Smooth Gowdy symmetric generalized Taub-NUT solutions

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Abstract. We study a class of $\mathbb{S}^3$ Gowdy vacuum models with a regular past Cauchy horizon which we call smooth Gowdy symmetric generalized Taub-NUT solutions. In particular, we prove existence of such solutions by formulating a singular initial value problem with asymptotic data on the past Cauchy horizon. The result of our investigations is that a future Cauchy horizon exists for generic asymptotic data. Moreover, we derive an explicit expression for the metric on the future Cauchy horizon in terms of the asymptotic data on the past horizon. This complements earlier results about $\mathbb{S}^1 \times \mathbb{S}^2$ Gowdy models.

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1. Introduction

Studies of cosmological solutions of Einstein's field equations have a long tradition and led to astonishing results about our own universe. In particular, observations indicate that there was a big bang in the distant past, and indeed, the simplest cosmological models, namely the Friedmann solutions for reasonable matter fields, predict precisely this behavior. The question arises whether such curvature singularities occur for generic solutions of Einstein's field equations or if the strong symmetry assumptions underlying the Friedmann models are necessary for this. The Hawking-Penrose singularity theorems [17] shed some light on this question. They predict incompleteness of causal geodesics in a wide class of situations. However, the information about the reason for the incompleteness provided by these theorems is very limited, and it is indeed not true that it is always caused by a geometric singularity.

Let us restrict all of our investigations to vacuum with a vanishing cosmological constant and to four spacetime dimensions. The corresponding Cauchy problem for Einstein's field equations is well-posed and leads to the notion of the maximal globally hyperbolic development (MGHD) of a given Cauchy data set [1, 35]. The example of the Taub solution [37] shows that incompleteness of causal geodesics, as predicted by the singularity theorems, signals a different kind of phenomenon where it is possible to extend the MGHD. The extended solutions are not globally hyperbolic. There exist closed causal curves and indeed, there are several non-equivalent extensions. This unexpected property has caused an ongoing debate in the literature. Are such pathological phenomena a typical feature of Einstein's theory of gravity – in which case it could not be considered as a "proper" physical theory – or do such phenomena only occur under very strong and special assumptions, for example the high symmetry of the Taub solutions? An interesting and intensively debated hypothesis in this context is the strong cosmic censorship conjecture whose widely accepted formulation was given for the first time in [11], based on ideas by Eardly and Moncrief [22] and Penrose [32]. More details and references can be found in [35, 34]. This conjecture states that the MGHD of *generic* Cauchy data is *inextendible*[2]. Roughly speaking, this implies that, in the generic situation, incompleteness of causal geodesics is indeed caused by a geometric singularity in some sense.

In his effort to generalize the family of Taub solutions and hence to show that there is a large (but still non-generic) class of solutions of the field equations with similar "undesired" properties, Moncrief defines the family of *generalized Taub-NUT spacetimes* in [24]. He is able to prove an existence result under an analyticity assumption. Our motivation in this paper is two-fold. First, we want to extend the existence result to the smooth case[3] and formulate it as a singular initial value problem with "asymptotic data" on the Cauchy horizon. However, in this paper, we restrict ourselves to the case with Gowdy symmetry. Secondly, we want to study the global behavior of such solutions. By means of so-called soliton methods, it turns out that in the generic case, the existence of a past Cauchy horizon implies the existence of a future Cauchy horizon, at least under certain topological assumptions on the horizons discussed later. The reader should compare these results to the $\mathbb{S}^1 \times \mathbb{S}^2$ case in [19].

The paper is organized as follows. In Section 2 we discuss some background material, in particular the symmetry reduction introduced by Geroch which is necessary to be able to write the metric and the field equations in a useful way. Section 3 is devoted to the definition and discussion of our class of *smooth Gowdy symmetric generalized Taub-NUT solutions* based on Moncrief's earlier mentioned class. The existence and uniqueness theory and the corresponding singular initial value problem is considered in Section 3.3. The basic ingredients for these investigations are Fuchsian methods which we describe briefly in the appendix. The next main part is Section 4 where we discuss the global-in-time properties of smooth Gowdy symmetric generalized Taub-NUT solutions. We finish the paper with conclusions in Section 5.

[2] This conjecture, of course, only makes sense if one is able to give a precise and reasonable meaning to the terms "generic" and "inextendible". At this stage, however, this has not been found for general situations. See e.g. [35].
[3] We use the term "smooth" for infinitely differentiable objects, as opposed to "analytic" for which the Taylor series converges in addition.

2. Geometric preliminaries

2.1. Symmetry reduction by Geroch

We briefly present here the symmetry reduction introduced by Geroch in [14]. Let $M = \mathbb{R} \times H$ be an oriented and time-oriented globally hyperbolic 4-dimensional spacetime endowed with a metric $g_{\mu\nu}$ of signature $(-,+,+,+)$, a global time-function t and a Cauchy surface H. We denote the volume form of $g_{\mu\nu}$ by $\epsilon_{\mu\nu\rho\sigma}$ and the hypersurfaces given by $t = t_0$ for any constant t_0 by H_{t_0}. Each H_{t_0} is homeomorphic to H.

Now, let ξ be a smooth spacelike[1] Killing vector field which is tangent to the hypersurfaces H_t and set

$$\lambda := g(\xi, \xi).$$

The twist 1-form of ξ is

$$\Omega_\mu := \epsilon_{\mu\nu\rho\sigma}\xi^\nu \nabla^\rho \xi^\sigma,$$

and ∇ is the covariant derivative compatible with g. The field ξ is hypersurface forming if and only if $\Omega_\mu \equiv 0$, which will, however, not be assumed in the following. We define the "3-metric"

$$h_{\mu\nu} := g_{\mu\nu} - \frac{1}{\lambda}\xi_\mu \xi_\nu$$

on M, and, by raising indices with the inverse of g, we define also $h^\mu{}_\nu$ and $h^{\mu\nu}$ on M. The first of these tensors is the projector to the space orthogonal to ξ in T_pM, $p \in M$. From the volume form $\epsilon_{\mu\nu\rho\sigma}$ of g, we furthermore introduce the tensor

$$\epsilon_{\mu\nu\rho} := \frac{1}{\sqrt{\lambda}}\epsilon_{\mu\nu\rho\sigma}\xi^\sigma.$$

Let α_μ be any 1-form. One defines the derivative operator D as

$$D_\mu \alpha_\nu := h^{\mu'}{}_\mu h^{\nu'}{}_\nu \nabla_{\mu'} \alpha_{\nu'}.$$

Note that at this stage we are only interested in local patches of M. Then, the flow generated by ξ induces a map π from M to the space of orbits S, i.e. π maps every $p \in M$ to the (locally) uniquely determined integral curve of ξ starting at p. The requirement that π is a smooth map induces a differentiable structure on S, and hence S can be considered as a smooth manifold. Using $\mathcal{L}_\xi h = 0$ and $h(\xi, \cdot) = 0$, Geroch shows that there is a unique metric on S which pulls back to $h_{\mu\nu}$ along π. We call this metric on S again $h_{\mu\nu}$. The same can be done for $h^\mu{}_\nu$ and $h^{\mu\nu}$, which are henceforth considered as objects on the quotient space S. The first is just the identity operator, while the second is the inverse of the 3-metric $h_{\mu\nu}$ on S. We can proceed in the same way with the function λ, the 1-form Ω_μ, the tensor $\epsilon_{\mu\nu\rho}$ and the covariant derivative operator D_μ, which can henceforth be considered as objects on S. Then $\epsilon_{\mu\nu\rho}$ becomes the volume form of $h_{\mu\nu}$ and D_μ the covariant derivative operator compatible with $h_{\mu\nu}$.

[1] Note that Geroch considers the case of a timelike Killing vector field.

Geroch shows that Einstein's vacuum field equations on (M, g) imply that the 1-form Ω is closed, $\mathrm{d}\Omega = 0$. Our local considerations allow us to introduce the twist potential ω so that $\Omega = \mathrm{d}\omega$. The basic quantities λ, ω and $h_{\mu\nu}$ on S completely characterize the geometry of (M, g).

Now Geroch introduces a conformal rescaling

$$\hat{h}_{\mu\nu} := \lambda h_{\mu\nu}.$$

We refer to the associated covariant derivative operator as $\hat{D}$, Ricci tensor as $\hat{S}_{\mu\nu}$ etc. He shows that the vacuum field equations for (M, g) (and certain geometric identities) are equivalent to the following set of equations

$$\hat{D}^2\lambda = \frac{1}{\lambda}\left(\hat{D}^\mu\lambda\hat{D}_\mu\lambda - \hat{D}^\mu\omega\hat{D}_\mu\omega\right), \tag{1}$$

$$\hat{D}^2\omega = \frac{2}{\lambda}\hat{D}^\mu\lambda\hat{D}_\mu\omega, \tag{2}$$

$$\hat{S}_{\mu\nu} = \frac{1}{2\lambda^2}\left(\hat{D}_\mu\lambda\hat{D}_\nu\lambda + \hat{D}_\mu\omega\hat{D}_\nu\omega\right). \tag{3}$$

These equations are the Euler-Lagrange equations of the Lagrangian density

$$\mathcal{L} = \sqrt{-\det\hat{h}}\left[\hat{S} + \frac{1}{2\lambda^2}\left(\hat{D}^\mu\lambda\hat{D}_\mu\lambda + \hat{D}^\mu\omega\hat{D}_\mu\omega\right)\right].$$

Hence, the equations can be interpreted as 3-dimensional gravity on S coupled to a wave map $u : S \to \mathcal{H}$ where $\mathcal{H}$ is the 2-dimensional hyperbolic space represented by the components (λ, ω).

We point out that the quotient manifold S can be identified with the surfaces orthogonal to ξ if and only if ξ is hypersurface forming, i.e. $\Omega_\mu = 0$. However, in general this is not the case and the manifold S cannot be interpreted as a submanifold of M.

2.2. Spacetimes of spatial 3-sphere topology with isometry group $U(1)$ or $U(1) \times U(1)$

We specialize the previous general discussion to the case $M = \mathbb{R} \times \mathbb{S}^3$. Now, the Cauchy surfaces are $H = \mathbb{S}^3$. We think of $\mathbb{S}^3$ as the submanifold of $\mathbb{R}^4$ determined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. The Euler coordinates of $\mathbb{S}^3$ are written as

$$x_1 = \cos\frac{\theta}{2}\cos\lambda_1, \quad x_2 = \cos\frac{\theta}{2}\sin\lambda_1,$$

$$x_3 = \sin\frac{\theta}{2}\cos\lambda_2, \quad x_4 = \sin\frac{\theta}{2}\sin\lambda_2,$$

where $\theta \in (0, \pi)$ and $\lambda_1, \lambda_2 \in (0, 2\pi)$. Clearly, these coordinates break down at the *poles* $\theta = 0$ and π. We will also use the coordinates (θ, ρ_1, ρ_2) determined by

$$\lambda_1 =: (\rho_1 + \rho_2)/2, \quad \lambda_2 =: (\rho_1 - \rho_2)/2. \tag{4}$$

Note that the coordinate fields ∂_{ρ_1} and ∂_{ρ_2} are smooth non-vanishing vector fields on $\mathbb{S}^3$, while the fields ∂_{λ_1} and ∂_{λ_2} vanish at certain places. Note that these fields can be characterized geometrically (without making reference to coordinates) in terms of left- and right-invariant vector fields with respect to the standard action of $SU(2)$ on $\mathbb{S}^3$, see for example [4, 5].

We specialize now to the case of a metric g with a spacelike Killing vector field ξ which generates a smooth effective action of the group $U(1)$ on M such that ξ is tangent to the level sets of the time function t. In particular, the integral curves of ξ are closed in H_t for each t. Moreover, we assume that H, and hence H_t for each t, is a Hopf bundle generated by ξ. This means that the map π, defined in the previous section, is a Hopf map from $M = \mathbb{R} \times \mathbb{S}^3$ to $S = \mathbb{R} \times \mathbb{S}^2$. In order to give an explicit representation of this map, we can introduce, on H_t for every t, either Euler coordinates or the coordinates x_1, x_2, x_3, x_4 of the embedding into $\mathbb{R}^4$. There is no loss of generality in assuming $\xi = \partial_{\rho_1}$ because we can apply a diffeomorphism from $\mathbb{S}^3$ to itself to achieve this for each t. However, from the point of view of the initial value problem, the assumption $\xi = \partial_{\rho_1}$ is not compatible with all coordinate gauges; see below. In any case, this allows to represent the map π as

$$\begin{aligned}\pi: \qquad & \mathbb{R} \times \mathbb{S}^3 \to \mathbb{R} \times \mathbb{S}^2, \\ & \quad (t, x_1, x_2, x_3, x_4) \mapsto (t, y_1, y_2, y_3) \\ & = (t, 2(x_1x_3 + x_2x_4), 2(x_2x_3 - x_1x_4), x_1^2 + x_2^2 - x_3^2 - x_4^2) \\ & = (t, \sin\theta\cos\rho_2, \sin\theta\sin\rho_2, \cos\theta).\end{aligned} \tag{5}$$

Here we consider $\mathbb{S}^2$ as the submanifold of $\mathbb{R}^3$ determined by $y_1^2 + y_2^2 + y_3^2 = 1$. When we introduce standard polar coordinates on $\mathbb{S}^2$, namely

$$y_1 = \sin\vartheta\cos\phi, \quad y_2 = \sin\vartheta\sin\phi, \quad y_3 = \cos\vartheta, \tag{6}$$

then we have that π is simply

$$(t, \theta, \rho_1, \rho_2) \mapsto (t, \vartheta, \phi) = (t, \theta, \rho_2).$$

In particular, the push-forward of ∂_{ρ_1} to $\mathbb{R} \times \mathbb{S}^2$ along π vanishes and the push-forward of ∂_{ρ_2} equals the coordinate vector field ∂_ϕ on $\mathbb{S}^2$.

We point out that Eq. (5) yields a global representation of the map π. Since both S and M are simply connected, the twist potential ω is defined globally. The vector fields ∂_{ρ_1} and ∂_{ρ_2} will play a major role in the following discussion. The fields ∂_{λ_1} and ∂_{λ_2}, on the other hand, cannot be used for Geroch's reduction directly because they vanish at either $\theta = 0$ or $\theta = \pi$.

Let us now consider a global smooth effective action of the group[2] $U(1) \times U(1)$ with a second spatial Killing vector field η commuting with ξ, under the same assumptions as before. We can[3] suppose $\xi = \partial_{\rho_1}$ and $\eta = \partial_{\rho_2}$, since it can be shown that all smooth effective actions of $U(1) \times U(1)$ on $\mathbb{S}^3$ are equivalent in the sense that any other smooth effective action equals the previous action after possibly applying a diffeomorphism of $\mathbb{S}^3$ into itself and an automorphism of $U(1) \times U(1)$ to itself [10]. This group action degenerates, in the sense that the group orbits become 1-dimensional, precisely at $\theta = 0$ and at $\theta = \pi$. When $\theta = 0$, the field ∂_{λ_2} vanishes, while ∂_{λ_1} vanishes at $\theta = \pi$. Note

[2] This is the direct product of the group $U(1)$ with itself.
[3] But this assumption leads to further restrictions on the possible choices of coordinate gauges for the initial value problem, see below.

that the fields ∂_{ρ_1} and ∂_{ρ_2} never vanish, but both become parallel at $\theta = 0$ and $\theta = \pi$. The action of the group $U(1)$ discussed in the previous section can be associated with the action of the subgroup $\{e\} \times U(1)$ of $U(1) \times U(1)$.

We consider Geroch's symmetry reduction only with respect to ξ and the corresponding projection map π according to Eq. (5). Since ξ and η commute, we are allowed, in principle, to apply the symmetry reduction successively one after the other for both fields. However, the result is then not a smooth manifold, but rather a manifold with boundary. In order to avoid the corresponding complications, we perform the reduction only with respect to ξ and obtain the orbit manifold $S = \mathbb{R} \times \mathbb{S}^2$ with the 3-metric h with Killing field η. Here, the push-forward of η along π to S is again denoted by η. Recall that the push-forward of ξ is 0. Now, according to Geroch [15] and Gowdy [16], Einstein's vacuum field equations imply that the twist quantities of the $U(1) \times U(1)$-action on M,

$$\kappa_1 := \epsilon_{\mu\nu\rho\sigma}\eta^\mu\xi^\nu\nabla^\rho\xi^\sigma, \quad \kappa_2 := \epsilon_{\mu\nu\rho\sigma}\eta^\mu\xi^\nu\nabla^\rho\eta^\sigma,$$

vanish. The geometrical interpretation is that the 2-space orthogonal to the 2-space spanned by ξ and η in M is integrable to a 2-surface everywhere. This allows us to make the following general ansatz for the metric g on M,

$$g = g_{AB}\mathrm{d}x^A\mathrm{d}x^B + R\left[\mathrm{e}^L(\mathrm{d}\rho_1 + Q\mathrm{d}\rho_2)^2 + \mathrm{e}^{-L}\mathrm{d}\rho_2{}^2\right], \tag{7}$$

where $A, B = 0, 1$ label coordinates on the submanifolds orthogonal to the Killing vector fields; the metric g_{AB} is so far unspecified. The functions R, L and Q only depend on t and θ, i.e. are constant along the Killing vector fields. It also follows that λ and ω, as objects on S, are constant along η. Moreover, η is a hypersurface orthogonal vector field in S (but not necessarily in M). We can compute

$$\lambda = R\mathrm{e}^L, \tag{8}$$

from Eq. (7), and

$$\partial_t\omega = -R\mathrm{e}^{2L}\sqrt{|\det(g_{CD})|}g^{\theta A}\partial_A Q, \quad \partial_\theta\omega = -R\mathrm{e}^{2L}\sqrt{|\det(g_{CD})|}g^{tA}\partial_A Q. \tag{9}$$

Finally, the 3-metric is

$$h = g_{AB}\mathrm{d}x^A\mathrm{d}x^B + R\mathrm{e}^{-L}\mathrm{d}\rho_2{}^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B + \frac{R^2}{\lambda}\mathrm{d}\rho_2{}^2. \tag{10}$$

Chruściel [9] defines the notion of smooth *generic vacuum Cauchy data* on a spacelike hypersurface of (M, g). He shows that there exist a smooth function[4] M, a constant $R_0 > 0$, and functions Q and L as above, on $(0, \pi) \times \mathbb{S}^3$ which are constant along ξ and η such that the spacetime $(\tilde{M}, \tilde{g})$ with $\tilde{M} = (0, \pi) \times \mathbb{S}^3$ and $\tilde{g}$ of the form Eq. (7) with

$$R = R_0 \sin t \sin\theta, \quad (g_{AB}) = \mathrm{e}^M \mathrm{diag}\,(-1, 1), \tag{11}$$

can be isometrically embedded into the maximal globally hyperbolic development of such data. We will assume that Chruściel's *genericity* condition is satisfied and henceforth

[4] The function M is not to be confused with the manifold M.

identify (M, g) with $(\tilde{M}, \tilde{g})$. The question whether (M, g) is isometric to the maximal globally hyperbolic development in general is, in some sense, the main content of this paper. One calls such a spacetime (M, g) a *Gowdy spacetime* and the time coordinate t is called *areal time*. We will assume that $t \in I := (0, \pi)$.

2.3. Smoothness conditions for the metric components

Of particular importance for our further studies is the behavior of the metric coefficients of h at the poles of the 2-sphere. We know that h is smooth on $I \times \mathbb{S}^2$ and invariant along η. Since η is purely spatial, it is hence a rotationally invariant metric. Moreover, η is hypersurface orthogonal with respect to h. We thus consider the standard embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$ in a neighborhood of the north pole $(y_1, y_2, y_3) = (0, 0, 1)$. We introduce a local coordinate patch (y_1, y_2) there which is related to spherical coordinates (ϑ, ϕ) by

$$y_1 = \sin\vartheta\cos\phi, \quad y_2 = \sin\vartheta\sin\phi.$$

A general smooth metric l has the form

$$l = l_{11}\mathrm{d}y_1^2 + 2l_{12}\mathrm{d}y_1\mathrm{d}y_2 + l_{22}\mathrm{d}y_2^2.$$

We want to study the implications of two conditions now:

(i) The metric is diagonal with respect to the (ϑ, ϕ)-coordinates (because $\eta = \partial_\phi$ is hypersurface orthogonal).

(ii) The metric is invariant under rotations around the polar axis (because η is a Killing vector field).

This easily implies that there are smooth functions F and G which only depend on $\cos\vartheta$ such that

$$\begin{aligned}
l_{11}(\vartheta, \phi) &= G(\cos\vartheta) - F(\cos\vartheta)\sin^2\vartheta\cos(2\phi),\\
l_{22}(\vartheta, \phi) &= G(\cos\vartheta) + F(\cos\vartheta)\sin^2\vartheta\cos(2\phi),\\
l_{12}(\vartheta, \phi) &= -F(\cos\vartheta)\sin^2\vartheta\sin(2\phi).
\end{aligned}$$

With respect to spherical coordinates, the metric then has the form

$$l = \cos^2\vartheta\left[G(\cos\vartheta) - F(\cos\vartheta)\sin^2\vartheta\right]\mathrm{d}\vartheta^2 + \sin^2\vartheta\left[G(\cos\vartheta) + F(\cos\vartheta)\sin^2\vartheta\right]\mathrm{d}\phi^2.$$

This representation is clearly only valid in a neighborhood of the north pole, in particular it breaks down at the equator due to the $\cos^2\vartheta$-factor. If we choose the functions F and G such that

$$l = \mathrm{e}^M\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{e}^{2U}\mathrm{d}\phi^2, \tag{12}$$

then M and U must be two smooth functions of $\cos\vartheta$ alone, so that the following condition holds

$$\mathrm{e}^M = \mathrm{e}^{2U} + \hat{M}(\cos\vartheta)\sin^2\vartheta \tag{13}$$

for some smooth function $\hat{M}$.

According to Eq. (10), the general metric h on $S = I \times \mathbb{S}^2$ is of the form

$$h = \mathrm{e}^M(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + \sin^2\theta\, \mathrm{e}^{2U}\mathrm{d}\rho_2{}^2, \tag{14}$$

in areal gauge given by Eq. (11). Hence

$$U = \ln R_0 + \ln\sin t - \frac{1}{2}\ln\lambda = (\ln R_0 - L - \ln\sin\theta + \ln\sin t)/2. \tag{15}$$

This is finite everywhere on $I\times\mathbb{S}^2$ under the previous assumptions because the quantity λ must be finite and bounded away from zero at the poles for all $t \in (0,\pi)$, and hence

$$\mathrm{e}^L = O(R^{-1}) = O(\sin^{-1}\theta).$$

In particular, the smoothness condition Eq. (13) translates to

$$\mathrm{e}^M = \frac{R^2}{\lambda\sin^2\theta} + \hat{M}(\cos\vartheta)\sin^2\vartheta. \tag{16}$$

A consequence is that e^M is bounded and non-vanishing at the poles for all $t \in (0,\pi)$.

Our choice of $\{\xi,\eta\} = \{\partial_{\rho_1},\partial_{\rho_2}\}$ as the Gowdy Killing basis has further important consequences due to the fact that $\partial_{\rho_1} = \partial_{\rho_2}$ at $\theta = 0$ and $\partial_{\rho_1} = -\partial_{\rho_2}$ at $\theta = \pi$. At the poles we must have $g(\xi,\xi) = g(\eta,\eta) = \pm g(\xi,\eta)$. Therefore there must exist a smooth function $\hat{Q}$ which only depends on t and $\cos\theta$ so that the function Q in Eq. (7) satisfies

$$Q(t,\theta) = \cos\theta + \hat{Q}(t,\cos\theta)\sin^2\theta. \tag{17}$$

In particular it follows from Eq. (9) that for each $t \in (0,\pi)$

$$-2 = Q(t,\pi) - Q(t,0) = \int_0^\pi Q_\theta\,\mathrm{d}\theta = -\int_0^\pi R^{-1}\mathrm{e}^{-2L}\partial_t\omega\,\mathrm{d}\theta = -\int_0^\pi R\lambda^{-2}\partial_t\omega\,\mathrm{d}\theta.$$

2.4. Reparametrizations of the Gowdy orbits

All of our discussions so far are based on the choice $\{\partial_{\rho_1},\partial_{\rho_2}\}$ as the Gowdy Killing basis on M. Now we study general reparametrizations of the Gowdy Killing orbits in M, i.e. arbitrary bases of the Gowdy Killing algebra. For example, the standard basis normally used in the literature is $\{\partial_{\lambda_1},\partial_{\lambda_2}\}$.

Let $(\phi_1,\phi_2) \in \mathbb{R}^2$ be coordinates on the Killing orbits so that a Gowdy invariant metric has the form analogous to Eq. (7), i.e.

$$g = \mathrm{e}^M(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + R\left[\mathrm{e}^L(\mathrm{d}\phi_1 + Q\mathrm{d}\phi_2)^2 + \mathrm{e}^{-L}\mathrm{d}\phi_2^2\right].$$

We are allowed to reparametrize the orbits by means of constants $a,b,c,d \in \mathbb{R}$, so that

$$ad - bc \neq 0$$

and

$$\phi_1 = a\tilde{\phi}_1 + b\tilde{\phi}_2, \quad \phi_2 = c\tilde{\phi}_1 + d\tilde{\phi}_2.$$

The coordinates t and θ are not changed. In terms of the new coordinates, we want to write the metric as

$$g = \mathrm{e}^M(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + \tilde{R}\left[\mathrm{e}^{\tilde{L}}(\mathrm{d}\tilde{\phi}_1 + \tilde{Q}\mathrm{d}\tilde{\phi}_2)^2 + \mathrm{e}^{-\tilde{L}}\mathrm{d}\tilde{\phi}_2^2\right].$$

One finds that

$$\tilde{R} = |ad - bc|R, \tag{18}$$

$$e^{\tilde{L}} = \frac{(a + c\,Q)^2 e^L + c^2 e^{-L}}{|ad - bc|}, \tag{19}$$

$$\tilde{Q} = \frac{(a + c\,Q)(b + d\,Q)e^L + c\,d\,e^{-L}}{(a + c\,Q)^2 e^L + c^2 e^{-L}}. \tag{20}$$

A particularly useful transformation is the *inversion*, i.e. the interchange of the Killing basis fields. Then we have $a = d = 0$, $b = c = 1$, and hence

$$\tilde{R} = R,$$
$$e^{\tilde{L}} = e^L Q^2 + e^{-L},$$
$$\tilde{Q} = \frac{e^L Q}{e^L Q^2 + e^{-L}}.$$

Let us now consider a metric in the parametrization (ρ_1, ρ_2) of the Killing orbits as given by Eq. (7). We pick $\phi_1 = \rho_1$ and $\phi_2 = \rho_2$. Now let $\tilde{\phi}_1 = \lambda_1$ and $\tilde{\phi}_2 = \lambda_2$ and hence

$$g = e^M(-dt^2 + d\theta^2) + \tilde{R}\left[e^{\tilde{L}}(d\lambda_1 + \tilde{Q}d\lambda_2)^2 + e^{-\tilde{L}}d\lambda_2^2\right].$$

For this we must choose $a = 1$, $b = 1$, $c = 1$, $d = -1$ from Eq. (4). It follows that

$$\tilde{R} = 2R, \tag{21}$$

$$e^{\tilde{L}} = \frac{(1 + Q)^2 e^L + e^{-L}}{2}, \tag{22}$$

$$\tilde{Q} = \frac{-(1 - Q^2)e^L + e^{-L}}{(1 + Q)^2 e^L + e^{-L}}. \tag{23}$$

The inverse of this reparametrization is

$$R = \frac{\tilde{R}}{2},$$
$$e^L = \frac{(1 + \tilde{Q})^2 e^{\tilde{L}} + e^{-\tilde{L}}}{2},$$
$$Q = \frac{(1 - \tilde{Q}^2)e^{\tilde{L}} - e^{-\tilde{L}}}{(1 + \tilde{Q})^2 e^{\tilde{L}} + e^{-\tilde{L}}}.$$

From this and the discussion in Section 2.3, we can easily derive the behavior of the functions $\tilde{R}$, $\tilde{L}$, $\tilde{Q}$ at the poles for $t \in (0, \pi)$ in areal coordinates,

$$\tilde{R} = \hat{R}\sin\theta, \tag{24}$$

$$e^{\tilde{L}} = e^{\hat{L}}\cot\frac{\theta}{2}, \tag{25}$$

$$\tilde{Q} = (1 - \cos\theta)\hat{Q}\,, \tag{26}$$

$$e^M = \frac{\hat{R}}{4}\left[e^{\hat{L}}(1 - \cos\theta) + e^{-\hat{L}}(1 + \cos\theta)\right] + \hat{M}\,\sin^2\theta. \tag{27}$$

with smooth functions $\hat{R}$, $\hat{L}$, $\hat{Q}$ and $\hat{M}$ which only depend on t and on $\cos\theta$.

Note the following interesting general fact about *polarized Gowdy spacetimes*. A Gowdy metric is called polarized if there exists an everywhere orthogonal basis of Gowdy

Killing fields. With respect to this basis, the function Q must hence vanish identically. Now, Eq. (17) shows that this can never happen for the Killing basis $\{\partial_{\rho_1}, \partial_{\rho_2}\}$, but it is possible for the basis $\{\partial_{\lambda_1}, \partial_{\lambda_2}\}$ according to Eq. (26). Indeed, one can show that Q can only vanish identically for a smooth Gowdy symmetric metric on $\mathbb{S}^3$ if the Killing basis is chosen such that one of the two fields is proportional to ∂_{λ_1} and the other to ∂_{λ_2}.

3. The class of smooth Gowdy symmetric generalized Taub-NUT solutions

3.1. The Taub solutions

The *Taub solutions* were discovered by Taub [37] as a family of cosmological solutions of the vacuum field equation with spatial $\mathbb{S}^3$-topology. They are a two-parameter family of spacetimes

$$g = l^2\left(-\frac{4(1+\tau^2)}{V(\tau)}\mathrm{d}\tau^2 + (1+\tau^2)(\omega_1\otimes\omega_1+\omega_2\otimes\omega_2) + \frac{V(\tau)}{1+\tau^2}\omega_3\otimes\omega_3\right),$$

with $l>0$, $m\in\mathbb{R}$ and

$$V(\tau) := -4\tau^2 - 8\frac{m}{l}\tau + 4.$$

Here,

$$\begin{aligned}
\omega_1 &= \sin\rho_1\mathrm{d}\theta - \cos\rho_1\sin\theta\mathrm{d}\rho_2,\\
\omega_2 &= \cos\rho_1\mathrm{d}\theta + \sin\rho_1\sin\theta\mathrm{d}\rho_2,\\
\omega_3 &= \mathrm{d}\rho_1 + \cos\theta\mathrm{d}\rho_2,
\end{aligned}$$

are the standard left-invariant one-forms[5] on $\mathbb{S}^3$. The metric is smooth and globally hyperbolic for

$$\tau\in(\tau_-,\tau_+),\quad \tau_\pm := -m/l \pm \sqrt{1+m^2/l^2}.$$

It was demonstrated for the first time in [31] that these solutions can be continued analytically through the apparently singular times $\tau_\pm$. These extensions were christened *Taub-NUT solutions*. The extended solutions are not globally hyperbolic; in particular there exist closed causal curves. Moreover, there are several non-equivalent analytic extensions.

We easily find that the Taub solutions are Gowdy symmetric (but not polarized), and we can bring them to the form Eq. (7) with (11)

$$g = \mathrm{e}^M(-\mathrm{d}t^2+\mathrm{d}\theta^2) + R\left[\mathrm{e}^L(\mathrm{d}\rho_1+Q\mathrm{d}\rho_2)^2 + \mathrm{e}^{-L}\mathrm{d}{\rho_2}^2\right].$$

For arbitrary parameters $l>0$ and $m\in\mathbb{R}$, the Taub solutions are then given by

$$\begin{aligned}
R &= 2l\sqrt{l^2+m^2}\sin t\sin\theta,\\
\mathrm{e}^M &= l^2 + (m+\sqrt{l^2+m^2}\cos t)^2,
\end{aligned}$$

[5] The term *left-invariance* here is understood with respect to the standard action of SU(2) on $\mathbb{S}^3$. Note that SU(2) is a subgroup of the isometry group of the Taub solutions.

$$\begin{aligned} e^L &= \frac{R}{e^M \sin^2\theta}, \\ Q &= \cos\theta. \end{aligned}$$

3.2. Generalizations of the Taub-NUT solutions

Moncrief [24] introduces the family of *generalized Taub-NUT spacetimes* as globally hyperbolic spacetimes $(0, \delta] \times \mathbb{S}^3$ with a smooth time function t for a sufficiently small $\delta > 0$. The level sets of t are Cauchy surfaces diffeomorphic to $\mathbb{S}^3$. Moncrief wants to study the situation where global hyperbolicity breaks down at $t = 0$, in the sense that the spacetimes can be extended through $t = 0$ as non-globally hyperbolic spacetimes so that the points corresponding to $t = 0$ form a smooth compact null hypersurface, i.e. a Cauchy horizon. It was shown in [23] that if the spacetime is an analytic solution of the vacuum field equations and if the Cauchy horizon is ruled by closed null generators in the sense of an $\mathbb{S}^1$-bundle, i.e. the null generator coincides with the generators of the bundle, then the spacetime necessarily has a 1-dimensional isometry group and the corresponding Killing field is proportional to the null generators of the Cauchy horizon on the horizon. The result was generalized to the smooth case in [13].

Motivated by this result, Moncrief focuses his discussion of generalized Taub-NUT solutions on the case of spacetimes above for which the Cauchy horizon is generated in the sense of a Hopf bundle with $U(1)$ isometry group. The metrics of all such spacetimes can be written as

$$g = e^{-2\gamma}(-\tilde{N}^2 dt^2 + \tilde{g}_{ab} dx^a dx^b) + \sin^2 t \, e^{2\gamma}[k(d\rho_1 + \cos\theta \, d\rho_2) + \beta_a dx^a]^2,$$

for sufficiently small $t > 0$, where we can assume as before that the corresponding Killing field is ∂_{ρ_1}. The functions γ and $\tilde{N}$ only depend on t, θ and ρ_2. The index a takes the values 1 (corresponding to the coordinate θ) and 3 (corresponding to the coordinate ρ_2). The field $\tilde{g}_{ab}$ is a symmetric 2-tensor field and the β_a a 1-form. Moreover, $k > 0$ is a constant. One assumes that all fields γ, $\tilde{N}$, $\tilde{g}_{ab}$ and β_a are smooth on $(0, \delta] \times \mathbb{S}^3$ for some small $\delta > 0$ and have a unique smooth extension through $t = 0$. We express this as the requirement that these can be considered[6] as smooth fields on $[-\delta, \delta] \times \mathbb{S}^3$. Here, we assume that $\tilde{N}$ is strictly positive.

In the following, we restrict to the case of Gowdy symmetry. Therefore the metric coefficients above are in addition independent of ρ_2. As discussed before, the metric can then be assumed to be of the form Eq. (7) with Eq. (11) for $t \in (0, \delta]$ for some $\delta \in (0, \pi)$. Let us define[7] the function N by

$$e^L = \frac{R e^{-M}}{\sin^2\theta} N^2. \tag{28}$$

[6] This is not to be confused with the requirement that the metric g is smooth on $[-\delta, \delta] \times \mathbb{S}^3$. Indeed, g, in the form above, degenerates at $t = 0$ and hence cannot be extended as a Lorentzian metric to all $t \in [-\delta, \delta]$.

[7] Eqs. (16) and (8) imply that $N(t, 0) = N(t, \pi) = 1$ for all $t > 0$.

Then we must set

$$
\begin{aligned}
&\tilde{g}_{\theta\theta} = \mathrm{e}^M, \quad \tilde{g}_{\theta\rho_1} = 0, \quad \tilde{g}_{\rho_1\rho_1} = \frac{\mathrm{e}^M \sin^2\theta}{N^2},\\
&\beta_\theta \ = 0, \quad \beta_\phi = R_0(Q - \cos\theta),\\
&\tilde{N} = \frac{R_0 N}{k}, \quad \gamma = -\frac{M}{2} + \ln\frac{R_0 N}{k}.
\end{aligned}
$$

The next task is to identify the conditions under which these spacetimes, in particular the metrics g, are extendible smoothly through $t = 0$. We introduce new coordinates $(t', \theta', {\rho_1}', {\rho_2}')$ by

$$
t = \arcsin\sqrt{t'}, \quad \theta = \theta', \quad \rho_1 = {\rho_1}' + \frac{\kappa}{R_0}\ln t', \quad \rho_2 = {\rho_2}'. \tag{29}
$$

Here, we assume that $t > 0$ is sufficiently small and hence $t' > 0$ is small. The quantity κ is a constant which has, so far, not been fixed. In these new coordinates, the metric becomes

$$
\begin{aligned}
g = &-\left(\frac{\mathrm{e}^M}{4(1-t')t'} - \frac{\mathrm{e}^{-M}N^2\kappa^2}{t'}\right)\mathrm{d}t'^2 + \mathrm{e}^M \mathrm{d}\theta^2\\
&+\mathrm{e}^{-M}N^2\left[2R_0\kappa(\mathrm{d}{\rho_1}' + Q\mathrm{d}{\rho_2}')\mathrm{d}t' + R_0^2 t'\left(\mathrm{d}{\rho_1}' + Qd{\rho_2}'\right)^2\right] + \frac{\mathrm{e}^M\sin^2\theta}{N^2}\mathrm{d}{\rho_2}'^2.
\end{aligned}
$$

The metric extends smoothly through $t' = 0$ if the functions M, N^2 (strictly positive) and Q are smooth with respect to t' in a neighborhood of $t' = 0$ and if $(4\kappa^2N^2 - \mathrm{e}^{2M})/t'$ is smooth for some choice of the constant κ. In this case, the field $\partial_{\rho_1'} = \partial_{\rho_1}$ is a null generator of the surface given by $t' = 0$. This surface is therefore a smooth null hypersurface with $\mathbb{S}^3$-topology and closed null generators and so is a smooth past Cauchy horizon.

For the Taub solutions for example, we choose

$$
\kappa = \pm\left(l^2 + m(m + \sqrt{l^2 + m^2})\right).
$$

Then the metric is extendible analytically and hence the $t = 0$-surface is a past Cauchy horizon[8]. The analogous argument applied for $t = \pi$ implies the existence of a future Cauchy horizon. It is shown in [31, 12] that several non-equivalent extensions through both horizons exist.

We will call spacetimes with all the above properties *smooth Gowdy symmetric generalized Taub-NUT spacetimes*. The name is motivated by Moncrief's before mentioned notion of *generalized Taub-NUT spacetimes*. We will restrict to Gowdy symmetry, but in contrast to Moncrief, we will not assume analyticity. Note that if an analytic spacetime as above solves Einstein's field equations in vacuum with vanishing cosmological constant for $t > 0$, then the analytically extended spacetimes are necessarily also solutions of the vacuum field equations. In the non-analytic smooth case, we do not know in general whether the extensions are vacuum solutions. We will not address this problem in this paper. As an interesting side-remark: Chruściel et al.

[8] Often in this paper, we will speak sloppily of the "$t = 0$-surface" when we actually mean the $t' = 0$-surface.

note in [9] that there are no smooth extensions through a Cauchy horizon of $\mathbb{S}^3$-topology – solution of the field equations or not – in the *polarized* Gowdy case. As we mentioned before, however, none of the spacetimes which we consider in the following are polarized.

3.3. Existence of smooth Gowdy symmetric generalized Taub-NUT solutions

3.3.1. The main existence result. In this section we address the question as to whether smooth Gowdy symmetric generalized Taub-NUT spacetimes, which solve Einstein's vacuum field equations, exist. Our techniques here are based on the particular Fuchsian method introduced in [7, 6]; a quick summary can be found in the appendix. Similar existence results can be obtained by the Fuchsian techniques in [21, 33, 36].

In the following we call a function *rotationally symmetric* on $\mathbb{S}^2$ if it does not depend on the azimuthal angle ϕ in standard spherical coordinates Eq. (6). The Hopf map allows to lift any such function to a smooth $U(1)\times U(1)$-invariant function on $\mathbb{S}^3$ (or $\mathbb{R}\times\mathbb{S}^3$ if the function also depends on t).

Theorem 3.1 *Let S_{**} and Q_* be rotationally symmetric functions in $C^\infty(\mathbb{S}^2)$ so that*

$$S_{**}(0) = S_{**}(\pi), \tag{30}$$

and R_0 a positive constant. Then there exists a unique smooth Gowdy symmetric generalized Taub-NUT solution for all $t\in(0,\pi)$ with

$$R(t,\theta) = R_0 \sin t \sin\theta,$$

satisfying the following uniform expansions at $t=0$:

$$\begin{aligned} R(t,\theta)\mathrm{e}^{L(t,\theta)} &= t^2\mathrm{e}^{S_{**}(\theta)} + O(t^4),\\ Q(t,\theta) &= \cos\theta + Q_*(\theta)\sin^2\theta + O(t^2),\\ M(t,\theta) &= S_{**}(\theta) - 2S_{**}(0) + 2\ln R_0 + O(t^2). \end{aligned}$$

Corresponding expansions hold for all derivatives.

Let us make a couple of comments.

(i) This result means that we can construct a unique smooth Gowdy symmetric generalized Taub-NUT solution from any given *asymptotic data* functions S_{**} and Q_* subject to the conditions (30). We have thus obtained the same number of free functions as in Moncrief's class of generalized Taub-NUT solution (after factoring out gauge transformations in his class).

(ii) The Taub-NUT solutions are determined by the asymptotic data

$$\begin{aligned} R_0 &= 2l\sqrt{l^2+m^2},\\ S_{**} &= 2\ln R_0 - \ln\left(l^2 + (m+\sqrt{l^2+m^2})^2\right),\\ Q_* &= 0. \end{aligned}$$

(iii) We shall now show that none of the solutions of Theorem 3.1 are polarized. Recall from Section 2.4 that a smooth Gowdy symmetric spacetime is polarized if Q vanishes with respect to the (λ_1, λ_2)-parametrization of the symmetry orbits. Eq. (23) yields that the spacetime is polarized if and only if $1-(1-Q^2)\mathrm{e}^{2L} \equiv 0$ with respect to the (ρ_1, ρ_2)-parametrization. However, for our solutions, Q is bounded in a neighborhood of $t = 0$ and e^{2L} is $O(t^4)$ for every θ, which is therefore a contradiction.

(iv) We compute the uniform limit of the quantity N (defined in Eq. (28)) at $t = 0$,

$$N(0, \theta) = \mathrm{e}^{2(S_{**}(\theta)-S_{**}(0))}.$$

From that it is easy to determine that value of the constant κ (defined in Eq. (29)) which allows the solution to be extended through $t = 0$:

$$\kappa = \pm\frac{R_0^2}{2}\mathrm{e}^{-S_{**}(0)}.$$

3.3.2. Equations and unknowns. In order to attempt the proof of Theorem 3.1, let us make the following convenient choices. Instead of the unknown L, we shall use

$$S := \ln\lambda = L + \ln R.$$

Moreover, Q is replaced by the twist potential ω (defined in Section 2). Eqs. (1) and (2), together with (14) and (15) imply the following equations for S and ω,

$$D^2S - t^2\Delta_{\mathbb{S}^2}S = (1 - t\cot t)DS - \mathrm{e}^{-2S}[(D\omega)^2 - (t\omega_\theta)^2], \tag{31}$$

$$D^2\omega - 4D\omega - t^2\Delta_{\mathbb{S}^2}\omega = (1 - t\cot t)D\omega + 2(DS - 2)D\omega - 2(t\partial_\theta S)(t\partial_\theta\omega). \tag{32}$$

Note that we have added a term $-4D\omega$ to the second equation for later convenience. We use the notation $D := t\partial_t$ and $D^2 = t\partial_t(t\partial_t)$. The operator $\Delta_{\mathbb{S}^2}$ is the Laplace operator of the standard metric on the unit sphere

$$\Delta_{\mathbb{S}^2} = \partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2.$$

In our case all unknowns are independent of the azimuthal angle ϕ, i.e. are rotationally symmetric. Therefore these two equations are geometric wave equations with respect to the standard metric on the unit sphere as long as $t \in (0, \pi)$. Indeed, these equations form a second-order hyperbolic Fuchsian system, see below.

Eq. (3) gives us the remaining equations. On the one hand, we obtain a wave equation for M,

$$0 = -M_{tt} + \Delta_{\mathbb{S}^2}M + \cot t(M_t + 2S_t) - S_t^2 - \mathrm{e}^{-2S}\omega_x^2 + 2 =: H. \tag{33}$$

On the other hand, we get two first-order equations

$$0 = 4R_{\pm,\theta} + R(S_\pm^2 + \mathrm{e}^{-2S}\omega_\pm^2) - 2R_\pm(S_\pm + M_\pm) =: 2R_\pm C_\pm, \tag{34}$$

with[9] $\partial_\pm := \partial_\theta \pm \partial_t$. The quantities H, C_+ and C_- are introduced for the later discussion.

[9] Lower indices $\pm$ of any function previously defined without an index mean the directional derivatives along the vector fields $\partial_\pm$.

3.3.3. Steps of the proof of Theorem 3.1. The main step in the proof is the construction of solutions of (31) and (32) in a small time neighborhood of $t = 0$, which are compatible with the notion of smooth Gowdy symmetric generalized Taub-NUT solutions given in Section 3.2. The main local existence result is as follows.

Proposition 3.2 *Let $\omega_* \in \mathbb{R}$, and $\omega_{**}, S_{**} \in H^{k+1}(\mathbb{S}^2)$ be rotationally symmetric functions with $k \geq 3$. Choose constants $\alpha = (\alpha_1, \alpha_2)$ with $1 < \alpha_1 < 2$ and $0 < \alpha_2 < \alpha_1$. Then there exists a unique solution (S, ω) of the Gowdy equations (31) and (32) with*

$$S(t, \theta) = 2 \ln t + S_{**}(\theta) + w_1(t, \theta),$$
$$\omega(t, \theta) = \omega_* + \omega_{**}(\theta) t^4 + w_2(t, \theta),$$

where $w := (w_1, w_2) \in \tilde{X}_{\delta,\alpha,k}$ for a sufficiently small $\delta > 0$. The functions w_1, w_2 are rotationally symmetric for each $t \in (0, \delta]$. If ω_{**} and S_{**} are in $C^\infty(\mathbb{S}^2)$, then $w \in X_{\delta,\alpha,\infty}$.

The spaces $X_{\delta,\alpha,k}$ and $\tilde{X}_{\delta,\alpha,k}$ are introduced in the appendix. The spaces H^k are the standard Sobolev spaces.

Here are some comments about this result:

(i) The problem addressed in this proposition has many similarities to the *standard singular initial value problem* of Fuchsian equations. Hence Theorem 3.6 of [6] should apply directly. However, in contrast to the standard singular initial value problem, it is crucial here to assume that $w_* =$ constant. Only if this is the case, are conditions (1) and (2) of that theorem satisfied. Still, condition (3) fails. We find that condition (3) has been derived in [6] from Proposition 3.5 for non-constant asymptotic data. Given that $w_* =$ constant, however, condition (3) weakens and it is straightforward to check that these assumptions are sufficient.

(ii) We stress that the singular initial value problem considered here is significantly different than a standard Cauchy problem. Since the equations are singular at $t = 0$, but of Fuchsian type, we obtain the fourth power of t in the leading-order term of ω and the logarithm in the expansion of S on the one hand, and we do not obtain the full expected number of free functions on the other hand. However, as mentioned before, the number of free functions here is the same as in Moncrief's results.

(iii) We must assume $k \geq 3$ (as opposed to $k \geq 2$ in [6]). Because of the Sobolev inequalities, the condition $k \geq 3$ is required when we check condition (2) of Theorem 3.6 of [6] for the 2-dimensional spatial manifold $\mathbb{S}^2$ here. It is crucial that Eqs. (31) and (32) are geometric wave equations in order to derive the energy estimates in the same way as in [7, 6].

(iv) The last statement in the theorem means that if ω_{**} and S_{**} are *smooth* rotationally symmetric asymptotic data functions on $\mathbb{S}^2$, then the corresponding solution is smooth on the time interval $(0, \delta]$. Furthermore, w and all of its derivatives decay with the same rate in t at each spatial point.

(v) The constants α_1 and α_2 control the decay of the remainders w_1 and w_2 at $t = 0$. Roughly speaking, the statement $(w_1, w_2) \in \tilde{X}_{\delta,\alpha,k}$ means that $w_1 = O(t^{\alpha_1})$ and $w_2 = O(t^{4+\alpha_2})$ at $t = 0$. We will argue below that in fact $\alpha_1 = \alpha_2 = 2$ in the smooth case. The fact that the statement about these constants in Proposition 3.2 is weaker than this is related to the following property of our general Fuchsian theory. Roughly speaking, the theory assumes that the solutions might have additional logarithmic terms in t. Those terms would show up in particular if the exponents of t in the leading-order terms at $t = 0$ are spatially dependent. This is generically the case for many Fuchsian problems, however, not for us here.

The proof of Proposition 3.2 follows roughly the same steps as in the Gowdy case with spatial $\mathbb{T}^3$-topology discussed in [8, 6].

Let us restrict to the smooth case $\omega_{**}, S_{**} \in C^\infty(\mathbb{S}^2)$ from now on. We see easily from Eqs. (31), (32) and the expansions in Proposition 3.2 that e^S and ω can be extended as smooth solutions through $t = 0$ such that both are even in t. Hence, we can assume in all of what follows that both are solutions of the equations on, say, $[-\delta, \delta] \times \mathbb{S}^2$.

We can compute Q by means of Eqs. (9)

$$\partial_\theta Q = -R\mathrm{e}^{-2S}\partial_t\omega, \quad \partial_t Q = -R\mathrm{e}^{-2S}\partial_\theta\omega. \tag{35}$$

The integrability condition for these equations follows from the fact that ω is a solution of Eq. (32). Let us suppose that a solution of Proposition 3.2 is given. Then all terms in Eqs. (35) are smooth on $[-\delta, \delta] \times \mathbb{S}^2$. Hence these equations determine Q uniquely up to a constant and Q is a smooth function on $[-\delta, \delta] \times \mathbb{S}^2$. From

$$\partial_\theta Q(0, \theta) = -4R_0\omega_{**}(\theta)\mathrm{e}^{-2S_{**}(\theta)} \sin\theta,$$

it follows that

$$Q(0, \theta) = Q_0 - 4R_0 \int_0^\theta \omega_{**}(x)\mathrm{e}^{-2S_{**}(x)} \sin x \, \mathrm{d}x,$$

for some constant Q_0. In order to guarantee the smoothness condition Eq. (17), namely $Q(t, 0) = 1$, $Q(t, \pi) = -1$ for all $t > 0$, we must choose $Q_0 = 1$ and the asymptotic data must satisfy

$$\int_0^\pi \omega_{**}(x)\mathrm{e}^{-2S_{**}(x)} \sin x \, \mathrm{d}x = \frac{1}{2R_0}. \tag{36}$$

Then it follows that $Q(0, 0) = 1$, $Q(0, \pi) = -1$. Hence from the second of Eqs. (35) we conclude that $Q(t, 0) = 1$, $Q(t, \pi) = -1$ for all $t \in [-\delta, \delta]$. This is summarized in the following lemma.

Lemma 3.3 *Let $R_0 > 0$ be a constant and ω_{**} and S_{**} be rotationally symmetric functions in $C^\infty(\mathbb{S}^2)$ satisfying Eq. (36). Suppose that (S, ω) is the corresponding solution of (31) and (32) according to Proposition 3.2. Then there is a unique smooth solution Q of Eqs. (35) on $[-\delta, \delta] \times \mathbb{S}^2$ which is rotationally invariant at each t and $Q(t, 0) = 1$, $Q(t, \pi) = -1$.*

It follows easily that Q has the expansion

$$Q(t,\theta) = 1 - 4R_0 \int_0^\theta \omega_{**}(x)\mathrm{e}^{-2S_{**}(x)} \sin x \,\mathrm{d}x + \frac{1}{2} R_0\, \omega'_{**}(\theta)\mathrm{e}^{-2S_{**}(\theta)} \sin\theta\, t^2 + \dots,$$

in t at $t=0$ and that Q is an even function in t. We can introduce a smooth rotationally symmetric function $Q_*(\theta)$ such that

$$Q(t,\theta) = \cos\theta + Q_*(\theta)\sin^2\theta + O(t^2).$$

Hence, instead of prescribing the function w_{**} as one of the asymptotic data functions in Proposition 3.2, we can equivalently prescribe the function Q_* and then set

$$w_{**}(\theta) = \mathrm{e}^{2S_{**}(\theta)} \frac{1 - \partial_x Q_*(\theta)\sin\theta - 2Q_*(\theta)\cos\theta}{4R_0}.$$

In the next step of the proof of Theorem 3.1 we construct the function M from the constraint equations (34) for given solutions (S,ω) of Proposition 3.2. The result is as follows.

Proposition 3.4 *Let $R_0 > 0$ be a constant and ω_{**} and S_{**} be rotationally symmetric functions in $C^\infty(\mathbb{S}^2)$ satisfying*

$$S_{**}(0) = S_{**}(\pi).$$

Suppose that (S,ω) is the corresponding solution of (31) and (32) according to Proposition 3.2. Then there is a unique smooth solution M of Eqs. (37) on $[-\delta,\delta]\times\mathbb{S}^2$ which is rotationally invariant at each t and satisfies the smoothness condition $\mathrm{e}^{M(t,\theta)} = (R_0^2\sin^2 t)\mathrm{e}^{-S(t,\theta)}$ for all $(t,\theta)\in[-\delta,\delta]\times\{0,\pi\}$. Moreover, M is an even function of t. The expansion of M at $t=0$ in t is

$$M(t,\theta) = S_{**}(\theta) - 2S_{**}(0) + 2\ln R_0 + O(t^2).$$

The proof of this is maybe the trickiest part of this section. Let us define the functions

$$\mu_\pm := 4R_{\pm,\theta} + R(S_\pm^2 + \mathrm{e}^{-2S}\omega_\pm^2).$$

Then the constraint equations can be written as

$$M_t = -S_t + \frac{\mu_+}{4R_+} - \frac{\mu_-}{4R_-}, \quad M_\theta = -S_\theta + \frac{\mu_+}{4R_+} + \frac{\mu_-}{4R_-}. \tag{37}$$

Since $R_\pm = R_0\sin(t\pm\theta)$, the terms on the right-hand sides are potentially dangerous along the diagonals of the Gowdy square where $R_\pm = 0$. Moreover, the functions $\mu_\pm$ are not bounded everywhere in the limit $t\to 0$. We proceed as follows. Instead of interpreting the functions in Eqs. (37) as rotationally symmetric functions on $\mathbb{R}\times\mathbb{S}^2$, we consider them for the moment as functions on $\mathbb{R}\times\mathbb{R}$ where we extend θ to $\mathbb{R}$, 2π-periodically. Smooth rotationally symmetric functions on $\mathbb{R}\times\mathbb{S}^2$ will then be identified with functions which are even periodic in θ. Since S is such a smooth function, it follows that S_t is even and S_θ is odd in this sense. Since we want to construct a smooth solution M of Eqs. (37), it follows that

$$F^{\text{even}} := \frac{\mu_+}{4R_+} - \frac{\mu_-}{4R_-} - \frac{2}{t}, \quad F^{\text{odd}} := \frac{\mu_+}{4R_+} + \frac{\mu_-}{4R_-}, \tag{38}$$

are even and odd functions of θ, respectively. Indeed this is true and it follows directly from their definition. The term $2/t$ is introduced for later convenience. This renders Eqs. (37) into

$$M_t = -(S_t - 2/t) + F^{\text{even}}, \quad M_\theta = -S_\theta + F^{\text{odd}}. \tag{39}$$

The fact that (S, ω) solves Eqs. (31) and (32) implies the following linear symmetric hyperbolic system

$$\partial_t F^{\text{even}} - \partial_\theta F^{\text{odd}} = G^{\text{even}}, \quad \partial_t F^{\text{odd}} - \partial_\theta F^{\text{even}} = 0, \tag{40}$$

with the even source function

$$G^{\text{even}} = \frac{2}{t^2} + \frac{1}{2}(S_+ S_- + \mathrm{e}^{-2S}\omega_+\omega_-).$$

Note that this function extends smoothly through $t = 0$ and hence the system Eq. (40) has a well-posed Cauchy problem with data on $t = 0$. We call those data functions $F_*^{\text{even}}(\theta)$ and $F_*^{\text{odd}}(\theta)$, and hence must suppose that these are even and odd, respectively, periodic smooth functions. Let us write F_f^{even} and F_f^{odd} for the smooth solution of Eq. (40) for any given such data F_*^{even} and F_*^{odd} in order to distinguish them from the functions F^{even} and F^{odd} determined by Eq. (38) from the given functions S and ω. It is straightforward to solve Eqs. (39) for M when F^{even} is substituted by F_f^{even} and F^{odd} by F_f^{odd}. Then the integrability condition is satisfied as a consequence of Eq. (40), and it follows

$$M(t, \theta) = M_* - (S(t, \theta) - 2\ln t) + \int_0^\theta F_*^{\text{odd}}(x)\,\mathrm{d}x + \int_0^t F_f^{\text{even}}(\tau, \theta)\,\mathrm{d}\tau, \tag{41}$$

for some constant M_*. Therefore, M can be considered as a smooth rotationally symmetric function on $[-\delta, \delta] \times \mathbb{S}^2$. It is an even function in t if and only if $F_*^{\text{even}} = 0$.

Now we must study the conditions for which $F^{\text{even}} \equiv F_f^{\text{even}}$ and $F^{\text{odd}} \equiv F_f^{\text{odd}}$ in order to give a meaning to Eq. (41). We find that under the conditions for S and ω of Proposition 3.2, the function F^{even} converges pointwise to 0 at $t = 0$, while F^{odd} goes to $2S'_{**}$, for every $\theta \in [0, \pi]$. Indeed, F^{even} and F^{odd} can be extended uniquely as continuous functions to all the points given by $t = 0$ and $\theta \in (0, \pi)$. Let us hence choose $F_*^{\text{even}} = 0$ and $F_*^{\text{odd}} = 2S'_{**}$. On the interior of the domain of dependence for Eqs. (40) of the subinterval $(0, \pi)$ of the $t = 0$-surface, it follows that $F^{\text{even}} \equiv F_f^{\text{even}}$ and $F^{\text{odd}} \equiv F_f^{\text{odd}}$ because both sets of functions are smooth and satisfy Eq. (40) with the same data at $t = 0$. Let us call this domain Ω. Recall, however, that F^{even} and F^{odd} are not defined in those points of $\partial\Omega$ where $R_\pm = 0$. However, we can extend the functions F^{even} and F^{odd} to those points using the continuous values of F_f^{even} and F_f^{odd} there. From the definition of F^{even} and F^{odd} involving only quotients of smooth functions, which become zero simultaneously at the same points, we obtain that these extensions of F^{even} and F^{odd} are continuous on the whole domain, in particular through $\partial\Omega$, possibly except for the points $(0, 0)$ and $(0, \pi)$. The continuity in the points $(0, 0)$ and $(0, \pi)$ follows directly from the asymptotic behavior of S and ω there. Therefore, F^{even} and F^{odd} are uniformly continuous functions on the whole domain. Now, both sets of functions F^{even}, F^{odd}, and F_f^{even}, F_f^{odd}, are solutions of Eqs. (40) with the same smooth data at $t = 0$. It is a standard fact about symmetric hyperbolic systems that corresponding solutions are smooth and that there can only exist one smooth solution for given smooth data. Therefore, it follows $F^{\mathrm{even}} \equiv F_f^{\mathrm{even}}$ and $F^{\mathrm{odd}} \equiv F_f^{\mathrm{odd}}$ everywhere. So, Eq. (41) represents the actual solution of Eq. (39), which takes the form

$$M(t,\theta) = M_* + 2S_{**}(\theta) - 2S_{**}(0) - (S(t,\theta) - 2\ln t) + \int_0^t F^{\mathrm{even}}(\tau,\theta)\,\mathrm{d}\tau. \quad (42)$$

Now let us consider the smoothness condition $\mathrm{e}^{M(t,\theta)} = (R_0^2 \sin^2 t)\mathrm{e}^{-S(t,\theta)}$ for all $t \in (0,\delta]$ and $\theta = 0,\pi$. We compute

$$\partial_t(M - 2\ln R_0 + S - 2\ln\sin t)|_{\theta=0,\pi} = F^{\mathrm{even}}|_{\theta=0,\pi} + \frac{2}{t} - 2\cot t.$$

Evaluating F^{even} from the definition Eq. (38) at $\theta = 0,\pi$ (where in particular $R = 0$), we conclude that

$$\partial_t(M - 2\ln R_0 + S - 2\ln\sin t)|_{\theta=0,\pi} = 0.$$

Therefore, the smoothness condition for M is satisfied for all times if and only if it is satisfied at $t = 0$. From Eq. (42), we see that this is the case provided $M_* = 2\ln R_0$ and $S_{**}(0) = S_{**}(\pi)$. This concludes the proof of the last proposition.

Now we define $C_1 := (C_+ + C_-)/2$ and $C_2 := (C_+ - C_-)/2$ from Eqs. (34), and H is defined in Eqs. (33). The system (31) and (32) implies the *subsidiary system*

$$\partial_t C_1 - \partial_x C_2 = 0, \quad \partial_t C_2 - \partial_x C_1 = H + \cot t\, C_2 + \cot\theta\, C_1.$$

Since M is a solution of Eqs. (37), it follows that the quantities C_1 and C_2 vanish identically. Then H must also be zero. We conclude that our solutions indeed solve the full set of Einstein's field equations.

All the so far constructed functions can now be lifted to smooth functions on $\mathbb{S}^3$ which are invariant along ∂_{ρ_1} and ∂_{ρ_2} at each $t = 0$. All these functions are smooth functions on $[-\tilde{\delta}, \tilde{\delta}] \times \mathbb{S}^3$ with respect to the coordinates $(t', \theta', \rho_1', \rho_2')$ for some small $\tilde{\delta} > 0$. We have thus obtained smooth Gowdy symmetric generalized Taub-NUT solutions in a (possibly small) time interval $0 < t \leq \delta$. The global existence theorem of Chruściel [10] can now be used to extend the spacetimes to the whole time t interval $(0,\pi)$ as smooth globally hyperbolic Gowdy solutions.

4. The linear problem and global-in-time properties

We have seen above that for given smooth asymptotic data at $t = 0$ (e.g. the values of S_{**} and ω_{**}) a smooth Gowdy-symmetric generalized Taub-NUT solution exists in a vicinity of $t = 0$. Moreover, using Chruściel's theorem (theorem 6.3 in [10]), we see that this solution can even be extended smoothly to the whole time interval $(0,\pi)$. However, the surface $t = \pi$ itself is expected to contain either singularities or Cauchy horizons. On the other hand, Chruściel's result also allows the case that the $t = \pi$-surface is regular, but just the coordinates break down there. It is the purpose of the following considerations to find out what happens at $t = \pi$. In particular, we construct explicitly the metric potentials at this boundary (as well as on the axes $\theta = 0, \pi$) in terms of the asymptotic data. For that purpose, we apply the so-called *soliton methods*, which were used in [19] for the investigation of $\mathbb{S}^2 \times \mathbb{S}^1$ Gowdy spacetimes[10], to the present $\mathbb{S}^3$-symmetric case.

In all of what follows we make the same hypotheses as in Theorem 3.1. These assumptions are consistent with those listed in Section 3.2, and hence we consider "smooth Gowdy symmetric generalized Taub-NUT solutions".

4.1. Einstein's field equations and the Ernst formulation

The first important step for the following considerations is the introduction of the complex Ernst formulation of the Einstein equations which will be described in this subsection.

Again we start from the metric

$$g = \mathrm{e}^M(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + R_0 \sin t \sin\theta \left[\mathrm{e}^L(\mathrm{d}\rho_1 + Q\mathrm{d}\rho_2)^2 + \mathrm{e}^{-L}\mathrm{d}\rho_2^2\right] \tag{43}$$

in the Killing basis $\{\partial_{\rho_1}, \partial_{\rho_2}\}$. Here, we express L in terms of a metric potential u via

$$\mathrm{e}^L = \frac{\sin t}{\sin\theta}\,\mathrm{e}^u. \tag{44}$$

In this way, we arrive at

$$g = \mathrm{e}^M(-\mathrm{d}t^2 + \mathrm{d}\theta^2) + R_0\left[\sin^2\!t\,\mathrm{e}^u(\mathrm{d}\rho_1 + Q\mathrm{d}\rho_2)^2 + \sin^2\!\theta\,\mathrm{e}^{-u}\mathrm{d}\rho_2^2\right]. \tag{45}$$

Note that u is related to the quantity S (defined in Sec. 3.3.2) via

$$u(t,\theta) = S(t,\theta) - \ln(R_0) - 2\ln\sin t,$$

i.e. the singularity of S at $t = 0$ (S behaves as $2\ln t$ for $t \to 0$, see Prop. 3.2) is removed by subtracting the term $2\ln\sin t$.

Now we reformulate the Einstein equations as equations for u, Q and M. We obtain two second-order equations for the metric potentials u and Q,

$$-\partial_t^2 u - \cot t\,\partial_t u + \partial_\theta^2 u + \cot\theta\,\partial_\theta u + \mathrm{e}^{2u}\frac{\sin^2\!t}{\sin^2\!\theta}\left[(\partial_t Q)^2 - (\partial_\theta Q)^2\right] + 2 = 0, \tag{46}$$

$$-\partial_t^2 Q - 3\cot t\,\partial_t Q + \partial_\theta^2 Q - \cot\theta\,\partial_\theta Q - 2[(\partial_t u)(\partial_t Q) - (\partial_\theta u)(\partial_\theta Q)] = 0 \tag{47}$$

and two first-order equations for M,

$$\begin{aligned}(\cos^2\!t - \cos^2\!\theta)\partial_t M = {}& \frac{1}{2}\mathrm{e}^{2u}\frac{\sin^3\!t}{\sin\theta}\Big[\cos t\sin\theta[(\partial_t Q)^2 + (\partial_\theta Q)^2] - 2\sin t\cos\theta(\partial_t Q)(\partial_\theta Q)\Big]\\ &+ \frac{1}{2}\sin t\sin\theta\Big[\cos t\sin\theta[(\partial_t u)^2 + (\partial_\theta u)^2] - 2\sin t\cos\theta(\partial_t u)(\partial_\theta u)\Big]\\ &- (2\cos^2\!t\,\cos^2\!\theta - \cos^2\!t - \cos^2\!\theta)\,\partial_t u\\ &- 2\sin t\cos t\sin\theta\cos\theta(\partial_\theta u + \tan\theta),\end{aligned} \tag{48}$$

[10] The methods described [19] have also been applied for studying the interior region of axisymmetric and stationary black holes with surrounding matter, see [2, 3, 18].

$$
\begin{aligned}
(\cos^2 t - \cos^2\theta)\partial_\theta M = &-\frac{1}{2}\mathrm{e}^{2u}\frac{\sin^3 t}{\sin\theta}\Big[\sin t\cos\theta[(\partial_t Q)^2 + (\partial_\theta Q)^2] - 2\cos t\sin\theta(\partial_t Q)(\partial_\theta Q)\Big]\\
&-\frac{1}{2}\sin t\sin\theta\Big[\sin t\cos\theta[(\partial_t u)^2 + (\partial_\theta u)^2] - 2\cos t\sin\theta(\partial_t u)(\partial_\theta u)\Big]\\
&- 2\sin t\cos t\sin\theta\cos\theta(\partial_t u - \tan t)\\
&- (2\cos^2 t\,\cos^2\theta\, - \cos^2 t - \cos^2\theta)\,\partial_\theta u. \qquad (49)
\end{aligned}
$$

Since M does not appear in (46) and (47) and since we assume the genericity condition of Chruściel, these equations may be solved in a first step. Afterwards, (48) and (49) can be used to calculate M via a line integral. Note that the integrability condition $\partial_t\partial_\theta M = \partial_\theta\partial_t M$ of the system (48), (49) is satisfied as a consequence of (46), (47). Hence, M does not depend on the path of integration.

It turns out that the two Einstein equations (46), (47) are equivalent to a single complex equation, namely to the Ernst equation

$$f\left(-\partial_t^2\mathcal{E} - \cot t\,\partial_t\mathcal{E} + \partial_\theta^2\mathcal{E} + \cot\theta\,\partial_\theta\mathcal{E}\right) = -(\partial_t\mathcal{E})^2 + (\partial_\theta\mathcal{E})^2 \qquad (50)$$

for the complex Ernst potential $\mathcal{E} = f + \mathrm{i}b$. Here, the real part f of $\mathcal{E}$ is defined in terms of the Killing vector ∂_{ρ_2} by

$$f := \frac{1}{R_0}g(\partial_{\rho_2}, \partial_{\rho_2}) = Q^2\mathrm{e}^u\sin^2 t + \mathrm{e}^{-u}\sin^2\theta \qquad (51)$$

and the imaginary part b is given by

$$\partial_t a = \frac{1}{f^2}\sin t\sin\theta\,\partial_\theta b, \quad \partial_\theta a = \frac{1}{f^2}\sin t\sin\theta\,\partial_t b \qquad (52)$$

with

$$a := \frac{g(\partial_{\rho_1}, \partial_{\rho_2})}{g(\partial_{\rho_2}, \partial_{\rho_2})} = \frac{Q}{f}\mathrm{e}^u\sin^2 t. \qquad (53)$$

Note that for smooth functions u and Q the Ernst potential $\mathcal{E}$ is also smooth: For the real part f, smoothness is clear from definition (51). In the case of the imaginary part b, it can be shown by solving the two equations in (52) for $\partial_t b$ and $\partial_\theta b$ and replacing a and f via (53) and (51). The resulting expressions for $\partial_t b$ and $\partial_\theta b$ in terms of Q and u (and their first order derivatives) turn out to be smooth functions, if we use the fact that Q behaves as given in (17). Hence, integration will lead to a smooth function b. Therefore, we can conclude from the previous local existence results and Chruściel's global existence theorem that for any given set of asymptotic data (as described in Theorem 3.1) the corresponding Ernst potential $\mathcal{E}$ is a smooth complex function on $(-T, \pi)\times\mathbb{S}^2$ for some $T > 0$. In the following we investigate under which conditions $\mathcal{E}$ can be extended smoothly to the boundary $t = \pi$ and beyond. Note that our assumptions imply that $f > 0$ holds in the entire Gowdy square with the exception of the points A and B and with the possible exception of the future boundary $\mathcal{H}_\mathrm{f}$ (see Fig. 1) which is important since we will divide by f in some of the following formulae.

Once we have obtained an Ernst potential $\mathcal{E}$ as a solution to the Ernst equation (50), we can calculate the corresponding metric potentials from it. It turns out that the integrability condition $\partial_t\partial_\theta a = \partial_\theta\partial_t a$ of (52) is satisfied as a consequence of the

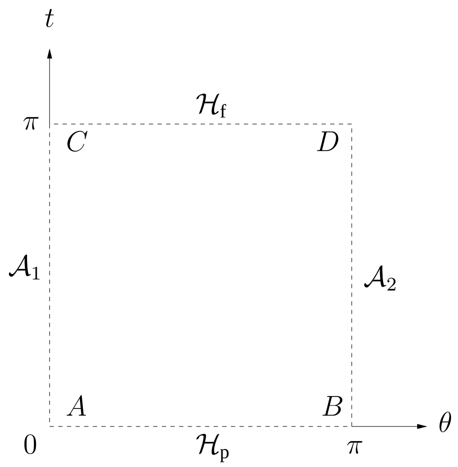

Figure 1. We integrate the LP along the boundaries of the Gowdy square (dashed path) in order to investigate for which asymptotic data the solution can be regularly extended up to the future boundary $\mathcal{H}_\mathrm{f}$ $(t = \pi)$.

Ernst equation. Therefore, a may be calculated via line integration from $\mathcal{E}$. The metric potentials u and Q can then be obtained from a and f. With (51) and (53) we find

$$\mathrm{e}^u = \frac{fa^2}{\sin^2 t} + \frac{\sin^2\theta}{f}, \quad Q = \frac{f^2 a}{f^2a^2 + \sin^2 t \sin^2\theta}. \tag{54}$$

Finally, M may be calculated using (48) and (49), as mentioned earlier.

As an example, we give the Ernst potential for the Taub solution:

$$f = \frac{2l}{X}\sin^2 t\,\cos^2\theta + \frac{X}{2l}\sin^2\theta, \tag{55}$$

$$b = \frac{1}{X}\left[\cos t(\cos^2 t - 3)\sqrt{m^2+l^2} - 2m\right]\cos^2\theta + \cos t \tag{56}$$

with $X := (1+\cos^2 t)\sqrt{m^2+l^2} + 2m\cos t$. (Here we have set an arbitrary additive integration constant in b to zero.)

4.2. The linear problem

Interestingly, the Ernst equation (50) belongs to a remarkable class of nonlinear partial differential equations for which an associated linear problem (LP) exists which is equivalent to the nonlinear equation via its integrability condition. For applications of this LP in the context of *axisymmetric and stationary* spacetimes we refer the reader to, e.g., [28, 30]. In the Gowdy setting, we use the LP in the form [25, 26], which reads in our coordinates[11] as

$$
\begin{aligned}
\partial_x \mathbf{\Phi} &= \left[\begin{pmatrix} B_x & 0 \\ 0 & A_x \end{pmatrix} + \lambda \begin{pmatrix} 0 & B_x \\ A_x & 0 \end{pmatrix} \right] \mathbf{\Phi}, \\
\partial_y \mathbf{\Phi} &= \left[\begin{pmatrix} B_y & 0 \\ 0 & A_y \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & B_y \\ A_y & 0 \end{pmatrix} \right] \mathbf{\Phi},
\end{aligned}
\tag{57}
$$

where the *pseudopotential* $\mathbf{\Phi} = \mathbf{\Phi}(x, y, K)$ is a 2×2 matrix depending on the coordinates

$$x = \cos(t - \theta), \qquad y = \cos(t + \theta) \tag{58}$$

as well as on the *spectral parameter* $K \in \mathbb{C}$. The remaining ingredients of the LP are the function λ,

$$\lambda(x, y, K) := \sqrt{\frac{K - y}{K - x}}, \tag{59}$$

and the matrix elements A_x, A_y, B_x and B_y, defined in terms of the Ernst potential as

$$A_i = \frac{\partial_i \mathcal{E}}{2f}, \qquad B_i = \frac{\partial_i \bar{\mathcal{E}}}{2f}, \qquad i = x, y. \tag{60}$$

Due to the two possible signs of the square root in (59), $\lambda : \mathbb{C} \to \mathbb{C}$, $K \mapsto \lambda$ describes, for fixed values x, y, a mapping from a two-sheeted Riemann surface (K-plane) onto the complex λ-plane. The two K-sheets are connected at the branch points

$$K_1 = x \quad (\lambda = \infty), \qquad K_2 = y \quad (\lambda = 0). \tag{61}$$

In general, the pseudopotential $\mathbf{\Phi}$ will take on different values on the two K-sheets. Only at the branch points it has to be unique, since both Riemannian sheets coincide there. We will see below that this observation plays an important role for the calculation of the Ernst potential from the solution of the LP.

As already mentioned, the integrability condition $\partial_x \partial_y \mathbf{\Phi} = \partial_y \partial_x \mathbf{\Phi}$ of (57) is equivalent to the Ernst equation (50). Hence, the Ernst equation is a consequence of the LP and, on the other hand, for a given potential $\mathcal{E}$ as a solution to the Ernst equation, the matrix $\mathbf{\Phi}$ does not depend on the path of integration.

Finally, we note that for any solution $\mathbf{\Phi}$ to the LP (57), the product $\mathbf{\Phi}\mathbf{C}(K)$, where $\mathbf{C}(K)$ is an arbitrary 2×2 matrix, is also a solution (corresponding to the same Ernst potential). As shown by Neugebauer [28], it is always possible to choose $\mathbf{C}(K)$ in such

[11] The formal relation between our coordinates (describing Gowdy spacetimes with two spacelike Killing vectors) and the Weyl-Lewis-Papapetrou coordinates $(\rho, \zeta, \varphi, \tilde{t}\,)$ as used by Neugebauer (describing axisymmetric and stationary spacetimes with one spacelike and one timelike Killing field) is given by $\rho = \mathrm{i}R_0 \sin t \sin\theta$, $\zeta = R_0 \cos t \cos\theta$, $\varphi = \rho_1$, $\tilde{t} = \rho_2$.

a way that the transformed pseudopotential takes the form

$$\begin{aligned}
\mathbf{\Phi}^{>}(x,y,K) &= \begin{pmatrix} \psi_1^{>}(x,y,K) & \psi_1^{<}(x,y,K) \\ \psi_2^{>}(x,y,K) & -\psi_2^{<}(x,y,K) \end{pmatrix}, \\
\mathbf{\Phi}^{<}(x,y,K) &= \begin{pmatrix} \psi_1^{<}(x,y,K) & \psi_1^{>}(x,y,K) \\ \psi_2^{<}(x,y,K) & -\psi_2^{>}(x,y,K) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{\Phi}^{>}(x,y,K) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned} \tag{62}$$

where the superscripts "$>$" or "$<$" indicate whether the functions are evaluated on the "upper" ($\lambda = 1$ for $K = \infty$) or "lower" ($\lambda = -1$ for $K = \infty$) K-sheet. Hence, $\mathbf{\Phi}$ is completely determined by the values of two functions ψ_1 and ψ_2 on both K-sheets. In all of what follows we assume that we have already achieved this form of $\mathbf{\Phi}$.

4.3. Solution of the linear problem

4.3.1. Coordinate transformation. In the following we intend to integrate the LP along the boundaries of the Gowdy square. For that purpose, it turns out to be useful to study the situation not only in the coordinate system Σ, corresponding to the Killing basis $\{\partial_{\rho_1}, \partial_{\rho_2}\}$, but also in a coordinate frame $\tilde{\Sigma}$,

$$\tilde{\Sigma}: \quad \tilde{t} = t, \quad \tilde{\theta} = \theta, \quad \tilde{\rho}_1 = \rho_1 + q\rho_2, \quad \tilde{\rho}_2 = \rho_2 \tag{63}$$

with $q =$ constant. According to (18)-(20) and (44), the transformed metric potentials are

$$\tilde{R}_0 = R_0, \quad \tilde{u} = u, \quad \tilde{Q} = Q - q, \tag{64}$$

i.e. only Q is changed by subtracting a constant. In particular, we will choose the two systems $\tilde{\Sigma}$ with $q = 1$ or $q = -1$, in which $\tilde{Q}|_{\mathcal{A}_1} = 0$ or $\tilde{Q}|_{\mathcal{A}_2} = 0$ holds, respectively.

Since the coordinate transformation (63) is merely a change of the Killing basis, the Ernst equation (50) retains its form in $\tilde{\Sigma}$. This implies the existence of a LP (57) for a pseudopotential $\tilde{\mathbf{\Phi}}$ in this frame. As shown by Neugebauer [27, 28], the matrices $\tilde{\mathbf{\Phi}}$ and $\mathbf{\Phi}$ are connected by the transformation

$$\tilde{\mathbf{\Phi}} = \left[\begin{pmatrix} c_- & 0 \\ 0 & c_+ \end{pmatrix} + \mathrm{i}\frac{q}{f}(K - x) \begin{pmatrix} 1 & \lambda \\ -\lambda & -1 \end{pmatrix} \right] \mathbf{\Phi} \tag{65}$$

with

$$c_\pm := 1 - q\left(a \pm \frac{\mathrm{i}}{f}\sin t \sin\theta\right), \tag{66}$$

where all quantities on the right hand side of Eq. (65) belong to the original frame Σ. As we will see, this transformation becomes particularly simple at the boundaries of the Gowdy square for our choices $q = \pm 1$.

4.3.2. The LP on $\mathcal{H}_{\rm p}$, $\mathcal{A}_1$ and $\mathcal{A}_2$. From our previous discussions, namely from the local investigation of the singular initial value problem for the Einstein equations with Fuchsian methods (see Sec. 3.3) and from Chruściel's global existence theorem, we know that for any smooth set of asymptotic data on $\mathcal{H}_{\rm p}$ a corresponding smooth Gowdy symmetric generalized Taub-NUT solutions exists. Moreover, this solution is smooth both on the axes of symmetry $\mathcal{A}_1$, $\mathcal{A}_2$ and in the interior of the Gowdy square. The goal of this subsection is to find explicit expressions for the values of the solution on $\mathcal{A}_1$ and $\mathcal{A}_2$, which are determined by the data on $\mathcal{H}_{\rm p}$. Afterwards we may study the behavior as $t \to \pi$ and investigate whether a continuation of the solution to $\mathcal{H}_{\rm f}$ is possible.

Along the entire integration path, we have $x = y$ and therefore $\lambda = \pm 1$, cf. (59). However, it suffices to study the case $\lambda = 1$ alone, since the solution on the Riemannian sheet with $\lambda = -1$ can easily be obtained from the solution with $\lambda = 1$ using (62).

For $x = y$ and $\lambda = 1$, the LP (57) reduces to the ODE

$$\partial_x \mathbf{\Phi} = \frac{1}{2f} \begin{pmatrix} \partial_x \bar{\mathcal{E}} & \partial_x \bar{\mathcal{E}} \\ \partial_x \mathcal{E} & \partial_x \mathcal{E} \end{pmatrix} \mathbf{\Phi} \tag{67}$$

with the general solution[12]

$$\mathbf{\Phi} = \mathbf{E}\mathbf{C}(K), \qquad \mathbf{E} := \begin{pmatrix} \bar{\mathcal{E}} & 1 \\ \mathcal{E} & -1 \end{pmatrix} \tag{68}$$

in terms of the Ernst potential on the boundary, where the 2×2 matrix $\mathbf{C}$ is a K-dependent "integration constant". The solutions on all parts of the integration path have the form (68), but with different integration constants:

$$t = 0: \quad \mathbf{\Phi} = \mathbf{E}\mathbf{C}, \quad \mathbf{C} = \begin{pmatrix} C_1 & C_3 \\ C_2 & C_4 \end{pmatrix}, \tag{69}$$

$$\theta = 0: \quad \mathbf{\Phi} = \mathbf{E}\mathbf{D}, \quad \mathbf{D} = \begin{pmatrix} D_1 & D_3 \\ D_2 & D_4 \end{pmatrix}, \tag{70}$$

$$\theta = \pi: \quad \mathbf{\Phi} = \mathbf{E}\tilde{\mathbf{D}}, \quad \tilde{\mathbf{D}} = \begin{pmatrix} \tilde{D}_1 & \tilde{D}_3 \\ \tilde{D}_2 & \tilde{D}_4 \end{pmatrix}. \tag{71}$$

A further simplification can be achieved by normalizing $\mathbf{\Phi}$ at $t = 0$ via

$$t = 0: \quad \psi_1^< = \psi_2^< = \psi(K), \tag{72}$$

where $\psi(K)$ is an arbitrary gauge function (which will later be specified in such a way that the LP has a regular solution, see Eq. (83) below). This is possible since the form (62) of $\mathbf{\Phi}$ is invariant under the transformation [30]

$$\mathbf{\Phi} \to \mathbf{\Phi} \cdot \begin{pmatrix} \alpha(K) & \beta(K) \\ \beta(K) & \alpha(K) \end{pmatrix}. \tag{73}$$

[12] By plugging the solution (68) into (67), we see that the matrix on the right hand side of (67) is proportional to f, i.e. the factor $1/f$ is cancelled and the solution extends smoothly over points with $f = 0$. However, it follows from (51) that f does not vanish on $\mathcal{H}_{\rm p}$, $\mathcal{A}_1$, $\mathcal{A}_2$ with the exception of the corners A, B of the Gowdy square (as well as C and D, provided $Q^2 e^u$ is bounded for $t \to \pi$).

The two degrees of freedom α, β can be used to achieve the two conditions in (72). As a consequence, we obtain

$$C_3 = 0, \quad C_4 = \psi \tag{74}$$

in this gauge.

From (69)-(71) we can now calculate the solution of the LP in the frame $\tilde{\Sigma}$ (cf. (63)) using the transformation formula (65). It follows from (53) that a takes on the boundary values[13]

$$\mathcal{H}_\mathrm{p}: \quad a = 0, \quad \mathcal{A}_1: \quad a = \frac{1}{Q} = 1, \quad \mathcal{A}_2: \quad a = \frac{1}{Q} = -1. \tag{75}$$

Plugging this into (65), we obtain for $\lambda = 1$

$$t = 0: \ \tilde{\mathbf{\Phi}} = \begin{pmatrix} \bar{\mathcal{E}} \pm 2\mathrm{i}(K-x) & 1 \\ \mathcal{E} \mp 2\mathrm{i}(K-x) & -1 \end{pmatrix} \mathbf{C} \quad \text{in } \tilde{\Sigma} \text{ with } q = \pm 1, \tag{76}$$

$$\theta = 0: \ \tilde{\mathbf{\Phi}} = +2\mathrm{i}(K-x) \begin{pmatrix} D_1 & D_3 \\ -D_1 & -D_3 \end{pmatrix} \quad \text{in } \tilde{\Sigma} \text{ with } q = 1, \tag{77}$$

$$\theta = \pi: \ \tilde{\mathbf{\Phi}} = -2\mathrm{i}(K-x) \begin{pmatrix} \tilde{D}_1 & \tilde{D}_3 \\ -\tilde{D}_1 & -\tilde{D}_3 \end{pmatrix} \quad \text{in } \tilde{\Sigma} \text{ with } q = -1. \tag{78}$$

As we will see below, $C_1(K)$ and $C_2(K)$ are determined completely by the data at $t = 0$. Now we intend to express the components of the matrices $\mathbf{D}$ and $\tilde{\mathbf{D}}$ in terms of C_1, C_2. For that purpose, we use that $\mathbf{\Phi}$ has to be continuous at the corners A and B of the Gowdy square (see Fig. 1). This condition leads to an algebraic system of 4 equations which, however, is not sufficient to calculate the 8 unknowns $D_1, \dots, D_4$, $\tilde{D}_1, \dots, \tilde{D}_4$. This is the reason for introducing the coordinate frame $\tilde{\Sigma}$. From the requirement that also $\tilde{\mathbf{\Phi}}$ (in $\tilde{\Sigma}$ with $q = 1$) is continuous at A and $\tilde{\mathbf{\Phi}}$ (in $\tilde{\Sigma}$ with $q = -1$) is continuous at B we find another 4 algebraic equations[14]. In this way we obtain an algebraic system of 8 equations for the 8 unknowns with the following solution for the matrices $\mathbf{D}(K)$ and $\tilde{\mathbf{D}}(K)$ in terms of $\mathbf{C}(K)$:

$$D_1 = C_1 - \frac{b_A C_1 + \mathrm{i}C_2}{2(K-1)}, \quad D_2 = C_2 - \frac{\mathrm{i}b_A(b_A C_1 + \mathrm{i}C_2)}{2(K-1)},$$

$$D_3 = -\frac{\mathrm{i}\psi}{2(K-1)}, \qquad D_4 = \psi\left(1 + \frac{b_A}{2(K-1)}\right),$$

$$\tilde{D}_1 = C_1 + \frac{b_B C_1 + \mathrm{i}C_2}{2(K+1)}, \quad \tilde{D}_2 = C_2 + \frac{\mathrm{i}b_B(b_B C_1 + \mathrm{i}C_2)}{2(K+1)},$$

[13] Note that a is automatically discontinuous at the points A, B, C, D as a consequence of the definition (53). In contrast, a is a smooth function in the remaining part of the Gowdy square.

[14] The reason why just the usage of a different coordinate system can lead to independent algebraic equations is the following. The boundary values of the quantity a enter the transformation law (65). Therefore, the additional algebraic equations found in $\tilde{\Sigma}$ ensure that a indeed takes on the boundary values (75) (and, as a consequence, also Q takes on the correct boundary values $Q = 1$ on $\mathcal{A}_1$ and $Q = -1$ on $\mathcal{A}_2$). From (52) alone it would only follow that a is constant on the boundaries without specification of these constants.

$$\tilde{D}_3 = \frac{\mathrm{i}\psi}{2(K+1)}, \qquad \tilde{D}_4 = \psi\left(1 - \frac{b_B}{2(K+1)}\right),$$

In the following subsection we utilize these results in order to determine the Ernst potential and the metric potentials on $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of the data on $\mathcal{H}_\mathrm{p}$.

4.3.3. Ernst potential on $\mathcal{A}_1$ and $\mathcal{A}_2$. From the solution of the LP obtained in the previous subsection we may, as a first step, calculate the Ernst potential on $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of the initial potential on $\mathcal{H}_\mathrm{p}$. To this end, we start by expressing C_1 and C_2 in terms of $\mathcal{E}$ on $\mathcal{H}_\mathrm{p}$.

As mentioned in Sec. 4.2, the mapping $K \mapsto \lambda$ in (59) defines a two-sheeted Riemannian K-surface. At the branch points K_1 and K_2, where both sheets are connected, any function of K has to be unique, i.e. the values on the upper and lower sheet have to be the same. For t-θ-values on the boundaries of the Gowdy square, we have confluent branch points, i.e. $K_1 = K_2 = x$. The uniqueness of $\mathbf{\Phi}$ at $K = K_1 = K_2$ leads to the conditions (see (62))

$$\mathcal{H}_\mathrm{p}, \mathcal{A}_1, \mathcal{A}_2: \quad \psi_1^> = \psi_1^< \quad \text{and} \quad \psi_2^> = \psi_2^< \quad \text{for} \quad K = x. \tag{79}$$

In particular, on $\mathcal{H}_\mathrm{p}$ we obtain the two equations

$$\bar{\mathcal{E}}_\mathrm{p} C_1 + C_2 = \psi, \quad \mathcal{E}_\mathrm{p} C_1 - C_2 = \psi \tag{80}$$

with the solution

$$C_1(x) = \frac{2\psi(x)}{\mathcal{E}_\mathrm{p}(x) + \bar{\mathcal{E}}_\mathrm{p}(x)} \equiv \frac{\psi(x)}{f_\mathrm{p}(x)}, \quad C_2(x) = \frac{\mathcal{E}_\mathrm{p}(x) - \bar{\mathcal{E}}_\mathrm{p}(x)}{\mathcal{E}_\mathrm{p}(x) + \bar{\mathcal{E}}_\mathrm{p}(x)}\psi \equiv \frac{\mathrm{i}b_\mathrm{p}(x)}{f_\mathrm{p}(x)}\psi(x), \tag{81}$$

where

$$\mathcal{E}_\mathrm{p}(x) = f_\mathrm{p}(x) + \mathrm{i}b_\mathrm{p}(x) = \mathcal{E}(t = 0, \theta = \arccos x). \tag{82}$$

Now we can suggest a possible choice for the gauge function ψ, which was introduced in (72). If we set

$$\psi(K) = (K^2 - 1)^2, \tag{83}$$

then the solution $\mathbf{\Phi}$ (as well as $\tilde{\mathbf{\Phi}}$) is regular, because ψ compensates for the poles that the matrices $\mathbf{D}$ and $\tilde{\mathbf{D}}$ would otherwise have at $K = \pm 1$. Note that C_1 and C_2 are also regular because $f_\mathrm{p}(x) = \mathrm{e}^{-u}(1 - x^2)$, cf. (51). But of course, as we will see below, the Ernst potential on $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{H}_\mathrm{f}$ is independent of this gauge choice.

With these expressions for C_1 and C_2 together with the solution of the LP from the previous section, we can also evaluate the condition (79) on $\mathcal{A}_1$ and $\mathcal{A}_2$ to obtain explicit formulae for the Ernst potential there. The result that we find independently of the particular choice for the gauge function ψ is

$$\mathcal{A}_1: \quad \mathcal{E}_1(x) := \mathcal{E}(t = \arccos x, \theta = 0) = \frac{\mathrm{i}[b_A - 2(1-x)]\mathcal{E}_\mathrm{p}(x) + b_A^2}{\mathcal{E}_\mathrm{p}(x) - \mathrm{i}[b_A + 2(1-x)]}, \tag{84}$$

$$\mathcal{A}_2: \quad \mathcal{E}_2(x) := \mathcal{E}(t = \arccos(-x), \theta = \pi) = \frac{\mathrm{i}[b_B - 2(1+x)]\mathcal{E}_\mathrm{p}(x) + b_B^2}{\mathcal{E}_\mathrm{p}(x) - \mathrm{i}[b_B + 2(1+x)]}. \tag{85}$$

From the latter equations we can conclude that $\mathcal{E}_1$ and $\mathcal{E}_2$ are smooth functions of x. To see this, recall that we assume smooth data at $t=0$. For smooth initial functions $u(0,\theta)$ and $Q(0,\theta)$, the initial Ernst potential $\mathcal{E}_\mathrm{p}$ will also be smooth, cf. Eqs. (86), (87) below. As a consequence, the numerators and denominators of the fractions in (84), (85) are also smooth and an irregularity in the Ernst potentials could only occur if the denominators became zero for some $x\in[-1,1]$. However, it follows from (86), (87) below together with Eq. (17) that the only zeros are at $x=1$ or at $x=-1$. Moreover, these equations show that the numerators have zeros of at least the same multiplicity at these x-values. Hence, the zeros in the numerators and denominators cancel each other out and the fractions are smooth functions of x for all $x\in[-1,1]$. The only exceptional cases occur for asymptotic data with $b_B=b_A+4$ or $b_B=b_A-4$. In the first case, $\mathcal{E}_1$ diverges at point C and in the second case $\mathcal{E}_2$ diverges at D, cf. Fig. 1.

4.3.4. Metric potentials on $\mathcal{A}_1$ and $\mathcal{A}_2$. In the previous subsection we have provided explicit formulae for the Ernst potential on the axes $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of the initial potential on $\mathcal{H}_\mathrm{p}$. Now we will see how the metric potentials u, Q and M can be obtained from the Ernst potential on these boundaries.

We assume that asymptotic data $u(0,\theta)$ and $Q(0,\theta)$ (or, equivalently, $S_{**}(\theta)$ and $\omega_{**}(\theta)$) and a constant $R_0>0$ are given. From these data, we may calculate the initial Ernst potential $\mathcal{E}_\mathrm{p}=f_\mathrm{p}+\mathrm{i}b_\mathrm{p}$. The real part can be obtained from (51),

$$f_\mathrm{p}(\theta)=\mathrm{e}^{-u(0,\theta)}\sin^2\theta, \tag{86}$$

and the imaginary part can be calculated by integrating the first equation in (52) with respect to θ, using (53). We obtain

$$b_\mathrm{p}(\theta)=b_A+2\int_0^\theta Q(0,\theta')\sin\theta'\,\mathrm{d}\theta', \tag{87}$$

where $b_A=b(0,0)$ is an arbitrary integration constant.

From $\mathcal{E}_\mathrm{p}$ we may calculate $\mathcal{E}_1$ and $\mathcal{E}_2$ via (84), (85). Afterwards, we can use these results to determine the potentials u, Q and M on $\mathcal{A}_1$ and $\mathcal{A}_2$. Using again (51)-(53) together with (13), (17) and (44), we find

$$\mathcal{A}_1:\quad \mathrm{e}^{u(t,0)}=\frac{f_1(t)}{\sin^2 t},\quad \mathrm{e}^{M(t,0)}=\frac{R_0\sin^2 t}{f_1(t)},\quad Q(t,0)=1, \tag{88}$$

$$\mathcal{A}_2:\quad \mathrm{e}^{u(t,\pi)}=\frac{f_2(t)}{\sin^2 t},\quad \mathrm{e}^{M(t,0)}=\frac{R_0\sin^2 t}{f_2(t)},\quad Q(t,\pi)=-1. \tag{89}$$

4.4. Situation on $\mathcal{H}_\mathrm{f}$

So far we have seen that we can prescribe arbitrary smooth data at $t=0$ and will always find smooth potentials for $t<\pi$ as solution to the field equations. In particular, we have derived explicit formulae for the Ernst potential and the metric potentials on the axes $\mathcal{A}_1$ and $\mathcal{A}_2$. It remains to study under which conditions the solution can even be extended smoothly to the future boundary $\mathcal{H}_\mathrm{f}$. In order to answer this question, we tentatively solve the LP on $\mathcal{H}_\mathrm{f}$ and investigate whether this solution can be attached continuously to the solutions on $\mathcal{A}_1$ and $\mathcal{A}_2$.

The LP on $\mathcal{H}_\mathrm{f}$ reduces to the same ODE as on the other boundaries of the Gowdy square, namely to Eq. (67). We write the solution in Σ as

$$t=\pi: \quad \mathbf{\Phi}=\mathbf{E}\tilde{\mathbf{C}}, \quad \tilde{\mathbf{C}}=\begin{pmatrix} \tilde{C}_1 & \tilde{C}_3 \\ \tilde{C}_2 & \tilde{C}_4 \end{pmatrix}. \tag{90}$$

In order to obtain the solution in the coordinate frame $\tilde{\Sigma}$ too, we need to calculate the quantity a on $\mathcal{H}_\mathrm{f}$ so we can apply the transformation formula (65). It follows from (53) that, if the metric potentials u and Q remain bounded for $t\to\pi$, $a=0$ then holds on $\mathcal{H}_\mathrm{f}$ (provided f does not vanish on $\mathcal{H}_\mathrm{f}$ with exception of the boundary points C, D). However, it is not yet clear how u and Q behave for $t\to\pi$. Therefore, so far we can only say that a is constant on $\mathcal{H}_\mathrm{f}$, cf. (52),

$$t=\pi: \quad a=a_0=\text{constant}. \tag{91}$$

Using (65), we find therefore in $\tilde{\Sigma}$

$$t=\pi: \quad \tilde{\mathbf{\Phi}}=\begin{pmatrix} (1\mp a_0)\bar{\mathcal{E}}\pm 2\mathrm{i}(K-x) & 1\mp a_0 \\ (1\mp a_0)\mathcal{E}\mp 2\mathrm{i}(K-x) & -(1\mp a_0) \end{pmatrix}\tilde{\mathbf{C}} \quad \text{in } \tilde{\Sigma} \text{ with } q=\pm 1, \tag{92}$$

Now we can investigate whether $\mathbf{\Phi}$ in (90) and $\tilde{\mathbf{\Phi}}$ in (92) can be attached continuously to the corresponding solutions on $\mathcal{A}_1$ and $\mathcal{A}_2$. This question is equivalent to the solvability of an algebraic system of 8 equations. It turns out that this system can be solved if and only if the initial parameters b_A and b_B satisfy

$$b_B\neq b_A+4, \quad \text{and} \quad b_B\neq b_A-4. \tag{93}$$

(It was already discussed at the end of Sec. 4.3.3 that the Ernst potential diverges at C or D if one of these conditions is violated.) The algebraic equations then fix the values of b_C, b_D and a_0 in terms of the initial quantities b_A and b_B,

$$b_C=\frac{4b_B+b_A(b_A-b_B)}{b_A-b_B+4} \tag{94}$$

$$b_D=\frac{-4b_A+b_B(b_A-b_B)}{b_A-b_B-4} \tag{95}$$

$$a_0=\frac{8(b_B-b_A)}{16+(b_B-b_A)^2}. \tag{96}$$

With these results we can calculate the Ernst potential on $\mathcal{H}_\mathrm{f}$. With the same considerations as in Sec. 4.3.3 we obtain

$$\mathcal{E}_\mathrm{f}:=\mathcal{E}(t=\pi,\theta=\arccos(-x))=\frac{a_1(x)\mathcal{E}_\mathrm{p}(x)+a_2(x)}{b_1(x)\mathcal{E}_\mathrm{p}(x)+b_2(x)}, \tag{97}$$

where

$$\begin{aligned} a_1 = &-\mathrm{i}\left[\left((b_A-b_B)^2+16\right)x^2-2(b_A-b_B)(b_A+b_B-4)x\right. \\ &\qquad \left.+\,(b_A-b_B)^2+8(b_A+b_B-2)\right] \end{aligned} \tag{98}$$

$$a_2=4(b_A-b_B)(b_Ab_B-2b_A-2b_B)x-8(b_A^2+b_B^2) \tag{99}$$

$$\begin{aligned} b_1 &= 4\left[(b_A-b_B)x-4\right] && (100)\\ b_2 &= -\mathrm{i}\left[\left((b_A-b_B)^2+16\right)x^2+2(b_A-b_B)(b_A+b_B-4)x\right.\\ &\qquad\left.+\,(b_A-b_B)^2-8(b_A+b_B+2)\right]. && (101)\end{aligned}$$

Similarly to the discussion in Sec. 4.3.3 we find that $\mathcal{E}_\mathrm{f}$ is a smooth function on the entire boundary $\mathcal{H}_\mathrm{f}$ (with our assumption (93)).

As already mentioned, the auxiliary quantity a would satisfy the boundary condition $a=0$ on $\mathcal{H}_\mathrm{f}$ if the metric potentials u and Q were bounded for $t\to\pi$. From (96) we can read off that this is only the case if the initial parameters b_A and b_B satisfy the condition

$$b_A=b_B, \tag{102}$$

which can also be expressed in terms of the metric potential Q,

$$\int_0^\pi Q(0,\theta)\sin\theta\,\mathrm{d}\theta=0, \tag{103}$$

cf. (87). In the following subsections, we study separately the cases $b_A=b_B$ and $b_A\neq b_B$.

4.4.1. Initial data with $b_A=b_B$. Such data lead to a solution of the field equations with $a=0$ on $\mathcal{H}_\mathrm{f}$. As a consequence, the metric potentials u and Q are regular at $\mathcal{H}_\mathrm{f}$.

The formula (97) for $\mathcal{E}_\mathrm{f}$ simplifies in this case to

$$\mathcal{H}_\mathrm{f}:\quad \mathcal{E}_\mathrm{f}(x)=\frac{\mathrm{i}(b_A-1+x^2)\mathcal{E}_\mathrm{p}(x)+b_A^2}{\mathcal{E}_\mathrm{p}(x)-\mathrm{i}(b_A+1-x^2)} \tag{104}$$

and in terms of this Ernst potential, the metric potentials are given by

$$\mathcal{H}_\mathrm{f}:\quad \mathrm{e}^{u(\pi,\theta)}=\frac{\sin^2\theta}{f_\mathrm{f}(\theta)},\quad \mathrm{e}^{M(\pi,\theta)}=R_0\mathrm{e}^{2u_A}\frac{\sin^2\theta}{f_\mathrm{f}(\theta)},\quad Q(\pi,\theta)=-\frac{\partial_\theta b_\mathrm{f}(\theta)}{2\sin\theta}, \tag{105}$$

where $f_\mathrm{f}=\Re\mathcal{E}_\mathrm{f}$, $b_\mathrm{f}=\Im\mathcal{E}_\mathrm{f}$. Here we have used that $M-u$ is constant on $\mathcal{H}_\mathrm{f}$ as a discussion of Eq. (49) in the limit $t\to\pi$ reveals.

It follows from these results that $\mathcal{H}_\mathrm{f}$ is a regular Cauchy horizon, generated by the Killing vector ∂_{ρ_1} (just like the past horizon $\mathcal{H}_\mathrm{p}$). To see this, we can use a modification of the transformation (29) to regular coordinates in a vicinity of this boundary,

$$\pi-t=\arcsin\sqrt{t'},\quad \theta=\theta',\quad \rho_1={\rho_1}'+\frac{\kappa}{R_0}\ln t',\quad \rho_2={\rho_2}'. \tag{106}$$

As a consequence of (105), the constant κ can always be chosen such that the metric is regular in terms of t', θ', ρ_1', ρ_2'. Moreover, $(\partial_{\rho_1},\partial_{\rho_1})=R_0\mathrm{e}^u\sin^2 t$ tends to zero for $t\to\pi$, i.e. $\mathcal{H}_\mathrm{f}$ is indeed a regular null hypersurface and therefore a Cauchy horizon.

4.4.2. Initial data with $b_A\neq b_B$. Now we study the case $b_A\neq b_B$ and assume that in addition $b_B\neq b_A\pm 4$ holds. In this case, the auxiliary quantity a tends to $a_0\neq 0$ as given in (96) for $t\to\pi$. As a consequence of (53), we see that at least one of the metric potentials Q and u cannot be bounded in this limit. Indeed, we can read off from (54) that e^u diverges as $1/\sin^2 t$ for $t\to\pi$ for all $\theta\in(0,\pi)$.

However, it turns out that this divergence is only a peculiarity of our special choice of metric potentials. A better quantity for discussing regularity is the Ernst potential $\mathcal{E}$ which is defined invariantly in terms of the Killing vectors. And indeed, the Ernst potential also remains regular in the entire Gowdy square for $b_A \neq b_B$. Moreover, from $\mathcal{E}$ one can calculate the Kretschmann scalar $R_{ijkl}R^{ijkl}$ on $\mathcal{H}_\mathrm{f}$ and find that it remains bounded — with the exception of the earlier discussed special cases $b_B = b_A \pm 4$ which we have excluded here. (For $b_B = b_A + 4$, the Kretschmann scalar on $\mathcal{A}_1$ behaves as $1/(\pi - t)^{12}$ for $t \to \pi$, and it has the same behavior on $\mathcal{A}_2$ for $b_B = b_A - 4$, i.e. there occur scalar curvature singularities at the points C or D.)

In order to obtain the metric potentials on $\mathcal{H}_\mathrm{f}$ in terms of the Ernst potential and the constant a_0, we replace the potential u by a potential v in a neighborhood of $\mathcal{H}_\mathrm{f}$ via

$$\mathrm{e}^{u(t,\theta)} = \frac{\mathrm{e}^{v(t,\theta)}}{\sin^2 t}. \tag{107}$$

Then we find

$$\mathcal{H}_\mathrm{f}: \quad Q = \frac{1}{a_0}, \quad \mathrm{e}^{v(\pi,\theta)} = a_0^2 f_\mathrm{f}(\theta), \quad \mathrm{e}^{M(\pi,\theta)} = c\,\frac{\sin^2\theta}{f_\mathrm{f}}, \tag{108}$$

where the integration constant c in the expression for M can be determined from continuous transition to the axes.

In the present case $b_A \neq b_B$, it turns out that $\mathcal{H}_\mathrm{f}$ is a regular Cauchy horizon, generated by the linear combination $\partial_{\rho_1} - a_0\partial_{\rho_2}$ of the two Killing vectors. Regular coordinates can be introduced via

$$\pi - t = \arcsin\sqrt{t'}, \quad \rho_1 = \rho_1' + \frac{\kappa_1}{R_0}\ln t' \quad \rho_2 = \rho_2' + \frac{\kappa_2}{R_0}\ln t', \tag{109}$$

where κ_1 and κ_2 are two constants that can always be chosen such that the resulting metric potentials are regular.

Finally, we may look again at the singular cases $b_B = b_A \pm 4$. As mentioned earlier, the corresponding Ernst potential and the Kretschmann scalar on $\mathcal{A}_1$ (for $b_B = b_A + 4$) or $\mathcal{A}_2$ (for $b_B = b_A - 4$) diverge in the limit $t \to \pi$. Since we therefore cannot find a solution of the LP on $\mathcal{H}_\mathrm{f}$ that is continuously connected to the axes, it is not possible to construct the Ernst potential on $\mathcal{H}_\mathrm{f}$ in these two singular cases directly. However, in order to study the situation on $\mathcal{H}_\mathrm{f}$ in these cases too, we can consider a sequence of solutions with $b_B \neq b_A \pm 4$. Then, for each element of the sequence, the LP can be solved along all four boundaries of the Gowdy square and the corresponding expression for the Ernst potential $\mathcal{E}_\mathrm{f}$ on $\mathcal{H}_\mathrm{f}$, constructed from this solution, is valid. It turns out that the limit $t \to \pi$ of $\mathcal{E}_\mathrm{f}$ remains regular for $0 < \theta < \pi$, whereas $\mathcal{E}_\mathrm{f}$ diverges as expected at C or D. Hence we can conclude that only the boundary points C or D of $\mathcal{H}_\mathrm{f}$ become singular and the interior of $\mathcal{H}_\mathrm{f}$ is still a regular null hypersurface.

5. Discussion

In this paper we have studied the class of *smooth Gowdy symmetric generalized Taub-NUT solutions* as interesting examples of Gowdy spacetimes with spatial $\mathbb{S}^3$ topology.

This class is characterized by a special behavior of the metric potentials in a vicinity of the initial surface $\mathcal{H}_\mathrm{p}$ ($t = 0$) which, in particular, implies that $\mathcal{H}_\mathrm{p}$ is a smooth (past) Cauchy horizon. Utilizing Fuchsian methods, we were able to show that for smooth asymptotic data, describing the spacetime at $\mathcal{H}_\mathrm{p}$, there always exists a unique smooth Gowdy symmetric generalized Taub-NUT spacetime as solution to the Einstein equations for $t \in (0, \pi)$. In a second step, we have investigated the behavior of these solutions on the symmetry axes $\mathcal{A}_1$ ($\theta = 0$) and $\mathcal{A}_2$ ($\theta = \pi$). Using the complex Ernst formulation of the field equations and its reformulation in terms of an equivalent linear problem, we have constructed explicit formulae for the metric potentials on $\mathcal{A}_1$ and $\mathcal{A}_2$ in terms of the data on $\mathcal{H}_\mathrm{p}$. Afterwards, it was possible to extend the solution to the future boundary $\mathcal{H}_\mathrm{f}$ ($t = \pi$) of the Gowdy square and to find explicit expressions for the metric potentials there, too. It followed from these expressions, that we have to distinguish between four types of asymptotic data, which are characterized by the values b_A and b_B of the imaginary part b of the Ernst potential at the points A ($t = \theta = 0$) and B ($t = 0$, $\theta = \pi$) and which lead to solutions with a completely different behavior on $\mathcal{H}_\mathrm{f}$:

(i) $b_B = b_A + 4$:
In this case a scalar curvature singularity occurs at point C ($t = \pi$, $\theta = 0$).

(ii) $b_B = b_A - 4$:
Here, a scalar curvature singularity occurs at point D ($t = \theta = \pi$).

(iii) $b_B = b_A$:
The spacetime is regular in the entire Gowdy square. In particular, the Ernst potential $\mathcal{E}$ and the metric potentials u, Q and M are smooth. Moreover, $\mathcal{H}_\mathrm{f}$ is a smooth Cauchy horizon, generated by the Killing vector ∂_{ρ_1}.

(iv) $b_B \neq b_A$ and $b_B \neq b_B \pm 4$:
The spacetime is regular in the entire Gowdy square and the Ernst potential $\mathcal{E}$ is smooth, however the metric potential u is not well adapted to describe this case and blows up at $\mathcal{H}_\mathrm{f}$. However, there is no physical singularity at $\mathcal{H}_\mathrm{f}$. Instead, $\mathcal{H}_\mathrm{f}$ is a smooth Cauchy horizon, generated by the null vector $\partial_{\rho_1} - a_0 \partial_{\rho_2}$.

This shows that — with exception of the two singular cases (i) and (ii) — smooth Gowdy symmetric generalized Taub-NUT solutions (with a past Cauchy horizon at $t = 0$) always develop a second Cauchy horizon at $t = \pi$. This future Cauchy horizon, in the same way as the one in the past, is homeomorphic to $\mathbb{S}^3$ and its null generator has closed integral curves. Hence our results can in particular be understood as a partial resolution of a problem which remained open in [10]. Namely, at least in our class of spacetimes, the union of the Gowdy square and its interior is isomorphic to the closure of the MGHD of corresponding Cauchy data.

It is interesting to compare these results with the situation of spatial $\mathbb{S}^2 \times \mathbb{S}^1$ topology as investigated in [19]. For this case, it was shown that Gowdy spacetimes with a regular past Cauchy horizon $\mathcal{H}_\mathrm{p}$ develop a regular future horizon $\mathcal{H}_\mathrm{f}$ if and only if a particular quantity J, which can be read-off from the asymptotic data[15], does not vanish. In the limit $J \to 0$, $\mathcal{H}_\mathrm{p}$ transforms into a scalar curvature singularity. Hence, the behavior is similar to the $\mathbb{S}^3$ case: with exception of singular cases, spacetimes with a past Cauchy horizon generically develop a future Cauchy horizon. However, the nature of the singular cases is slightly different: In the $\mathbb{S}^2 \times \mathbb{S}^1$ case, the curvature blows up along the entire future boundary $\mathcal{H}_\mathrm{f}$, whereas we find only singularities at the isolated points C or D on $\mathcal{H}_\mathrm{f}$ for $\mathbb{S}^3$ symmetry.

Do our assumptions rule out important cases? This is not clear and probably difficult to answer. For example our assumptions do not allow solutions with past Cauchy horizons ruled by non-closed generators. There is no reason why such solutions should not exist. Another possibility excluded by our assumptions are solutions with a non-compact or incomplete Cauchy horizon. Indeed such solutions can be constructed, in the polarized case, by the techniques of Moncrief and Isenberg in [20].

Can our results be generalized to situations with less symmetry and eventually maybe even to generic solutions with Cauchy horizons? We have employed two, in principle, independent techniques for the two main steps of our discussion: the Fuchsian method for the basic existence proof and the soliton method for the study of the global properties of the solutions. As far as the Fuchsian method and the underlying singular initial value problem and hence the existence and uniqueness of solutions with prescribed "data" on a Cauchy horizon is concerned, we can say the following. In general, it cannot be expected that such an "initial value problem" for equations of hyperbolic type is well-posed. It seems at least necessary that the generator of the horizon, being a null hypersurface, is a Killing field. This is the case here and also in the more general $U(1)$-symmetric case discussed by Moncrief [24]. Therefore under Moncrief's assumptions, our Fuchsian technique should apply in the smooth $U(1)$-symmetric case. The results in [23, 13] suggest that there must be a $U(1)$-symmetry in a neighborhood of a Cauchy horizon in general vacuum spacetimes, at least if the horizon is compact and the generator has closed integral curves. So, there is hope that a similar singular initial value problem can be formulated under quite general assumptions and that the existence and uniqueness proof based on Fuchsian methods goes through. We also want to mention that most of the results here can easily be generalized to solutions with only finitely many derivatives, cf. Proposition 3.2.

As far as the Ernst formulation and the associated linear problem are concerned there is probably little hope of a generalization to situations with fewer symmetries. The introduction of the complex Ernst potential relies essentially on the existence of two Killing vectors, cf. (51), (53), and the linear problem makes use of the special structure of the Ernst equation. However, it should be quite straightforward to apply the methods to Gowdy symmetric spacetimes with additional *electromagnetic fields.* In that case one has to study the coupled system of the Einstein-Maxwell equations,

[15] In terms of the Ernst potential at $\mathcal{H}_\mathrm{p}$, J is defined as $J = -\frac{1}{8Q_\mathrm{p}^2}(b_A - b_B - 4Q_\mathrm{p})$, where Q_p denotes the constant value of the metric potential Q on $\mathcal{H}_\mathrm{p}$ in the $\mathbb{S}^2 \times \mathbb{S}^1$ case.

for which, remarkably, a complex Ernst formulation and an associated linear problem exist as well. The corresponding calculations would follow closely the investigation of axisymmetric and stationary black hole spacetimes with electromagnetic fields as presented in [18].

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Appendix A. Theory of second-order hyperbolic Fuchsian equations

Here is a quick summary of the theory of second-order hyperbolic Fuchsian equations outlined in [6]; a more detailed presentation can be found in [7].

A *second-order hyperbolic Fuchsian system* is a set of partial differential equations of the form

$$D^2 v + 2A\, Dv + B\, v - t^2 K^2 \Delta v = f[v],$$

in which the function $v : (0, \delta] \times U \to \mathbb{R}^n$ is the main unknown (defined for some $\delta > 0$ and some spatial domain U), while the coefficients $A = A(x)$, $B = B(x)$, $K = K(t, x)$ are diagonal $n \times n$ matrix-valued maps and are smooth in $x \in U$ and t in the half-open interval $(0, \delta]$, and $f = f[v](t, x)$ is an n-vector-valued map of the following form

$$f[v](t, x) := f\Big(t, x, v(t, x), Dv(t, x), tK(t, x)\partial_x v(t, x)\Big).$$

Here, we can assume that K is the identity matrix and that U is $\mathbb{S}^2$; the condition that U is a one-dimensional periodicity domain in the references above can easily be generalized. Of particular convenience for this is that the equations are geometric wave equations with spatial domain $\mathbb{S}^2$. We assume that the time variable t satisfies $t > 0$ and use the operator $D := t\partial_t$ to write the equations. We denote the eigenvalues of A and B by $a^{(1)}, \dots, a^{(n)}$ and $b^{(1)}, \dots, b^{(n)}$, respectively. When it is not necessary to specify the superscripts, we just write a, b to denote any eigenvalues of A, B. With this convention, we introduce:

$$\lambda_1 := a + \sqrt{a^2 - b}, \quad \lambda_2 := a - \sqrt{a^2 - b}.$$

It turns out that these coefficients, which might be complex in general, are important to describe the expected behavior at $t = 0$ of general solutions.

Consider a second-order hyperbolic Fuchsian system with coefficients $a, b, \lambda_1, \lambda_2$. To simplify the presentation, we restrict attention to scalar equations ($n = 1$) and shortly comment on the general case in the course of the discussion. Fix some integers $l, m \geq 0$

and constants $\alpha, \delta > 0$; indeed α can also be a smooth positive function on U. For $w \in C^l((0,\delta], H^m(U))$, we define the norm

$$\|w\|_{\delta,\alpha,l,m} := \sup_{0<t\leq\delta} \left(\sum_{p=0}^{l} \sum_{q=0}^{m} \int_U t^{2(\Re\lambda_2(x)-\alpha)} \left|\partial_x^q D^p w(t,x)\right|^2 dx \right)^{1/2},$$

and denote by $X_{\delta,\alpha,l,m}$ the space of all such functions with finite norm $\|w\|_{\delta,\alpha,l,m} < \infty$. Throughout, $H^m(U)$ denotes the standard Sobolev space. To cover a system of $n \geq 1$ second-order Fuchsian equations, the norm above is defined by summing over all vector components with different exponents used for different components. Recall that each equation in the system will have, in general, a different root function λ_2. We allow that $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(n)})$ is a vector of different positive constants for each equation. The constant δ, however, is assumed to be common for all equations in the system. The motivation for including the quantity λ_2 into the definition of the norms is the *standard singular initial value problem* in [6]. Throughout it is assumed that $\Re\lambda_2$ is continuous and it is then easy to check that $(X_{\delta,\alpha,l,m}, \|\cdot\|_{\delta,\alpha,l,m})$ is a Banach space. For each non-negative integer l and real numbers $\delta, \alpha > 0$, we define $X_{\delta,\alpha,l} := \bigcap_{p=0}^{l} X_{\delta,\alpha,p,l-p}$, and introduce the norm

$$\|f\|_{\delta,\alpha,l} := \left(\sum_{p=0}^{l} \|f\|_{\delta,\alpha,p,l-p}^2 \right)^{1/2}, \qquad f \in X_{\delta,\alpha,l}.$$

It turns out that we must also use spaces $(\tilde{X}_{\delta,\alpha,l}, \|\cdot\|_{\widetilde{\delta,\alpha,l}})$. These are defined as before, but in the norm $\|f\|_{\widetilde{\delta,\alpha,l}}$ of some function f, the highest spatial derivative term $\partial_x^l f$ is weighted with the additional factor t (in our case here). It is easy to see under the earlier conditions that also $(\tilde{X}_{\delta,\alpha,l}, \|\cdot\|_{\widetilde{\delta,\alpha,l}})$ are Banach spaces. Let us also define $X_{\delta,\alpha,\infty} := \bigcap_{l=0}^{\infty} X_{\delta,\alpha,l}$.

Let us choose a function u on $(0,\delta] \times U$ (whose regularity is fixed in the main existence theorem). The *singular initial value problem* associated with u is then defined as follows. We ask whether there exists a solution v of the given second-order hyperbolic Fuchsian system so that the *remainder*

$$w(t,x) := v(t,x) - u(t,x),$$

can be interpreted as "higher order" in t at $t = 0$, where u is interpreted as the *leading-order term*. By this we mean that w is an element in $X_{\delta,\alpha,l}$ for some (sufficiently large) $\alpha > 0$. Often u will be parametrized by certain free functions which we call *asymptotic data*. An example of a singular initial value problem with a leading-order term parametrized by asymptotic data is Proposition 3.2. In that case λ_1 is zero for both equations and λ_2 is zero for the first equation and -4 for the second one.

The existence and uniqueness result of the singular initial value problem is Proposition 3.5 of [6]. The existence proof is based on a new approximation scheme which can also be implemented numerically. See in particular [8] for more details.

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