THE CAUCHY PROBLEM FOR METRICS WITH PARALLEL SPINORS

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Abstract. We show that in the analytic category, given a Riemannian metric g on a hypersurface $M \subset \mathcal{Z}$ and a symmetric tensor W on M, the metric g can be locally extended to a Riemannian Einstein metric on $\mathcal{Z}$ with second fundamental form W, provided that g and W satisfy the constraints on M imposed by the Codazzi equations. We use this fact to study the Cauchy problem for metrics with parallel spinors in the real analytic category and give an affirmative answer to a question raised in [4].

1. Introduction

This paper attempts to solve two problems: the question of existence of Riemannian Einstein metrics prescribed on a hypersurface together with their second fundamental form, and the extension problem for spinors from a hypersurface to parallel spinors on the total space. These problems are related: parallel spinors can only exist over Ricci-flat manifolds.

In the Lorentzian setting, Ricci-flat or more generally Einstein metrics form the central objects of general relativity. Given a space-like hypersurface, a Riemannian metric, and a symmetric tensor which plays the role of the second fundamental form, a local extension to a Lorentzian Einstein metric always exists [13], [9], provided that the local conditions given by the Gauss equation are satisfied, see (2.11), (2.12) below. One crucial step in the proof is the reduction to an evolution equation which is hyperbolic due to the signature of the metric. The corresponding equations in the Riemannian setting are elliptic and no general local existence results are available.

In the Riemannian setting, DeTurck [10] analyzed the related problem of finding a metric with prescribed nonsingular Ricci tensor. The Ricci-flat case is at the opposite spectrum of degeneracy, while the general Einstein case is reminiscent of DeTurck's setting. Despite some formal similarities with [10], the Cauchy problem for Einstein metrics studied here is in essence quite different.

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In order to introduce the second problem, we need to recall some basic facts about restrictions of spin bundles to hypersurfaces. If $\mathcal{Z}$ is a spin manifold, any oriented hypersurface $M \subset \mathcal{Z}$ inherits a spin structure and it is well-known that the restriction to M of the complex spin bundle $\Sigma\mathcal{Z}$ if n is even (resp. $\Sigma^+\mathcal{Z}$ if n is odd) is canonically isomorphic to the complex spin bundle ΣM (cf. [4]). If W denotes the Weingarten tensor of M, the spin covariant derivatives $\nabla^{\mathcal{Z}}$ and ∇^g are related by ([4, Eq. (8.1)])

$$(\nabla^{\mathcal{Z}}_X \Psi)|_M = \nabla^g_X(\Psi|_M) - \frac{1}{2}W(X)\cdot(\Psi|_M), \qquad \forall\ X \in TM, \tag{1.1}$$

for all spinors (resp. half-spinors for n odd) Ψ on $\mathcal{Z}$. We thus see that if Ψ is a parallel spinor on $\mathcal{Z}$, its restriction ψ to any hypersurface M is a *generalized Killing spinor* on M, i.e. it satisfies the equation

$$\nabla^g_X \psi = \frac{1}{2}W(X)\cdot\psi, \qquad \forall\ X \in TM, \tag{1.2}$$

and the symmetric tensor W, called the stress-energy tensor of ψ, is just the Weingarten tensor of the hypersurface M. It is natural to ask whether the converse holds:

> *(Q): If ψ is a generalized Killing spinor on M^n, does there exist an isometric embedding of M into a spin manifold $(\mathcal{Z}^{n+1}, g^{\mathcal{Z}})$ carrying a parallel spinor Ψ whose restriction to M is ψ?*

This question is precisely the Cauchy problem for metrics with parallel spinors asked in [4].

The answer is known to be positive in several special cases: if the stress-energy tensor W of ψ is the identity [3], if W is parallel [15] and if W is a Codazzi tensor [4]. The common feature of each of these cases is that one can actually construct in an explicit way the "ambient" metric $g^{\mathcal{Z}}$ on the product $M \times (-\varepsilon, \varepsilon)$.

Our aim is to show that the same is true more generally, under the sole additional assumption that (M, g) and W are analytic.

Theorem 1.1. *Let ψ be a spinor field on a analytic spin manifold (M^n, g), and W a analytic field of endomorphisms of TM. Assume that ψ is a generalized Killing spinor with respect to W, i.e. it satisfies* (1.2)*. Then there exists a unique metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ on a sufficiently small neighborhood $\mathcal{Z}$ of $M \times \{0\}$ inside $M \times \mathbb{R}$ such that $(\mathcal{Z}, g^{\mathcal{Z}})$, endowed with the spin structure induced from M, carries a parallel spinor Ψ whose restriction to M is ψ.*

In particular, the solution $g^{\mathcal{Z}}$ must be Ricci-flat. Einstein manifolds are real-analytic but of course hypersurfaces can lose this structure so our hypothesis is restrictive. We leave open the general smooth case, note however that local Einstein metrics with smooth initial data can be constructed as constant sectional curvature metrics when the second fundamental form is a Codazzi tensor, see [4]. In particular in dimensions $1+1$ and $2+1$ the above theorem remains valid in the C^3 category. What we can achieve in general is to solve the Einstein equation (and the parallel spinor equation) in Taylor series near the initial hypersurface.

Related problems have been studied starting with the work of Fefferman-Graham [12] concerning asymptotically hyperbolic Poincaré-Einstein metrics. The initial hypersurface (M, g) is then at infinite distance from the manifold, the metric being conformal to a smooth metric $\bar{g}$ on a manifold with boundary

$$\mathcal{Z} = (0, \varepsilon) \times M, \qquad g^{\mathcal{Z}} = x^{-2}\bar{g}$$

such that the conformal factor x is precisely the distance function to the boundary $x = 0$ with respect to $\bar{g}$. The metric is required to be Einstein of negative curvature up to an error term which vanishes with all derivatives at infinity. Such a metric exists when n is odd, and its Taylor series at infinity is determined by the initial metric g and the symmetric transverse traceless tensor appearing on position $2n$, while in even dimensions some logarithmic terms must be allowed.

In our setting, starting from a smooth hypersurface (M, g) with prescribed Weingarten tensor W we prove that there exist formal Einstein metrics $g^{\mathcal{Z}}$ such that W is the second fundamental form at $t = 0$, i.e., we solve the Einstein equation modulo rapidly vanishing errors. Guided by the analytic and the low dimensional ($n = 1$ or $n = 2$) cases, we are tempted to guess that germs of Einstein metrics do exist for any smooth initial data.

Let us stress that existence results of Einstein metrics with prescribed first fundamental form and Weingarten tensor cannot hold globally in general, see Example 2.5.

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2. The Cauchy problem for Einstein metrics

Let $(\mathcal{Z}, g^{\mathcal{Z}})$ be an oriented Riemannian manifold of dimension $n + 1$ and let M be an oriented hypersurface with induced Riemannian metric $g := g^{\mathcal{Z}}|_M$. We start by fixing some notations. Denote by $\nabla^{\mathcal{Z}}$ and ∇^g the Levi-Civita covariant derivatives on $(\mathcal{Z}, g^{\mathcal{Z}})$ and (M, g), by ν the unit normal vector field along M compatible with the orientations, and by $W \in \mathrm{End}(TM)$ the Weingarten tensor defined by

$$\nabla^{\mathcal{Z}}_X \nu = -W(X), \qquad \forall\, X \in TM. \tag{2.1}$$

Using the normal geodesics issued from M, the metric on $\mathcal{Z}$ can be expressed in a neighborhood $\mathcal{Z}_0$ of M as $g^{\mathcal{Z}} = dt^2 + g_t$, where t is the distance function to M and g_t is a family of Riemannian metrics on M with $g_0 = g$ (cf. [4]). The vector field ν extends to $\mathcal{Z}_0$ as $\nu = \partial/\partial t$ and (2.1) defines a symmetric endomorphism on $\mathcal{Z}_o$ which can be viewed as a family W_t of endomorphisms of M, symmetric with respect to g_t, and satisfying (cf. [4, Equation (4.1)]):

$$g_t(W_t(X),Y) = -\frac{1}{2}\dot g_t(X,Y), \qquad \forall\ X,Y\in TM. \tag{2.2}$$

By [4, Equations (4.5)–(4.8)], the Ricci tensor and the scalar curvature of $\mathcal{Z}$ satisfy for every vectors $X,Y\in TM$

$$\mathrm{Ric}^{\mathcal{Z}}(\nu,\nu) = \mathrm{tr}(W_t^2) - \frac{1}{2}\mathrm{tr}_{g_t}(\ddot g_t), \tag{2.3}$$

$$\mathrm{Ric}^{\mathcal{Z}}(\nu,X) = d\mathrm{tr}(W_t)(X) + \delta^{g_t}(W)(X), \tag{2.4}$$

$$\mathrm{Ric}^{\mathcal{Z}}(X,Y) = \mathrm{Ric}^{g_t}(X,Y) + 2g_t(W_tX,W_tY) + \frac{1}{2}\mathrm{tr}(W_t)\dot g_t(X,Y) - \frac{1}{2}\ddot g_t(X,Y), \tag{2.5}$$

$$\mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^{g_t} + 3\mathrm{tr}(W_t^2) - \mathrm{tr}^2(W_t) - \mathrm{tr}_{g_t}(\ddot g_t). \tag{2.6}$$

Using (2.3) and (2.6) we get

$$-2\mathrm{Ric}^{\mathcal{Z}}(\nu,\nu) + \mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^{g_t} + \mathrm{tr}(W_t^2) - \mathrm{tr}^2(W_t), \tag{2.7}$$

where $\delta^g : \mathrm{End}(TM)\to T^*M$ is the divergence operator defined in a local g-orthonormal basis $\{e_i\}$ of TM by

$$\delta^g(A)(X) = -\sum_{i=1}^n g((\nabla^g_{e_i}A)(e_i),X). \tag{2.8}$$

A straightforward calculation yields

$$\delta^g(fA) = f\delta^g(A) - A(\nabla^g f) \tag{2.9}$$

for all functions f.

For later use, let us recall that the second Bianchi identity implies the following relation between the divergence of the Ricci tensor and the exterior derivative of the scalar curvature:

$$\delta^g(\mathrm{Ric}^g) = -\frac{1}{2}d\mathrm{Scal}^g \tag{2.10}$$

for every Riemannian metric g (cf. [5, Prop. 1.94]).

Assume now that the metric $g^{\mathcal{Z}}$ is Einstein with scalar curvature $(n+1)\lambda$, i.e. $\mathrm{Ric}^{\mathcal{Z}} = \lambda g^{\mathcal{Z}}$. Evaluating (2.4) and (2.7) at $t=0$ yields

$$d\mathrm{tr}(W) + \delta^g W = 0, \tag{2.11}$$

$$\mathrm{Scal}^g + \mathrm{tr}(W^2) - \mathrm{tr}^2(W) = (n-1)\lambda. \tag{2.12}$$

If $g_t : \mathrm{End}(TM)\to T^*M\otimes T^*M$ is the isomorphism defined by $g_t(A)(X,Y) := g_t(A(X),Y)$ and $g_t^{-1} : T^*M\otimes T^*M \to \mathrm{End}(TM)$ denotes its inverse, then taking (2.3) into account, (2.5) reads

$$\ddot g_t = 2\mathrm{Ric}^{g_t} + \dot g_t(g_t^{-1}(\dot g_t)\cdot,\cdot) - \mathrm{tr}(g_t^{-1}(\dot g_t))\dot g_t - 2\lambda g_t, \tag{2.13}$$

which can also be written

$$\dot{W}_t = -g_t^{-1}\mathrm{Ric}^{g_t} + W_t \mathrm{tr}(W_t) - 2\lambda \mathrm{Id}. \tag{2.14}$$

In the rest of this section we prove an existence and unique continuation result for Einstein metrics.

Theorem 2.1. *Let (M^n, g) be an analytic Riemannian manifold and let W be an analytic symmetric endomorphism field on M satisfying (2.11) and (2.12). Then for $\varepsilon > 0$, there exists a unique germ near $M \times \{0\}$ of an Einstein metric $g^{\mathcal{Z}}$ with scalar curvature $(n+1)\lambda$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ on $\mathcal{Z} := M \times \mathbb{R}$ whose Weingarten tensor at $t = 0$ is W.*

Proof. In equation (2.13) the only term involving partial derivatives of the metric g_t along M is Ric^{g_t}, which is an analytic expression in g_t and its first and second order derivatives along M which does not involve any derivative with respect to t. Indeed, in local coordinates x_i on M, with the usual summation convention one has

$$\mathrm{Ric}^g(\partial_i, \partial_j) = \partial_k \Gamma^i_{jk} - \partial_j \Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{il}\Gamma^l_{kj}, \qquad \Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{jk}).$$

The second order Cauchy-Kowalewskaya theorem (see e.g. [8]) shows that for every $x \in M$ there exists a neighborhood $V_x \ni x$ and some $\varepsilon_x > 0$ such that the Cauchy problem (2.13) with initial data

$$\begin{cases} g_0 = g \\ \dot{g}_0 = -2W \end{cases}$$

has a unique analytic solution on $V_x \times (-\varepsilon_x, \varepsilon_x)$. Let $g^{\mathcal{Z}} = dt^2 + g_t$ be the metric defined on $V_x \times (-\varepsilon_x, \varepsilon_x)$ by this solution. We claim that $g^{\mathcal{Z}}$ is Einstein with scalar curvature $(n+1)\lambda$.

Consider the 1-parameter family of functions and 1-forms on M:

$$f_t := \frac{1}{2}((n-1)\lambda - \mathrm{Scal}^{g_t} - \mathrm{tr}(W_t^2) + \mathrm{tr}^2(W_t)), \qquad \omega_t := d\mathrm{tr}(W_t) + \delta^{g_t} W_t, \tag{2.15}$$

where W_t is defined as before by (2.2). Using (2.14) and the formula for the first variation of the scalar curvature ([5, Thm. 1.174 (e)]) we get

$$\begin{aligned}
\frac{df_t}{dt} =& \Delta^{g_t}(\mathrm{tr}(W_t)) + \delta^{g_t}(\delta^{g_t} W_t) - g_t(\mathrm{Ric}^{g_t}, g_t(W_t)) - \mathrm{tr}(W_t \circ \dot{W}_t) + \mathrm{tr}(W_t)\mathrm{tr}(\dot{W}_t) \\
=& \delta^{g_t}\omega_t - g_t(\mathrm{Ric}^{g_t}, g_t(W_t)) - \mathrm{tr}(W_t \circ (-g_t^{-1}\mathrm{Ric}^{g_t} + W_t\mathrm{tr}(W_t) - \lambda \mathrm{Id}) \\
& + \mathrm{tr}(W_t)(-\mathrm{Scal}^{g_t} + \mathrm{tr}^2(W_t) - n\lambda) \\
=& \delta^{g_t}\omega_t + \mathrm{tr}(W_t)(-\mathrm{Scal}^{g_t} + \mathrm{tr}^2(W_t) - \mathrm{tr}(W_t^2) + (n-1)\lambda),
\end{aligned}$$

whence

$$\frac{df_t}{dt} = \delta^{g_t}\omega_t + 2\mathrm{tr}(W_t) f_t. \tag{2.16}$$

(note that the symmetric tensor h in [5] is $h = \dot{g}_t = -2g_t(W_t)$ in our notations).

In order to compute the time derivative of ω_t we need to compute the variation of δ^{g_t}. The computation being rather lengthy, we postpone it to Lemma 2.2 below. Taking $A_t = W_t$ in that lemma yields

$$\frac{d(\delta^{g_t} W_t)}{dt} = -\frac{1}{2} d\mathrm{tr}(W_t^2) + W_t(\nabla^{g_t}(\mathrm{tr}(W_t)) + \delta^{g_t}(\dot{W}_t).$$

Using (2.9), (2.10) and (2.14) we get

$$\begin{aligned}
\frac{d\omega_t}{dt} &= d\mathrm{tr}(\dot{W}_t) + \frac{d(\delta^{g_t} W_t)}{dt} \\
&= d(-\mathrm{Scal}^{g_t} + \mathrm{tr}^2(W_t)) - \frac{1}{2} d\mathrm{tr}(W_t^2) + W_t(\nabla^{g_t}(\mathrm{tr}(W_t)) + \delta^{g_t}(\dot{W}_t) \\
&= -d\mathrm{Scal}^{g_t} + d\mathrm{tr}^2(W_t) - \frac{1}{2} d\mathrm{tr}(W_t^2) + W_t(\nabla^{g_t}(\mathrm{tr}(W_t)) + \delta^{g_t}(-g_t^{-1}\mathrm{Ric}^{g_t} + W_t \mathrm{tr}(W_t)) \\
&= -\frac{1}{2} d\mathrm{Scal}^{g_t} - \frac{1}{2} d\mathrm{tr}(W_t^2) + d\mathrm{tr}^2(W_t) + \mathrm{tr}(W_t)\delta^{g_t}(W_t),
\end{aligned}$$

which implies

$$\frac{d\omega_t}{dt} = df_t + \mathrm{tr}(W_t)\omega_t. \tag{2.17}$$

Denoting by H the analytic function $\mathrm{tr}(W_t)$, Equations (2.16) and (2.17) show that the pair (f_t, ω_t) satisfies the first order linear system

$$\begin{cases} \partial_t f_t = \delta^{g_t}\omega_t + 2H f_t \\ \partial_t \omega_t = df_t + H\omega_t. \end{cases} \tag{2.18}$$

Moreover, the constraints (2.11) and (2.12) show that (f_t, ω_t) vanishes at $t = 0$. By the Cauchy-Kowalewskaya theorem, (f_t, ω_t) vanishes for all t.

Using (2.4), (2.5), (2.7) and (2.13), we see that the metric $g^{\mathcal{Z}} := dt^2 + g_t$ constructed in this way satisfies

$$\begin{cases} \mathrm{Ric}^{\mathcal{Z}}(\nu, X) = 0 & \forall X \perp \nu \\ \mathrm{Ric}^{\mathcal{Z}}(X, Y) = \lambda g^{\mathcal{Z}}(X, Y) & \forall X, Y \perp \nu \\ \mathrm{Scal}^{\mathcal{Z}} - 2\mathrm{Ric}^{\mathcal{Z}}(\nu, \nu) = (n-1)\lambda. \end{cases}$$

On the other hand we clearly have $\mathrm{Scal}^{\mathcal{Z}} = \mathrm{Ric}^{\mathcal{Z}}(\nu, \nu) + n\lambda$ and therefore $\mathrm{Ric}^{\mathcal{Z}} = \lambda g^{\mathcal{Z}}$, thus proving our claim.

To end the proof of the theorem, we note that the local metric $g_x^{\mathcal{Z}}$ constructed above on $V_x \times (-\varepsilon_x, \varepsilon_x)$ is unique, thus $g_x^{\mathcal{Z}}$ and $g_y^{\mathcal{Z}}$ coincide on the intersection $(V_x \cap V_y) \times (-\varepsilon, \varepsilon)$ for $\varepsilon := \min\{\varepsilon_x, \varepsilon_y\}$. Hence $g^{\mathcal{Z}}$ is well-defined on a neighborhood of M in $M \times \mathbb{R}$. ☐

Lemma 2.2. *If g_t is a family of Riemannian metrics on a manifold M and A_t is a family of endomorphism fields of TM symmetric with respect to g_t, then*

$$\frac{d(\delta^{g_t} A_t)}{dt}(X) = g_t(A_t(\nabla^{g_t}\mathrm{tr}(W_t)), X) - g_t(\nabla_X^{g_t} W_t, A_t) + (\delta^{g_t}\dot{A}_t)(X), \tag{2.19}$$

where W_t is defined by (2.2).

Proof. Let vol_t denote the volume form of the metric g_t. A straightforward computation yields

$$\frac{d(\mathrm{vol}_t)}{dt} = -\mathrm{tr}(W_t)\mathrm{vol}_t. \tag{2.20}$$

In the computations below we will drop the subscripts t for an easier reading and use the dot sign for differentiation with respect to t. From [5, Thm. 1.174 (a)] we get

$$g(\dot\nabla_X Y, Z) = g((\nabla_Z W)X, Y) - g((\nabla_X W)Y, Z) - g((\nabla_Y W)X, Z). \tag{2.21}$$

Differentiating with respect to t the formula valid for every compactly supported vector field X on M

$$\int_M (\delta^{g_t} A_t)(X)\mathrm{vol}_t = \int_M \mathrm{tr}(A_t \circ \nabla^{g_t} X)\mathrm{vol}_t \tag{2.22}$$

and using (2.20) yields

$$\int_M (\dot\delta A + \delta \dot A - (\delta A)\mathrm{tr}(W))(X)\mathrm{vol} = \int_M \mathrm{tr}(\dot A \circ \nabla X + A \circ \dot\nabla X - \mathrm{tr}(W) A \circ \nabla X)\mathrm{vol}.$$

Subtracting (2.22) applied to $\dot A$ from this last equation gives

$$\int_M (\dot\delta A - (\delta A)\mathrm{tr}(W))(X)\mathrm{vol} = \int_M \mathrm{tr}(A \circ \dot\nabla X - \mathrm{tr}(W) A \circ \nabla X)\mathrm{vol}. \tag{2.23}$$

From (2.21) and the fact that A and W are symmetric with respect to g we obtain

$$\mathrm{tr}(A \circ \dot\nabla X) = -g(\nabla_X W, A), \tag{2.24}$$

Using (2.24) and (2.22) again, but this time applied to $-\mathrm{tr}(W)A$, (2.23) becomes

$$\int_M (\dot\delta A - (\delta A)\mathrm{tr}(W))(X)\mathrm{vol} = \int_M -g(\nabla_X W, A) - \delta(\mathrm{tr}(W)A)(X)\mathrm{vol},$$

so from (2.9) we get

$$\int_M (\dot\delta A)(X)\mathrm{vol} = \int_M -g(\nabla_X W, A) + g(A(\nabla \mathrm{tr}(W)), X)\mathrm{vol},$$

Since this holds for every compactly supported vector field X, the integrand must vanish identically, i.e.

$$(\dot\delta A)(X) = -g(\nabla_X W, A) + g(A(\nabla \mathrm{tr}(W)), X),$$

which is equivalent to (2.19). ☐

2.1. **Formal solution in the smooth case.** Without the hypothesis that g and W are analytic we are not able to solve the nonlinear PDE system (2.13). However, it is rather evident from (2.13) that the full Taylor series of $g^{\mathcal{Z}}$ is recursively determined by its first two coefficients, which are g and W. Let $\dot{C}^\infty(\mathcal{Z})$ denote the space of tensors vanishing at M together with all their derivatives. By the Borel lemma (see e.g. [14]), there exists a metric $g^{\mathcal{Z}}$ such that its Ricci tensor satisfies the Einstein equation in the tangential directions modulo $\dot{C}^\infty(\mathcal{Z})$. Then the system (2.18) remains valid modulo $\dot{C}^\infty(\mathcal{Z})$ and we can easily show recursively that the right-hand sides of Equations (2.4) and (2.7) vanish modulo $\dot{C}^\infty(\mathcal{Z})$. Thus $g^{\mathcal{Z}}$ is Einstein modulo $\dot{C}^\infty(\mathcal{Z})$.

Proposition 2.3. *Let (M^n, g) be a smooth Riemannian manifold and let W be a smooth symmetric field of endomorphisms of TM satisfying* (2.11) *and* (2.12)*. Then there exists on $\mathcal{Z} := M \times (-\varepsilon, \varepsilon)$ a metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ whose Weingarten tensor at $t = 0$ is W, and such that*

$$\mathrm{Ric}^{\mathcal{Z}} - \lambda g^{\mathcal{Z}} \in \dot{C}^\infty(\mathcal{Z}).$$

Moreover, $g^{\mathcal{Z}}$ is unique up to $\dot{C}^\infty(\mathcal{Z})$.

2.2. **Existence and uniqueness for smooth initial data.** The small-time uniqueness of the Ricci-flat metric, or more generally of an Einstein metric follows under milder assumptions (e.g. when the g and W are only C^∞), see [1] or [6, Thm. 4].

The small-time existence is known to fail in general for elliptic Cauchy problems with C^∞ initial data, even in the linear case. Note however that in dimension $n+1 = 2+1$ the C^3 initial value problem can always be solved for small time:

Proposition 2.4. *Let M be a surface with C^3 Riemannian metric g, and let W be a C^3 symmetric field of endomorphisms on M satisfying* (2.11) *and* (2.12) *for some $\lambda \in \mathbb{R}$. Then there exists a constant sectional curvature metric $g^{\mathcal{Z}}$ on a neighborhood of $M \times \{0\}$ inside $\mathcal{Z} := M \times \mathbb{R}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$, whose Weingarten tensor at $t = 0$ is W.*

Proof. Direct application of [4, Theorem 7.2]. Namely, in dimension 2 the hypotheses (2.11), (2.12) are equivalent to [4, Eq. (7.3)] resp. [4, Eq. (7.4)] with $\kappa = 2\lambda$. It follows, at least in the smooth case, that g_t can be constructed explicitly in terms of g and W such that $g^{\mathcal{Z}}$ has constant sectional curvature κ. It remains to note that the proof of [4, Theorem 7.2] remains valid when g, W are of class C^3. □

Similarly, in dimension $1 + 1$ we can embed the curve (M, g) in a constant curvature surface with prescribed curvature function W. In this case, the constraint equations are empty, and the metric is again explicitly given by [4, Theorem 7.2].

2.3. **Global existence.** The preceding case of dimension $2 + 1$ hints that in general the Einstein metric $g^{\mathcal{Z}}$ cannot be extended on a complete manifold containing M as a hypersurface (or even half-complete, in the sense that geodesics pointing in one side of M can be extended until they meet again M). This sort of question is rather different from the arguments of this paper so we will only give an counterexample in dimension $1+1$ where global existence for the solution to the Cauchy problem fails. We restrict ourselves to the case of Ricci-flat metrics, which means vanishing Gaussian curvature in this dimension.

Example 2.5. Let $\mathcal{Z}$ be the incomplete flat surface obtained from $\mathbb{C}^*$ (or from the complement of a small disk in $\mathbb{C}$) by the following cut-and-paste procedure: cut along the positive real axis, then glue again after a translation of length $l > 0$. More precisely, x_+ is identified with $(x+l)_-$ for all $x > \varepsilon$. The resulting surface $\mathcal{Z}$ is clearly smooth and has a smooth flat metric including along the gluing locus. The unit circle in $\mathbb{R}^2$ gives rise to a curve in $\mathcal{Z}$ of curvature 1 and length 2π with different endpoints 1_- and $(1+l)_-$. In a complete flat surface, a curve of curvature 1 and length 2π must be closed (in fact smooth, since its lift to the universal cover must be a circle). Therefore, the surface $\mathcal{Z}$ cannot be embedded in any complete flat surface. In particular, for any closed curve in $\mathcal{Z}$ circling around the singular locus, the interior cannot be continued to a compact (or half-complete) flat surface with boundary.

3. Spinors on Ricci-flat manifolds

We keep the notations from the previous section. Our starting point is the following corollary of Theorem 2.1:

Corollary 3.1. *Assume that (M^n, g) is an analytic spin manifold carrying a non-trivial generalized Killing spinor ψ with analytic stress-energy tensor W. Then in a neighborhood of $M \times \{0\}$ in $\mathcal{Z} := M \times \mathbb{R}$ there exists a unique Ricci-flat metric $g^{\mathcal{Z}}$ of the form $g^{\mathcal{Z}} = dt^2 + g_t$ whose Weingarten tensor at $t = 0$ is W.*

Proof. We just need to check that the constraints (2.11), (2.12) are a consequence of (1.2). In order to simplify the computations, we will drop the reference to the metric g and denote respectively by ∇, R, Ric and Scal the Levi-Civita covariant derivative, curvature tensor, Ricci tensor and scalar curvature of (M, g). As usual, $\{e_i\}$ will denote a local g-orthonormal basis of TM.

We will use the following two classical formulas in Clifford calculus. The first one is the fact that the Clifford contraction of a symmetric tensor A only depends on its trace:

$$\sum_{i=1}^{n} e_i \cdot A(e_i) = -\mathrm{tr}(A). \tag{3.1}$$

The second formula expresses the Clifford contraction of the spin curvature in terms of the Ricci tensor ([2], p. 16):

$$\sum_{i=1}^{n} e_i \cdot R_{X,e_i}\psi = -\frac{1}{2}\mathrm{Ric}(X)\cdot\psi, \qquad \forall X \in TM,\ \forall \psi \in \Sigma M. \tag{3.2}$$

Let now ψ be a non-trivial generalized Killing spinor satisfying (1.2). Being parallel with respect to a modified connection on ΣM, ψ is nowhere vanishing (and actually of constant norm).

Taking a further covariant derivative in (1.2) and skew-symmetrizing yields

$$R_{X,Y}\psi = \frac{1}{4}\left(W(Y)\cdot W(X) - W(X)\cdot W(Y)\right)\cdot\psi + \frac{1}{2}\left((\nabla_X W)(Y) - (\nabla_Y W)(X)\right)\cdot\psi$$

for all $X, Y \in TM$. In this formula we set $Y = e_i$, take the Clifford product with e_i and sum over i. From (3.1) and (3.2) we get

$$\begin{aligned}\mathrm{Ric}(X)\cdot\psi =& -\frac{1}{2}\sum_{i=1}^{n} e_i\cdot\left(W(e_i)\cdot W(X) - W(X)\cdot W(e_i)\right)\cdot\psi \\ & - \sum_{i=1}^{n} e_i\cdot\left((\nabla_X W)(e_i) - (\nabla_{e_i} W)(X)\right)\cdot\psi \\ =& \frac{1}{2}\mathrm{tr}(W)W(X)\cdot\psi + \frac{1}{2}\sum_{i=1}^{n}\big(-W(X)\cdot e_i - 2g(W(X), e_i)\big)\cdot W(e_i)\cdot\psi \\ & + \nabla_X(\mathrm{tr}(W))\psi + \sum_{i=1}^{n} e_i\cdot(\nabla_{e_i} W)(X)\cdot\psi.\end{aligned}$$

whence

$$\mathrm{Ric}(X)\cdot\psi = \mathrm{tr}(W)W(X)\cdot\psi - W^2(X)\cdot\psi + X(\mathrm{tr}(W))\psi + \sum_{i=1}^{n} e_i\cdot(\nabla_{e_i} W)(X)\cdot\psi. \tag{3.3}$$

We set $X = e_j$ in (3.3), take the Clifford product with e_j and sum over j. Using (3.1) again we obtain

$$\begin{aligned}-\mathrm{Scal}\,\psi &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + \nabla(\mathrm{tr}(W))\cdot\psi + \sum_{i,j=1}^{n} e_j\cdot e_i\cdot(\nabla_{e_i} W)(e_j)\cdot\psi \\ &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + d\mathrm{tr}(W)\cdot\psi + \sum_{i,j=1}^{n}(-e_i\cdot e_j - 2\delta_{ij})\cdot(\nabla_{e_i} W)(e_j)\cdot\psi \\ &= -\mathrm{tr}^2(W)\psi + \mathrm{tr}(W^2)\psi + 2d\mathrm{tr}(W)\cdot\psi + 2\delta W\cdot\psi,\end{aligned}$$

which implies simultaneously (2.11) and (2.12) (indeed, if $f\psi = X\cdot\psi$ for some real f and vector X, then $-|X|^2\psi = X\cdot X\cdot\psi = X\cdot(f\psi) = f^2\psi$, so both f and X vanish). $\square$

Theorem 3.2. *Let $(\mathcal{Z}, g^{\mathcal{Z}})$ be a Ricci-flat spin manifold with Levi-Civita connection $\nabla^{\mathcal{Z}}$ and let $M \subset \mathcal{Z}$ be any oriented analytic hypersurface. Assume there exists some spinor $\psi \in C^\infty(\Sigma\mathcal{Z}|_M)$ which is parallel along M:*

$$\nabla^{\mathcal{Z}}_X\psi = 0, \qquad \forall X \in TM \subset T\mathcal{Z}. \tag{3.4}$$

Assume moreover that the application $\pi_1(M) \to \pi_1(\mathcal{Z})$ induced by the inclusion is surjective. Then there exists a parallel spinor $\Psi \in C^\infty(\Sigma\mathcal{Z})$ such that $\Psi|_M = \psi$.

Proof. Any Ricci-flat manifold is analytic, cf. [11], [5], thus the analyticity of M makes sense. The proof is split in two parts.

Local extension. Let ν denote the unit normal vector field along M. Every $x \in M$ has an open neighborhood V in M such that the exponential map $V \times (-\varepsilon, \varepsilon) \to \mathcal{Z}$, $(y,t) \mapsto \exp_y(t\nu)$ is well-defined for some $\varepsilon > 0$. Its differential at $(x, 0)$ being the identity, one can assume, by shrinking V and choosing a smaller ε if necessary, that it maps $V \times (-\varepsilon, \varepsilon)$ diffeomorphically onto some open neighborhood U of x in $\mathcal{Z}$. We extend the spinor ψ to a spinor Ψ on U by parallel transport along the normal geodesics $\exp_y(t\nu)$ for every fixed y. It remains to prove that Ψ is parallel on U in horizontal directions.

Let $\{e_i\}$ be a local orthonormal basis along M. We extend it on U by parallel transport along the normal geodesics, and notice that $\{e_i, \nu\}$ is a local orthonormal basis on U. More generally, every vector field X along V gives rise to a unique horizontal vector field, also denoted X, on U satisfying $\nabla_\nu X = 0$. For every such vector field we get

$$\nabla^{\mathcal{Z}}_\nu(\nabla^{\mathcal{Z}}_X \Psi) = R^{\mathcal{Z}}(\nu, X)\Psi + \nabla^{\mathcal{Z}}_{[\nu,X]}\Psi = R^{\mathcal{Z}}(\nu, X)\Psi + \nabla^{\mathcal{Z}}_{W(X)}\Psi. \tag{3.5}$$

Since $\mathcal{Z}$ is Ricci-flat, (3.2) applied to the local orthonormal basis $\{e_i, \nu\}$ of $\mathcal{Z}$ yields

$$0 = \frac{1}{2}\mathrm{Ric}^{\mathcal{Z}}(X)\cdot\Psi = \sum_{i=1}^n e_i\cdot R^{\mathcal{Z}}(e_i, X)\Psi + \nu\cdot R^{\mathcal{Z}}(\nu, X)\Psi\,. \tag{3.6}$$

We take the Clifford product with ν in this relation, differentiate again with respect to ν and use the second Bianchi identity to obtain:

$$\begin{aligned}\nabla^{\mathcal{Z}}_\nu(R^{\mathcal{Z}}(\nu, X)\Psi) =& \nabla^{\mathcal{Z}}_\nu\left(\nu\cdot\sum_{i=1}^n e_i\cdot R^{\mathcal{Z}}(e_i, X)\Psi\right) = \nu\cdot\sum_{i=1}^n e_i\cdot(\nabla^{\mathcal{Z}}_\nu R^{\mathcal{Z}})(e_i, X)\Psi\\ =& \nu\cdot\sum_{i=1}^n e_i\cdot\big((\nabla^{\mathcal{Z}}_{e_i} R^{\mathcal{Z}})(\nu, X)\Psi + (\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}})(e_i, \nu)\Psi\big),\end{aligned}$$

whence

$$\begin{aligned}\nabla^{\mathcal{Z}}_\nu(R^{\mathcal{Z}}(\nu, X)\Psi) = \nu\cdot\sum_{i=1}^n e_i\cdot\big(&\nabla^{\mathcal{Z}}_{e_i}(R^{\mathcal{Z}}(\nu, X)\Psi) + R^{\mathcal{Z}}(W(e_i), X)\Psi - R^{\mathcal{Z}}(\nu, \nabla^{\mathcal{Z}}_{e_i}X)\Psi\\ &- R^{\mathcal{Z}}(\nu, X)\nabla^{\mathcal{Z}}_{e_i}\Psi + \nabla^{\mathcal{Z}}_X(R^{\mathcal{Z}}(e_i, \nu)\Psi) - R^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X e_i, \nu)\Psi\\ &+ R^{\mathcal{Z}}(e_i, W(X))\Psi - R^{\mathcal{Z}}(e_i, \nu)\nabla^{\mathcal{Z}}_X\Psi\big)\,.\end{aligned} \tag{3.7}$$

Let $\nu^\perp$ denote the distribution orthogonal to ν on U and consider the sections $A, B \in C^\infty((\nu^\perp)^* \otimes \Sigma U)$ and $C \in C^\infty(\Lambda^2(\nu^\perp)^* \otimes \Sigma U)$ defined for all $X, Y \in \nu^\perp$ by

$$A(X) := \nabla^{\mathcal{Z}}_X\Psi, \qquad B(X) := R^{\mathcal{Z}}(\nu, X)\Psi, \qquad C(X, Y) := R^{\mathcal{Z}}(X, Y)\Psi.$$

We have noted that the metric $g^{\mathcal{Z}}$ is real-analytic since it is Ricci-flat. From the assumption that M is analytic and that ψ is parallel along M it follows that Ψ, and thus the tensors A, B and C, are analytic.

Equations (3.5) and (3.7) read in our new notation:

$$(\nabla^{\mathcal{Z}}_\nu A)(X) = B(X) + A(W(X)), \tag{3.8}$$

and

$$\begin{aligned}(\nabla^{\mathcal{Z}}_\nu B)(X) = \nu\cdot \sum_{i=1}^n e_i\cdot\big((\nabla^{\mathcal{Z}}_{e_i} B)(X) + C(W(e_i),X) - R^{\mathcal{Z}}(\nu,X)A(e_i)\\ - (\nabla^{\mathcal{Z}}_X B)(e_i) + C(e_i, W(X)) - R^{\mathcal{Z}}(e_i,\nu)A(X)\big).\end{aligned} \tag{3.9}$$

Moreover, the second Bianchi identity yields

$$\begin{aligned}(\nabla^{\mathcal{Z}}_\nu C)(X,Y) =&(\nabla^{\mathcal{Z}}_\nu R^{\mathcal{Z}})(X,Y)\Psi = (\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}})(\nu,Y)\Psi + (\nabla^{\mathcal{Z}}_Y R^{\mathcal{Z}})(X,\nu)\Psi\\ =&\nabla^{\mathcal{Z}}_X(R^{\mathcal{Z}}(\nu,Y)\Psi) - R^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_X\nu,Y)\Psi - R^{\mathcal{Z}}(\nu,\nabla^{\mathcal{Z}}_X Y)\Psi - R^{\mathcal{Z}}(\nu,Y)\nabla^{\mathcal{Z}}_X\Psi\\ &- \nabla^{\mathcal{Z}}_Y(R^{\mathcal{Z}}(\nu,X)\Psi) + R^{\mathcal{Z}}(\nabla^{\mathcal{Z}}_Y\nu,X)\Psi + R^{\mathcal{Z}}(\nu,\nabla^{\mathcal{Z}}_Y X)\Psi + R^{\mathcal{Z}}(\nu,X)\nabla^{\mathcal{Z}}_Y\Psi\\ =&(\nabla^{\mathcal{Z}}_X B)(Y) + C(W(X),Y) - R^{\mathcal{Z}}(\nu,Y)\nabla^{\mathcal{Z}}_X\Psi\\ &- (\nabla^{\mathcal{Z}}_Y B)(X) + C(X,W(Y)) + R^{\mathcal{Z}}(\nu,X)\nabla^{\mathcal{Z}}_Y\Psi,\end{aligned}$$

thus showing that

$$\begin{aligned}(\nabla^{\mathcal{Z}}_\nu C)(X,Y) =&(\nabla^{\mathcal{Z}}_X B)(Y) + C(W(X),Y) - R^{\mathcal{Z}}(\nu,Y)(A(X))\\ &- (\nabla^{\mathcal{Z}}_Y B)(X) + C(X,W(Y)) + R^{\mathcal{Z}}(\nu,X)(A(Y)).\end{aligned} \tag{3.10}$$

The hypothesis (3.4) is equivalent to $A = 0$ for $t = 0$. Differentiating this again in the direction of M and skew-symmetrizing yields $C = 0$ for $t = 0$. Finally, (3.6) shows that $B = 0$ for $t = 0$. We thus see that the section $S := (A, B, C)$ vanishes on along the hypersurface $V \times \{0\}$ of U.

The system (3.9)–(3.10) is a linear PDE for S and the hypersurfaces $t =$ constant are clearly non-characteristic. The Cauchy-Kowalewskaya theorem shows that S vanishes everywhere on U. In particular, $A = 0$ on U, thus proving our claim.

Global extension. Now we prove that there exists a parallel spinor $\Psi \in C^\infty(\Sigma\mathcal{Z})$ such that $\Psi|_M = \psi$. Take any $x \in M$ and an open neighborhood U like in Theorem 3.2 on which a parallel spinor Ψ extending ψ is defined. The spin holonomy group $\widetilde{\mathrm{Hol}}(U,x)$ thus preserves Ψ_x. Since any Ricci-flat metric is analytic (cf. [5, p. 145]), the restricted spin holonomy group $\widetilde{\mathrm{Hol}}_0(\mathcal{Z},x)$ is equal to $\widetilde{\mathrm{Hol}}_0(U,x)$ for every $x \in \mathcal{Z}$ and for every open neighborhood U of x. By the local extension result proved above, $\widetilde{\mathrm{Hol}}_0(U,x)$ acts trivially on Ψ_x, thus showing that Ψ_x can be extended (by parallel transport along every curve in $\tilde{\mathcal{Z}}$ starting from x) to a parallel spinor $\tilde\Psi$ on the universal cover $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$. The deck transformation group acts trivially on $\tilde\Psi$ since every element in $\pi_1(\mathcal{Z},x)$ can be represented by a curve in M (here we use the surjectivity hypothesis) and Ψ was assumed to be parallel along M. Thus $\tilde\Psi$ descends to $\mathcal{Z}$ as a parallel spinor. □

This result, together with Corollary 3.1 yields the solution to the analytic Cauchy problem for parallel spinors stated in Theorem 1.1.

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