b_2 b_3} \right\rangle_c - \bar\lambda \left\langle \varphi^3_{.\mu} \cdot [\bar\varphi^0 \bar\varphi^2 \varphi^1]_{.\nu} \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\nu b_2 b_3} \right\rangle_c \\
&+ \lambda \left\langle \bar\varphi^0_{\nu.} \cdot [\varphi^1 \bar\varphi^2 \varphi^3]_{\mu.} \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\nu b_2 b_3} \right\rangle_c + \bar\lambda \left\langle \bar\varphi^3_{.\nu} \cdot [\varphi^0 \varphi^2 \bar\varphi^1]_{.\mu} \, \bar\varphi^0_{\nu a_2 a_3} \varphi^0_{\nu b_2 b_3} \right\rangle_c
\end{aligned} \tag{52}
$$

which is a non-trivial relation. This WT identity and the kind with more external legs have been illustrated in Figure 1 where a field is graphically represented by three parallel strands each of which are associated with a field argument.

V. CONCLUSION

Using a generic unitary field transformation, first starting from the two-point and four-point functions, WT identities for any even-point functions of the Boulatov model have been identified. In particular, there exists a particular class of operators (called right invariant unitary operators) under which the interaction becomes invariant. This class of operators has allowed to refine the formalism and to identify WT identities associated with this symmetry with a non-trivial content. A combination of two-point functions can be expanded versus four-point functions with an insertion. The analysis performed here could be useful for both perturbative and nonperturbative renormalization.

APPENDIX

Appendix A: Gaussian integration for tensor models

In this appendix, a series of lemmas are introduced. These pertain to the properties of the GFT Gaussian measure and are extensively used in the text. We will denote the fields $\varphi_{[h]} = \varphi(h_1, h_2, h_3)$, and covariance associated with Feynman Gaussian measure as $C_{[g][h]} = C_{g_1 g_2 g_3;\, h_1 h_2 h_3}$ and for all field, $C_{[g][h]}\varphi_{[h]} := \int_{[h]} C_{[g][h]}\varphi_{[h]}$. In a colored theory, fields are equipped with an extra index $\varphi^i_{[h]}$ and the covariance reads $C^{ij}_{[g][h]} = \delta^{ij} C_{[g][h]}$. The subsequent analysis admits a straightforward generalization in any GFT dimension.

Definition 1. *A Gaussian measure of covariance C is defined by its non zero correlations*

$$\int d\mu_C(\bar\varphi, \varphi)\; \varphi_{[a_1]} \dots \varphi_{[a_n]} \bar\varphi_{[b_1]} \dots \bar\varphi_{[b_n]} = \sum_{\pi} \prod_{i=1}^{n} C_{[a_i][b_{\pi(i)}]} \,, \tag{A.1}$$

where the sum is taken over all permutations π of n elements. For a colored theory, we have

$$\int d\mu_C(\bar\varphi^{\ell}, \varphi^{\ell})\; \varphi^{k_1}_{[a_1]} \dots \varphi^{k_1}_{[a_n]} \bar\varphi^{j_1}_{[b_1]} \dots \bar\varphi^{j_n}_{[b_n]} = \sum_{\pi} \prod_{i=1}^{n} C^{k_i j_{\pi(i)}}_{[a_i][b_{\pi(i)}]} \,. \tag{A.2}$$

In the following, the developments hold in general, i.e. without colors using the definition (A.1). However being interested in colored theory, we will give the corresponding result in that particular instance. In order to alleviate notations, the source term will be denoted as

$$\bar\eta\varphi + \bar\varphi\eta := \int_{[h_i]} (\bar\eta_{h_1 h_2 h_3}\, \varphi_{h_1 h_2 h_3} + \bar\varphi_{h_1 h_2, h_3}\, \eta_{h_1 h_2 h_3}) \,. \tag{A.3}$$

Lemma 1. *We have*

$$\int d\mu_C(\bar\varphi, \varphi)\; e^{\bar\varphi\eta + \bar\eta\varphi} = e^{\bar\eta C \eta} \,. \tag{A.4}$$

Proof: By direct evaluation using the Wick theorem (A.1), we obtain

$$\int d\mu_C(\bar\varphi, \varphi)\; e^{\bar\varphi\eta + \bar\eta\varphi} = \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \sum_{\pi} \prod_{i=1}^{n} \int_{[a_i][b_{\pi(i)}]} \bar\eta_{[a_i]} C_{[a_i][b_{\pi(i)}]} \eta_{[b_{\pi(i)}]} = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\bar\eta C \eta\right]^n \,. \tag{A.5}$$

□

Lemma 2. *[Integration by parts] Introducing the functional derivative $\delta_{\varphi_{[h]}}(\cdot) = \delta(\cdot)/\delta\varphi_{[h]}$ for any functional $H(\varphi, \bar\varphi)$, we have*

$$\int d\mu_C(\bar\varphi, \varphi)\; \Big(\varphi_{[a]} H(\bar\varphi, \varphi) - C_{[a][b]} \delta_{\bar\varphi_{[b]}} H(\bar\varphi, \varphi)\Big) = 0 \,. \tag{A.6}$$

Proof: This relation can be shown by first introducing source terms. We rewrite

$$\begin{aligned} &\int d\mu_C(\bar\varphi, \varphi)\; \Big(\varphi_{[a]} H(\bar\varphi, \varphi) - C_{[a][b]} \delta_{\bar\varphi_{[b]}} H(\bar\varphi, \varphi)\Big) \\ &= \int d\mu_C(\bar\varphi, \varphi)\; \Big[\varphi_{[a]} H(\delta_\eta, \delta_{\bar\eta}) e^{\bar\varphi\eta + \bar\eta\varphi}|_{\bar\eta=\eta=0} - C_{[a][b]} \delta_{\bar\varphi_{[b]}} \Big(H(\delta_\eta, \delta_{\bar\eta}) e^{\bar\varphi\eta + \bar\eta\varphi}|_{\bar\eta=\eta=0}\Big)\Big] \\ &= H(\delta_\eta, \delta_{\bar\eta}) \Big[\int d\mu_C(\bar\varphi, \varphi)\; \Big(\varphi_{[a]} - C_{[a][b]} \eta_{[b]}\Big) e^{\bar\varphi\eta + \bar\eta\varphi}\Big]\Big|_{\bar\eta=\eta=0} \,. \end{aligned} \tag{A.7}$$

The latter expression can be calculated using Lemma 1 and the fact that number of fields φ and $\bar\varphi$ should be the same in order to give a nonvanishing amplitude:

$$\begin{aligned} &\int d\mu_C(\bar\varphi, \varphi)\; \Big(\varphi_{[a]} - C_{[a][b]} \eta_{[b]}\Big) e^{\bar\varphi\eta + \bar\eta\varphi} = \\ &\int d\mu_C(\bar\varphi, \varphi)\; \Big(\varphi_{[a]} \sum_n \frac{1}{(n+1)!n!} (\bar\varphi\eta)^{n+1} (\bar\eta\varphi)^n - C_{[a][b]} \eta_{[b]} \sum_n \frac{1}{(n!)^2} (\bar\varphi\eta)^n (\bar\eta\varphi)^n\Big) \end{aligned}$$

$$= \sum_n \frac{1}{(n!)^2} C_{[a][b]}\eta_{[b]}\; n!\; (\bar\eta C\eta)^n - C_{[a][b]}\eta_{[b]} \sum_n \frac{1}{(n!)^2}\; n!\; (\bar\eta C\eta)^n = 0\;. \tag{A.8}$$

□

Lemma 2 can be translated in terms of colored fields as

$$\int d\mu_C(\bar\varphi,\varphi)\, \Big(\varphi^i_{[a]} H(\bar\varphi,\varphi) - C^{ij}_{[a][b]}\delta_{\bar\varphi^j_{[b]}} H(\bar\varphi,\varphi)\Big) = 0\;. \tag{A.9}$$

Indeed, following step by step the previous proof, we have

$$\begin{aligned}
&\int d\mu_C(\bar\varphi^\ell,\varphi^\ell)\, \Big(\varphi^i_{[a]} - C^{ij}_{[a][b]}\eta^j_{[b]}\Big) e^{\bar\varphi^j\eta^j+\bar\eta^j\varphi^j} \prod_{k\neq j}\; e^{\bar\varphi^k\eta^k+\bar\eta^k\varphi^k} = \\
&\int d\mu_C(\bar\varphi^\ell,\varphi^\ell)\, \Big(\varphi^i_{[a]} \sum_n \frac{1}{(n+1)!n!}(\bar\varphi^j\eta^j)^{n+1}(\bar\eta^j\varphi^j)^n - C^{ij}_{[a][b]}\eta^j_{[b]} \sum_n \frac{1}{(n!)^2}(\bar\varphi^j\eta^j)^n(\bar\eta^j\varphi^j)^n\Big) \prod_{k\neq j} e^{\bar\varphi^k\eta^k+\bar\eta^k\varphi^k} \\
&= \int d\mu_C(\bar\varphi^{\ell'},\varphi^{\ell'})\Big(\sum_n \frac{1}{(n!)^2} C^{ij}_{[a][b]}\eta^j_{[b]}\; n!\; (\bar\eta C\eta)^n - C^{ij}_{[a][b]}\eta^j_{[b]} \sum_n \frac{1}{(n!)^2}\; n!\; (\bar\eta C\eta)^n = 0\Big) \prod_{k\neq j}\; e^{\bar\varphi^k\eta^k+\bar\eta^k\varphi^k}\;.
\end{aligned} \tag{A.10}$$

Lemma 3. *Modifying the covariance C for $C+A$, the following relation holds for any functional $H(\bar\varphi,\varphi)$*

$$\int d\mu_{C+A}(\bar\varphi,\varphi) H(\bar\varphi,\varphi) = \int d\mu_C(\bar\varphi,\varphi)\; e^{\delta_\varphi A \delta_{\bar\varphi}} H(\bar\varphi,\varphi)\;. \tag{A.11}$$

Proof: First, one performs the expansion using again the source term and uses Lemma 1 in order to obtain

$$\begin{aligned}
&H(\delta_\eta,\delta_{\bar\eta})\Big[\int d\mu_{C+A}(\bar\varphi,\varphi) e^{\bar\varphi\eta+\bar\eta\varphi}\Big]\Big|_{\bar\eta=\eta=0} = H(\delta_\eta,\delta_{\bar\eta})\Big[e^{\bar\eta C\eta} e^{\bar\eta A\eta}\Big|_{\bar\eta=\eta=0} \\
&= H(\delta_\eta,\delta_{\bar\eta})\Big[\int d\mu_C(\bar\varphi,\varphi) e^{\bar\eta A\eta} e^{\bar\varphi\eta+\bar\eta\varphi}\Big]\Big|_{\bar\eta=\eta=0} = \int d\mu_C(\bar\varphi,\varphi)\; e^{\delta_\varphi A\delta_{\bar\varphi}} H(\delta_\eta,\delta_{\bar\eta})\Big[e^{\bar\varphi\eta+\bar\eta\varphi}\Big]\Big|_{\bar\eta=\eta=0}
\end{aligned} \tag{A.12}$$

which is the desired result. □

Appendix B: Unitary transformations

1. Left/Right invariant unitary operators

Let us recall first some basics facts of $G = SU(2)$ representation theory. A Wigner matrix element of a $SU(2)$ group element g in the representation j will be denoted by $D^j_{mn}(g)$. Note the properties of these representation matrices given by $\bar D^j_{mn}(g) = D^j_{nm}(g^{-1}) = (-)^{m-n} D^j_{-m-n}(g)$ and $\int dg\, D^j_{mn}(g)\bar D^{j'}_{m'n'}(g) = (1/d_j)\delta^{jj'}\delta_{mm'}\delta_{nn'}$, where $d_j = 2j+1$. In the following, we will use the symbol $\int_h := \int dh$ for denoting the Haar integral with respect to the variable h. Dumb sums like $\sum$ without specifying the arguments mean that these sums are performed over all repeated discrete variables.

Any function of one variable over G can be expanded in representations via Peter-Weyl theorem as

$$f(g) = \sum \sqrt{d_j} f^j_{mn} D^j_{mn}(g) := \sum_{j,m,n} \sqrt{d_j} f^j_{mn} D^j_{mn}(g)\;, \qquad f^j_{mn} = \sqrt{d_j}\int_g f(g)\bar D^j_{mn}(g)\;. \tag{B.1}$$

An operator over the one variable functions is defined by a kernel

$$U(\alpha,\beta) = \sum \sqrt{d_{j_1}d_{j_2}}\, U^{j_1j_2}_{m_1n_1\; m_2n_2} D^{j_1}_{m_1n_1}(\alpha)\bar D^{j_2}_{m_2n_2}(\beta)\;, \quad \forall \alpha,\beta\in G\;. \tag{B.2}$$

The normalization is justified by the fact that

$$\begin{aligned}
U(f)(\alpha) &= \int_h U(\alpha,h)f(h) = \sum \sqrt{d_{j_1}d_{j_2}}\, U^{j_1j_2}_{m_1n_1\; m_2n_2} D^{j_1}_{m_1n_1}(\alpha) \sum \sqrt{d_j} f^j_{mn} \int_h \bar D^{j_2}_{m_2n_2}(h) D^j_{mn}(h) \\
&= \sum \sqrt{d_{j_1}} \Big[U^{j_1j_2}_{m_1n_1\; m_2n_2} f^{j_2}_{m_2n_2}\Big] D^{j_1}_{m_1n_1}(\alpha)\;.
\end{aligned} \tag{B.3}$$

For instance, the identity operator $\mathbb{I}$ possesses the tensor components $\mathbb{I}^{j_1j_2}_{m_1n_1\; m_2n_2} = \delta^{j_1j_2}\delta_{m_1m_2}\delta_{n_1n_2}$ and the kernel

$$\mathbb{I}(\alpha,\beta) = \sum \sqrt{d_{j_1}d_{j_2}}\,\delta^{j_1j_2}\delta_{m_1m_2}\delta_{n_1n_2} D^{j_1}_{m_1n_1}(\alpha)\bar D^{j_2}_{m_2n_2}(\beta) = \sum_j d_j D^j_{mn}(\alpha)\bar D^j_{mn}(\beta) = \sum_j d_j \chi^j(\alpha\beta^{-1}) = \delta(\alpha\beta^{-1})\;.$$

Note that we introduce the symbol $\chi^j(g) := \sum_m D^j_{mm}(g)$ denoting the character of the group element g in the representation j.

The adjoint of an operator U is denoted by $U^\dagger$ and its kernel is defined by $[U^\dagger](\alpha,\beta) = \overline{U(\beta,\alpha)}$. Hence, an operator U is unitary if the following relation holds

$$[U^\dagger U](\alpha,\beta) = [UU^\dagger](\alpha,\beta) = \int_h U(\alpha,h)\overline{U(\beta,h)} = \mathbb{I}(\alpha,\beta) = \delta(\alpha\beta^{-1})\ . \tag{B.4}$$

Furthermore, by noting that

$$\begin{aligned}&\sum \sqrt{d_{j_1}d_{j_2}}U^{j_1j_2}_{m_1n_1\ m_2n_2}D^{j_1}_{m_1n_1}(\alpha)\sqrt{d_{j_1'}d_{j_2'}}\ \overline{U^{j_1'j_2'}_{m_1'n_1'\ m_2'n_2'}}\bar{D}^{j_1'}_{m_1'n_1'}(\beta)\int_h \bar{D}^{j_2}_{m_2n_2}(h)D^{j_2'}_{m_2'n_2'}(h)\\ &= \sum\sqrt{d_{j_1}d_{j_1'}}\Big[U^{j_1j_2}_{m_1n_1\ m_2n_2}\overline{U^{j_1'j_2}_{m_1'n_1'\ m_2n_2}}\Big]D^{j_1}_{m_1n_1}(\alpha)\bar{D}^{j_1'}_{m_1'n_1'}(\beta)\ ,\end{aligned} \tag{B.5}$$

as expected, an operator is unitary if and only if

$$\sum_{j_2,m_2,n_2} U^{j_1j_2}_{m_1n_1\ m_2n_2}\overline{U^{j_1'j_2}_{m_1'n_1'\ m_2n_2}} = \delta^{j_1j_1'}\delta_{m_1m_1'}\delta_{n_1n_1'}\ . \tag{B.6}$$

Among the unitary operators there exists a special class of unitaries, those invariant under left group action

$$\forall p,\alpha,\beta\in G\ ,\qquad A(p\alpha,p\beta) = A(\alpha,\beta)\ ,\qquad \int_h A(\alpha,h)\overline{A(\beta,h)} = \delta(\alpha^{-1}\beta)\ . \tag{B.7}$$

Due to that invariance, we have

$$\begin{aligned}A(\alpha,\beta) &= \int_h A(h\alpha,h\beta) = \sum\sqrt{d_{j_1}d_{j_2}}U^{j_1j_2}_{m_1m_2\ n_1n_2}D^{j_1}_{k_1n_1}(\alpha)\bar{D}^{j_2}_{k_2n_2}(\beta)\int_h D^{j_1}_{m_1k_1}(h)\bar{D}^{j_2}_{m_2k_2}(h)\\ &= \sum U^{j_1j_2}_{m_1m_2\ n_1n_2}\bar{D}^{j_1}_{n_1k_1}(\alpha^{-1})\bar{D}^{j_2}_{k_2n_2}(\beta)\delta^{j_1j_2}\delta_{m_1m_2}\delta_{k_1k_2}\\ &= \sum\Big[\sum_{m_1}U^{j_1j_1}_{m_1m_1\ n_1n_2}\Big]\bar{D}^{j_1}_{n_1n_2}(\alpha^{-1}\beta) \equiv \sum d_jA^j_{pq}D^j_{pq}(\alpha^{-1}\beta)\end{aligned} \tag{B.8}$$

where we read off $A^j_{pq} = [(-)^{p-q}/d_j]\sum_m U^{jj}_{mm\ -p-q}$. Imposing unitarity on these operators yields

$$\begin{aligned}\int_h A(\alpha,h)\overline{A(\beta,h)} &= \sum d_jd_{j'}A^j_{pq}\overline{A^{j'}_{p'q'}}D^j_{pr}(\alpha^{-1})\bar{D}^{j'}_{p'r'}(\beta^{-1})\int_h D^j_{rq}(h)\bar{D}^{j'}_{r'q'}(h)\ ,\\ &= \sum d_j\Big[\sum_q A^j_{pq}\overline{A^j_{p'q}}\Big]D^j_{pr}(\alpha^{-1})\bar{D}^j_{p'r}(\beta^{-1}) = \sum d_j\Big[\sum_q A^j_{pq}\overline{A^j_{p'q}}\Big]D^j_{pp'}(\alpha^{-1}\beta)\ .\end{aligned} \tag{B.9}$$

Therefore the invariant unitary operators are represented by unitary matrices in each dimension d_j of the representation

$$\sum_q A^j_{pq}\overline{A^j_{p'q}} = \delta_{pp'}\ . \tag{B.10}$$

We call such an A as a *left invariant unitary*. For *right invariant unitary* operator, a similar definition can be given and a little computation leads to

$$A(\alpha,\beta){=}\int_h A(\alpha h,\beta h) = \sum\Big[\sum_n U^{j_1j_1}_{m_1m_2\ nn}\Big]D^{j_1}_{m_1m_2}(\alpha\beta^{-1}) \equiv \sum d_jA^j_{pq}D^j_{pq}(\alpha\beta^{-1})\ , \tag{B.11}$$

with $A^j_{pq} := (1/d_j)\sum_n U^{jj}_{pq\ nn}$. Further imposing unitarity yields some conditions on the coefficients A^j_{pq}:

$$\begin{aligned}\int dhA(\alpha,h)\overline{A(\beta,h)} &= \sum d_jd_{j'}A^j_{pq}\overline{A^{j'}_{p'q'}}\int_h D^j_{pq}(\alpha h^{-1})\overline{D^j_{p'q'}(\beta h^{-1})}\\ &= \sum d_jA^j_{pq}\overline{A^j_{p'q}}D^j_{pr}(\alpha)\overline{D}^j_{p'r}(\beta) = \sum d_j[\sum_q A^j_{pq}\overline{A^j_{p'q}}]D^j_{pp'}(\alpha\beta^{-1})\ ,\end{aligned} \tag{B.12}$$

so that $\sum_q A^j_{pq}\overline{A^j_{p'q}} = \delta_{pp'}$ which is a similar to left invariant unitary condition (B.10). However, assuming that we impose that the following is unitary

$$\int dhA(h,\alpha)\overline{A(h,\beta)} = \sum d_j[\sum_p A^j_{pq}\overline{A^j_{pq'}}]D^j_{q'q}(\beta^{-1}\alpha)\ , \tag{B.13}$$

one could get another condition on the A^j_{pq}'s that is

$$\sum_p A^j_{pq}\overline{A^j_{pq'}} = \delta_{qq'}\ . \tag{B.14}$$

2. Unitary transformation of fields

1-action on fields - Consider the right invariant unitary operator as detailed in Appendix B 1 which is of the form

$$A(g,h) = \sum d_j A^j_{mn} D^j_{mn}(gh^{-1}) \ . \tag{B.15}$$

The condition $\sum_n A^j_{mn}\bar{A}^j_{m'n} = \delta_{mm'}$, ensures that, for all j, A^j is a unitary matrix of rank $2j+1$, i.e. $A^j \in U(2j+1)$.
Given a D-dimensional GFT, the 1-action of A on a field means that the said field transforms with respect to its first argument, namely

$${}^A\varphi(g_1,g_2,\dots,g_D) = \int_h A(g_1,h)\varphi(h,g_2,\dots,g_D) \ , \quad {}^A\bar{\varphi}(g_1,g_2,\dots,g_D) = \int dh \ \overline{A(g_1,h)\varphi(h,g_2,\dots,g_D)} \ , \tag{B.16}$$

this is, using the mode expansion (and equivalent in term of matrices and tensors),

$$\begin{aligned}
{}^A\varphi(g_1,g_2,\dots,g_D) &= \sum d_j\sqrt{d_{j_1}}A^j_{pq}\varphi^{j_1 j_i}_{m_1n_1\ m_in_i}\int_h D^j_{pq}(g_1h^{-1})D^{j_1}_{m_1n_1}(h)\prod_{i\neq 1}\sqrt{d_{j_i}}D^{j_i}_{m_in_i}(g_i)\\
&= \sum A^{j_1}_{m_1q}\varphi^{j_1j_i}_{qn_1\ m_in_i}\ \sqrt{d_{j_1}}D^{j_1}_{m_1n_1}(g_1)\prod_{i\neq 1}[\sqrt{d_{j_i}}D^{j_i}_{m_in_i}(g_i)] \ ,\\
{}^A\bar{\varphi}(g_1,g_2,\dots,g_D) &= \sum d_j\sqrt{d_{j_1}}\bar{A}^j_{pq}\bar{\varphi}^{j_ij_1}_{m_1n_1\ m_in_i}\int_h \bar{D}^j_{pq}(g_1h^{-1})\bar{D}^{j_1}_{m_1n_1}(h)\prod_{i\neq 1}\sqrt{d_{j_i}}\bar{D}^{j_i}_{m_in_i}(g_i)\\
&= \sum \bar{A}^{j_1}_{m_1q}\bar{\varphi}^{j_ij_1}_{qn_1\ m_in_i}\ \sqrt{d_{j_1}}\bar{D}^{j_1}_{m_1n_1}(g_1)\prod_{i\neq 1}[\sqrt{d_{j_i}}\bar{D}^{j_i}_{m_in_i}(g_i)] \ .
\end{aligned} \tag{B.17}$$

Hence, the modes of the transformed field ${}^A\varphi$ can be related to the modes of the prime field as

$${}^A\varphi^{j_1j_i}_{m_1n_1\ m_in_i} = \sum_p A^{j_1}_{m_1p}\varphi^{j_1j_i}_{pn_1\ m_in_i} \ , \qquad {}^A\bar{\varphi}^{j_ij_1}_{m_in_i\ m_1n_1} = \sum_p \bar{A}^{j_1}_{m_1p}\bar{\varphi}^{j_ij_1}_{pn_1\ m_in_i} \ , \tag{B.18}$$

with the notable feature that only the first set of labels coined by 1, j_1, m_1 and n_1, is actually involved under this transformation. Returning to the group formulation, this field transformation will be referred to the equivalent forms when no possible confusion may occur

$$A_{gh}\varphi_{h.} := {}^A\varphi_{g.} = \int_h A(g,h)\varphi(h,(.)) \ , \qquad \bar{\varphi}_{h.}A^{-1}_{hg} := {}^A\bar{\varphi}_{g.} = \int_h \overline{A(g,h)\varphi(h,(.))} \ . \tag{B.19}$$

Infinitesimal transformation - Given a right invariant unitary A, its component $A^j \in U(d_j)$, and therefore there exists $(B^j)^\dagger = B^j$ an Hermitian matrix of the same dimension d_j, such that

$$A^j_{mn} = \delta^j_{mn} + \imath B^j_{mn} \ , \qquad \bar{B}^j_{nm} = B^j_{mn} \ . \tag{B.20}$$

We can expand A infinitesimally at first order in B:

$$A(h,g) \simeq \sum_j d_j\sum_{mn}(\delta^j_{mn} + \imath B^j_{mn})D^j_{mn}(h^{-1}g) = \delta(gh^{-1}) + \imath B(g,h) \ . \tag{B.21}$$

B is a Hermitian kernel in the sense that

$$\overline{B(h,g)} = \sum_j d_j\sum_{mn}\bar{B}^j_{mn}\bar{D}^j_{mn}(hg^{-1}) = \sum_j d_j\sum_{mn}B^j_{nm}D^j_{nm}(gh^{-1}) = B(g,h) \ . \tag{B.22}$$

Appendix C: Calculation of infinitesimal variations under unitary transformations

1. General unitary transformation

We start by considering a general unitary operator U which satisfies $U^{ij}_{[a][b]}\bar{U}^{kj}_{[c][b]} = \delta^{ik}\delta_{[a][c]}$ where $[a] := (a_1,a_2,a_3)$, and $\delta_{[a][b]}$ stands for the kernel of the unit operator identifying each field arguments. Appendix B 2 provides a particular type of this unitary operator of the form $U^{jk} = \delta^{jk}A\otimes\mathbb{I}\otimes\mathbb{I}$ that we will discuss in detail in the next subsection. In this appendix, we assume a formal and general expression for this operator and infer the infinitesimal variations for the action, the covariance and source term.

We assume that, under U, the colored fields φ^i transform as

$$ {}^U\varphi^i_{[a]} = U^{ij}_{[a][b]}\varphi^j_{[b]} \; , \qquad {}^U\bar\varphi^i_{[a]} = \bar\varphi^j_{[b]}\bar U^{ij}_{[a][b]} \; , \qquad \frac{\delta}{\delta\, {}^U\varphi^i_{[a]}} = \frac{\delta}{\delta\varphi^j_{[b]}}(U^{-1})^{ji}_{[b][a]} \; , \qquad \frac{\delta}{\delta\, {}^U\bar\varphi^i_{[a]}} = U^{ij}_{[a][b]}\frac{\delta}{\delta\bar\varphi^j_{[b]}} \quad . \tag{C.1} $$

A sum (integration on arguments and discrete sum on colors) is understood over all repeated indices. This transformation therefore mixes both colors and group arguments of the fields.

$S^{\rm int}$ becomes after this field transformation

$$ \begin{aligned} {}^U S^{\rm int} = & \frac{\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\int_{[a][b][c][d]} U^{0i_0}_{h_{03}h_{02}h_{01}\,[a]}\varphi^{i_0}_{[a]}\,\bar\varphi^{i_1}_{[b]}\bar U^{1i_1}_{h_{10}h_{13}h_{12}\,[b]}\,U^{2i_2}_{h_{21}h_{20}h_{23}\,[c]}\,\varphi^{i_2}_{[c]}\,\bar\varphi^{i_3}_{[d]}\bar U^{3i_3}_{h_{32}h_{31}h_{30}\,[d]} \\ & +\frac{\bar\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\int_{[a][b][c][d]} \bar\varphi^{i_0}_{[a]}\bar U^{0i_0}_{h^{03}h^{02}h^{01}[a]}\varphi^{i_1}_{[b]}U^{1i_1}_{h^{10}h^{13}h^{12}[b]}\,\bar\varphi^{i_2}_{[c]}\bar U^{2i_2}_{h^{21}h^{20}h^{23}\,[c]}\,U^{3i_3}_{h^{32}h^{31}h^{30}\,[d]}\varphi^{i_3}_{[d]} \; . \end{aligned} \tag{C.2} $$

Expanding the unitary operator around the identity, one has $U^{ij}_{[a][b]} = \delta^{ij}\delta_{[a][b]} + \imath B^{ij}_{[a][b]}$, where B is an Hermitian kernel i.e. ought to satisfy $\bar B^{ij}_{[b][a]} = B^{ji}_{[a][b]}$. At first order in B, the variation of the interaction part can be computed as follows:

$$ \begin{aligned} \delta_B S^{\rm int} = & \frac{\imath\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\Big[-\int_{[d]} B^{i_3 3}_{[d][h_{3i}]}\varphi^0_{[h_{0i}]}\,\bar\varphi^1_{[h_{1i}]}\varphi^2_{[h_{2i}]}\,\bar\varphi^{i_3}_{[d]} + \int_{[a]} B^{0i_0}_{[h_{0i}][a]}\varphi^{i_0}_{[a]}\,\bar\varphi^1_{[h_{1i}]}\varphi^2_{[h_{2i}]}\,\bar\varphi^3_{[h_{3i}]} \\ & -\int_{[b]} B^{i_1 1}_{[b][h_{1i}]}\varphi^0_{[h_{0i}]}\,\bar\varphi^{i_1}_{[b]}\varphi^2_{[h_{2i}]}\,\bar\varphi^3_{[h_{3i}]} + \int_c B^{2i_2}_{[h_{2i}][c]}\,\varphi^0_{[h_{0i}]}\,\bar\varphi^1_{[h_{1i}]}\varphi^{i_2}_{[c]}\,\bar\varphi^3_{[h_{3i}]}\Big] \\ & +\frac{\imath\bar\lambda}{\sqrt{\delta^N(e)}}\int_{h^{ij}}\int_{[d]}\Big[B^{3i_3}_{[h^{3i}][d]}\bar\varphi^0_{[h^{0i}]}\varphi^1_{[h^{1i}]}\bar\varphi^2_{[h^{2i}]}\varphi^{i_3}_{[d]} - \int_{[a]} B^{i_0 0}_{[a][h^{0i}]}\bar\varphi^{i_0}_{[a]}\varphi^1_{[h^{1i}]}\bar\varphi^2_{[h^{2i}]}\varphi^3_{[h^{3i}]} \\ & +\int_{[b]} B^{1i_1}_{[h^{1i}][b]}\bar\varphi^0_{[h^{0i}]}\varphi^{i_1}_{[b]}\bar\varphi^2_{[h^{2i}]}\varphi^3_{[h^{3i}]} - \int_{[c]} B^{i_2 2}_{[c][h^{2i}]}\bar\varphi^0_{[h^{0i}]}\varphi^1_{[h^{1i}]}\bar\varphi^{i_2}_{[c]}\varphi^3_{[h^{3i}]}\Big] \; , \end{aligned} \tag{C.3} $$

that can be denoted compactly by

$$ \begin{aligned} & \delta_B S^{\rm int} := \imath\lambda\Big[[B\varphi]^0\bar\varphi^1\varphi^2\bar\varphi^3 - \varphi^0[\bar\varphi B]^1\varphi^2\bar\varphi^3 + \varphi^0\bar\varphi^1[B\varphi]^2\bar\varphi^3 - \varphi^0\bar\varphi^1\varphi^2[\bar\varphi B]^3\Big] \; + \; \imath\bar\lambda\{\bar\varphi\} \; , \\ & \bar\lambda\{\bar\varphi\} = -[\bar\varphi B]^0\varphi^1\bar\varphi^2\varphi^3 + \bar\varphi^0[B\varphi]^1\bar\varphi^2\varphi^3 - \bar\varphi^0\varphi^1[\bar\varphi B]^2\varphi^3 + \bar\varphi^0\varphi^1\bar\varphi^2[B\varphi]^3 \; . \end{aligned} \tag{C.4} $$

Meanwhile, the source terms have the infinitesimal variations

$$ \begin{aligned} & \delta_B(\bar\eta\varphi + \bar\varphi\eta) = \sum_i\int_{[g]}\Big({}^U\bar\varphi^i_{[g]}\eta^i_{[g]} + \bar\eta^i_{[g]}\;{}^U\varphi^i_{[g]}\Big) - (\bar\eta\varphi + \bar\varphi\eta) \\ & = \sum_i\int_{[g_i][a]}\Big((\delta^{ij}\delta_{[g][a]} - \imath\bar B^{ij}_{[g][a]})\bar\varphi^j_{[a]}\eta^i_{[g]} + \bar\eta^i_{[g]}(\delta^{ij}\delta_{[g][a]} + \imath B^{ij}_{[g][a]})\varphi^j_{[a]}\Big) - \sum_i\int_{[g]}(\bar\eta^i_{[g]}\varphi^i_{[g]} + \bar\varphi^i_{[g]}\eta^i_{[g]}) \\ & = \imath\sum_i\int_{[g][a]}\Big(-\bar\varphi^j_{[a]}B^{ji}_{[a][g]}\eta^i_{[g]} + \bar\eta^i_{[g]}B^{ij}_{[g][a]}\varphi^j_{[a]}\Big) =: \imath(-\bar\varphi B\eta + \bar\eta B\varphi) \; . \end{aligned} \tag{C.5} $$

Under (C.1), the partition function transforms according to

$$ Z(\bar\eta,\eta) = \int d\mu_{UCU^{-1}}(\bar\varphi,\varphi)e^{-S^{\rm int}\,(U\varphi,\bar\varphi U^{-1}) + \bar\varphi U^{-1}\eta + \bar\eta U\varphi} \; . \tag{C.6} $$

We have used the fact that the covariance varies as

$$ \begin{aligned} C^{kk'}_{[h][h']} &= \int d\mu_C(\,{}^U\bar\varphi^i,\;{}^U\varphi^i)\int_{[a][b]}\bar U^{kj}_{[h][a]}\,\bar\varphi^j_{[a]}\,U^{k'j'}_{[h'][b]}\,\varphi^{j'}_{[b]} \\ \bar U^{k'l'}_{[h'][b']}C^{kk'}_{[h][h']}U^{kl}_{[h][a']} &= \int d\mu_C(\,{}^U\bar\varphi^i,\;{}^U\varphi^i)\int_{[a][b][h][h']}U^{kl}_{[h][a']}\bar U^{kj}_{[h][a]}U^{k'j'}_{[h'][b]}\bar U^{k'l'}_{[h'][b']}\,\bar\varphi^j_{[a]}\varphi^{j'}_{[b]} \\ &= \int d\mu_C(\,{}^U\bar\varphi^i,\;{}^U\varphi^i)\int_{ab}\bar\varphi^l_{[a']}\varphi^{l'}_{[b']} := \,[UCU^{-1}]^{ll'}_{[a'][b']} := [\,{}^U C]^{ll'}_{[a'][b']} \; . \end{aligned} \tag{C.7} $$

Seeking the infinitesimal variation of the covariance, one finds

$$ \begin{aligned} [UCU^{-1}]^{ij}_{[a][b]} - C^{ij}_{[a][b]} &= (\delta^{li}\delta_{[c][a]} + \imath B^{li}_{[c][a]})C^{ll'}_{[c][c']}(\delta^{jl'}\delta_{[b][c']} - \imath B^{jl'}_{[b][c']}) - C^{ij}_{[a][b]} \\ &= \imath[-C^{il'}_{[a][c']}B^{jl'}_{[b][c']} + B^{li}_{[c][a]}C^{lj}_{[c][b]}] =: \imath[BC - CB]^{ij}_{[a][b]} \; . \end{aligned} \tag{C.8} $$

2. Right invariant unitary transformation

Working with a right invariant unitary in the sense of the 1-action of Appendix B 2 Eq. (B.19), we get a change of variables such that

$$
\begin{aligned}
&{}^{U}\varphi^0_{a.} = U_{ab}\varphi^0_{b.}\ , \qquad {}^{U}\bar\varphi^0_{a.} = \bar\varphi^0_{b.}(U^{-1})_{ba} = \bar U_{ab}\bar\varphi^0_{b.}\ , \qquad \frac{\delta}{\delta\, {}^{U}\varphi^0_{a.}} = \frac{\delta}{\delta\varphi^0_{b.}}(U^{-1})_{ba}\ , \qquad \frac{\delta}{\delta\, {}^{U}\bar\varphi^0_{a.}} = U_{ab}\frac{\delta}{\delta\bar\varphi^0_{b.}}\ , \\
&{}^{U}\varphi^1_{.a} = U_{ab}\varphi^1_{.b}\ , \qquad {}^{U}\bar\varphi^1_{.a} = \bar\varphi^1_{.b}(U^{-1})_{ba} = \bar U_{ab}\bar\varphi^1_{.b}\ , \qquad \frac{\delta}{\delta\, {}^{U}\varphi^1_{.a}} = \frac{\delta}{\delta\varphi^1_{.b}}(U^{-1})_{ba}\ , \qquad \frac{\delta}{\delta\, {}^{U}\bar\varphi^1_{.a}} = U_{ab}\frac{\delta}{\delta\bar\varphi^1_{.b}}\ . \quad (C.9)
\end{aligned}
$$

meanwhile colors 2 and 3 transform like 0 and 1, respectively. The subscripts a, b should be considered here as a unique group element (and not a triplet) and the dot notifies the position of the remaining arguments of the field. Hence fields 0 and 2 are transformed with respect to their first argument whereas fields 1 and 3 to their last argument.

Under U the term $S^{\rm int}$ transforms as

$$
\begin{aligned}
{}^{U}S^{\rm int} &= \frac{\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\int_{abcd} U_{h_{03}a}\varphi^0_{ah_{02}h_{01}}\ \bar\varphi^1_{h_{10}h_{13}b}\bar U_{h_{12}b}\ U_{h_{21}c}\ \varphi^2_{ch_{20}h_{23}}\ \bar\varphi^3_{h_{32}h_{31}d}\bar U_{h_{30}d} \\
&+\frac{\bar\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\int_{abcd} \bar U_{h^{03}a}\bar\varphi^0_{ah^{02}h^{01}}\varphi^1_{h^{10}h^{13}b}U_{h^{12}b}\ \bar U_{h^{21}c}\bar\varphi^2_{ch^{20}h^{23}}\ \varphi^3_{h^{32}h^{31}d}U_{h^{30}d}\ , \qquad (C.10)
\end{aligned}
$$

using the orthogonality relation the unitary operators, namely $U_{hb}\bar U_{hc} = \delta_{bc}$, we have after a proper renaming of variables

$$
{}^{U}S^{\rm int} = S^{\rm int}\ . \qquad (C.11)
$$

Remark that this symmetry can be even decomposed in two *minimal* and independent symmetries: one performed on the couple $(0, 3)$ and another one performed on $(1, 2)$. Each of these latter symmetries does not modify $S^{\rm int}$ and can be used to determine all the subsequent developments without loss of generality. These simpler symmetries can be useful to reduce the generic WT identities and to obtain particular graphical equations.

Under (C.9), the partition function undergoes the following modification

$$
Z(\bar\eta,\eta) = \int d\mu_{UCU^{-1}}(\bar\varphi,\varphi)e^{-S^{\rm int}\,(U\varphi,\bar\varphi U^{-1})+\bar\varphi U^{-1}\eta+\bar\eta U\varphi}\ . \qquad (C.12)
$$

We have used the fact that the covariance transforms as follows

$$
\begin{aligned}
C^{ii=0,2}_{h_0h_1h_2;\,h'_0h'_1h'_2} &= \int d\mu_C(\,{}^{U}\bar\varphi^i,\ {}^{U}\varphi^i)\int_{ab}\bar U_{h_0a}\ \bar\varphi^{i=0,2}_{ah_1h_2}\ U_{h'_0b}\ \varphi^{i=0,2}_{bh'_1h'_2} \\
\bar U_{h'_0c'}C^{ii=0,2}_{h_0h_1h_2;\,h'_0h'_1h'_2}U_{h_0c} &= \int_{abh_0h'_0} U_{h_0c}\bar U_{h_0a}\bar U_{h'_0c'}U_{h'_0b}\ \bar\varphi^{i=0,2}_{ah_1h_2}\ \varphi^{i=0,2}_{bh'_1h'_2} := [UCU^{-1}]_{ch_1h_2;\,c'h'_1h'_2} \\
C^{ii=1,3}_{h_0h_1h_2;\,h'_0h'_1h'_2} &= \int d\mu_C(\,{}^{U}\bar\varphi^i,\ {}^{U}\varphi^i)\int_{ab}\bar\varphi^{i=1,3}_{h_0h_1a}\bar U_{h_2a}\ \ \varphi^{i=1,3}_{h'_0h'_1b}U_{h'_2b} \\
\bar U_{h'_2c'}C^{ii=1,3}_{h_0h_1h_2;\,h'_0h'_1h'_2}U_{h_2c} &= \int_{abh_2h'_2} U_{h_2c}\bar U_{h_2a}\bar U_{h'_2c'}U_{h'_2b}\bar\varphi^{i=1,3}_{h_0h_1a}\varphi^{i=1,3}_{h'_0h'_1b} := [UCU^{-1}]_{h_0h_1c;\,h'_0h'_1c'}\ . \qquad (C.13)
\end{aligned}
$$

For a small B, we decompose $U_{ab} = \delta_{ab} + \imath B_{ab}$, hence at first order in B, we can explicitly check that the variation of the interaction part vanishes. After a straightforward computation, one has

$$
\begin{aligned}
\delta_B S^{\rm int} &= \frac{\imath\lambda}{\sqrt{\delta^N(e)}}\int_{h_{ij}}\Big[-\int_d \bar B_{h_{30}d}\varphi^0_{h_{03}h_{02}h_{01}}\bar\varphi^1_{h_{10}h_{13}h_{12}}\varphi^2_{h_{21}h_{20}h_{23}}\ \bar\varphi^3_{h_{32}h_{31}d} \\
&\quad+\int_a B_{h_{03}a}\varphi^0_{ah_{02}h_{01}}\bar\varphi^1_{h_{10}h_{13}h_{12}}\varphi^2_{h_{21}h_{20}h_{23}}\ \bar\varphi^3_{h_{32}h_{31}h_{30}} \\
&\quad-\int_b \bar B_{h_{12}b}\varphi^0_{h_{03}h_{02}h_{01}}\bar\varphi^1_{h_{10}h_{13}b}\varphi^2_{h_{21}h_{20}h_{23}}\ \bar\varphi^3_{h_{32}h_{31}h_{30}} \\
&\quad+\int_c B_{h_{21}c}\ \varphi^0_{h_{03}h_{02}h_{01}}\bar\varphi^1_{h_{10}h_{13}h_{12}}\varphi^2_{ch_{20}h_{23}}\ \bar\varphi^3_{h_{32}h_{31}h_{30}}\Big] \\
&\quad+\frac{\imath\bar\lambda}{\sqrt{\delta^N(e)}}\int_{h^{ij}}\Big[\int_d B_{h^{30}d}\varphi^0_{h^{03}h^{02}h^{01}}\bar\varphi^1_{h^{10}h^{13}h^{12}}\varphi^2_{h^{21}h^{20}h^{23}}\ \bar\varphi^3_{h^{32}h^{31}d} \\
&\quad-\int_a \bar B_{h^{03}a}\varphi^0_{ah^{02}h^{01}}\bar\varphi^1_{h^{10}h^{13}h^{12}}\varphi^2_{h^{21}h^{20}h^{23}}\ \bar\varphi^3_{h^{32}h^{31}h^{30}}
\end{aligned}
$$

$$+\int_b B_{h^{12}b}\varphi^0_{h^{03}h^{02}h^{01}}\bar\varphi^1_{h^{10}h^{13}b}\varphi^2_{h^{21}h^{20}h^{23}}\,\bar\varphi^3_{h^{32}h^{31}h^{30}}$$
$$-\int_c \bar B_{h^{21}c}\,\varphi^0_{h^{03}h^{02}h^{01}}\bar\varphi^1_{h^{10}h^{13}h^{12}}\varphi^2_{ch^{20}h^{23}}\,\bar\varphi^3_{h^{32}h^{31}h^{30}}\Big]\Big]\,, \tag{C.14}$$

then using the Herimiticity of the kernel $\bar B_{ab} = B_{ba}$ this term cancels. Thus, $\delta_B S^{\rm int} = 0$.

The infinitesimal variations of the covariance are given by

$$\begin{aligned}
[UCU^{-1}]^{ii=0,2}_{[a][b]} - C^{ii=0,2}_{[a][b]} &= (\delta_{ca_0} + \imath B_{ca_0})C^{ii=0,2}_{ca_1a_2;\,c'b_1b_2}(\delta_{c'b_0} - \imath\bar B_{c'b_0}) - C^{ii=0,2}_{[a][b]} \\
&= \imath[-C^{ii=0,2}_{[a];\,c'b_1b_2}B_{b_0c'} + B_{ca_0}C^{ii=0,2}_{ca_1a_2;\,[b]}] =: \imath[BC-CB]^{ii=0,2}_{[a][b]}\,, \\
[UCU^{-1}]^{ii=1,3}_{[a][b]} - C_{[a][b]} &= (\delta_{ca_2} + \imath B_{ca_2})C_{a_0a_1c;\,b_0b_1c'}(\delta_{c'b_2} - \imath\bar B_{c'b_2}) - C_{[a][b]} \\
&= \imath[-C^{ii=0,2}_{[a];\,b_0b_1c'}B_{b_2c'} + B_{ca_2}C^{ii=0,2}_{a_0a_1c;\,[b]}] =: \imath[BC-CB]^{ii=1,3}_{[a][b]}\,,
\end{aligned} \tag{C.15}$$

whereas the source terms can be varied as follows

$$\begin{aligned}
&\delta_B(\bar\eta\varphi + \bar\varphi\eta) = \sum_i \int_{[g]} \Big({}^U\bar\varphi^i_{[g]}\eta^i_{[g]} + \bar\eta^i_{[g]}\;{}^U\varphi^i_{[g]}\Big) - (\bar\eta\varphi + \bar\varphi\eta) \\
&= \imath\int_{g_i a}\Big\{\sum_{i=0,2}\big(-\bar\varphi^i_{a.}B_{ag_0}\eta^i_{g_0.} + \bar\eta^i_{g_0.}B_{g_0a}\varphi^i_{a.}\big) + \sum_{i=1,3}\big(-\bar\varphi^i_{.a}B_{ag_2}\eta^i_{.g_2} + \bar\eta^i_{.g_2}B_{g_2a}\varphi^i_{.a}\big)\Big\} \\
&=: \imath(-\bar\varphi B\eta + \bar\eta B\varphi)\,.
\end{aligned} \tag{C.16}$$

Appendix D: Free energy evaluations

1. General unitary transformation

We start by giving the variation of free energy (22) under the infinitesimal transformation generated by a general unitary operator[4]

$$\begin{aligned}
&\frac{\delta\ln Z(\eta,\bar\eta)}{\imath\delta B^{ij}_{[\mu][\nu]}} = 0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{\delta_{\varphi^j_{[\nu]}}C^{il}_{[\mu][\alpha]}\delta_{\bar\varphi^l_{[\alpha]}} - \delta_{\varphi^l_{[\alpha]}}C^{lj}_{[\alpha][\nu]}\delta_{\bar\varphi^i_{[\mu]}} - \bar\varphi^i_{[\mu]}\eta^j_{[\nu]} + \bar\eta^i_{[\mu]}\varphi^j_{[\nu]} \\
&-\lambda\Big[\delta^{i0}\varphi^j_{[\nu]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]} - \delta^{j1}\bar\varphi^i_{[\mu]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]} + \delta^{i2}\varphi^j_{[\nu]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]} - \delta^{j3}\bar\varphi^i_{[\mu]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big] \\
&-\bar\lambda\{\bar\varphi\}\Big\}\,e^{-S^{\rm int}\,(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\,,
\end{aligned} \tag{D.1}$$

where the remaining arguments of fields which do not appear are integrated (see (23)). Using the Hermiticity of the covariance $\bar C^{ij}_{[b][a]} = C^{ji}_{[a][b]}$ (as this is the two-point correlation function and the latter is a necessarily Hermitian), (D.1) is again

$$\begin{aligned}
&\frac{\delta\ln Z(\eta,\bar\eta)}{\imath\delta B^{ij}_{[\mu][\nu]}} = 0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{\delta_{\varphi^j_{[\nu]}}[C\delta_{\bar\varphi}]^i_{[\mu]} - \delta_{\bar\varphi^i_{[\mu]}}[\delta_\varphi C^\dagger]^j_{[\nu]} - \bar\varphi^i_{[\mu]}\eta^j_{[\nu]} + \bar\eta^i_{[\mu]}\varphi^j_{[\nu]} \\
&-\lambda\Big[\delta^{i0}\varphi^j_{[\nu]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]} - \delta^{j1}\bar\varphi^i_{[\mu]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]} + \delta^{i2}\varphi^j_{[\nu]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]} - \delta^{j3}\bar\varphi^i_{[\mu]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big] \\
&-\bar\lambda\{\bar\varphi\}\Big\}e^{-S^{\rm int}\,(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\,,
\end{aligned} \tag{D.2}$$

where $\bar\lambda\{\bar\varphi\}$ can be obtained from the term in λ by multiplying it by (-1), the symmetry $(j,\nu)\leftrightarrow(i,\mu)$ and complex conjugation.

Let us multiply (D.2) and sum over repeated indices by $\bar C^{j'j}_{[a][\nu]}$ and $C^{i'i}_{[b][\mu]}$, we get, after renaming i', j' by i, j:

$$0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{-[C\bar\varphi]^i_{[b]}\,[\eta C^\dagger]^j_{[a]} + [C\bar\eta]^i_{[b]}\,[\varphi C^\dagger]^j_{[a]}$$

[4] At each step of calculation and for simplicity purpose, we will not display the term $\bar\lambda\{\bar\varphi\}$. However we will provide an explicit symmetry of its analogous, i.e. the term with coefficient λ, from which $\bar\lambda\{\bar\varphi\}$ can be determined without ambiguity.

$$-\lambda\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]} - \bar C^{j1}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}$$
$$+C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]} - \bar C^{j3}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]_{[\nu]}\Big] \;-\; \bar\lambda\{\bar\varphi\}$$
$$+C^{ii'}_{[b][\mu]}[\delta_\varphi C^\dagger]^j_{[a]}[C\delta_{\bar\varphi}]^{i'}_{[\mu]} - \bar C^{jj'}_{[a][\nu]}[C\delta_{\bar\varphi}]^i_{[b]}[\delta_\varphi C^\dagger]^{j'}_{[\nu]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;, \tag{D.3}$$

here $\bar\lambda\{\bar\varphi\}$ can be obtained from the term in λ by multiplication by (-1), complex conjugation of fields and symmetry

$$(i,a,\nu,[\varphi C^\dagger]) \leftrightarrow (j,b,\mu,[C\bar\varphi])\;. \tag{D.4}$$

Using Lemma 2, we have:

$$\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{C^{ii'}_{[b][\mu]}[\delta_\varphi C^\dagger]^j_{[a]}[C\delta_{\bar\varphi}]^{i'}_{[\mu]} - \bar C^{jj'}_{[a][\nu]}[C\delta_{\bar\varphi}]^i_{[b]}[\delta_\varphi C^\dagger]^{j'}_{[\nu]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}$$
$$=\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{C^{ii'}_{[b][\mu]}\bar\varphi^j_{[a]}[C\delta_{\bar\varphi}]^{i'}_{[\mu]} - \bar C^{jj'}_{[a][\nu]}\varphi^i_{[b]}[\delta_\varphi C^\dagger]^{j'}_{[\nu]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \tag{D.5}$$

Another integration by parts and Lemma 2 yield

$$\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{\delta_{\bar\varphi^l_{[h]}}C^{ii'}_{[b][\mu]}\bar\varphi^j_{[a]}C^{i'l}_{[\mu][h]} - \delta_{\bar\varphi^l_{[h]}}[C^{ii'}_{[b][\mu]}\bar\varphi^j_{[a]}C^{i'l}_{[\mu][h]}]$$
$$-\delta_{\varphi^l_{[h']}}\bar C^{jj'}_{[a][\nu]}\varphi^i_{[b]}\bar C^{j'l}_{[\nu][h']} + \delta_{\varphi^l_{[h']}}[\bar C^{jj'}_{[a][\nu]}\varphi^i_{[b]}\bar C^{j'l}_{[\nu][h']}]\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \tag{D.6}$$

Using again the fact that C is Hermitian and performing some differentiations leads to

$$\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{\delta_{\bar\varphi^l_{[h]}}C^{ii'}_{[b][\mu]}\bar\varphi^j_{[a]}C^{i'l}_{[\mu][h]} - [C^2]^{ij}_{[b][a]} - \delta_{\varphi^l_{[h']}}\bar C^{jj'}_{[a][\nu]}\varphi^i_{[b]}\bar C^{j'l}_{[\nu][h']} + [\bar C^2]^{ji}_{[a][b]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \tag{D.7}$$

and since $[C^2]^{ij}_{[a][b]} = [\bar C^2]^{ji}_{[b][a]}$, these two terms cancel and the last expression (D.7) assumes now the form

$$\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{C^{ii'}_{[b][\mu]}[C\delta_{\bar\varphi}]^{i'}_{[\mu]}\bar\varphi^j_{[a]} - \bar C^{jj'}_{[a][\nu]}[\delta_\varphi C^\dagger]^{j'}_{[\nu]}\varphi^i_{[b]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \tag{D.8}$$

Making use, a third time, of Lemma 2, gives

$$\frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^j_{[a]} - \bar C^{jj'}_{[a][\nu]}\bar\varphi^{j'}_{[\nu]}\varphi^i_{[b]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \tag{D.9}$$

Plugging this into the variation of the free energy and one obtains

$$\frac{\delta\ln Z(\eta,\bar\eta)}{\imath\delta B_{[\mu][\nu]}} = 0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big\{-[C\bar\varphi]^i_{[b]}\,[\eta C^\dagger]^j_{[a]} + [C\bar\eta]^i_{[b]}\,[\varphi C^\dagger]^j_{[a]}$$
$$-\lambda\Big[C^{i0}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\bar\varphi^1\varphi^2\bar\varphi^3]_{[\mu]} - \bar C^{j1}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\varphi^2\bar\varphi^3]_{[\nu]}$$
$$+C^{i2}_{[b][\mu]}[\varphi C^\dagger]^j_{[a]}[\varphi^0\bar\varphi^1\bar\varphi^3]_{[\mu]} - \bar C^{j3}_{[a][\nu]}[C\bar\varphi]^i_{[b]}[\varphi^0\bar\varphi^1\varphi^2]\Big] \;-\; \bar\lambda\{\bar\varphi\}$$
$$+C^{ii'}_{[b][\mu]}\varphi^{i'}_{[\mu]}\bar\varphi^j_{[a]} - \bar C^{jj'}_{[a][\nu]}\bar\varphi^{j'}_{[\nu]}\varphi^i_{[b]}\Big\}e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \tag{D.10}$$

which is the bottom line for deriving the WT identities using a general unitary transformation.

2. 1-action

We denote (42) in the following way (recalling that $[\alpha]$ is a triple index, while μ,ν are single indices, and a dot notifies the position of remaning indices which, below, are integrated)

$$0 = \frac{1}{Z(\eta,\bar\eta)}\int d\mu_C(\bar\varphi,\varphi)\Big(-\bar\varphi^{0,2}_{\mu.}\eta^{0,2}_{\nu.} + \bar\eta^{0,2}_{\mu.}\varphi^{0,2}_{\nu.} - \bar\varphi^{1,3}_{.\mu}\eta^{1,3}_{.\nu} + \bar\eta^{1,3}_{.\mu}\varphi^{1,3}_{.\nu}$$
$$+\delta_{\varphi^{0,2}_{\nu.}}C^{0,2}_{\mu.\,[h]}\delta_{\bar\varphi^{0,2}_{[h]}} - \delta_{\varphi^{0,2}_{[h]}}C^{0,2}_{[h]\,\nu.}\delta_{\bar\varphi^{0,2}_{\mu.}} + \delta_{\varphi^{1,3}_{.\nu}}C^{1,3}_{.\mu\,[h]}\delta_{\bar\varphi^{1,3}_{[h]}} - \delta_{\varphi^{1,3}_{[h]}}C^{1,3}_{[h]\,.\nu}\delta_{\bar\varphi^{1,3}_{.\mu}}\Big)e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \tag{D.11}$$

Using $C^{ij}_{[a][b]} = \overline{C}^{ij}_{[b][a]}$, the above computes to

$$
\begin{aligned}
0 &= \frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big(-\bar\varphi^{0,2}_{\mu.}\eta^{0,2}_{\nu.} + \bar\eta^{0,2}_{\mu.}\varphi^{0,2}_{\nu.} - \bar\varphi^{1,3}_{.\mu}\eta^{1,3}_{.\nu} + \bar\eta^{1,3}_{.\mu}\varphi^{1,3}_{.\nu} \\
&+\delta_{\varphi^{0,2}_{\nu.}} C^{0,2}_{\mu.\;[h]}\delta_{\bar\varphi^{0,2}_{[h]}} - \delta_{\bar\varphi^{0,2}_{\mu.}} \bar C^{0,2}_{\nu.\;[h]}\delta_{\varphi^{0,2}_{[h]}} + \delta_{\varphi^{1,3}_{.\nu}} C^{1,3}_{.\mu\;[h]}\delta_{\bar\varphi^{1,3}_{[h]}} - \delta_{\bar\varphi^{1,3}_{.\mu}} \bar C^{1,3}_{.\nu\;[h]}\delta_{\varphi^{1,3}_{[h]}}\Big) e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \\
0 &= \frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big(-\bar\varphi^{0,2}_{\mu.}\eta^{0,2}_{\nu.} + \bar\eta^{0,2}_{\mu.}\varphi^{0,2}_{\nu.} - \bar\varphi^{1,3}_{.\mu}\eta^{1,3}_{.\nu} + \bar\eta^{1,3}_{.\mu}\varphi^{1,3}_{.\nu} \\
&+\delta_{\varphi^{0,2}_{\nu.}}[C\delta_{\bar\varphi}]^{0,2}_{\mu.} - \delta_{\bar\varphi^{0,2}_{\mu.}}[\delta_\varphi C^\dagger]^{0,2}_{\nu.} + \delta_{\varphi^{1,3}_{.\nu}}[C\delta_{\bar\varphi}]^{1,3}_{.\mu} - \delta_{\bar\varphi^{1,3}_{.\mu}}[\delta_\varphi C^\dagger]^{1,3}_{.\nu}\Big) e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \qquad \text{(D.12)}
\end{aligned}
$$

Let us evaluate by Lemma 2 the functional derivative terms and that we can write

$$
\begin{aligned}
&\frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big([C\delta_{\bar\varphi}]^{0,2}_{\mu.}\delta_{\varphi^{0,2}_{\nu.}} - [\delta_\varphi C^\dagger]^{0,2}_{\nu.}\delta_{\bar\varphi^{0,2}_{\mu.}} + [C\delta_{\bar\varphi}]^{1,3}_{.\mu}\delta_{\varphi^{1,3}_{.\nu}} - [\delta_\varphi C^\dagger]^{1,3}_{.\nu}\delta_{\bar\varphi^{1,3}_{.\mu}}\Big) e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \\
&= \frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big(\varphi^{0,2}_{\mu.}\delta_{\varphi^{0,2}_{\nu.}} - \bar\varphi^{0,2}_{\nu.}\delta_{\bar\varphi^{0,2}_{\mu.}} + \varphi^{1,3}_{.\mu}\delta_{\varphi^{1,3}_{.\nu}} - \bar\varphi^{1,3}_{.\nu}\delta_{\bar\varphi^{1,3}_{.\mu}}\Big) e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \\
&= \frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big\{ \int_{h_{ij}h^{ij}} \\
&\varphi^0_{\mu.}\Big[\frac{-\lambda}{\sqrt{\delta^N(e)}}\bar\varphi^1_{.h_{13}h_{12}}\varphi^2_{h_{21}.h_{23}}\bar\varphi^3_{h_{32}h_{31}\nu} + \bar\eta^0_{\nu.}\Big] + \varphi^2_{\mu.}\Big[\frac{-\lambda}{\sqrt{\delta^N(e)}}\varphi^0_{h_{03}.h_{01}}\bar\varphi^1_{h_{10}h_{13}\nu}\bar\varphi^3_{.h_{31}h_{30}} + \bar\eta^2_{\nu.}\Big] \\
&+\varphi^1_{.\mu}\Big[\frac{-\bar\lambda}{\sqrt{\delta^N(e)}}\bar\varphi^0_{h^{03}h^{02}.}\bar\varphi^2_{\nu h^{20}h^{23}}\varphi^3_{h^{32}.h^{30}} + \bar\eta^1_{.\nu}\Big] + \varphi^3_{.\mu}\Big[\frac{-\bar\lambda}{\sqrt{\delta^N(e)}}\bar\varphi^0_{\nu h^{02}h^{01}}\varphi^1_{h^{10}.h^{12}}\bar\varphi^2_{h^{21}h^{20}.} + \bar\eta^3_{.\nu}\Big] \\
&-\bar\varphi^0_{\nu.}\Big[\frac{-\lambda}{\sqrt{\delta^N(e)}}\varphi^1_{.h^{13}h^{12}}\bar\varphi^2_{h^{21}.h^{23}}\varphi^3_{h^{32}h^{31}\mu} + \eta^0_{\mu.}\Big] - \bar\varphi^2_{\nu.}\Big[\frac{-\lambda}{\sqrt{\delta^N(e)}}\bar\varphi^0_{h^{03}.h^{01}}\varphi^1_{h^{10}h^{13}\mu}\varphi^3_{.h^{31}h^{30}} + \eta^2_{\mu.}\Big] \\
&-\bar\varphi^1_{.\nu}\Big[\frac{-\bar\lambda}{\sqrt{\delta^N(e)}}\varphi^0_{h_{03}h_{02}.}\varphi^2_{\mu h_{20}h_{23}}\bar\varphi^3_{h_{32}.h_{30}} + \eta^1_{.\mu}\Big] - \bar\varphi^3_{.\nu}\Big[\frac{-\bar\lambda}{\sqrt{\delta^N(e)}}\varphi^0_{\mu h_{02}h_{01}}\bar\varphi^1_{h_{10}.h_{12}}\varphi^2_{h_{21}h_{20}.} + \eta^3_{.\mu}\Big] \\
&\Big\} e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \qquad \text{(D.13)}
\end{aligned}
$$

that will be shortly denoted by

$$
\begin{aligned}
&\frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big(\varphi^{0,2}_{\mu.}\bar\eta^{0,2}_{\nu.} - \lambda\,\varphi^{0,2}_{\mu.}\cdot[\bar\varphi^1\varphi^{2,0}\bar\varphi^3]_{.\nu} + \varphi^{1,3}_{.\mu}\bar\eta^{1,3}_{.\nu} - \bar\lambda\,\varphi^{1,3}_{.\mu}\cdot[\bar\varphi^0\bar\varphi^2\varphi^{3,1}]_{.\nu} \\
&-\bar\varphi^{0,2}_{\nu.}\eta^{0,2}_{\mu.} + \lambda\,\bar\varphi^{0,2}_{\nu.}\cdot[\varphi^1\bar\varphi^{2,0}\varphi^3]_{\mu.} - \bar\varphi^{1,3}_{.\nu}\eta^{1,3}_{.\mu} + \bar\lambda\,\bar\varphi^{1,3}_{.\nu}\cdot[\varphi^0\varphi^2\bar\varphi^{3,1}]_{.\mu}\Big) e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi}\;. \qquad \text{(D.14)}
\end{aligned}
$$

Inserting this last relation into (D.12), we get

$$
\begin{aligned}
0 = \frac{1}{Z(\eta,\bar\eta)} \int d\mu_C(\bar\varphi,\varphi)\Big[&-\bar\varphi^{0,2}_{\mu.}\eta^{0,2}_{\nu.} + \bar\eta^{0,2}_{\mu.}\varphi^{0,2}_{\nu.} - \bar\varphi^{1,3}_{.\mu}\eta^{1,3}_{.\nu} + \bar\eta^{1,3}_{.\mu}\varphi^{1,3}_{.\nu} + \varphi^{0,2}_{\mu.}\bar\eta^{0,2}_{\nu.} - \bar\varphi^{0,2}_{\nu.}\eta^{0,2}_{\mu.} + \varphi^{1,3}_{.\mu}\bar\eta^{1,3}_{.\nu} - \bar\varphi^{1,3}_{.\nu}\eta^{1,3}_{.\mu} \\
&-\lambda\,\varphi^{0,2}_{\mu.}\cdot[\bar\varphi^1\varphi^{2,0}\bar\varphi^3]_{.\nu} - \bar\lambda\,\varphi^{1,3}_{.\mu}\cdot[\bar\varphi^0\bar\varphi^2\varphi^{3,1}]_{.\nu} + \lambda\,\bar\varphi^{0,2}_{\nu.}\cdot[\varphi^1\bar\varphi^{2,0}\varphi^3]_{\mu.} + \bar\lambda\,\bar\varphi^{1,3}_{.\nu}\cdot[\varphi^0\varphi^2\bar\varphi^{3,1}]_{.\mu}\Big] e^{-S^{\rm int}(\varphi,\bar\varphi)+\bar\varphi\eta+\bar\eta\varphi} \\
&\qquad \text{(D.15)}
\end{aligned}
$$

where repeated colors are summed as well as arguments in dot. Eq.(D.15) is the starting equation for deriving the WT identities in the case of unitary 1-action.

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