# Markov processes of infinitely many nonintersecting random walks

Alexei Borodin* Vadim Gorin†

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**Abstract**

Consider an $N$-dimensional Markov chain obtained from $N$ one-dimensional random walks by Doob $h$-transform with the $q$-Vandermonde determinant. We prove that as $N$ becomes large, these Markov chains converge to an infinite-dimensional Feller Markov process. The dynamical correlation functions of the limit process are determinantal with an explicit correlation kernel.

The key idea is to identify random point processes on $\mathbb{Z}$ with $q$-Gibbs measures on Gelfand-Tsetlin schemes and construct Markov processes on the latter space.

Independently, we analyze the large time behavior of PushASEP with finitely many particles and particle-dependent jump rates (it arises as a marginal of our dynamics on Gelfand-Tsetlin schemes). The asymptotics is given by a product of a marginal of the GUE-minor process and geometric distributions.

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*California Institute of Technology; Massachusetts Institute of Technology; Institute for Information Transmission Problems of Russian Academy of Sciences. e-mail: borodin@caltech.edu

†Institute for Information Transmission Problems of Russian Academy of Sciences. e-mail: vadicgor@gmail.com

# 1 Introduction

Let $X(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{Z}^N$ be $N \geq 1$ independent rate one Poisson processes started at $X(0) = (0, 1, \dots, N-1)$ and conditioned to finish at $X(T) = Y$ while taking mutually distinct values for all $0 \leq t \leq T$. Thus, $\{X_i(t)\}_{i \geq 1}$ form $N$ *nonintersecting paths* in $\mathbb{Z} \times [0, T]$.

A result of [KOR], based on a classical theorem of Karlin and McGregor [KM], says that as $T \to \infty$ with $Y$ being asymptotically linear, $Y \sim T\xi$ with a collection of asymptotic speeds $\xi = (\xi_1 \leq \cdots \leq \xi_N) \in \mathbb{R}^N_{>0}$, the process $(X(t), 0 \leq t \leq T)$ has a limit $(\mathcal{X}(t), t \geq 0)$, which is a homogeneous Markov process on $\mathbb{Z}^N$ with initial condition $X(0)$ and transition probabilities over time interval $t$ given by

$$P_t\big((x_1, \dots, x_N) \to (y_1, \dots, y_N)\big) = const \cdot \frac{\Xi_N(y)}{\Xi_N(x)} \det_{i,j=1,\dots,N} \left[ \frac{t^{y_i - x_j}}{(y_i - x_j)!} \right]$$

with $\Xi_N(u) = \det\big[\xi_i^{u_j}\big]_{i,j=1}^N$. [1] The process $\mathcal{X}(t)$ is the Doob $h$-transform of $N$ independent Poisson processes with respect to the harmonic function $h = \Xi_N$.

The case of equal speeds $\xi_j \equiv const$ is special. In that case, the distribution of $X(t)$ for a fixed $t > 0$ is known as the *Charlier orthogonal polynomial ensemble*, which is the basic discrete probabilistic model of random matrix type. If one considers its limiting behavior as $N \to \infty$, in different parts of the state space via suitable scaling limits one uncovers discrete sine, sine, and Airy determinantal point processes which play a fundamental role in Random Matrix Theory, cf. [J1], [BO1].

The procedure of passing to a limit by increasing the number of particles and scaling the space appropriately in models of random matrix type is very well developed. In many cases such a limit can also be successfully performed for joint distributions at finitely many time moments, if the original model undergoes a Markov dynamics. The most common approach is to control the limiting behavior of local *correlation functions* for the model at hand.

It is much less clear what happens to the Markovian structure of the dynamics under such limit transitions. While there are no *a priori* reasons for the Markov property to be preserved, it is a common belief that in many cases it

[1] If some of $\xi_j$'s coincide, for the formula to make sense one needs to perform a limit transition from the case of different speeds.

survives the limit. However, providing a proof for such a statement seems to be difficult.

One goal of the present paper is to prove that if $\xi_j$'s form a geometric progression with ratio $q^{-1} > 1$, the limit $\mathfrak{X}(t)$ of $\mathcal{X}(t)$ (and other similar processes, see below) as $N \to \infty$ can be viewed as a Feller Markov process on all point configurations in $\mathbb{Z}_{\geq 0}$ started from the densely packed initial condition. Note that no scaling is needed in the limiting procedure. As all Feller processes, our limiting process can be started from any initial condition, and it has a modification with càdlàg sample paths.

We also show that the dynamical correlation functions of $\mathfrak{X}(t)$ started from the packed initial condition are determinantal with an explicit correlation kernel, and they are the limits of the corresponding correlation functions for $\mathcal{X}(t)$.

The process $\mathfrak{X}(t)$ can be interpreted as a restriction of a 'local' Markov process on the infinite Gelfand-Tsetlin schemes that falls into a class of such processes constructed in [BF2]. This interpretation implies, in particular, that the distribution of the leftmost particle of $\mathfrak{X}(t)$ coincides with the asymptotic displacement of the $m$th particle, as $m \to \infty$, in the totally asymmetric simple exclusion process (TASEP) with jump rates depending on particles, particle $j$ has rate $\xi_j$, and step initial condition. [2] Inspired by this connection, we derive large time asymptotics for TASEP, and more general PushASEP of [BF1], with finitely many particles that have arbitrary jumps rates.

In the situation when the correlation functions have a determinantal structure, the question of Markovianity for systems with infinite many particles has been previously addressed in [Spo], [KT1], [Os] for the sine process, in [KT2] for the Airy process, in [Jon] for nonintersecting Bessel processes, in [Ol2] for the Whittaker process, and in [BO2] for a process arising in harmonic analysis on the infinite-dimensional unitary group.

Our work seems to be the first one that deals with the discrete limiting state space. Structurally, we adopt the approach of [BO2]: We use the fact (proved earlier by one of us, see [G2]) that infinite point configurations in $\mathbb{Z}$ with finitely many particles to the left of the origin can be identified with ergodic $q$-Gibbs measures on infinite Gelfand-Tsetlin schemes. We further show that the Markov processes $\mathcal{X}(t)$ for different $N$'s are consistent with respect to natural projections from the $N$-particle space to the $(N-1)$-particle one; the projections are uniquely determined by the $q$-Gibbs property. Together with certain (nontrivial) estimates, this leads to the existence of the limiting Feller Markov process. One interesting feature of our construction is that we need to add Gelfand-Tsetlin schemes with infinite entries in order to make the space of the ergodic $q$-Gibbs measures locally compact.

It is worth noting what happens in the limit $q \to 1$. The space of ergodic 1-Gibbs measures has countably many continuous parameters (as opposed to discrete ones for $q < 1$), and it is naturally isomorphic to the space of indecomposable characters of the infinite-dimensional unitary group, and to the space of totally positive doubly infinite Toeplitz matrices (see e.g. [VK], [OO], [Ol1] and references therein.) The $q \to 1$ limit of our Feller Markov process ends up being a *deterministic* (in fact, linear) flow on this space.

It is plausible that our results on the existence of limit Markov processes can be extended to the case of arbitrary $\{\xi_j\}_{j\geq 1}$ which grow sufficiently fast, but at

[2] The step initial condition means that we place the particle $j$ at location $-j$ at $t = 0$.

the moment our proofs of a number of estimates rely on the fact that $\{\xi_j\}_{j\geq 1}$ form a geometric progression.

We now proceed to a more detailed description of our work.

### 1.1 Extended Gelfand-Tsetlin schemes and $q$-Gibbs measures

Following [Wey], for $N \geq 1$ we define a *signature* of length $N$ as an $N$-tuple of nonincreasing integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$, and we denote by $\mathbb{GT}_N$ the set of all such signatures. For $\lambda \in \mathbb{GT}_N$ and $\nu \in \mathbb{GT}_{N+1}$, we say that $\lambda \prec \nu$ if $\nu_{j+1} \leq \lambda_j \leq \nu_j$ for all meaningful values of indices. We agree that $\mathbb{GT}_0$ is a singleton $\varnothing$, and $\varnothing \prec \nu$ for any $\nu \in \mathbb{GT}_1$.

A *Gelfand-Tsetlin scheme* of order $M \in \{0, 1, 2, \dots\} \cup \{\infty\}$ is a length $M$ sequence

$$\lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \dots, \qquad \lambda^{(j)} \in \mathbb{GT}_j.$$

Equivalently, such a sequence can be viewed as an array of numbers $\{\lambda_i^{(j)}\}$ satisfying the inequalities $\lambda_{i+1}^{(j+1)} \leq \lambda_i^{(j)} \leq \lambda_i^{(j+1)}$. An interpretation of Gelfand-Tsetlin schemes in terms of lozenge tilings or stepped surfaces can be found in the introduction to [BF2].

Define an *extended* signature of length $N$ as an $N$-tuple $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$ with $\lambda_j \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$[3], and define an *extended* Gelfand-Tsetlin scheme as a sequence of extended signatures $\lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \dots$, $\lambda^{(j)} \in \overline{\mathbb{GT}}_j$, with the condition that the number of infinite coordinates $m_j$ of $\lambda^{(j)}$ has the property that if $m_j > 0$ then $m_{j-1} = m_j - 1$.

Fix $q \in (0, 1)$. We say that a probability measure on (extended) Gelfand-Tsetlin schemes is *$q$-Gibbs* if for any $N \geq 2$, the conditional distribution of $\lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N-1)}$ given $\lambda^{(N)}$ has weights that are proportional to $q^{|\lambda^{(1)}|+|\lambda^{(2)}|+\cdots+|\lambda^{(N-1)}|}$, where $|\lambda|$ denotes the sum of all finite coordinates of $\lambda$.

There are at least two motivations for our interest in $q$-Gibbs measures. On one hand, if one views Gelfand-Tsetlin schemes as stepped surfaces, the conditional distribution above has weights proportional to $q^{volume}$, where the exponent is the volume underneath the surface, and measures of such type have various interesting features, cf. [V1], [CK], [OR1], [BGR]. On the other hand, the notion of 1-Gibbsianness naturally arises in the representation theory of the infinite-dimensional unitary group, and we hope that $q$-Gibbs measures will arise in suitably defined representations of inductive limits of quantum groups.

In [G2], one of the authors obtained a classification of $q$-Gibbs measures on (non-extended infinite) Gelfand-Tsetlin schemes. The result says that the space of such $q$-Gibbs measures is isomorphic to the space of Borel probability measures on the space

$$\mathcal{N} = \{\nu_1 \leq \nu_2 \leq \dots, \ \nu_i \in \mathbb{Z}\}$$

equipped with the topology of coordinate-wise convergence. Moreover, for any *extreme* $q$-Gibbs measure $\mu$ (that corresponds to the delta measure at a point $\nu \in \mathcal{N}$), the value of $\nu_k$, $k \geq 1$, is the almost sure limit of $\lambda_{N-k+1}^{(N)}$ as $N \to \infty$, with $\{\lambda^{(j)}\}_{j\geq 0}$ distributed according to $\mu$.

[3]We assume that $+\infty > k$ for every integer $k$.

In this paper we generalize this statement and prove that the space of $q$-Gibbs measures on extended infinite Gelfand-Tsetlin schemes is isomorphic to the space of Borel probability measures on the space

$$\overline{\mathcal{N}} = \{\nu_1 \le \nu_2 \le \dots,\ \nu_i \in \overline{\mathbb{Z}}\}$$

equipped with the topology of coordinate-wise convergence. Observe that $\overline{\mathcal{N}}$ is a locally compact topological space, and it should be viewed as a completion of $\mathcal{N}$.

Under the one-to-one correspondence

$$\{\nu_1 \le \nu_2 \le \nu_3 \le \dots\} \longleftrightarrow \{\nu_1 < \nu_2 + 1 < \nu_3 + 2 < \dots\}, \tag{1.1}$$

$\mathcal{N}$ turns into the set of all infinite subsets of $\mathbb{Z}$ that are bounded from below, and $\overline{\mathcal{N}}$ can be viewed as the set of all (finite and infinite) subsets of $\mathbb{Z}$ bounded from below.

## 1.2 Consistent $N$-dimensional random walks

Let $g(x)$ be a finite product of elementary factors of the form

$$(1-\alpha x^{-1})^{-1},\ 0<\alpha<1, \qquad (1+\beta x^{\pm 1}),\ \beta>0; \qquad \exp(\gamma x^{\pm 1}),\ \gamma>0,$$

and $g(x) = \sum_{k\in\mathbb{Z}} g_k x^k$ be its Laurent series in $\{x \in \mathbb{C} : 1 < |x| < \infty\}$. This means that $\{g_k\}_{k\in\mathbb{Z}}$ is, up to a constant, a convolution of geometric, Bernoulli, and Poisson distributions.

For any such $g(x)$, define a transition probability on $\mathbb{GT}_N$ by

$$P_N(\lambda \to \mu; g(x)) = const \cdot \det_{i,j=1,\dots,N}[g_{\mu_i - i - \lambda_j + j}] \prod_{1\le i<j\le N} \frac{q^{i-\lambda_i} - q^{j-\lambda_j}}{q^{i-\mu_i} - q^{j-\mu_j}}, \tag{1.2}$$

and extend it to a transition probability $\overline{P}_N(\lambda \to \mu; g(x))$ on $\overline{\mathbb{GT}}_N$ by applying $P_k(\,\cdot\,; g(x))$ with appropriate $k < N$ to all $k$ finite coordinates of extended signatures.

We prove that if one starts with a $q$-Gibbs measure on extended Gelfand-Tsetlin schemes and applies $\overline{P}_N(\,\cdot\,, g(x))$ to its projections on $\overline{\mathbb{GT}}_N$, $N \ge 1$, then the resulting distributions are also projections of a new $q$-Gibbs measure. This gives rise to a Markov transition kernel $\overline{P}_\infty(g(x))$ on $\overline{\mathcal{N}}$ or, via (1.1), on point configurations in $\mathbb{Z}$ that do not have $-\infty$ as an accumulation point.

## 1.3 Correlation functions

Let $g_0(x), g_1(x), g_2(x), \dots$ be a sequence of functions as above, and let $\mathcal{Z}(t)$, $t = 0, 1, 2, \dots$, be the discrete time Markov process on $\overline{\mathcal{N}}$ with initial condition given by the delta measure at $\mathbf{0} = (0, 0, \dots)$ and transition probabilities $\overline{P}_\infty(g_0(x)), \overline{P}_\infty(g_1(x)), \dots$. One easily sees that the process is in fact supported by $\mathcal{N}$.

Define the $n$th correlation function $\rho_n$ of $\mathcal{Z}(t)$ at $(x_1, t_1), \dots, (x_n, t_n)$ as the probability that $Z(t_j)$, viewed via (1.1) as a subset of $\mathbb{Z}$, contains $x_j$ for every $j = 1, \dots, n$. We prove that these correlation functions are determinantal: For any $n \ge 1$, $t_j \in \mathbb{Z}_{\ge 0}$, $x_j \in \mathbb{Z}$, we have

$$\rho_n(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = \det_{i,j=1,\dots,n}[K(x_i, t_i; x_j, t_j)],$$

where

$$K(x_1,t_1;x_2,t_2) = -\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{dw}{w^{x_1-x_2+1}} \prod_{t=t_2}^{t_1-1} g_t(w)\mathbf{1}_{t_1>t_2}$$
$$+ \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}} dw \oint_{\mathcal{C}'} dz \frac{\prod_{t=0}^{t_1-1} g_t(w)}{\prod_{t=0}^{t_2-1} g_t(z)} \prod_{j\geq 0} \frac{1-wq^j}{1-zq^j} \frac{z^{x_2}}{w^{x_1+1}} \frac{1}{w-z},$$

$\mathcal{C}$ is a positively oriented contour that includes only the pole 0 of the integrand, and $\mathcal{C}'$ goes from $+i\infty$ to $-i\infty$ between $\mathcal{C}$ and point 1.

## 1.4 The Feller property

A Markov transition kernel on a locally compact state space is said to have the *Feller property* if its natural action on functions on the state space preserves the space $C_0$ of continuous functions that converge to zero at infinity, cf. [EK].

We prove that for any function $g(x)$ as above, the kernel $\overline{P}_\infty(g(x))$ on $\overline{\mathcal{N}}$ enjoys the Feller property.

We also prove that for any $\gamma^+,\gamma^- \geq 0$ and any probability measure on $\overline{\mathcal{N}}$, the continuous time Markov process on $\overline{\mathcal{N}}$ with initial condition $\mu$ and transition probabilities

$$\overline{P}_\infty\big(\exp(t(\gamma_+ x + \gamma^-/x))\big), \qquad t \geq 0,$$

is Feller. In addition to the Feller property of the transition kernels, this also requires their strong continuity in $t$ when they are viewed as an operator semigroup in $C_0(\overline{\mathcal{N}})$.

Note that such a statement would have made no sense for non-extended Gelfand-Tsetlin schemes because $\mathcal{N}$ is not locally compact. This is the place where the extended theory becomes a necessity.

## 1.5 Asymptotics of PushASEP

Fix parameters $\zeta_1,\dots,\zeta_N > 0$, $a,b \geq 0$, and assume that at least one of the numbers $a$ and $b$ does not vanish. Consider $N$ particles in $\mathbb{Z}$ located at different sites and enumerated from left to right. The particle number $n$, $1 \leq n \leq N$, has two exponential clocks - the "right clock" of rate $a\zeta_n$ and the "left clock" of rate $b/\zeta_n$. When the right clock of particle number $n$ rings, one checks whether the position to the right of the particle is empty. If yes, then the particle jumps to the right by 1, otherwise it stays put. When the left clock of particle number $n$ rings, it jumps to the left by 1 and pushes the (maybe empty) block of particles sitting next to it. All $2N$ clocks are independent.

This interacting particle system was introduced in [BF1] under the name of *PushASEP*. It interpolates between well-known TASEP and long-range TASEP, cf. [Spi].

Results of [BF2] imply that if $\zeta_1 = q^{1-N}, \zeta_2 = q^{2-N}, \dots, \zeta_N = 1$, and the PushASEP is started from the initial configuration $(1-N, 2-N, \dots, -1, 0)$, then at time $t$ it describes the behavior of the coordinates $(\lambda_N^{(N)} + 1 - N, \lambda_{N-1}^{(N-1)} + 2 - N, \dots, \lambda_1^{(1)})$ of the random infinite Gelfand-Tsetlin scheme distributed according to the $q$-Gibbs measure corresponding to the distribution at the same time $t$

of the Markov process on $\overline{\mathcal{N}}$ with transitional probabilities $\{\overline{P}_\infty\big(\exp(t(ax + b/x))\big)\}_{t\geq 0}$ and the delta-measure at $\mathbf{0}$ as the initial condition.

We prove, for the $N$-particle PushASEP with any jump rates and any deterministic initial condition, and independently of the rest of the paper, that at large times the PushASEP particles demonstrate the following asymptotic behavior: In each asymptotic cluster, particles with the lowest values of $\zeta_k$ fluctuate on $\sqrt{t}$ scale, and the fluctuations are given by the distribution of the smallest eigenvalues in an appropriate GUE-minor process, while faster particles remain at $O(1)$ distances from the blocking slower ones, with geometrically distributed distances between neighbors. The GUE-governed fluctuations and the geometric distributions are asymptotically independent.

Note that for TASEP, which arises when $b = 0$, $\zeta_1 = \zeta_2 = \cdots = \zeta_N$, the relation of large time asymptotics with marginals of the GUE-minor process is well known, see [Bar].

### 1.6 Organization of the paper

In Section 2 we discuss $N$-dimensional random walks with transition probabilities $P_N(\lambda \to \mu; g(x))$. In Section 3 we prove that these random walks are consistent for different values of $N$. In Section 4 we explain how the family of consistent transition probabilities define the transition kernel $\overline{P}_\infty(\lambda \to \mu; g(x))$ on $\overline{\mathcal{N}}$ and describe basic properties of this kernel. The properties of the corresponding Markov processes are studied in Sections 5 and 6.

The main results of Section 5 are Theorem 5.1, where we prove that finite-dimensional distributions of the constructed processes are $N \to \infty$ limits of those for the $N$-dimensinal processes; and Theorem 5.6, where we prove the corresponding result for the correlation functions. In Section 5.3 we describe the connection between processes with transition probabilities $\overline{P}_\infty(\lambda \to \mu; g(x))$ and stochastic dynamics of [BF2].

Section 6 is devoted to the study of extended Gelfand–Tsetlin schemes. In Theorem 6.3 we prove that the space of $q$-Gibbs measures on extended infinite Gelfand-Tsetlin schemes is isomorphic to the space of Borel probability measures on the space $\overline{\mathcal{N}}$. Theorems 6.14 and 6.18 contain the proofs of the results on Feller properties mentioned above. This is the most technical part of the paper.

Finally, in Section 7 we prove Theorem 7.2 describing the asymptotic behavior of PushASEP with finitely many particles.

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## 2 $N$–dimensional random walks

In this section we introduce a family of Markov chains on the set of $N$-point configurations in $\mathbb{Z}$. In a variety of situations they can viewed as collections of $N$

independent identically distributed random walks conditioned not to intersect and to have prescribed asymptotic speeds at large times.

The state space of our processes is the set $\mathbb{M}_N$ of $N$-point configurations in $\mathbb{Z}$:

$$\mathbb{M}_N = \{a(1) < a(2) < \cdots < a(N) \mid a(i) \in \mathbb{Z}\}.$$

We identify elements of $\mathbb{M}_N$ with weakly decreasing sequences of integers, which we call *signatures*. The set of all signatures of size $N$ is denoted by $\mathbb{GT}_N$,

$$\mathbb{GT}_N = \{\lambda_1 \geq \cdots \geq \lambda_N \mid \lambda_i \in \mathbb{Z}\}.$$

We use the following correspondence between the elements of $\mathbb{M}_N$ and $\mathbb{GT}_N$:

$$a(1) < \cdots < a(N) \longleftrightarrow a(N)-N+1 \geq a(N-1)-N+2 \geq \cdots \geq a(2)-1 \geq a(1).$$

We also agree that $\mathbb{M}_0$ ($\mathbb{GT}_0$) consists of a singleton — the empty configuration (the empty signature $\varnothing$).

'Signature' is the standard term for the label of an irreducible representation of the unitary group $U(N)$ over $\mathbb{C}$, and the letters $\mathbb{GT}$ stand for 'Gelfand-Tsetlin' as in Gelfand-Tsetlin bases of the same representations. Although the material of this paper is not directly related to representation theory, we prefer not to change the notation of related previous works, cf. [OO, G2].

In studying probability measures and Markov processes on $\mathbb{GT}_N$ we extensively use *rational Schur functions*. These are Laurent polynomials $s_\lambda \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ indexed by $\lambda \in \mathbb{GT}_N$ and defined by

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det_{i,j=1,\ldots,N}\left[x_i^{\lambda_j+N-j}\right]}{\prod_{i<j}(x_i - x_j)}.$$

Let us introduce certain generating functions of probability measures on $\mathbb{GT}_N$ that may be viewed as analogues of characteristic functions but that are more suitable for our purposes.

Fix a non-decreasing sequence of positive reals $\{\xi_i\}_{i=1,2,\ldots}$. Let $T_N$ be the $N$-dimensional torus

$$T_N = \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = \xi_i\}.$$

Denote by $\mathcal{F}_N$ a class of functions on $T_N$ which can be decomposed as

$$f(x_1, \ldots, x_N) = \sum_{\lambda \in \mathbb{GT}_N} c_\lambda(f) \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(\xi_1, \ldots, \xi_N)} \tag{2.1}$$

with $c_\lambda(f) \geq 0$ and $\sum_\lambda c_\lambda(f) = 1$. Note that the latter condition is equivalent to $f(\xi_1, \ldots, \xi_N) = 1$.

**Lemma 2.1.** *The series* (2.1) *converges uniformly on* $T_N$ *and its sum* $f(x_1, \ldots, x_N)$ *is a real analytic function on* $T_N$.

*Proof.* This fact immediately follows from a simple observation

$$\sup_{(x_1,\ldots,x_N)\in T_N} \left| \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(\xi_1, \ldots, \xi_N)} \right| = 1,$$

which, in turn, follows from the combinatorial formula for Schur functions (see e.g. Section I.5 of [Mac]). $\square$

We note without proof that $\mathcal{F}_N$ is a closed subset of the Banach space of continuous function on $T_N$.

Let $P$ be a probability measure on $\mathbb{GT}_N$. Its *Schur generating function* is a function $\mathcal{S}(x_1,\dots,x_N;P)\in\mathcal{F}_N$ with coefficients $c_\lambda(\mathcal{S})$ defined through

$$c_\lambda(\mathcal{S})=P(\lambda),$$

where $P(\lambda)$ stays for the measure of the singleton $\{\lambda\}$. Let $\mathcal{L}_N$ be the map sending probability measures on $\mathbb{GT}_N$ to the corresponding functions in $\mathcal{F}_N$. Clearly this is an isomorphism of convex sets. We agree that $\mathcal{F}_0$ contains a single function (constant 1) which corresponds to a unique probability measure on the singleton $\mathbb{GT}_0$.

Our next goal is to construct a family of stochastic matrices with rows and columns enumerated by elements of $\mathbb{GT}_N$. Let $Q$ be one such stochastic matrix. Then $Q$ acts on probability measures on $\mathbb{GT}_N$

$$P\mapsto QP.$$

We will always identify stochastic matrices with the corresponding operators and use the same notations for them.

Let $\widetilde{Q}$ be a bounded linear operator in the Banach space of continuous functions on $T_N$ such that $\widetilde{Q}(\mathcal{F}_N)\subset\mathcal{F}_N$. Then $\mathcal{L}_N^{-1}\widetilde{Q}\mathcal{L}_N$ is a Markovian linear operator or a stochastic matrix.

For a function $g(x)$ on $\bigcup_{i=1}^N\{x\in\mathbb{C}:|x|=\xi_i\}$, define an operator $\widetilde{Q}_N^g$ via

$$\widetilde{Q}_N^g: f(x_1,\dots,x_N)\mapsto f(x_1,\dots,x_N)\prod_{i=1}^N\frac{g(x_i)}{g(\xi_i)},$$

(here we agree that $Q_0^g$ is the identity operator). Clearly, if the function $g(x)$ is continuous then $\widetilde{Q}_N^g$ is a bounded linear operator in the space of continuous functions on $T_N$.

**Lemma 2.2.** *Suppose that $g(x)$ can be decomposed into a converging power series in annulus $\mathbf{K}=\{x\in\mathbb{C}:r<|x|<R\}$:*

$$g(x)=\sum_{k\in\mathbb{Z}}c_kx^k,\quad x\in\mathbf{K}.$$

*Then for $(x_1,\dots,x_N)\in\mathbf{K}^N$ we have*

$$s_\lambda(x_1,\dots,x_N)g(x_1)\cdots g(x_N)=\sum_{\mu\in\mathbb{GT}_N}\det_{i,j=1,\dots,N}\left[c_{\mu_i-i-\lambda_j+j}\right]s_\mu(x_1,\dots,x_N).\tag{2.2}$$

*Proof.* Straightforward computation. One multiplies both sides of (2.2) by $\prod_{i<j}(x_i-x_j)$ and compares the coefficients of the monomials, cf. [Ol1, Lemma 6.5]. □

Denote by $\mathfrak{g}(N,\xi)$ the set of functions consisting of

$$\begin{array}{rrrr}
x; & & x^{-1}; & \\
(1+\beta x), & \beta>0; & (1+\beta x^{-1}), & \beta>0;\\
e^{\gamma x}, & \gamma>0; & e^{\gamma x^{-1}}, & \gamma>0;\\
(1-\alpha x)^{-1} & 0<\alpha<\xi_N^{-1}; & (1-\alpha x^{-1})^{-1}, & 0<\alpha<\xi_1.
\end{array}$$

Also denote $\mathfrak{g}(\infty,\xi)=\bigcap_N \mathfrak{g}(N,\xi)$. In what follows we call $g(x)\in\mathfrak{g}(\infty,\xi)$ an *elementary admissible function.* Let $\mathcal{G}(N,\xi)$ denote the set of all finite products of the functions of $\mathfrak{g}(N,\xi)$ and denote $\mathcal{G}(\infty,\xi)=\bigcap_N \mathcal{G}(N,\xi)$. We call $g(x)\in\mathcal{G}(\infty,\xi)$ an *admissible function.*

**Proposition 2.3.** *If $g\in\mathcal{G}(N,\xi)$, then*

$$\widetilde{Q}_N^g(\mathcal{F}_N)\subset\mathcal{F}_N.$$

*Proof.* For $g(x)\in\mathcal{G}(N,\xi)$ we can use Lemma 2.2. Furthermore, it is known that in these cases all determinants in the decomposition (2.2) are non-negative (see [Edr], [Voi], [Boy], [VK], [OO]).

Then we proceed as follows: Take a function $f\in\mathcal{F}_N$, decompose it into a sum of Schur polynomials. Then by Lemma 2.2 we obtain a double sum for $\widetilde{Q}_N^g(f)$. Changing the order of summation we see that $\widetilde{Q}_N^g(f)\in\mathcal{F}_N$. □

Set

$$P_N(g(x))=\mathcal{L}_N^{-1}\circ\widetilde{Q}_N^g\circ\mathcal{L}_N$$

and let $P_N(\lambda\to\mu;g(x))$ be the matrix element of the corresponding stochastic matrix.

**Proposition 2.4.** *Let*

$$g(x)=\sum_{k\in\mathbb{Z}}c_kx^k$$

*be a decomposition of $g(x)$ into a power series converging for all $x$ such that* $\min_i\xi_i\le|x|\le\ \max_i\xi_i$*, $i=1,2\dots,N$. Then*

$$P_N(\lambda\to\mu;g(x))=\left(\prod_{i=1}^N\frac{1}{g(\xi_i)}\right)\det_{i,j=1,\dots,N}\left[c_{\mu_i-i-\lambda_j+j}\right]\frac{s_\mu(\xi_1,\dots,\xi_N)}{s_\lambda(\xi_1,\dots,\xi_N)}.\tag{2.3}$$

*Proof.* This is an immediate consequence of Lemma 2.2 and definitions. □

**Remark 1.** For $g(x)=(1-\alpha x)^{-1}$ the determinants in the proposition above can be explicitly evaluated:

$$\det_{i,j=1,\dots,N}\left[c_{\mu_i-i-\lambda_j+j}\right]=\begin{cases}\alpha^{\sum_{i=1}^N(\mu_i-\lambda_i)}, & \text{if }\ \mu_{i-1}\le\lambda_i\le\mu_i,\ 1\le i\le N,\\ 0, & \text{otherwise.}\end{cases}\tag{2.4}$$

(The condition $\mu_0\le\lambda_1$ above is empty.)

For $g(x)=1+\beta x$, the evaluation takes the form

$$\det_{i,j=1,\dots,N}\left[c_{\mu_i-i-\lambda_j+j}\right]=\begin{cases}\beta^{\sum_{i=1}^N(\mu_i-\lambda_i)},\ \text{if }\mu_i-\lambda_i\in\{0,1\},\ 1\le i\le N,\\ 0,\ \text{otherwise.}\end{cases}$$

Similar formulas exist for $g(x)=(1-\alpha^{-1}x)^{-1}$ and $g(x)=1+\beta x^{-1}$ with $\lambda$ and $\mu$ interchanged.

**Remark 2.** When $\xi$ is a geometric progression, the Schur function $s_\lambda(\xi_1,\dots,\xi_N)$ can be evaluated as follows (see e.g. [Mac, Example 3.1])

$$s_\lambda(1,\dots,q^{1-N})=q^{-((N-1)\lambda_1+(N-2)\lambda_2+\dots+\lambda_{N-1})}\prod_{i<j}\frac{1-q^{\lambda_i-i-\lambda_j+j}}{1-q^{j-i}}.\tag{2.5}$$

Then the formula for the transition probability (2.3) turns into (1.2).

**Remark 3.** If $\xi_i = q^{1-i}$ and $g(x) = (1+\beta x^{\pm 1})$, then one can formally send $N \to \infty$ in formulas (1.2), (2.3) and obtain well-defined transition probabilities; while for $g(x) = \exp(\gamma x)$ such formal limit transition does not lead to anything meaningful.

Denote by $\mathcal{X}_{N,g}(t)$ a discrete time homogenous Markov chain on $\mathbb{GT}_N$ with transition probabilities $P_N(\lambda \to \mu; g(x))$, started from the delta-measure at the zero signature $\mathbf{0} = (0 \geq 0 \geq \cdots \geq 0)$.

Also let $\mathcal{Y}_{N,\gamma_+,\gamma_-}(t)$ be the continuous time homogenous Markov chain on $\mathbb{GT}_N$ with transition probabilities $P_N\left(\lambda \to \mu; \exp(t(\gamma_+ x + \gamma_- x^{-1}))\right)$, started from delta-measure on zero signature $\mathbf{0}$. (Clearly the corresponding stochastic matrices form a semigroup.)

A number of such Markov chains with various $g(x)$, $\gamma_+$, $\gamma_-$, have independent probabilistic interpretations. Let us list some of them.

- For $N=1$ and $\xi_1 = 1$, $\mathcal{X}_{1,1+\beta x}(t)$ and $\mathcal{X}_{N,1+\beta x^{-1}}(t)$ are simple Bernoulli random walks with jump probability $\beta(1+\beta)^{-1}$ and particle jumping to the right and to the left, respectively; $\mathcal{X}_{1,x^{\pm 1}}(t)$ is the deterministic shift of the particle to the right (left) by 1; $\mathcal{X}_{1,(1-\alpha x)^{-1}}(t)$ and $\mathcal{X}_{N,(1-\alpha x^{-1})^{-1}}(t)$ are random walks with geometrical jumps; $\mathcal{Y}_{1,\gamma_+,0}(t)$ is the Poisson process of intensity $\gamma_+$ and $\mathcal{Y}_{1,\gamma_+,\gamma_-}(t)$ is the homogenous birth and death process on $\mathbb{Z}$.
- For any $N \geq 1$ and an arbitrary sequence $\xi$, it is proved in [KOR] that $\mathcal{Y}_{N,\gamma_+,0}(t)$ can be viewed as $N$ independent rate 1 Poisson processes conditioned never to collide and to have asymptotic speeds of particles given by $\xi_i$. Similar interpretations could be given for $\mathcal{X}_{N,g}(t)$.
- If $\xi_1 = \xi_2 = \cdots = \xi_N = 1$, then $\mathcal{X}_{N,1+\beta x}(t)$ can be obtained as a limit of uniformly distributed 3d Young diagrams in $a \times b \times c$ box (see [J2], [J3], [JN1], [G1]) with $a = N$ and $b, c \to \infty$ in such a way that $c/b \to \beta$.
- The connection to exclusion processes is explained in Section 5.3 below.

Proposition 2.4 implies that the one-dimensional distribution of $\mathcal{X}_{N,g}(t)$ at a given time $t_0$ is a (possibly, two-sided) *Schur measure*, cf. [Bor],

$$\mathrm{Prob}(\mathcal{X}_{N,g}(t_0) = \lambda) = \left(\prod_{i=1}^{N} \frac{1}{g(\xi_i)}\right)^{t_0} \det_{i,j=1,\dots,N}\left[c^{t_0}_{\lambda_i - i + j}\right] s_\lambda(\xi_1, \dots, \xi_N),$$

where $c^{t_0}_k$ are the coefficients of the Laurent series for $\big(g(x)\big)^{t_0}$.

If we view $\mathcal{X}_{N,g}(t_0)$ as a point configuration in $\mathbb{Z}$, i.e. as an element of $\mathbb{M}_N$, then we may speak about its *correlation functions*:

$$\rho_n(x_1, \dots, x_n) = \mathrm{Prob}(x_1 \in \mathcal{X}_{N,g}(t_0), \dots, x_n \in \mathcal{X}_{N,g}(t_0)).$$

As shown in [BK], the correlation functions of the two-sided Schur measures have a determinantal form (for the one-sided Schur measure this was proved earlier, see [Ok])

$$\rho_n(x_1, \dots, x_n) = \det_{i,j=1,\dots n}[K(x_i, x_j)],$$

with *correlation kernel* $K$ given by a double contour integral

$$K(x_1;x_2) = \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}} dw \oint_{\mathcal{C}'} dz \left(\frac{g(w)}{g(z)}\right)^{t_0} \prod_{\ell=1}^{N} \frac{(1-w/\xi_\ell)}{(1-z/\xi_\ell)} \frac{z^{x_2}}{w^{x_1+1}} \frac{1}{w-z}, \tag{2.6}$$

where the (positively oriented) contour $\mathcal{C}$ includes only the pole at 0 and $\mathcal{C}'$ includes only the poles at $\xi_i$. A similar formula exists for dynamical correlation functions (see [BF2] and Section 5.2 below), describing finite-dimensional distributions of $\mathcal{X}_{N,g}(t)$ and also for $\mathcal{Y}_{1,\gamma_+,\gamma_-}(t)$.

Note that if $\xi_i$'s grow fast enough as $i \to \infty$, then one may formally pass to the limit $N \to \infty$ in (2.6). Thus it is natural to expect that there is an *infinite dimensional* Markov process which is a $N \to \infty$ limit of processes $\mathcal{X}_{N,g}(t)$ (or $\mathcal{Y}_{N,\gamma_+,\gamma_-}(t)$). One goal of this paper is to define this limiting process rigorously and to show that its finite-dimensional distributions are indeed given by limits of (2.6).

# 3 Commutation relations

In this section we show that transition probabilities $P_N(\lambda \to \mu; g(x))$ are in a certain way consistent for various $N$.

We start by introducing stochastic matrices with rows and columns indexed by elements of $\mathbb{GT}_N$ and $\mathbb{GT}_{N-1}$, $N \geq 1$. In other words, we want to define transition probabilities from $\mathbb{GT}_N$ to $\mathbb{GT}_{N-1}$. As above, it is convenient to pass from stochastic matrices to maps between spaces of Schur generating functions of probability measures. Thus, we want to introduce a map

$$\widetilde{P}_N^{\downarrow} : \mathcal{F}_N \to \mathcal{F}_{N-1}.$$

**Proposition 3.1.** *The specialization map*

$$\widetilde{P}_N^{\downarrow} : f(x_1,\dots,x_N) \to f(x_1,\dots,x_{N-1},\xi_N)$$

*is a bounded linear operator between appropriate spaces of continuous functions, and* $\widetilde{P}_N^{\downarrow}(\mathcal{F}_N) \subset \mathcal{F}_{N-1}$.

*Proof.* The fact that this is a bounded linear operator is straightforward. Using well-known branching rules for Schur functions (see e.g. [Mac]) we see that:

$$\widetilde{P}_N^{\downarrow}\left(\frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(\xi_1,\dots,\xi_N)}\right) = \sum_{\mu \prec \lambda} \frac{s_\mu(x_1,\dots,x_{N-1})}{s_\mu(\xi_1,\dots,\xi_{N-1})} \xi_N^{|\lambda|-|\mu|} \frac{s_\mu(\xi_1,\dots,\xi_{N-1})}{s_\lambda(\xi_1,\dots,\xi_N)},$$

where $\mu \prec \lambda$ means the following interlacing condition:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{N-1} \geq \lambda_N, \tag{3.1}$$

and $|\lambda| = \lambda_1 + \dots + \lambda_N$, $|\mu| = \mu_1 + \dots + \mu_{N-1}$. Since all the coefficients

$$\xi_N^{|\lambda|-|\mu|} \frac{s_\mu(\xi_1,\dots,\xi_{N-1})}{s_\lambda(\xi_1,\dots,\xi_N)}$$

are positive, we immediately conclude that $P_N^{\downarrow}(\mathcal{F}_N) \subset \mathcal{F}_{N-1}$. ☐

Let us denote by $P_N^{\downarrow}$ a stochastic matrix of transition probabilities corresponding to $\widetilde{P}_N^{\downarrow}$, i.e. $P_N^{\downarrow} = \mathcal{L}_{N-1}^{-1} \circ \widetilde{P}_N^{\downarrow} \circ L_N$. We call this matrix a *stochastic link* between levels $N$ and $N-1$.

Using the definition of Schur functions we conclude that the matrix elements $P_N^{\downarrow}(\lambda \to \mu)$ are given by the following formula:

$$P_N^{\downarrow}(\lambda \to \mu) = \begin{cases} \xi_N^{|\lambda|-|\mu|} \dfrac{\det_{i,j=1,\dots,N-1}\left[\xi_i^{\mu_j+N-1-j}\right]}{\det_{i,j=1,\dots,N}\left[\xi_i^{\lambda_j+N-j}\right]} \prod_{i=1}^{N-1}(\xi_i - \xi_N), & \mu \prec \lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Note that if $\xi_i$'s form a geometric progression then the determinants in (3.2) turn into $q$-Vandermonde determinants, cf. (2.5).

**Proposition 3.2.** *Matrices of transition probabilities $P_N(\cdot; g(x))$ commute with links $P_N^{\downarrow}$.*

*Proof.* This is equivalent to commutativity relations between maps $\widetilde{Q}_N^g$ and $\widetilde{P}_N^{\downarrow}$, which is straightforward. ☐

For our further constructions it is necessary to extend the space $\mathbb{GT}_N$ and the definition of links $P_N^{\downarrow}$.

The extended level $N$, $\overline{\mathbb{GT}}_N$ consists of all sequences $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N$, where $\lambda_i \in \mathbb{Z} \cup \{+\infty\}$. We identify every such sequence with a shorter sequence $\mu_1 \ge \mu_2 \ge \dots \ge \mu_k$ with $\mu_i = \lambda_{i+N-k}$ and $k$ being the smallest integer such that $\lambda_{N-k} = +\infty$, and with the corresponding $k$–point configuration in $\mathbb{Z}$.

We have

$$\overline{\mathbb{GT}}_N = \mathbb{GT}_N^{(0)} \cup \mathbb{GT}_N^{(1)} \cup \mathbb{GT}_N^{(2)} \cup \dots \cup \mathbb{GT}_N^{(N)}, \tag{3.3}$$

where

$$\mathbb{GT}_N^{(k)} = \{\lambda : \lambda_1 = \dots = \lambda_{N-k} = +\infty, \lambda_{N-k+1}, \dots, \lambda_N \in \mathbb{Z}\},$$

in particular, $\mathbb{GT}_N^{(0)}$ consists of a single signature with all infinite coordinates. It is convenient to use the obvious identification

$$\mathbb{GT}_N^{(k)} \simeq \mathbb{GT}_{N-1}^{(k)} \simeq \dots \simeq \mathbb{GT}_k^{(k)} = \mathbb{GT}_k.$$

In order to define the extended matrix of transition probabilities $\overline{P}_N^{\downarrow}(\lambda \to \mu)$, $\lambda \in \overline{\mathbb{GT}}_N$, $\mu \in \overline{\mathbb{GT}}_{N-1}$ we first introduce for any $k < N$ an auxiliary stochastic matrices $Q_N^k$ with rows and columns indexed by elements of $\mathbb{GT}_k$ by $Q_N^k = Q_k^g$ with $g = 1/(1 - \xi_N^{-1}x)$. And $Q_N^0$ is the unique $1 \times 1$ stochastic matrix.

Now we are ready to define $\overline{P}_N^{\downarrow}(\lambda \to \mu)$. This matrix has a block structure with respect to splittings (3.3) on levels $N$ and $N-1$. For $\lambda \in \mathbb{GT}_N^{(N)}$

$$\overline{P}_N^{\downarrow}(\lambda \to \mu) = \begin{cases} P_N^{\downarrow}(\lambda \to \mu), \text{ if } \mu \in \mathbb{GT}_{N-1}^{(N-1)}, \\ 0, \text{ otherwise.} \end{cases}$$

For $\lambda \in \mathbb{GT}_N^{(k)}$ with $k < N$, we define

$$\overline{P}_N^{\downarrow}(\lambda \to \mu) = \begin{cases} Q_N^k(\lambda \to \mu), \text{ if } \mu \in \mathbb{GT}_{N-1}^{(k)}, \\ 0, \text{ otherwise.} \end{cases}$$

Let us also extend the definition of stochastic matrices $P_N(\cdot; g(x))$ to larger matrices $\overline{P}_N(\cdot; g(x))$ with rows and columns indexed by elements of $\overline{\mathbb{GT}}_N$. These matrices also have a block structure with respect to (3.3). We define

$$\overline{P}_N(\lambda \to \mu; g(x)) = \begin{cases} P_k(\,\cdot\,; g(x)), \text{ if } \lambda \in \mathbb{GT}_N^{(k)} \text{ and } \mu \in \mathbb{GT}_N^{(k)}, \\ 0, \text{ otherwise}, \end{cases}$$

where the arguments of $P_k$ on the right are suitable truncations of $\lambda$ and $\mu$.

**Proposition 3.3.** *Matrices of transition probabilities $\overline{P}_N(\cdot; g(x))$ commute with links $\overline{P}_N^{\downarrow}$.*

*Proof.* This is immediate from the definitions. □

# 4 Infinite–dimensional dynamics.

In this section we introduce infinite–dimensional dynamics — the main object of study of the present paper. We start from a general theorem about sequences of probability measures.

For any topological space $W$ we denote by $\mathcal{M}(W)$ the Banach space of signed measures on $W$ with total variation norm, and by $\mathcal{M}_p(W)$ the closed convex subset of probability measures.

Suppose that we have a sequence of countable sets $\Gamma_0, \Gamma_1, \dots$, and for any $N \geq 0$ we have a stochastic matrix $\Lambda_N^{N+1}$ with rows enumerated by elements of $\Gamma_{N+1}$ and columns enumerated by elements of $\Gamma_N$. We call these matrices *links.* $\Lambda_N^{N+1}$ induces a linear operator mapping $\mathcal{M}(\Gamma_{N+1})$ to $\mathcal{M}(\Gamma_N)$ and we keep the same notation for this operator:

$$(\Lambda_N^{N+1}\pi)(y) = \sum_{x \in \Gamma_{N+1}} \pi(x) \Lambda_N^{N+1}(x, y), \quad \pi \in \mathcal{M}(\Gamma_N).$$

The projective limit $\varprojlim \mathcal{M}(\Gamma_N)$ with respect to the maps $\Lambda_N^{N+1}$ is a Banach space with norm

$$\|(\pi_0, \pi_1, \dots)\| = \sup_N \|\pi_N\|.$$

There is also another topology on $\varprojlim \mathcal{M}(\Gamma_N)$ that we call *discrete weak topology.* This is the minimal topology such that for every $N \geq 0$ and $x \in \Gamma_N$ the map

$$(\pi_0, \pi_1, \dots) \to \pi_N(x)$$

is continuous.

We equip $\varprojlim \mathcal{M}(\Gamma_N)$ with Borel $\sigma$–algebra spanned by open sets in the norm-topology. One proves that this is the same algebra as Borel $\sigma$–algebra spanned by open sets in the discrete weak topology.

A projective limit $\varprojlim \mathcal{M}_p(\Gamma_N)$ is a closed convex subset of $\varprojlim \mathcal{M}(\Gamma_N)$. Elements of $\varprojlim \mathcal{M}_p(\Gamma_N)$ are called *coherent systems.* Note that if $M$ is a coherent system, then

$$\|M_0\| = \|M_1\| = \dots = \|M\| = 1.$$

**Definition 4.1.** *A topological space $\mathcal{Q}$ is a* boundary *of a sequence $\{\Gamma_N, \Lambda_N^{N+1}\}$ if*

1. *There exists a bijective map*
$$E : \mathcal{M}(\mathcal{Q}) \to \varprojlim \mathcal{M}(\Gamma_N);$$
2. *$E$ and $E^{-1}$ are bounded linear operators in the corresponding norms;*
3. *$E$ maps $\mathcal{M}_p(\mathcal{Q})$ bijectively onto $\varprojlim \mathcal{M}_p(\Gamma_N)$;*
4. *$x \to E(\delta_x)$ is a bijection between $\mathcal{Q}$ and extreme points of the convex set of coherent systems, and this bijection is a homeomorphism, where we use the discrete weak topology on $\mathcal{M}(\mathcal{Q})$ .*

**Remark 1.** If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two boundaries, then they are homeomorphic.

**Remark 2.** As follows from 4. of the above definition, the boundary can be always identified with the set of all extreme coherent systems. However, finding a more explicit description of the boundary can be complicated.

**Remark 3.** Some authors *define* the boundary to be the set of all extreme coherent systems (see e.g. [Ker], [DF], [Ol1]), then 1.–4. become the *properties* of the boundary. Also note that a slightly different definition of the boundary was used in [BO2].

**Theorem 4.2.** *For any sequence $\{\Gamma_N, \Lambda_N^{N+1}\}$ there exists a boundary $\mathcal{Q}$.*

*Proof.* Statements similar to Theorem 4.2 were proved in [DF] and [Ol1]. We use [Ol1] as our main reference.

Let $\mathcal{Q}$ be the set of extreme points of the convex set $\varprojlim \mathcal{M}_p(\Gamma_N)$. We equip $\mathcal{Q}$ with discrete weak topology inherited from $\varprojlim \mathcal{M}(\Gamma_N)$. Note that condition 4. of Definition 4.1 is satisfied automatically.

Theorem 9.2 in [Ol1] says that $\mathcal{Q}$ is a Borel subset of $\varprojlim \mathcal{M}_p(\Gamma_N)$, and for any $M \in \varprojlim \mathcal{M}_p(\Gamma_N)$ there exist a unique probability measure $\pi_M$ on $\mathcal{Q}$ such that
$$M = \int_{\mathcal{Q}} M^q \pi_M(dq),$$
meaning that for any $N \geq 0$ and any subset $A$ of $\Gamma_N$ we have
$$M_N(A) = \int_{\mathcal{Q}} M_N^q(A) \pi_M(dq).$$

For any $\pi \in \mathcal{M}_p(\mathcal{Q})$, set
$$E(\pi) = \int_{\mathcal{Q}} M^q \pi(dq).$$
If $\pi$ is a signed measure, i.e. $\pi \in \mathcal{M}(\mathcal{Q})$, then there exist $\pi_1, \pi_2 \in \mathcal{M}_p(\mathcal{Q})$ and two non-negative numbers $c_1$, $c_2$ such that
$$\pi = c_1\pi_1 - c_2\pi_2$$
and $c_1 + c_2 = \|\pi\|$. We define
$$E(\pi) = c_1 E(\pi_1) - c_2 E(\pi_2).$$

Clearly, $E$ is a linear operator from $\mathcal{M}(\mathcal{Q})$ to $\varprojlim \mathcal{M}(\Gamma_N)$ of norm 1. Using Theorem 9.2 from [Ol1] we conclude that condition 3. of Definition 4.1 is satisfied and, moreover, $E$ is an injection.

In order to prove that $E$ is a surjection it is enough to show that for any $M \in \varprojlim \mathcal{M}(\Gamma_N)$ there exist $K, L \in \varprojlim \mathcal{M}_p(\Gamma_N)$ and $k, l \geq 0$ such that $M = kK - lL$. Without loss of generality we may assume that neither $M$, nor $-M$ are positive measures. Let $M_N = K_N^N - L_N^N$ be a decomposition of $M_N$ into a difference of two positive measures such that $\|M_N\| = \|K_N^N\| + \|L_N^N\|$. Set $K_{N-1}^N = \Lambda_{N-1}^N K_N^N$, $K_{N-2}^N = \Lambda_{N-2}^{N-1} K_{N-1}^N$ and so on, and similarly for $L_N^N$. Note that for any $N \geq 0$, $K_N^i$ monotonically increases as $i \to \infty$. Also $\|K_N^i\| \leq \|M\|$. Hence, there exists a limit

$$K_N^\infty = \lim_{i \to \infty} K_N^i.$$

In the same way there is a limit

$$L_N^\infty = \lim_{i \to \infty} L_N^i.$$

Note that for any $i$, $K_N^i$ and $L_N^i$ are positive measures and $M_N = K_N^i - L_N^i$. Therefore, similar statements hold for $M$, $K_N^\infty$ and $L_N^\infty$. Setting $k = \|K^\infty\|$, $l = \|L^\infty\|$ (neither $k$ nor $l$ can vanish because we assumed $M$ is not positive), $K = K^\infty/k$, $L = L^\infty/l$ we get the required decomposition of $M$.

We have proved that $E$ is a bounded linear operator in Banach spaces which is a bijection. Then it follows from Banach Bounded Inverse Theorem that $E^{-1}$ is also a bounded linear operator. $\square$

Now let us specialize the general situation by setting $\Gamma_N = \mathbb{GT}_N$ and $\Lambda_N^{N+1} = P_{N+1}^{\downarrow}$, where these matrices implicitly depend on the sequence $\{\xi_N\}$. Denote by $\mathcal{Q}(\xi_1, \xi_2, \dots)$ the boundary of the sequence $\{\mathbb{GT}_N, P_{N+1}^{\downarrow}\}$.

Theorem 4.2 and Proposition 3.2 provide a way of constructing Markov transition probabilities on the abstract space $\mathcal{Q}(\xi_1, \xi_2, \dots)$ starting from any sequence $\{\xi_N\}$ and an admissible function $g(x)$. Let us explain this more formally.

Let $\mathcal{W}$ be a topological space. A function $P(u, A)$ of a point $u \in \mathcal{W}$ and Borel subset $A$ of $\mathcal{W}$ is *a Markov transition kernel* if

1. For any $u \in \mathcal{W}$, $P(u, \cdot\,)$ is a probability measure on $\mathcal{W}$,
2. For any Borel subset $A \subset \mathcal{W}$, $P(\,\cdot\,, A)$ is a measurable function.

Given a sequence $\xi_1, \xi_2, \dots$ and an admissible function $g(x) \in \mathcal{G}(\infty, \xi)$, we define a kernel $P_\infty(u, A; g(x))$ on $\mathcal{Q}(\xi_1, \xi_2, \dots)$ in the following way. Take point $u \in \mathcal{Q}(\xi_1, \xi_2, \dots)$; by Theorem 4.2 Dirac $\delta$–measure $\delta_u$ corresponds to a certain coherent system $\{M_N^u\}$. Then Proposition 3.2 yields that $\{P_N(\cdot; g(x)) M_N\}$ is also a coherent system and, consequently, it corresponds to a probability measure on $\mathcal{Q}(\xi_1, \xi_2, \dots)$. We *define* $P_\infty(u, A; g(x))$ to be equal to this measure.

**Proposition 4.3.** *$P_\infty(u, A; g(x))$ is a Markov transition kernel.*

*Proof.* $P_\infty(u, \cdot; g(x))$ is a probability measure on $\mathcal{Q}(\xi_1, \xi_2, \dots)$ by the very definition. Thus, it remains to check that $P_\infty(\,\cdot\,, A; g(x))$ is a measurable function. Theorem 4.2 yields that the map $u \mapsto M_N^u(\lambda)$ is continuous and, thus, measurable for any $N \geq 0$ and any $\lambda \in \mathbb{GT}_N$. Then it follows from the definition of matrices $P_N(g(x))$ that the map $u \mapsto (P_N(g(x))M_N^u)(\lambda)$ is also measurable. Since, the last property holds for any $\lambda$, we conclude that $u \mapsto \{P_N(g(x))M_N^u\}$ is a measurable map from $\mathcal{Q}(\xi_1, \xi_2, \dots)$ to coherent systems. The definition of the boundary implies that the correspondence between probability measures on $\mathcal{Q}(\xi_1, \xi_2, \dots)$ and coherent systems is bi-continuous and, therefore, it is bi-measurable. We conclude that the correspondence $u \mapsto P_\infty(u, \cdot\,; g(x))$ is a measurable map between $\mathcal{Q}(\xi_1, \xi_2, \dots)$ and $\mathcal{M}_p(\mathcal{Q}(\xi_1, \xi_2, \dots))$ (the Borel structure in the latter space corresponds to the total variation norm). It remains to note that for any Borel $A \subset \mathcal{Q}(\xi_1, \xi_2, \dots)$ the map

$$\mathrm{Ev}_A : \mathcal{M}(\mathcal{Q}(\xi_1, \xi_2, \dots)) \to \mathbb{R}, \quad \mathrm{Ev}_A(P) = P(A)$$

is continuous (in the total variation norm topology), thus measurable. Therefore, for any Borel $A$, the map $u \mapsto P_\infty(u, A; g(x))$ is measurable. $\square$

We see that transition kernels $P_\infty(g(x))$ naturally define Markov chains on $\mathcal{Q}(\xi_1, \xi_2, \dots)$. However, Theorem 4.2 tells us nothing about the actual state space of these Markov chains, and their properties can be very different for different sequences $\xi_1, \xi_2, \dots$.

As far as the authors know, an explicit description of the set $\mathcal{Q}(\xi_1, \xi_2, \dots)$ is currently known in two special cases. Namely,

1. $\xi_1 = \xi_2 = \dots = 1$. This case was studied by Voiculescu [Voi], Boyer [Boy], Vershik–Kerov [VK], Okounkov–Olshanski [OO]. It is related to representation theory of the infinite–dimensional unitary group and to the classification of totally–positive Toeplitz matrices. However, it turns out that in this case the Markov operators $P_\infty(g(x))$ correspond to *deterministic* dynamics on $\Omega$, cf. [Bor].

2. $\xi_N = q^{1-N}$, $0 < q < 1$. This case was studied in [G2]. As we show below, $P_\infty(g(x))$ leads to a non-trivial stochastic dynamics. (Note that the case $q > 1$ is essentially the same as $0 < q < 1$ up to a certain simple transformation of spaces $\mathbb{GT}_N$.)

From now on we restrict ourselves to the case $\xi_N = q^{1-N}$, $N \geq 1$. The following theorem was proven in [G2].

**Theorem 4.4.** *For $\xi_N = q^{1-N}$, $0 < q < 1$, the boundary $\mathcal{Q}(\xi_1, \xi_2, \dots)$ is homeomorphic to the set $\mathcal{N}$ of infinite increasing sequences of integers*

$$\mathcal{N} = \{\nu_1 \leq \nu_2 \leq \dots,\, \nu_i \in \mathbb{Z}\}$$

*with the topology of coordinate–wise convergence.*

*Denote by $E_q$ the bijective map from (signed) measures on $\mathcal{N}$ to $\varprojlim \mathcal{M}(\mathbb{GT}_N)$, and let $\mathcal{E}^\nu = E_q(\delta^\nu)$. Then the coherent system $\mathcal{E}^\nu_N$, $N = 1, 2, \dots,$ has the following property: If we view $\lambda_N$, $\lambda_{N-1}$, $\dots$ as random variables on the probability space $(\mathbb{GT}_N, \mathcal{E}^\nu_N)$, then for every $k \geq 1$, $\lambda_{N-k+1}$ converges (in probability) to $\nu_k$ as $N \to \infty$.*

Similarly to $\mathbb{GT}_N$, we identify elements of $\mathcal{N}$ with point configurations in $\mathbb{Z}$:

$$\nu_1 \leq \nu_2 \leq \dots \longleftrightarrow \{\nu_i + i - 1,\, i = 1, 2, \dots\}.$$

Note that in this way we get all semiinfinite point configurations in $\mathbb{Z}$; we denote the set of such configurations by $\mathbb{M}_\infty$.

The space $\mathcal{N}$ is *not* locally compact. This introduces certain technical difficulties in studying continuous time Markov chains on this space. To avoid these difficulties we seek a natural local compactification of $\mathcal{N}$.

Let

$$\overline{\mathcal{N}} = \bigsqcup_{N=0}^{\infty} \mathbb{GT}_N \ \sqcup \mathcal{N}.$$

We identify elements of $\mathbb{GT}_N$ with infinite sequences $\nu_1 \le \nu_2 \dots$ such that $\nu_1, \dots, \nu_N$ are integers and $\nu_{N+1} = \nu_{N+2} = \dots = +\infty$. Thus,

$$\overline{\mathcal{N}} = \{\nu_1 \le \nu_2 \le \dots,\ \nu_i \in \mathbb{Z} \cup \{+\infty\}\}.$$

In the same way we set

$$\overline{\mathbb{M}_\infty} = \mathbb{M}_\infty \cup \mathbb{M}_0 \cup \mathbb{M}_1 \cup \dots$$

Clearly, $\overline{\mathbb{M}_\infty}$ is a set of all point configurations in $\mathbb{Z}$ which have finitely many points to the left from zero. There is a natural bijection between $\overline{\mathcal{N}}$ and $\overline{\mathbb{M}_\infty}$.

We equip $\overline{\mathcal{N}}$ with the following topology. The base consists of the neighborhoods

$$A_{\eta,k} = \{\nu \in \overline{\mathcal{N}} : \nu_1 = \eta_1, \dots, \nu_k = \eta_k\}, \quad \eta_i \in \mathbb{Z},$$

and

$$B_{\eta,k,\ell} = \{\nu \in \overline{\mathcal{N}} : \nu_1 = \eta_1, \dots, \nu_k = \eta_k, \nu_{k+1} \ge \ell\}, \quad \eta_i \in \mathbb{Z}, \quad l \in \mathbb{Z}.$$

The following proposition is straightforward.

**Proposition 4.5.** *Topological space $\overline{\mathcal{N}}$ is locally compact, the natural inclusion $\mathcal{N} \hookrightarrow \overline{\mathcal{N}}$ is continuous, and its image is dense in $\overline{\mathcal{N}}$.*

Now we are ready to define a kernel $\overline{P_\infty}(u, A; g(x))$ on $\overline{\mathcal{N}}$: If $u \in \mathbb{GT}_N \subset \overline{\mathcal{N}}$, then $\overline{P_\infty}(u, A; g(x))$ is a discrete probability measure concentrated on $\mathbb{GT}_N$ with weight of a signature $\lambda \in \mathbb{GT}_N$ given by $P_N(u \to \lambda; g(x))$; if $u \in \mathcal{N} \subset \overline{\mathcal{N}}$, then measure $\overline{P_\infty}(u, A; g(x))$ is concentrated on $\mathcal{N}$ and coincides with $P_\infty(u, A; g(x))$ on it.

**Proposition 4.6.** *$\overline{P_\infty}(u, A; g(x))$ is a Markov transition kernel on $\overline{\mathcal{N}}$.*

*Proof.* For every $u$, $\overline{P_\infty}(u, \cdot; g(x))$ is a probability measure by the definition. The measurability of $\overline{P_\infty}(\cdot, A; g(x))$ follows from the measurability of $P_\infty(\cdot, A; g(x))$ and $P_N(\cdot \to A; g(x))$. □

# 5 Description of the limiting processes

In this section we study finite–dimensional distributions of Markov processes that correspond to the Markov kernels $P_\infty(u, A; g(x))$.

## 5.1 A general convergence theorem

Let $g_k(x)$, $k = 0, 1, 2, \dots$, be a sequence of admissible functions, and let $\mathcal{Z}_0$ be an arbitrary probability distribution on $\mathcal{N}$.

Denote by $\mathcal{Z}_N(t)$, $t = 0, 1, 2, \dots$, a discrete time Markov chain on $\mathbb{GT}_N$ with initial distribution $\mathcal{Z}_N(0) \stackrel{D}{=} E_q(\mathcal{Z}_0)_N$ and transition probabilities

$$\mathrm{Prob}\{\mathcal{Z}_N(t+1) \in A \mid \mathcal{Z}_N(t)\} = P_N(\mathcal{Z}_N(t), A; g_t(x)).$$

Also let $\mathcal{Z}(t)$, $t = 0, 1, 2, \dots$, be a discrete time Markov process on $\mathcal{N}$ with initial distribution $\mathcal{Z}(0) \stackrel{D}{=} Z_0$ and transition measures

$$\mathrm{Prob}\{\mathcal{Z}(t+1) \in A \mid \mathcal{Z}(t)\} = P_\infty(\mathcal{Z}(t), A; g_t(x)).$$

Note that the processes $\mathcal{Z}(t)$ and $\mathcal{Z}_N(t)$ will always depend on $q$ and on the $\{g_k\}$, although we omit these dependencies in the notations.

We want to prove that finite-dimensional distributions of processes $\mathcal{Z}(t)$ are limits of the distributions of processes $\mathcal{Z}_N(t)$.

More formally, introduce embeddings:

$$\iota_N : \mathbb{GT}_N \hookrightarrow \mathcal{N}, \quad (\lambda_1 \geq \dots \geq \lambda_N) \mapsto (\lambda_N \leq \dots \leq \lambda_1 \leq \lambda_1 \leq \lambda_1 \dots).$$

We also use the same notations for the induced maps $\mathcal{M}(\mathbb{GT}_N) \to \mathcal{M}(\mathcal{N})$. Note that these maps are isometric in total variation norm.

Cylindrical subsets of $\mathcal{N}$ have the form

$$U = \{(\nu_1 \leq \nu_2 \leq \dots) \in \mathcal{N} \mid \nu_1 \in H_1, \dots, \nu_k \in H_k\}$$

for aribitrary subsets $H_1, \dots, H_k$ of $\mathbb{Z}$.

The following statement is the main result of this section.

**Theorem 5.1.** *For every $k \geq 1$, the joint distribution of random variables $\big(\iota_N(\mathcal{Z}_N(0)), \dots, \iota_N(\mathcal{Z}_N(k))\big)$ weakly converges to the joint distribution of $(\mathcal{Z}(0), \dots, \mathcal{Z}(k))$ as $N \to \infty$.*

*Equivalently, if $A_0, \dots, A_k$ are arbitrary cylindrical subsets of $\mathcal{N}$, then*

$$\lim_{N\to\infty} \mathrm{Prob}\{\iota_N(\mathcal{Z}_N(0)) \in A_0, \dots, \iota_N(\mathcal{Z}_N(k)) \in A_k\} \\ = \mathrm{Prob}\{\mathcal{Z}(0) \in A_0, \dots, \mathcal{Z}(k) \in A_k\}.$$

We start the proof with two lemmas.

**Lemma 5.2.** *Let $\mu$ be a finite measure on $\mathcal{N}$, and let $A$ be any cylindrical subset of $\mathcal{N}$. We have*

$$\mu(A) = \lim_{N\to\infty} \iota_N(E_q(\mu)_N)(A).$$

*Proof.* It suffices to prove the lemma for cylindrical sets of the form

$$A = A_{b(1),\dots,b(\ell)} = \{\nu_1 \leq \nu_2 \leq \dots \mid \nu_1 = b(1), \dots, \nu_\ell = b(\ell);\ b(j) \in \mathbb{Z}\}.$$

First, suppose that $\mu = \delta^\nu$ for a certain $\nu \in \mathcal{N}$, then $\mu(A) = 1$ if $\nu_1 = b(1), \dots, \nu_\ell = b(\ell)$, and $\mu(A) = 0$ otherwise. The statement of Lemma 5.2 follows from Theorem 4.4.

For an arbitrary measure we have

$$\mu(A) = \int_{\mathcal{N}} \delta^\nu(A)\mu(d\nu) = \int_{\mathcal{N}} \lim_{N\to\infty} \iota_N(E_q(\delta^\nu)_N)(A)\mu(d\nu)$$
$$\stackrel{(*)}{=} \lim_{N\to\infty} \int_{\mathcal{N}} \iota_N(E_q(\delta^\nu)_N)(A)\mu(d\nu) = \lim_{N\to\infty} \iota_N(E_q(\mu)_N)(A),$$

where the equality $(*)$ follows from the dominated convergence theorem. □

Let us denote by $I_A$ the indicator function of set $A$:

$$I_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise}. \end{cases}$$

If $\mu$ is a measure, then $I_A\mu$ stands for the measure given by

$$(I_A\mu)(B) = \mu(A \cap B).$$

**Lemma 5.3.** *Let $\mu$ be a probability measure on $\mathcal{N}$ and let $A$ be any cylindrical set. Then the total variation distance between measures $\iota_N(E_q(\mu I_A)_N)$ and $I_A\iota_N(E_q(\mu)_N)$ tends to zero as $N \to \infty$.*

*Proof.* First, suppose that $\mu = \delta^\nu$ for a certain sequence $\nu \in \mathcal{N}$. If $\nu \in A$, then $E_q(I_A\mu)_N = E_q(\mu)_N$, consequently,

$$\big\|\iota_N(E_q(I_A\mu)_N) - I_A\iota_N(E_q(\mu)_N)\big\| = \big\|(1-I_A)\iota_N(E_q(\mu)_N)\big\| = \iota_N(E_q(\mu)_N)(\bar{A}),$$

where $\bar{A} = \mathcal{N} \setminus A$. The right-hand side tends to zero by Lemma 5.2.

If $\nu$ does not belong to $A$, then $(I_A\mu)_N = 0$ and

$$\big\|\iota_N(E_q(I_A\mu)_N) - I_A\iota_N(E_q(\mu)_N)\big\| = \big\|I_A\iota_N(E_q(\mu)_N)\big\| = \iota_N(E_q(\mu)_N)(A) \to 0$$

by Lemma 5.2.

To prove the claim for a general measure $\mu$ we observe the following property of the total variation norm. Suppose that $\mathcal{W}$ and $\mathcal{V}$ are measurable spaces, and $f_N$ is a sequence of measurable maps from $\mathcal{W}$ to $\mathcal{M}(\mathcal{V})$, such that $\|f_N(w)\| \leq 1$ and $\|f_N(w)\| \to 0$ for any $w \in \mathcal{W}$. Then for any probability measure $\pi$ on $\mathcal{W}$, we have

$$\left\| \int_{\mathcal{W}} f_N(w)\pi(dw) \right\| \to 0.$$

We obtain

$$\big\|\iota_N(E_q(I_A\mu)_N) - I_A\iota_N(E_q(\mu)_N)\big\|$$
$$= \left\| \int_{\mathcal{N}} \Big( \iota_N(E_q(I_A\delta^\nu)_N) - I_A\iota_N(E_q(\delta^\nu)_N) \Big) \mu(d\nu) \right\| \to 0.$$

□

*Proof of Theorem 5.1.* Let us write $P^g_\infty$ for the linear operator on $\mathcal{M}(\mathcal{N})$ corresponding to the kernel $P_\infty(u, A; g(x))$:

$$(P^g_\infty\pi)(A) = \int_{\mathcal{N}} P_\infty(u, A; g(x))\pi(du).$$

Let $P_N^g$ be the operator acting on measures of the form $\iota_N(\mathcal{M}(\mathbb{GT}_N))$ via the kernel $P_N(g(x))$:

$$(P_N^g \pi)(\iota_N(\eta)) = \sum_{\lambda \in \mathbb{GT}_N} P_N(\lambda \to \eta; g(x))\pi(\lambda), \qquad \eta \in \mathbb{GT}_N.$$

Note that these operators are contractions in total variation norm.

We have

$$\mathrm{Prob}\{\mathcal{Z}(0) \in A_0, \dots, \mathcal{Z}(k) \in A_k\} = (I_{A_k} P_\infty^{g_{k-1}} \dots (I_{A_1} P_\infty^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)(\mathcal{N})$$

and

$$\begin{aligned} &\mathrm{Prob}\{\iota_N(\mathcal{Z}_N(0)) \in A_0, \dots, \iota_N(\mathcal{Z}_N(k)) \in A_k\} \\ &\qquad = (I_{A_k} P_N^{g_{k-1}} \dots (I_{A_1} P_N^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)(\mathcal{N}). \end{aligned}$$

Applying Lemmas 5.2, 5.3 and definitions we obtain

$$\begin{aligned} &(I_{A_k} P_\infty^{g_{k-1}} \dots (I_{A_1} P_\infty^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)(\mathcal{N}) \\ &= \lim_{N\to\infty} \iota_N(E_q(I_{A_k} P_\infty^{g_{k-1}} \dots (I_{A_1} P_\infty^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)_N)(\mathcal{N}) \\ &= \lim_{N\to\infty} (I_{A_k} \iota_N(E_q(P_\infty^{g_{k-1}} \dots (I_{A_1} P_\infty^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)_N))(\mathcal{N}) \\ &= \lim_{N\to\infty} (I_{A_k} P_N^{g_{k-1}} I_{A_{k-1}} \iota_N(E_q(P_\infty^{g_{k-2}}(\dots (I_{A_1} P_\infty^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)_N))(\mathcal{N}) = \dots \\ &= \lim_{N\to\infty} (I_{A_k} P_N^{g_{k-1}} \dots (I_{A_1} P_N^{g_1}(I_{A_0}\mathcal{Z}_0)) \dots)(\mathcal{N}) \end{aligned}$$

as desired. □

## 5.2 Correlation functions

Let $Q(t)$ be a stochastic process taking values in subsets of $\mathbb{Z}$ (= point configurations in $\mathbb{Z}$). For $n \geq 1$, the $n$th correlation function $\rho_n$ of $Q(t)$ is the following function of $n$ distinct pairs $(x_i, t_i)$:

$$\rho_n(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = \mathrm{Prob}\{x_1 \in Q(t_1), \dots, x_n \in Q(t_n)\}.$$

Recall that we identify $\mathbb{GT}_N$ with $N$–point configurations in $\mathbb{Z}$, and $\mathcal{N}$ is identified with infinite subsets of $\mathbb{Z}$ that do not have $-\infty$ as their limit point. The following statement is a corollary of Theorem 5.1.

**Theorem 5.4.** *The correlation functions of processes $\mathcal{Z}_N(t)$ pointwise converge as $N \to \infty$ to the correlation functions of processes $\mathcal{Z}(t)$.*

*Proof.* Let us proof this statement for the first correlation function, for all other correlation functions the proof is analogous. Let $\rho_1^N(x, t)$ denote the first correlation function of the point configuration corresponding to $\mathcal{Z}_N(t)$ and let $\rho_1(x, t)$ denote the first correlation function of the point configuration corresponding to $\mathcal{Z}(t)$.

Choose $\varepsilon > 0$. Let $m$ be a number such that $\mathrm{Prob}\{\mathcal{Z}(t)_1 > m\} > 1 - \varepsilon$. Since the distribution of $\iota_N(\mathcal{Z}_N(t))_1$ converges to that of $\mathcal{Z}(t)_1$, for sufficiently large $N$ we have $\mathrm{Prob}\{(\iota_N \mathcal{Z}_N(t))_1 > m\} > 1 - 2\varepsilon$. Now let $k > x - m$. Note that $(\iota_N \mathcal{Z}_N(t))_1 > m$ implies $(\iota_N \mathcal{Z}_N(t))_{k+1} + (k+1) - 1 > x$. Thus,

$$\left|\mathrm{Prob}\Big\{(\iota_N \mathcal{Z}_N(t))_1 = x \text{ or } \dots \text{ or } (\iota_N \mathcal{Z}_N(t))_k + k - 1 = x\Big\} - \rho_1^N(x,t)\right| < 2\varepsilon$$

and similarly for $\mathcal{Z}(t)$.

It follows from Theorem 5.1 that for sufficiently large $N$

$$\left|\mathrm{Prob}\Big\{(\iota_N \mathcal{Z}_N(t))_1 = x \text{ or } \dots \text{ or } (\iota_N \mathcal{Z}_N(t))_k + k - 1 = x\Big\}\right.$$
$$\left. - \mathrm{Prob}\Big\{\mathcal{Z}(t)_1 = x \text{ or } \dots \text{ or } \mathcal{Z}(t)_k + k - 1 = x\Big\}\right| < \varepsilon.$$

Therefore, for sufficiently large $N$ we have

$$\begin{aligned}
&\left|\rho_1^N(x,t) - \rho_1(x,t)\right| \\
&\quad \le \left|\rho_1^N(x,t) - \mathrm{Prob}\Big\{(\iota_N \mathcal{Z}_N(t))_1 = x \text{ or } \dots \text{ or } (\iota_N \mathcal{Z}_N(t))_k + k - 1\Big\}\right| \\
&\qquad + \left|\mathrm{Prob}\Big\{(\iota_N \mathcal{Z}_N(t))_1 = x \text{ or } \dots \text{ or } (\iota_N \mathcal{Z}_N(t))_k + k - 1 = x\Big\}\right. \\
&\qquad\qquad \left. - \mathrm{Prob}\Big\{\mathcal{Z}(t)_1 = x \text{ or } \dots \text{ or } \mathcal{Z}(t)_k + k - 1 = x\Big\}\right| \\
&\qquad + \left|\mathrm{Prob}\Big\{\mathcal{Z}(t)_1 = x \text{ or } \dots \text{ or } \mathcal{Z}(t)_k + k - 1 = x\Big\} - \rho_1(x,t)\right| \\
&\qquad\qquad < 2\varepsilon + \varepsilon + \varepsilon = 4\varepsilon.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, the proof is complete. □

Now we are in position to actually compute the correlation functions of $\mathcal{Z}(t)$. From now on we confine ourselves to the case $\mathcal{Z}_0 = \delta^{\mathbf{0}}$. In other words, $\mathcal{Z}(0) = \mathbf{0} = (0 \le 0 \le 0 \le \dots)$. As shown in [G2], this implies that for every $N \ge 1$, $\mathcal{Z}_N(0) = \mathbf{0}_N = (0 \le 0 \le \dots \le 0)$.

**Proposition 5.5.** *For any $n, N \ge 1$, the nth correlation function $\rho_n^N$ of the process $\mathcal{Z}_N(t)$ started from $\mathcal{Z}_N(0) = \mathbf{0}_N$ admits the following determinantal formula:*

$$\rho_n^N(x_1,t_1;x_2,t_2;\dots;x_n,t_n) = \det_{i,j=1,\dots,n}[K^N(x_i,t_i;x_j,t_j)],$$

*where*

$$\begin{aligned}
K^N(x_1,t_1;x_2,t_2) &= -\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{dw}{w^{x_1-x_2+1}} \prod_{t=t_2}^{t_1-1} g_t(w) \mathbf{1}_{t_1>t_2} \\
&\quad + \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}} dw \oint_{\mathcal{C}'_N} dz \frac{\prod_{t=0}^{t_1-1} g_t(w)}{\prod_{t=0}^{t_2-1} g_t(z)} \prod_{\ell=0}^{N-1} \frac{(1-q^\ell w)}{(1-q^\ell z)} \frac{z^{x_2}}{w^{x_1+1}} \frac{1}{w-z},
\end{aligned} \tag{5.1}$$

*the contours $\mathcal{C}$ and $\mathcal{C}'_N$ are closed and positively oriented; $\mathcal{C}$ includes only the pole 0 and $\mathcal{C}'_N$ includes only the poles $q^{-i}$, $i = 0, \dots, N-1$ of the integrand.*

*Proof.* See Theorem 2.25, Corollary 2.26, Remark 2.27 in [BF2], see also Proposition 3.4 in [BF1]. To match the notations, one needs to set $\alpha_\ell$ of [BF2] to be $q^{1-\ell}$, $\ell \geq 1$, set the symbol of the Toeplitz matrix $F_t(z)$ of [BF2] to $g_t(z^{-1})$, change the integration variables via $\zeta \to \zeta^{-1}$, and shift the particles of [BF2] to the right by $N$. $\square$

In what follows we use the standard notation

$$(w;q)_\infty = \prod_{i=0}^{\infty}(1 - wq^i).$$

**Theorem 5.6.** *For any $n \geq 1$, the nth correlation function $\rho_n$ of process $\mathcal{Z}(t)$ started from $\mathcal{Z}(0) = \mathbf{0}$ has the form*

$$\rho_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = \det_{i,j=1,\ldots,n}[K(x_i, t_i; x_j, t_j)],$$

*where*

$$K(x_1, t_1; x_2 t_2) = -\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{dw}{w^{x_1 - x_2 + 1}} \prod_{t=t_2}^{t_1 - 1} g_t(w) \mathbf{1}_{t_1 > t_2} + \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}} dw \oint_{\mathcal{C}'} dz \frac{\prod_{t=0}^{t_1-1} g_t(w)}{\prod_{t=0}^{t_2-1} g_t(z)} \frac{(w;q)_\infty}{(z;q)_\infty} \frac{z^{x_2}}{w^{x_1+1}} \frac{1}{w-z},$$

*$\mathcal{C}$ is positively oriented and includes only the pole $0$ of the integrand; $\mathcal{C}'$ goes from $+i\infty$ to $-i\infty$ between $\mathcal{C}$ and point $1$.*

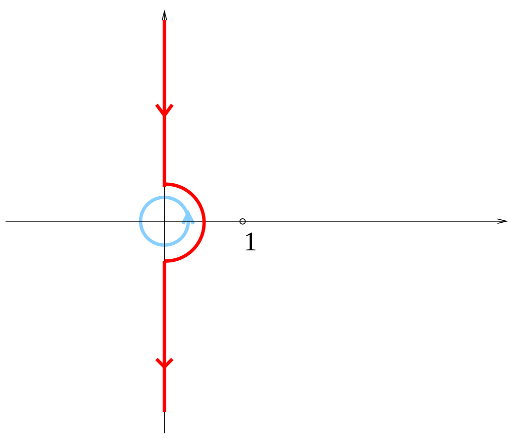


Figure 1: Contours of integration for Theorem 5.6: $\mathcal{C}$ in blue and $\mathcal{C}'$ in red.

*Proof.* It is sufficient to prove that for every $(x_1, t_1; x_2, t_2)$, $K^N(x_1, t_1; x_2, t_2) \to K(x_1, t_1; x_2, t_2)$ as $N \to \infty$.

Denote by $f_N(z, w)$ the integrand in the second (double) integral in (5.1). Note that if $N$ is large enough, then for every $w$, $|z^2 f_N(z, w)|$ goes to zero as

$|z| \to \infty$. Consequently, the replacement of the contour of integration $\mathcal{C}'_N$ by $\mathcal{C}'$ does not change the integral.

Observe that the integral

$$\oint_{\mathcal{C}'} dz f_N(z,w)$$

converges uniformly in $w \in \mathcal{C}$ because of the rapid decay of $f_N(z,w)$ when $z \to \pm i\infty$. Moreover, the functions $f_N(z,w)$ uniformly converge on $\mathcal{C} \times \mathcal{C}'$ as $N \to \infty$. Therefore,

$$\oint_{\mathcal{C}} dw \oint_{\mathcal{C}'} dz f_N(z,w) \to \oint_{\mathcal{C}} dw \oint_{\mathcal{C}'} \frac{\prod_{t=0}^{t_1-1} g_t(w)}{\prod_{t=0}^{t_2-1} g_t(z)} \frac{\phi(w)}{\phi(z)} \frac{z^{x_2}}{w^{x_1+1}} \frac{1}{w-z}.$$

□

## 5.3 First coordinate of the process

In this section we give an independent interpretation for the evolution of the first coordinate of $\mathcal{Z}(t)$. Similar interpretations also exist for the evolutions of first $k$ coordinates for every $k$. Theorems of this section are based on the results of [BF2] and we are not giving their proofs.

Although all constructions make sense for general processes introduced in the previous sections, for simplicity of the exposition we restrict ourselves to homogeneous Markov processes started from the delta measure at $\mathbf{0}$.

Denote by $\mathcal{X}_g(t)$ a discrete time homogenous Markov process on $\mathcal{N}$ with transition probabilities $P_\infty(u, A; g(x))$ started from the delta measure at $\mathbf{0}$.

Also let $\mathcal{Y}_{\gamma_+,\gamma_-}(t)$ be continuous time homogenous Markov process on $\mathcal{N}$ with transition probabilities $P_\infty(u, A; \exp(t(\gamma_+ x + \gamma_-/x)))$ started from the delta measure at $\mathbf{0}$.

Denote by $\mathcal{X}_g(t)_1$ and $\mathcal{Y}_{\gamma_+,\gamma_-}(t)_1$ projections of these processes to the first coordinate (i.e. we consider the position of the leftmost particle). Note that $\mathcal{X}_g(t)_1$ and $\mathcal{Y}_{\gamma_+,\gamma_-}(t)_1$ do not have to be Markov processes.

Similarly, denote by $\mathcal{X}_{N,g}(t)_N$ and $\mathcal{Y}_{N,\gamma_+,\gamma_-}(t)_N$ the projections of processes $\mathcal{X}_{N,g}(t)$ and $\mathcal{Y}_{N,\gamma_+,\gamma_-}(t)$ to the $N$th coordinate (which again corresponds to the position of the leftmost particle).

As in [BF2], we introduce a state space $\mathcal{S}$ of interlacing variables

$$\mathcal{S} = \left\{ \{x_k^m\}_{\substack{k=1,\dots,m \\ m=1,2,\dots}} \subset \mathbb{Z}^\infty \mid x_{k-1}^m < x_{k-1}^{m-1} \le x_k^m \right\}.$$

We interpret $x_1^m < x_2^m < \cdots < x_m^m$ as positions of $m$ particles at horizontal line $y = m$. An example is shown in Figure 2.

Introduce projection maps

$$\pi_N : \mathcal{S} \to \mathbb{M}_N \cong \mathbb{GT}_N,$$

$$\{x_k^m\}_{m \ge 1,\, 1 \le k \le m} \mapsto x_1^N + N - 1 < x_2^N + N - 1 < \cdots < x_N^N + N - 1.$$

Let $\mathcal{H}$ be a set of decreasing sequences of integers:

$$\mathcal{H} = \{y_1 > y_2 > y_3 > \cdots \mid y_i \in \mathbb{Z}\}.$$

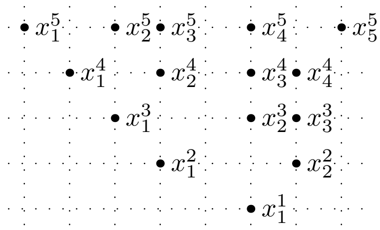


Figure 2: Interlacing particles

Denote by $\Pi$ the projection from $\mathcal{S}$ to $\mathcal{H}$:

$$\Pi(\{x_k^m\}) = x_1^1 > x_1^2 > x_1^3 > \dots .$$

Finally, let $\Pi^\infty$ denote the map from $\mathcal{S}$ to $\mathbb{Z} \cup \{-\infty\}$ given by

$$\Pi^\infty(\{x_k^m\}) = \lim_{N\to\infty} x_1^N + N - 1 = \lim_{N\to\infty} \pi_N(\{x_k^m\})[1],$$

where by $(\,\cdot\,)[1]$ we mean a coordinate of the leftmost particle (it corresponds to $N$th coordinate in $\mathbb{GT}_N$).

An algebraic formalism which leads to a family of Markov processes on $\mathcal{S}$ was introduced in [BF2]. Among the processes in [BF2] there were processes $\mathcal{X}_g^{\mathcal{S}}(t)$ and $\mathcal{Y}_{\gamma_+,\gamma_-}^{S}(t)$ such that the projections $\pi_N(\mathcal{X}_g^{\mathcal{S}}(t))$ and $\pi_N(\mathcal{Y}_{\gamma_+,\gamma_-}^{S}(t))$ coincide with $\mathcal{X}_{N,g}(t)$ and $\mathcal{Y}_{N,\gamma_+,\gamma_-}(t)$, respectively. Moreover, the projections $\Pi(\mathcal{X}_g^{\mathcal{S}}(t))$ and $\Pi(\mathcal{Y}_{\gamma_+,\gamma_-}^{S}(t))$ are also Markov chains that we explicitly describe below.

The following theorem explains the connection to our processes.

**Theorem 5.7.** *Finite dimensional distributions of the stochastic processes* $\Pi^\infty(\mathcal{X}_g^{\mathcal{S}}(t))$ *and* $\Pi^\infty(\mathcal{Y}_{\gamma_+,\gamma_-}^{\mathcal{S}}(t))$ *coincide with those of* $\mathcal{X}_g(t)_1$ *and* $\mathcal{Y}_{\gamma_+,\gamma_-}(t)_1$.

In other words, Theorem 5.7 states that the first coordinate in the stochastic process $\mathcal{X}_g(t)$ (or $\mathcal{Y}_{\gamma_+,\gamma_-}(t)$) evolves as the limiting value of coordinates of the Markov process $\Pi(\mathcal{X}_g^{\mathcal{S}}(t))$ (or $\Pi(\mathcal{Y}_{\gamma_+,\gamma_-}(t))$) .

Note that, in particular, Theorem 5.7 guarantees that if $\{x_m^k\}$ is distributed as $\mathcal{X}_{N,g}(t)_N$ or $\mathcal{Y}_{N,\gamma_+,\gamma_-}(t)_N$, then the limit $\lim_{N\to\infty}(x_1^N - N + 1)$ is almost surely finite.

*Proof of Theorem 5.7.* The proofs for $\mathcal{X}_g(t)_1$ and $\mathcal{Y}_{\gamma_+,\gamma_-}(t)_1$ are the same, and we will work with $\mathcal{X}_g(t)_1$.

Theorem 5.1 implies that finite dimensional distributions of the processes $\mathcal{X}_{N,g}(t)_N$ converge as $N \to \infty$ to finite dimensional distributions of $\mathcal{X}_g(t)_1$.

Therefore,

$$\begin{aligned}
&\mathrm{Prob}\{\Pi^\infty(\mathcal{X}_g^{\mathcal{S}}(t_1)) \in A_1, \dots, \Pi^\infty(\mathcal{X}_g^{\mathcal{S}}(t_k)) \in A_k\} \\
&\quad = \mathrm{Prob}\{\lim_{N\to\infty} \pi_N(\{x_k^m(t_1)\})[1] \in A_1, \dots, \pi_N(\{x_k^m(t_k)\})[1] \in A_k\} \\
&\quad = \lim_{N\to\infty} \mathrm{Prob}\{\mathcal{X}_{N,g}(t_1)_N \in A_1, \dots, \mathcal{X}_{N,g}(t_k)_N \in A_k\} \\
&\quad = \mathrm{Prob}\{\mathcal{X}_g(t_1)_1 \in A_1, \dots, \mathcal{X}_g(t_k)_1 \in A_k\}.
\end{aligned}$$

$\square$

It remains to describe the processes $\Pi(\mathcal{X}_g^{\mathcal{S}}(t))$ and $\Pi(\mathcal{Y}_{\gamma_+,\gamma_-}^{S}(t))$. We have $\mathcal{X}_g^{\mathcal{S}}(0) = \mathcal{Y}_{\gamma_+,\gamma_-}^{S}(0) = (0 > -1 > -2 > -3 > \dots)$. We view coordinates of the process as positions of particles in $\mathbb{Z}$. Thus, $\mathcal{X}_g^{\mathcal{S}}(0) = \mathcal{Y}_{\gamma_+,\gamma_-}^{S}(0)$ is the densely packed configuration of particles in $\mathbb{Z}_{\leq 0}$.

Let us start from $\Pi(\mathcal{X}_g^{\mathcal{S}}(t))$. The description depends on $g(x)$, and we discuss only the cases $g(x) = 1 + \beta x$ and $g(x) = 1 + \beta x^{-1}$; for other possibilities we refer to [BF2].

Given a configuration $\{y_i\}$ at time moment $t$, to construct a configuration at moment $t+1$ that we denote by $\{z_i\}$, we perform a sequential update from right to left.

For $g(x) = 1+\beta x$, the first particle jumps to the right by one with probability $p_1 := \beta/(1+\beta)$ and stays put with probability $1 - p_1$. In other words, $z_1 = y_1$ with probability $1 - p_1$ and $z_1 = y_1 + 1$ with probability $p_1$. For particle number $k$ we do the following: If $y_k = z_{k-1} - 1$, then this particle is *blocked* and we set $z_k = y_k$. Otherwise, the particle jumps to the right with probability $p_k := q^{1-k}\beta/(1+q^{1-k}\beta)$ and stays put with probability $1 - p_k$.

For $g(x) = 1+\beta/x$, the first particle jumps to the left by one with probability $p_1 := \beta/(1+\beta)$ and stays put with probability $1 - p_1$. For particle number $k$ we do the following: If $y_k = z_{k-1}$, then this particle is forced to jump to the left and we set $z_k = y_k - 1$ (one might say that particle number $k$ was pushed by particle number $k-1$). Otherwise, the particle jumps to the left by one with probability $p_k = q^{k-1}\beta/(1+q^{k-1}\beta)$ and stays put with probability $1 - p_k$.

Observe that the above two update rules are closely related. Indeed, $1+\beta x = (\beta x)(1 + \beta^{-1}/x)$. From the probabilistic viewpoint, this equality means that one stochastic step for $g(x) = 1 + \beta x$ is equivalent to the composition of the stochastic step with $g(x) = 1 + \beta^{-1}/x$ and deterministic shift $y_i \to y_i + 1$ for all $i \geq 1$.

For continuous time Markov processes $\Pi(\mathcal{Y}_{\gamma_+,\gamma_-}^{S}(t))$ the interpretation is quite similar. Particle number $k$ has two exponential clocks, "right clock" and "left clock" with parameters $q^{1-k}\gamma_+$ and $q^{k-1}\gamma_-$, respectively. These two numbers are *intensities* of the right and left jumps. When the right clock of a particle number $k$ rings, it checks whether the position to its right is occupied (i.e. if $y_k = y_{k-1} - 1$). If the answer is "Yes" then nothing happens, otherwise the particle jumps to the right by one. When the left clock of the particle number $k$ rings, then this particle jumps to the left and pushes the (maybe empty) block of particles sitting next to it.

The processes above with one-sided jumps are versions of the totally asymmetric simple exclusion process (known as TASEP) and long range TASEP, cf. [Spi], [L1] [L2]. The processes with two-sided jumps was defined and studied in [BF1] under the name of PushASEP.

# 6 The Feller property

In this section we show that transition probabilities

$$P_\infty(u; A; \exp(t(\gamma_+ x + \gamma_-/x)))$$

of the Markov process $\mathcal{Y}_{\gamma_+,\gamma_-}(t)$ satisfy the Feller property. The notion of a Feller process makes sense only for the processes in a locally compact space, and this is a reason why we embed $\mathcal{N}$ into the bigger locally compact space $\overline{\mathcal{N}}$ as in Section 4 above.

## 6.1 Extended boundary

The aim of this section is to identify the local compactification $\overline{\mathcal{N}}$ of $\mathcal{N}$ with a boundary of the sequence of sets $\overline{\mathbb{GT}_N}$ and links $\overline{P}_N^{\downarrow}(\lambda \to \mu)$ with the sequence of parameters $\xi_i = q^{1-i}$, $i \geq 1$. These sets and links were introduced in the second part of Section 3.

First, we want to specify the definition of the boundary given at the beginning of Section 4 for the case when the sets $\Gamma_N$ are equipped with some (other than discrete) topology. More precisely, we replace discrete weak topology in Definition 4.1 by *weak* topology. The weak topology on $\varprojlim \mathcal{M}(\Gamma_N)$ is the minimal topology such that for every $n \geq 1$ and every bounded continuous function $f(w)$ on $\Gamma_N$ the map

$$\{M_N\} \mapsto \sum_{w \in \Gamma_N} f(w) M_N(w)$$

is continuous. Note that if $\Gamma_N$ is equipped with discrete topology, then the weak topology and the discrete weak topology on the set of coherent systems $\varprojlim \mathcal{M}_p(\Gamma_N)$ coincide.

Let $M = (M_0, M_1, \dots)$ be an element of $\varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$. We say that $M$ is of class $k$ if for $N > k$ the support of $M_N$ is a subset of $\mathbb{GT}_N^{(k)}$. We say that $M$ is of class $\infty$ if for any $N$ the support of $M_N$ is a subset of $\mathbb{GT}_N^{(N)}$.

**Proposition 6.1.** *For any $M \in \varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$ there exist unique $M^\infty, M^0, M^1, M^2, \dots \in \varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$ such that $M^i$ is of class $i$ and $M = M^\infty + M^0 + M^1 + M^2 + \dots$.*

*If $M$ is a nonnegative then $M^\infty, M^0, M^1, M^2, \dots$ are also nonnegative.*

*Proof.* Decompose $M_N$ into the sum

$$M_N = M_N^{(0)} + M_N^{(1)} + \dots + M_N^{(N)}$$

with

$$\mathrm{supp}(M_N^{(k)}) \subset \mathbb{GT}_N^{(k)}.$$

Clearly, such decomposition is unique and

$$\|M_N\| = \sum_k \|M_N^{(k)}\|. \tag{6.1}$$

Note that

$$\overline{P}_N^{\downarrow} M_N^{(k)} = M_{N-1}^{(k)}$$

for $k \leq N-2$. Set $M_N^k = M_N^{(k)}$ for $N > k$, $M_k^k = \overline{P}_{k+1}^{\downarrow} M_{k+1}^k$, $M_k^{k-1} = \overline{P}_k^{\downarrow} M_k^k$, and so on. One proves that for every $k$, $M^k = (M_n^k)_{n \geq 1}$ is of class $k$.

Furthermore, (6.1) yields that $\sum_{k=0}^r \|M_N^k\| \leq \|M_{r+1}\| \leq \|M\|$. Consequently, the sum $\sum_{k=0}^\infty M^k$ is well defined. Set $M^\infty = M - \sum_{k=0}^\infty M^k$. It follows that $M^\infty$ is of class $\infty$ and, thus,

$$M = M^\infty + \sum_{k=0}^{\infty} M^k.$$

The uniqueness of the decomposition and nonnegativity are immediate. $\square$

Now we fix $k$ and aim to describe the set of all class $k$ elements of $\varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$. Observe that these are just elements of $\varprojlim \mathcal{M}(\Gamma_N)$ with $\Gamma_N = \mathbb{GT}_N^{(\min(k,N))}$ with links given by the restrictions of matrices $\overline{P}_N^{\downarrow}(\lambda \to \mu)$. Thus, we may use Theorem 4.2. Therefore, it remains to identify the set of all extreme coherent systems, i.e. the extreme points of $\varprojlim \mathcal{M}_p(\mathbb{GT}_N^{(\min(k,N))})$.

**Theorem 6.2.** *The extreme points of* $\varprojlim \mathcal{M}_p(\mathbb{GT}_N^{(\min(k,N))})$ *are enumerated by signatures* $\lambda \in \mathbb{GT}_k$. *Let* $\mathcal{E}^\lambda$ *be an element of* $\varprojlim \mathcal{M}_p(\mathbb{GT}_N^{(\min(k,N))})$ *corresponding to* $\lambda$. *Then for* $N \geq k$, *the Schur generating function of the measure* $\mathcal{E}_N^\lambda$ *is*

$$\mathcal{S}(x_1, \ldots, x_k; \mathcal{E}_N^\lambda) = \frac{s_\nu(x_1, \ldots, x_k)}{s_\nu(1, q^{-1}, \ldots, q^{1-k})} \prod_{n=N+1}^{\infty} G_n^k,$$

*and for* $N \in \{1, \ldots, k-1\}$ *we have*

$$\mathcal{S}(x_1, \ldots, x_N; \mathcal{E}_N^\lambda)$$
$$= \frac{s_\nu(x_1, \ldots, x_N, q^{1-N}, \ldots, q^{1-k})}{s_\nu(1, q^{-1}, \ldots, q^{1-k})} \prod_{n=k+1}^{\infty} G_n^k(x_1, \ldots, x_N, q^{1-N}, \ldots, q^{1-k}),$$

*where*

$$G_N^\ell(x_1, \ldots, x_\ell) = \prod_{i=1}^{\ell} \frac{1 - q^{N-1} q^{1-i}}{1 - q^{N-1} x_i}.$$

*For* $k = 0$, *the unique element of* $\varprojlim \mathcal{M}_p(\mathbb{GT}_N^{(\min(k,N))})$ *is* $\mathcal{E}^\varnothing$, *such that for every* $N$, $\mathcal{E}_N^\varnothing$ *is the unique probability measure on the singleton* $\mathbb{GT}_N^{(0)}$.

*Proof.* The case $k = 0$ is obvious. We use the general *ergodic method* to prove this theorem for $k \geq 1$ (see [V2], [OO, Section 6] and also [DF, Theorem 1.1]).

Choose $\lambda \in \mathbb{GT}_N^{(k)}$. There exists a unique system of measures $M_1^\lambda, \ldots, M_N^\lambda$ such that for every $i \leq N$, $M_i^\lambda$ is a measure on $\mathbb{GT}_i^{(\min(k,N))}$, $M_i^\lambda = \overline{P}_{i+1}^{\downarrow} M_{i+1}^\lambda$, and $M_N^\lambda$ is the delta-measure at $\lambda$. We call such a system of measures a *primitive* system of $\lambda$. The ergodic method states that every extreme coherent system $M \in \varprojlim \mathcal{M}_p(\mathbb{GT}_N^{(\min(k,N))})$ is a weak limit of primitive systems. In other words, there exist sequences $N_i \to \infty$ and $\lambda^i \in \mathbb{GT}_{N_i}^{(\min(N_i,k))}$ such that for every $N$ and every $\mu \in \mathbb{GT}_N^{(\min(k,N))}$, we have

$$M_N(\mu) = \lim_{i \to \infty} M_N^{\lambda^i}(\mu).$$

According to the definition of links $\overline{P}_{i+1}^{\downarrow}$, the Schur generating function of the measure $M_N^{\lambda_i}$ has the following form for $N_i \geq N \geq k$:

$$\mathcal{S}(x_1, \ldots, x_k; M_N^{\lambda_i}) = \frac{s_{\lambda^i}(x_1, \ldots, x_k)}{s_{\lambda^i}(1, q^{-1}, \ldots, q^{1-k})} \prod_{n=N+1}^{N_i} G_n^k.$$

Proposition 4.1 of [G2] implies that the weak convergence of $M_N^{\lambda^i}(\mu)$ as $i \to \infty$ is equivalent to the uniform convergence of the Schur generating functions of measures $M_N^{\lambda^i}$ to those of $M_N$. Observe that

$$\prod_{n=N+1}^{N_i} G_n^k \rightrightarrows \prod_{n=N+1}^{\infty} G_n^k$$

as $i \to \infty$. Therefore, functions

$$\frac{s_{\lambda^i}(x_1, \ldots, x_k)}{s_{\lambda^i}(1, q^{-1}, \ldots, q^{1-k})}$$

should also uniformly converge. But this happens if and only if the sequence of signatures $\lambda^i$ stabilize to a certain $\lambda \in \mathbb{GT}_k$. Then measures $M_N^{\lambda^i}$ converge precisely to $\mathcal{E}_N^{\lambda}$.

Thus, coherent systems $\mathcal{E}^{\lambda}$ contain all extreme points of $\varprojlim \mathcal{M}_p(\mathbb{GT}_N^{(\min(k,N))})$. It remains to prove that all of them are indeed extreme. But this follows from linear independence of the measures $\mathcal{E}_N^{\lambda}$ which, in turn, follows from linear independence of Schur polynomials $s_{\lambda}(x_1, \ldots, x_k)$. □

Now we are ready to describe the map $E$ from $\mathcal{M}(\overline{\mathcal{N}})$ to $\varprojlim M(\overline{\mathbb{GT}_N})$, cf. Definition 4.1. Let $\pi$ be a finite signed measure on $\overline{\mathcal{N}}$. There is a unique decomposition

$$\pi = \pi^{\infty} + \pi^0 + \pi^1 + \pi^2 + \ldots$$

such that $\operatorname{supp}(\pi^k) \subset \mathbb{GT}_N \subset \overline{\mathcal{N}}$ and $\operatorname{supp}(\pi^{\infty}) \subset \mathcal{N} \subset \overline{\mathcal{N}}$.

By Theorems 4.2 and 4.4, $\pi^{\infty}$ corresponds to a unique element of $\varprojlim \mathcal{M}(\mathbb{GT}_N)$ which can be viewed as an element $M^{\infty}$ of $\varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$. Similarly, by Theorems 4.2 and 6.2, $\pi^k$ corresponds to a unique element of $\varprojlim \mathcal{M}(\mathbb{GT}_N^{(\min(k,N))})$ which can be viewed as an element $M^k$ of $\varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$. Note that

$$\|M^{\infty}\| + \|M^0\| + \|M^1\| + \cdots = \|\pi^{\infty}\| + \|\pi^0\| + \|\pi^1\| + \cdots < \infty.$$

Therefore, we may define

$$M := M^{\infty} + M^0 + M^1 + \ldots.$$

Set $E(\pi) = M$.

**Theorem 6.3.** *The set $\overline{\mathcal{N}}$ and the map $E$ satisfy the first 3 conditions of Definition 4.1.*

*Proof.*

1. Proposition 6.1 implies that $E$ is surjective. $E$ is injective by the construction.
2. $E$ is a direct sum of norm 1 operators, thus it is a norm 1 operator. Furthermore, $E$ is a bijection, thus $E^{-1}$ is also bounded.
3. Second part of Proposition 6.1 guarantees that $E$ maps $\mathcal{M}_p(\overline{\mathcal{N}})$ bijectively onto $\varprojlim \mathcal{M}_p(\overline{\mathbb{GT}_N})$,

□

As for the fourth condition of Definition 4.1, its proof is nontrivial and we present it as a separate theorem in the next section.

## 6.2 The topology of the extended boundary

In this section we prove the following theorem

**Theorem 6.4.** *The map*

$$\nu \to \mathcal{E}^\nu = E(\delta^\nu)$$

*from* $\overline{\mathcal{N}}$ *to* $\varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$ *(equipped with weak topology) is a homeomorphism on its image.*[4]

We present a proof in a series of lemmas.

Let $a[1..N]$ be an element of $\mathbb{M}_N$, i.e. $a[1..N]$ is the $N$-point subset of $\mathbb{Z}$ consisting of points $a[1] < a[2] < \cdots < a[N]$. Set

$$A_N^a(x_1, \dots, x_N) = Alt(x_1^{a[1]} \cdots x_N^{a[N]}) = \sum_{\sigma \in S(N)} (-1)^\sigma x_1^{a[\sigma(1)]} \cdots x_N^{a[\sigma(N)]}.$$

**Lemma 6.5.** *The following factorization property holds:*

$$\begin{aligned}
&\frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1, \dots, q^{1-N})} \\
&\qquad = \frac{s_{\lambda_{N-k+1},\dots,\lambda_N}(x_1, \dots, x_k) s_{(\lambda_1+k,\dots,\lambda_{N-k}+k)}(x_{k+1}, \dots, x_N)}{\prod_{i=1,\dots,k} \prod_{j=k+1,\dots,N} (x_i - x_j)} \\
&\qquad \times \frac{\prod_{i=1,\dots,k} \prod_{j=k+1,\dots,N} (q^{1-i} - q^{1-j})}{s_{\lambda_{N-k+1},\dots,\lambda_N}(1, \dots, q^{1-k}) s_{(\lambda_1+k,\dots,\lambda_{N-k}+k)}(q^{-k}, \dots, q^{1-N})} + Q, \qquad (6.2)
\end{aligned}$$

*where if we expand* $Q$ *as a sum of monomials*

$$Q = \sum_{m_1,\dots,m_N} c^{m_1,\dots,m_N} x_1^{m_1} \cdots x_N^{m_N},$$

*then*

$$\sum_{m_1,\dots,m_N} |c^{m_1,\dots,m_N}| 1^{m_1} \cdots q^{(1-N)m_N} < R(k, \lambda_{N-k}, \lambda_{N-k+1}),$$

*and for any fixed* $k$ *and bounded* $\lambda_{N-k+1}$, $R \to 0$ *as* $\lambda_{N-k} \to \infty$.

**Remark 1.** The statement should hold for more general sequences of $\xi$'s, but our proof works only for geometric progressions.

*Proof.* We have (the first equality is the definition of the Schur polynomial)

$$\begin{aligned}
(-1)^{N(N-1)/2} s_\lambda(x_1, \dots, x_N) &= \frac{Alt(x_1^{\lambda_N} x_2^{\lambda_{N-1}+1} \cdots x_N^{\lambda_1+N-1})}{\prod_{i<j}(x_i - x_j)} \\
&= \frac{Alt(x_1^{\lambda_N} \cdots x_k^{\lambda_{N-k+1}+k-1}) Alt(x_{k+1}^{\lambda_{N-k}+k} \cdots x_N^{\lambda_1+N-1})}{\prod_{i<j}(x_i - x_j)} + B.
\end{aligned}$$

[4] Note that the image of this map consists of the extreme points of $\varprojlim \mathcal{M}_p(\overline{\mathbb{GT}_N})$.

Let us estimate the remainder $B$. By the definitions

$$B = \sum_{\{j(1),\dots,j(k)\}\neq\{N-k+1,\dots,N\}} \pm \frac{x_1^{\lambda_{j(1)}+N-j(1)} \cdots x_k^{\lambda_{j(k)}+N-j(k)}}{\prod_{i<j\le k}(x_i-x_j)}$$
$$\times \frac{A_{N-k}^{(\lambda+\delta)\setminus\{\lambda_{j(1)}+N-j(1),\dots,\lambda_{j(k)}+N-j(k)\}}(x_{k+1},\dots,x_N)}{\prod_{k<i<j}(x_i-x_j)}$$
$$\times \frac{1}{\prod_{i=1,\dots,k}\prod_{j=k+1,\dots,N}(x_i-x_j)}, \qquad (6.3)$$

where $(\lambda+\delta)\setminus\{\lambda_{j(1)}+N-j(1),\dots,\lambda_{j(k)}+N-j(k)\}$ stands for the $(N-k)$-element subset of $\mathbb{Z}$ which is the set-theoretical difference of $\lambda+\delta=\{\lambda_1+N-1,\lambda_2+N-2+\dots,\lambda_{N-1}+1,\lambda_N\}$ and $\{\lambda_{j(1)}+N-j(1),\dots,\lambda_{j(k)}+N-j(k)\}$.

Denote the three factors in (6.3) by $B_1$, $B_2$ and $B_3$, respectively. For any function $L$ on $T_N$ that has a Laurent expansion, let $Est(L)$ be the following number: Decompose $L$ into the sum of monomials

$$L = \sum_{m_1,\dots,m_N} \ell^{m_1,\dots,m_N} x_1^{m_1}\cdots x_N^{m_N}$$

and set

$$Est(L) = \sum_{m_1,\dots,m_N} |\ell^{m_1,\dots,m_N}| 1^{m_1}\cdots q^{(1-N)m_N}.$$

Clearly, $Est(L_1L_2)\le Est(L_1)Est(L_2)$. Therefore,

$$Est(B)\le Est(B_1)Est(B_2)Est(B_3).$$

In what follows $const_k$ denotes various constants depending solely on $k$. We have

$$Est(B_1)\le const_k \prod_{m=1}^{k} q^{(1-m)(\lambda_{j(m)}+N-j(m))}.$$

Observe that $B_2$ is a Schur polynomial. As follows from the combinatorial formula(see e.g. Section I.5 of [Mac]), these functions are sums of monomials with non-negative coefficients, thus $Est(B_2)=B_2(1,q^{-1},\dots,q^{1-N})$.

Finally, decomposing the denominators in $B_3$ into geometric series and then converting them back, we get

$$Est(B_3)\le \left(\prod_{i=1}^{k}\prod_{j=k+1,\dots,N}(q^{1-j}-q^{1-i})^{-1}\right)\le const_k \prod_{j=k+1}^{N} q^{(j-1)k}.$$

Now set $a[N-j+1]=\lambda_j+N-j$. We have

$$Est(B)\le \sum_{\{j(1),\dots j(k)\}\neq\{1,\dots,k\}} const_k \left(\prod_{j=k+1}^{N} q^{(j-1)k}\prod_{i=1}^{k} q^{(1-i)a[j(i)]}\right)$$
$$\times A_{N-k}^{a[1..N]\setminus\{a[j(1)],\dots,a[j(k)]\}}(q^{-k},\dots,q^{1-N})\frac{1}{\prod_{i<j\le N-k}(q^{-k-i+1}-q^{-k-j+1})}$$

Let $C(j(1),\dots,j(k))$ denote the right-hand side of the above inequality. Then

$$\frac{C(j(1),\dots,j(k))}{C(1,\dots,k)} = \prod_{i=1}^{k} q^{(1-i)(a[j(i)]-a[i])} \frac{A_{N-k}^{a[1..N]\setminus\{a[j(1)],\dots,a[j(k)]\}}(q^{-k},\dots,q^{1-N})}{A_{N-k}^{a[k+1..N]}(q^{-k},\dots,q^{1-N})}.$$

For any increasing sequence $b[1..N-k]$ we have

$$\begin{aligned} A_{N-k}^{b[1..N-k]}(q^{-k},\dots,q^{1-N}) &= q^{(1-N)(b[1]+\dots+b[N-k])} A_{N-k}^{b[1..N-k]}(1,\dots,q^{N-k-1}) \\ &\overset{(*)}{=} q^{(1-N)(b[1]+\dots+b[N-k])} \prod_{i<j\le(N-k)} (q^{b[i]}-q^{b[j]}) \\ &= q^{(1-N)(b[1]+\dots+b[N-k])+(N-k-1)b[1]+(N-k-2)b[2]+\dots+0b[N-k]} \prod_{i<j\le(N-k)} (1-q^{b[j]-b[i]}) \\ &= q^{(-k)b[1]+(-k-1)b[2]+\dots+(1-N)b[N-k]} \prod_{i<j\le(N-k)} (1-q^{b[j]-b[i]}), \end{aligned}$$

where the equality $(*)$ is the Vandermonde determinant evaluation.

To analyze $A_{N-k}^{a[1..N]\setminus\{a[j(1)],\dots,a[j(k)]\}}$ we think of the set

$$b[1..N-k] := a[1..N] \setminus \{a[j(1)],\dots,a[j(k)]\}$$

as of a small modification of the set $a[k+1..N]$. Note that under this modification only finite number of members (up to $k$) of the sequence $b[i]$ change. Using the finiteness of $\prod_{n\ge1}(1-q^n)$, one easily sees that the 'modified' product $\prod_{1\le i<j\le(N-k)}(1-q^{b[j]-b[i]})$ differs from the 'unmodified' one $\prod_{k+1\le i<j\le N}(1-q^{a[j]-a[i]})$ by a constant that is bounded away from 0 and $\infty$ (note that $k$ is fixed, while $N$ can be arbitrarily large).

Hence,

$$\begin{aligned} &\frac{A_{N-k}^{a[1..N]\setminus\{a[j(1)],\dots,a[j(k)]\}}(q^{-k},\dots,q^{1-N})}{A_{N-k}^{a[k+1..N]}(q^{-k},\dots,q^{1-N})} \\ &\quad \le const_k \cdot q^{(-k)b[1]+(-k-1)b[2]+\dots+(1-N)b[N-k]} q^{ka[k+1]+\dots+(N-1)a[N]} \end{aligned} \tag{6.4}$$

The next step is to estimate the exponent of $q$ above by

$$\begin{aligned} &k(a[k+1]-b[1])+\dots+(N-1)(a[N]-b[N-k]) \\ &\qquad \ge k\sum_{m=1}^{N-k}(a[m+k]-b[m]) \ge k\sum_{m:j(m)>k}(a[j(m)]-a[k]). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{C(j(1),\dots,j(k))}{C(1,\dots,k)} &\le const_k \prod_{i=1}^{k} q^{(1-i)(a[j(i)]-a[i])} q^{k\sum_{m:j(m)>k}(a[j(m)]-a[k])} \\ &\le const(k,a[k]) \cdot q^{\sum_{m:j(m)>k} a[j(m)]}. \end{aligned}$$

Here and below we use $const(k, a[k])$ to denote any constant depending only on $k$ and $a[k]$.

Thus, the sum over all $\{j(1), \dots, j(k)\} \neq \{1, \dots, k\}$ can be bounded by geometric series and

$$Est(B) \le C(1, \dots, k) \cdot const(k, a[k])\, q^{a[k+1]}.$$

Hence, substituting the definition of $C(1, \dots, k)$ and using (2.5),

$$\begin{aligned}
Est\left(\frac{B}{s_\lambda(1, \dots, q^{1-N})}\right) &\le const(k, a[k]) q^{a[k+1]} \prod_{j=k+1}^{N} q^{(j-1)k} \prod_{i=1}^{k} q^{(1-i)(\lambda_{N+1-i}+i-1)} \\
&\times q^{(-k)(\lambda_{N-k}+k)+\dots+(1-N)(\lambda_1+N-1)} \prod_{1\le i<j\le N-k} \frac{1-q^{\lambda_i-i-\lambda_j+j}}{q^{-k-i+1}-q^{-k-j+1}} \\
&\times q^{(N-1)(\lambda_1+\dots+\lambda_N)} q^{-0\lambda_1-\dots-(N-1)\lambda_N} \prod_{1\le i<j\le N} \frac{1-q^{j-i}}{1-q^{\lambda_i-i-\lambda_j+j}} \\
= const(k, a[k]) q^{a[k+1]} \prod_{j=k+1}^{N} q^{(j-1)k} \prod_{i=1}^{N} q^{-(i-1)^2} &\prod_{1\le i<j\le (N-k)} \frac{1-q^{j-i}}{q^{-k-i+1}-q^{-k-j+1}} \\
&\times \prod_{i=1}^{N-k} \prod_{j=N-k+1}^{N} \frac{1-q^{j-i}}{1-q^{\lambda_i-i-\lambda_j+j}} \prod_{(N-k+1)\le i<j\le N} \frac{1-q^{j-i}}{1-q^{\lambda_i-i-\lambda_j+j}} \\
&\le const(k, a[k]) q^{a[k+1]} \prod_{j=k+1}^{N} q^{(j-1)k} \prod_{i=1}^{N} q^{-(i-1)^2} \prod_{1\le i<j\le (N-k)} q^{k+j-1} \\
&= const(k, a[k]) q^{a[k+1]} \prod_{j=k+1}^{N} q^{(j-1)k} \prod_{i=1}^{N} q^{-(i-1)^2} \prod_{j=k+1}^{N} q^{(j-1)(j-1-k)} \\
&= const(k, a[k]) q^{a[k+1]}
\end{aligned}$$

The statement of the lemma immediately follows. □

**Lemma 6.6.** *Let $P$ be a signed finite measure on $\mathbb{GT}_N$ with Schur generating function $\mathcal{S}$:*

$$\mathcal{S}(x_1, \dots, x_N; P) = \sum_{\lambda \in \mathbb{GT}_N} P(\lambda) \frac{s_\lambda(x_1, \dots, x_N)}{s_\lambda(1, \dots, q^{1-N})}.$$

*The total variation norm of $P$ can be estimated as*

$$\|P\| \le const_N Est(\mathcal{S}).$$

*Proof.* As before, we use the notation $const_N$ below for various positive constants depending on $N$. We have

$$Est\left(\mathcal{S} \prod_{i<j} (x_i - x_j)\right) = \sum_{\lambda \in \mathbb{GT}_N} \frac{|P(\lambda)|}{s_\lambda(1, \dots, q^{1-N})} Est\left(Alt(x_1^{\lambda_1+N-1} \cdots x_N^{\lambda_N})\right).$$

Note that

$$Est\left(Alt(x_1^{\lambda_1+N-1} \cdots x_N^{\lambda_N})\right) \ge q^{0\lambda_N} q^{(-1)(\lambda_{N-1}+1)} \cdots q^{(1-N)(\lambda_1+N-1)}.$$

Also, using (2.5)

$$s_\lambda(1,\dots,q^{-N}) \le \frac{1}{const_N}\, q^{0\lambda_N} q^{(-1)(\lambda_{N-1}+1)} \cdots q^{(1-N)(\lambda_1+N-1)}$$

Therefore,

$$Est\left(\mathcal{S}\prod_{i<j}(x_i-x_j)\right) \ge const_N \sum_{\lambda\in\mathbb{GT}_N} |P(\lambda)| = const_N \|P\|.$$

It remains to observe that

$$Est\left(\mathcal{S}\prod_{i<j}(x_i-x_j)\right) \le Est(\mathcal{S}) Est\left(\prod_{i<j}(x_i-x_j)\right) \le const_N Est(\mathcal{S}).$$

□

**Lemma 6.7.** *Let $\nu^n$ be a sequence of points of $\mathcal{N}\subset\overline{\mathcal{N}}$ converging to $\nu\in\overline{\mathcal{N}}\backslash\mathcal{N}$. Then $\mathcal{E}^{\nu^n}$ weakly converges to $\mathcal{E}^{\nu}$.*

*Proof.* Since $\nu\in\overline{\mathcal{N}}\setminus\mathcal{N}$, there exists $k\ge 1$ such that $\nu\in\mathbb{GT}_k\subset\overline{\mathcal{N}}$. First, take $m>k$. Recall that the measure $\mathcal{E}^\nu_m$ is supported by $\mathbb{GT}_m$ and $\mathcal{S}(x_1,\dots,x_m;\mathcal{E}^\nu_m)$ is its Schur generating function.

Theorem 1.3 of [G2] yields

$$\mathcal{S}(x_1,\dots,x_m;\mathcal{E}^{\nu^n}_m) = \lim_{N\to\infty} S^{\nu^n}_N(x_1,\dots,x_m),$$

where

$$S^{\nu^n}_N(x_1,\dots,x_m) = \frac{s_{(\nu^n_N,\dots,\nu^n_1)}(x_1,\dots,x_m,q^{-m},\dots,q^{1-N})}{s_{(\nu^n_N,\dots,\nu^n_1)}(1,\dots,q^{1-N})}.$$

Since $\nu^n\to\nu$, we have $\nu^n_{k+1}\to\infty$, while $\nu^n_1,\dots,\nu^n_k$ stabilize. Thus, we may use Lemma 6.5

$$\begin{aligned}
&\frac{s_{(\nu^n_N,\dots,\nu^n_1)}(x_1,\dots,x_m,q^{-m},\dots,q^{1-N})}{s_{(\nu^n_N,\dots,\nu^n_1)}(1,\dots,q^{1-N})}\\
&\quad= \frac{s_{(\nu^n_k,\dots,\nu^n_1)}(x_1,\dots,x_k)}{s_{(\nu^n_k,\dots,\nu^n_1)}(1,\dots,q^{1-k})}\,\frac{\prod_{i=1,\dots,k}\prod_{j=m+1,\dots,N}(q^{1-i}-q^{1-j})}{\prod_{i=1,\dots,k}\prod_{j=m+1,\dots,N}(x_i-q^{1-j})}\\
&\qquad\times\frac{s_{(\nu^n_n+k,\dots,\nu^n_{k+1}+k)}(x_{k+1},\dots,x_m,q^{-m},\dots,q^{1-N})}{s_{(\nu^n_n+k,\dots,\nu^n_{k+1}+k)}(q^{-k},\dots,q^{1-N})}\\
&\qquad\qquad\times\frac{\prod_{i=1,\dots,k}\prod_{j=k+1,\dots,m}(q^{1-i}-q^{1-j})}{\prod_{i=1,\dots,k}\prod_{j=k+1,\dots,m}(x_i-x_j)} + Q.
\end{aligned}\tag{6.5}$$

Note that the measure with Schur generating function $S^{\nu^n}_N(x_1,\dots,x_m)$ is supported on signatures $\lambda$ such that $\lambda_{m-k}\ge\nu^n_{k+1}$. Indeed, this readily follows from formulas (3.1) and (3.2), see also Proposition 5.5 in [G2].

Let us find the projection of the measure with Schur generating function $S^{\nu^n}_N(x_1,\dots,x_m)$ to the last $k$ coordinates of the signature (all other coordinates tend to infinity as $n \to \infty$). We claim that the Schur generating function of this projection uniformly (in $x$'s and in $N$) tends (as $n \to \infty$) to

$$\frac{s_{(\nu_k^n,\dots,\nu_1^n)}(x_1,\dots,x_k)}{s_{(\nu_k^n,\dots,\nu_1^n)}(1,\dots,q^{1-k})} \frac{\prod_{i=1,\dots,k}\prod_{j=m+1,\dots,N}(q^{1-i}-q^{1-j})}{\prod_{i=1,\dots,k}\prod_{j=m+1,\dots,N}(x_i-q^{1-j})}. \tag{6.6}$$

To prove this claim, we first expand (6.6) into a *finite* sum of normalized Schur polynomials

$$\sum_{\lambda\in\Lambda} c_\lambda^N \frac{s_\lambda(x_1,\dots,x_k)}{s_\lambda(1,\dots,q^{1-k})} + Q_1$$

It is clear, that for any $\varepsilon$ we may choose such a finite set $\Lambda$ not depending on $N$, that $Est(Q_1) < \varepsilon$.

We also expand

$$\frac{s_{(\nu_n^n+k,\dots,\nu_{k+1}^n+k)}(x_{k+1},\dots,x_m,q^{-k},\dots,q^{1-N})}{s_{(\nu_n^n+k,\dots,\nu_{k+1}^n+k)}(q^{-k},\dots,q^{1-N})}$$

into the *full* sum of normalized Schur polynomials in variables $x_{k+1},\dots,x_m$ (with normalization in the point $(q^{-k},\dots,q^{1-m})$ as

$$\sum_{\mu\in\mathbb{GT}_{m-k}} u_\mu \frac{s_\mu(x_{k+1},\dots,x_m)}{s_\mu(q^{-k},\dots,q^{1-m})}.$$

Observe that $\sum_\mu u_\mu = 1$. Note that, if $n$ is large enough, then coordinates of all signatures in the support of $u_\mu$ has larger coordinates than those of $\Lambda$. Moreover, as $n \to \infty$ these coordinates tend to infinity.

We substitute these two expansions into (6.5) and use Lemma 6.5 yet again (in the reverse direction, for polynomials in $m$ variables). This gives

$$\sum_{\lambda\in\Lambda} c_\lambda^N \sum_\mu u_\mu \frac{s_{\lambda\cup\mu}(x_1,\dots,x_m)}{s_{\lambda\cup\mu}(1,\dots,q^{1-m})} + Q_2 \tag{6.7}$$

If we forget about $Q_2$, then we arrive at a measure on $\mathbb{GT}_m$ which assigns to signature $\lambda\cup\mu$ the weight $c_\lambda u_\mu$. Clearly, its projection to the lowest $k$ coordinates assigns to signature $\lambda$ the weight $c_\lambda$, as needed.

It remains to work out the impact of $Q_2$. Let us list all the terms contributing to $Q_2$.

1. $Q$ from (6.5). By Lemma 6.5, $Est(Q) \to 0$ as $n \to \infty$.
2. $Q_1$ gives the term

$$Q_1' = Q_1 \frac{s_{(\nu_N^n+k,\dots,\nu_N^{k+1}+k)}(x_{k+1},\dots,x_m,q^{-m},\dots,q^{1-N})}{s_{(\nu_N^n+k,\dots,\nu_N^{k+1}+k)}(q^{-k},\dots,q^{1-N})} \times \frac{\prod_{i=1,\dots,k}\prod_{j=k+1,\dots,m}(q^{1-i}-q^{1-j})}{\prod_{i=1,\dots,k}\prod_{j=k+1,\dots,m}(x_i-x_j)}.$$

We have $Est(Q_1') \le Est(Q_1) \le \varepsilon$.

3. When we use Lemma 6.5 the second time, we get a term $Q^{\lambda,\mu}$ for each pair $(\lambda,\mu)$. Thus, the total impact is

$$\sum_{\lambda,\mu} Q^{\lambda,\mu} u_\mu c_\lambda^N.$$

Since by Lemma 6.5 we have uniform bounds for $Est(Q^{\lambda,\mu})$ (here we use the fact that $\Lambda$ is finite) and $\sum_{\lambda,\mu} |u_\mu c_\lambda^N| \le 1$, thus, the contribution of these terms also tends to zero as $n\to\infty$.

Summing up, we have

$$Est(Q_2) \le \varepsilon + R,$$

where $R \to 0$ as $\nu^n \to \nu$ (uniformly in $N$). Consequently,

$$Est\Big(Q_2 \prod_{1\le i<j\le m} (x_i - x_j)\Big) \le const_m(\varepsilon + R).$$

Now let $H$ be a (signed) measure on $\mathbb{GT}_m$ with Schur–generating function $Q_2$. Using Lemma 6.6 we conclude that total variation norm of $H$ can be bounded by

$$\|H\| \le const_m(\varepsilon + R).$$

Since $\varepsilon$ is arbitrary and $R\to 0$ as $n\to\infty$, the influence of $H$ and, thus, the influence of $Q_2$ in (6.7) is negligible.

We have proved that the Schur generating function of the projection to the last $k$ coordinates of measure on $\mathbb{GT}_m$ with Schur generating function $S_N^{\nu^n}(x_1,\dots,x_m)$, tends as $n\to\infty$ to the function given by (6.6). Now sending $N$ to infinity we see that the Schur generating function of the projection to the last $k$ coordinates of measure $\mathcal{E}_m^{\nu_n}$ uniformly tends to the Schur generating function of $\mathcal{E}_m^{\nu}$. Next, we use Proposition 4.1 of [G2] which yields that weak convergence of measures is equivalent to uniform convergence of their Schur generating functions. We conclude that the projection to the last $k$ coordinates of measure $\mathcal{E}_m^{\nu^n}$ weakly tends to $\mathcal{E}_m^{\nu}$. On the other hand, all other coordinates of the signature distributed according to $\mathcal{E}_m^{\nu^n}$ tend to infinity as $n\to\infty$. It follows that $\mathcal{E}_m^{\nu^n} \to \mathcal{E}_m^{\nu}$.

For $m \le k$, Lemma 6.5 yields

$$S_N^{\nu^n}(x_1,\dots,x_m) = \frac{s_{(\nu_N^n,\dots,\nu_1^n)}(x_1,\dots,x_m,q^{-m},\dots,q^{1-N})}{s_{(\nu_N^n,\dots,\nu_1^n)}(1,\dots,q^{1-N})}$$
$$= \frac{s_{(\nu_k^n,\dots,\nu_1^n)}(x_1,\dots,x_m,q^{-m},\dots,q^{1-k})}{s_{(\nu_k^n,\dots,\nu_1^n)}(1,\dots,q^{1-k})} \frac{\prod_{i=1,\dots,m}\prod_{j=k+1,\dots,N}(q^{1-i}-q^{1-j})}{\prod_{i=1,\dots,m}\prod_{j=k+1,\dots,N}(x_i-q^{1-j})} + Q$$

and

$$\lim_{n\to\infty} \mathcal{S}(x_1,\dots,x_m;\mathcal{E}_m^{\nu^n}) = \lim_{n\to\infty}\lim_{N\to\infty} S_N^{\nu^n}(x_1,\dots,x_m) =$$
$$= \frac{s_{(\nu_k^n,\dots,\nu_1^n)}(x_1,\dots,x_m,q^{-m},\dots,q^{1-k})}{s_{(\nu_k^n,\dots,\nu_1^n)}(1,\dots,q^{1-k})} \frac{\prod_{i=1}^m\prod_{j=k+1}^\infty(q^{1-i}-q^{1-j})}{\prod_{i=1}^m\prod_{j=k+1}^\infty(x_i-q^{1-j})}$$
$$= \mathcal{S}(x_1,\dots,x_m;\mathcal{E}_m^{\nu}).$$

By Proposition 4.1 of [G2] we conclude that $\mathcal{E}_m^{\nu^n} \to \mathcal{E}_m^{\nu}$. $\square$

**Lemma 6.8.** *Let $\nu^n$ be a sequence of points of $\overline{\mathcal{N}} \setminus \mathcal{N}$ converging to $\nu \in \overline{\mathcal{N}} \setminus \mathcal{N}$. Then $\mathcal{E}^{\nu^n}$ weakly converges to $\mathcal{E}^{\nu}$.*

The proof is similar to that of Lemma 6.7 and we omit it.

**Lemma 6.9.** *Let $\nu^n$ be a sequence of points of $\overline{\mathcal{N}} \setminus \mathcal{N}$ converging to $\nu \in \mathcal{N}$. Then $\mathcal{E}^{\nu^n}$ weakly converges to $\mathcal{E}^{\nu}$.*

*Proof.* Fix $m \geq 1$. Since $\nu^n \to \nu \in \mathcal{N}$, for large enough $n$ we have $\nu^n \in \mathbb{GT}_{k_n}$ with $k_n \geq m$. Moreover, $k_n \to \infty$ as $n \to \infty$. Therefore, $\mathcal{E}_m^{\nu^n}$ is supported on $\mathbb{GT}_m$, and its Schur generating function is

$$\mathcal{S}(x_1, \dots, x_m; \mathcal{E}_m^{\nu^n}) = \frac{s_{\nu^n}(x_1, \dots, x_m, q^{-m}, \dots, q^{1-k_n})}{s_{\nu^n}(1, \dots, q^{1-k_n})} \prod_{q=k_n+1}^{\infty} G_q^m.$$

Therefore,

$$\lim_{n \to \infty} \mathcal{S}(x_1, \dots, x_m; \mathcal{E}_m^{\nu^n}) = \lim_{n \to \infty} \frac{s_{\nu^n}(x_1, \dots, x_m, q^{-m}, \dots, q^{1-k_n})}{s_{\nu^n}(1, \dots, q^{1-k_n})}$$
$$= \mathcal{S}(x_1, \dots, x_m; \mathcal{E}_m^{\nu}),$$

where the last equality follows from Theorem 1.3 of [G2]. Using Proposition 4.1 of [G2] we conclude that $\mathcal{E}_m^{\nu^n} \to \mathcal{E}_m^{\nu}$. □

**Lemma 6.10.** *Let $\nu^n$ be a sequence of points of $\mathcal{N} \subset \overline{\mathcal{N}}$ converging to $\nu \in \mathcal{N}$. Then $\mathcal{E}^{\nu^n}$ weakly converges to $\mathcal{E}^{\nu}$.*

*Proof.* See Proposition 5.16 of [G2]. □

**Lemma 6.11.** *Let $\nu^n$ be a sequence of points of $\overline{\mathcal{N}}$ and $\nu \in \overline{\mathcal{N}}$. If $\mathcal{E}^{\nu^n}$ weakly converges to $\mathcal{E}^{\nu}$, then there exists $m \in \mathbb{Z}$ such that $\nu_1^n \geq m$ for every $n \geq 1$.*

*Proof.* We start by proving that that there exists a constant $c > 0$ such that $\mathcal{E}_1^{\nu^n}(\{\nu_1^n\}) > c$ for every $n \geq 1$. More exactly, one can take $c = (q;q)_\infty^2$.

For $\nu \in \mathcal{N}$ this was proved in Lemma 5.15 of [G2]. If $\nu^n \in \mathbb{GT}_k$, then the support of $\mathcal{E}_1^{\nu_n}$ consists of numbers greater or equal then $\nu_1^n$. Therefore, $x^{-\nu_1^n}\mathcal{S}(x; \mathcal{E}_1^{\nu^n})$ is a power series (without negative powers of $x$) and $\mathcal{E}_1^{\nu^n}(\{\nu_1^n\})$ equal the value of this series at $x = 0$.

We have

$$\mathcal{S}(x; \mathcal{E}_1^{\nu^n}) = \frac{s_{\nu^n}(x, q^{-1}, \dots, q^{1-k})}{s_{\nu^n}(1, q^{-1}, \dots, q^{1-k})} \prod_{\ell=k+1}^{\infty} G_\ell^1.$$

Observe that $s_{\mu+q}(x_1, \dots, x_k) = (x_1, \dots, x_k)^q s_\mu(x_1, \dots, x_k)$ and use this equality for $\mu = \nu^n$, $q = \nu_1^n$. We obtain

$$\mathcal{S}(x; \mathcal{E}_1^{\nu^n}) = x^{\nu_1^n} \frac{s_\lambda(x, q^{-1}, \dots, q^{1-k})}{s_\lambda(1, q^{-1}, \dots, q^{1-k})} \prod_{\ell=k+1}^{\infty} G_\ell^1.$$

where $\lambda = \nu^n - \nu_1^n$ is a signature with $\lambda_k = 0$. Therefore

$$\mathcal{E}_1^{\nu^n}(\{\nu_1^n\}) = \frac{s_\lambda(0, q^{-1}, \dots, q^{1-k})}{s_\lambda(1, q^{-1}, \dots, q^{1-k})} \prod_{\ell=k+1}^{\infty} G_\ell^1(0).$$

Using (2.5) we obtain

$$\frac{s_\lambda(q^{-1},\dots,q^{1-k})}{s_\lambda(1,q^{-1},\dots,q^{1-k})} = \frac{q^{-|\lambda|}\prod_{1\le i<j\le k-1}\dfrac{1-q^{-\lambda_i+\lambda_j+i-j}}{1-q^{i-j}}}{\prod_{1\le i<j\le k}\dfrac{1-q^{-\lambda_i+\lambda_j+i-j}}{1-q^{i-j}}} = \frac{q^{-|\lambda|}}{\prod_{i=1}^{k-1}\dfrac{1-q^{-\lambda_i+i-k}}{1-q^{i-k}}}$$
$$= \prod_{i=1}^{k-1}\frac{1-q^{k-i}}{1-q^{\lambda_i-i+k}} \ge \prod_{i=1}^{k-1}(1-q^{k-i}) \ge \prod_{i=1}^{\infty}(1-q^i).$$

Also

$$\prod_{\ell=k+1}^{\infty} G^1_\ell(0) = (1-q^k)(1-q^{k+1})\cdots.$$

Hence,

$$\mathcal{E}_1^{\nu^n}(\{\nu_1^n\}) \ge \prod_{i=1}^{\infty}(1-q^i)\prod_{i=k}^{\infty}(1-q^i).$$

Now let $C_q(u)$ be a function on $\overline{\mathbb{GT}_1} = \mathbb{Z}\cup\{\infty\}$ that vanishes at all points $> q$ and equals 1 at all points $\le q$. Choose $q$ so that

$$\sum_{u\in\overline{\mathbb{GT}_1}} C_q(u)\mathcal{E}_1^{\nu}(u) < c/2.$$

Note that $C_q$ is a continuous function on $\overline{\mathbb{GT}_1}$. Thus,

$$\sum_{u\in\overline{\mathbb{GT}_1}} C_q(u)\mathcal{E}_1^{\nu}(u) = \lim_{n\to\infty}\sum_{u\in\overline{\mathbb{GT}_1}} C_q(u)\mathcal{E}_1^{\nu^n}(u).$$

Hence, for large enough $n$ we have

$$\sum_{u\in\overline{\mathbb{GT}_1}} C_q(u)\mathcal{E}_1^{\nu^n}(u) < c.$$

But if $\nu_1^n < q$ then

$$\sum_{u\in\overline{\mathbb{GT}_1}} C_q(u)\mathcal{E}_1^{\nu^n}(u) \ge \mathcal{E}_1^{\nu^n}(\{\nu_1^n\}) > c.$$

This contradiction proves that $\nu_1^n \ge q$. □

Now we are ready to give a proof of Theorem 6.4.

*Proof of Theorem 6.4.* Let $\{\nu^n\}_{n\ge1}\subset\overline{\mathcal{N}}$ and $\nu\in\overline{\mathcal{N}}$. Our goal is to prove that $\lim_{n\to\infty}\nu_n = \nu$ if and only if $\mathcal{E}^{\nu_n}$ weakly converges to $\mathcal{E}^{\nu}$ as $n\to\infty$.

First, suppose that $\nu_n\to\nu$. Without loss of generality we may assume that either for every $n$ we have $\nu_n\in\mathcal{N}$, or for every $n$, $\nu_n\in\overline{\mathcal{N}}\setminus\mathcal{N}$. Also either $\nu\in\mathcal{N}$ or $\nu\in\overline{\mathcal{N}}\setminus\mathcal{N}$. Thus, we have four cases and they are covered by Lemmas 6.7-6.10.

Now suppose that $\mathcal{E}^{\nu^n}$ weakly converges to $\mathcal{E}^{\nu}$. By Lemma 6.11, there exists $m\in\mathbb{Z}$ such that $\nu_n\in D_m$ for every $n\ge1$, where

$$D_m = \{\nu\in\overline{\mathcal{N}} : \nu_1\ge m\}.$$

Observe that the set $D_m$ is compact. Therefore, the sequence $\nu^n$ has a converging subsequence $\nu^{n_h} \to \nu'$. But then $\mathcal{E}^{\nu^n} \to \mathcal{E}^{\nu'}$ and, thus, $\nu = \nu'$. We see that $\{\nu^n\}$ is a sequence in a compact set such that all its converging subsequences converge to $\nu$. This implies $\nu_n \to \nu$ as $n \to \infty$. $\square$

## 6.3 Feller Markov processes on the boundary

Let $\mathcal{W}$ be a locally compact topological space with a countable base. Let $\mathcal{B}(\mathcal{W})$ be the Banach space of real valued bounded measurable functions on $\mathcal{W}$, and let $C_0(\mathcal{W})$ be a closed subspace of all continuous function tending to zero at infinity. In other words, these are continuous function $f(w)$ such that for any $\varepsilon > 0$ there exist a compact set $V \subset \mathcal{W}$ such that $|f(w)| < \varepsilon$ for all $w \in \mathcal{W} \setminus V$. Note that $C_0(\mathcal{W})$ is separable and its Banach dual is $\mathcal{M}(\mathcal{W})$.

Let $P(u, A)$ be a Markov transition kernel. It induces a linear contracting operator $P^*$ in $\mathcal{B}(\mathcal{W})$:

$$(P^* f)(x) = \int_{\mathcal{W}} f(w) P(x, dw).$$

We say that $P(u, A)$ is a *Feller kernel* if $P^*$ maps $C_0(\mathcal{W})$ to $C_0(\mathcal{W})$.

Now let $X(t)$ be a homogenous continuous time Markov process on $\mathcal{W}$ with transition probabilities given by a semigroup of kernels $P_t(u, A)$. $X(t)$ is a *Feller process* if the following conditions are satisfied:

1. $P_t(u, A)$ is a Feller kernel, i.e. $P_t^*$ preserves $C_0(\mathcal{W})$.
2. For every $f \in C_0(\mathcal{W})$ the map $f \to P_t^* f$ is continuous at $t = 0$.

Feller processes have certain good properties. For instance, they have a modification with càdlàg sample paths, they are strongly Markovian, they have an infinitesimal generator, see e.g. [EK, Section 4.2].

In this section we prove that for any admissible function $g(x)$, $\overline{P_\infty}(u, A; g)$ is a Feller kernel on $\overline{\mathcal{N}}$. Moreover, we show that a Markov process on $\overline{\mathcal{N}}$ with semigroup of transition probabilities $\overline{P_\infty}(u, A; \exp(t(\gamma_+ x + \gamma_-/x)))$ and arbitrary initial distribution is Feller.

The section is organized as follows: First we prove in Proposition 6.12 that for any $N \geq 1$ and any admissible $g(x)$, $\overline{P_N}(\lambda \to \mu; g)$ is a Feller kernel on $\overline{\mathbb{GT}_N}$. As a corollary, we show in Theorem 6.14 that $\overline{P_\infty}(u, A; g)$ is a Feller kernel on $\overline{\mathcal{N}}$. Finally, we prove (Proposition 6.17 and Theorem 6.18) that Markov processes with semigroups of transition probabilities $\overline{P_N}(\mu \to \lambda; \exp(t(\gamma_+ x + \gamma_-/x)))$ and $\overline{P_\infty}(u, A; \exp(t(\gamma_+ x + \gamma_-/x)))$ are Feller.

**Proposition 6.12.** *For any admissible function $g(x)$, $\overline{P_N}(\mu \to \lambda; g)$ is a Feller kernel on $\overline{\mathbb{GT}_N}$.*

First, we prove a technical lemma.

**Lemma 6.13.** *For any elementary admissible function $g(x)$, the transition probabilities $P_N(\mu \to \lambda; g(x))$ admit an exponential tail estimate: There exist positive constants $a_1$ and $a_2$ such that*

$$P_N(\mu \to \lambda; g(x)) < a_1 \exp\bigl(-a_2 \max_{1 \leq i \leq N} |\lambda_i - \mu_i|\bigr).$$

*Proof.* Proposition 2.4 gives

$$P_N(\mu \to \lambda; g(x)) = \left( \prod_{i=1}^{N} \frac{1}{g(q^{1-i})} \right) \det_{i,j=1,\ldots,N} \left[ c_{\lambda_i - i - \mu_j + j} \right] \frac{s_\lambda(1,\ldots,q^{1-N})}{s_\mu(1,\ldots,q^{1-N})},$$

where

$$g(x) = \sum_{k\in\mathbb{Z}} c_k x^k.$$

If $g(x) = (1+\beta x^{\pm 1})$, then $P_N(\mu \to \lambda; g(x)) = 0$ as soon as $|\lambda_i - \mu_i| > 1$ for any $i$, and we are done.

For the remaining two cases note that (2.5) implies

$$\frac{s_\lambda(1,\ldots,q^{1-N})}{s_\mu(1,\ldots,q^{1-N})} < const_N \cdot q^{\sum_i (\lambda_i - \mu_i)(i-N)} < const_N \cdot q^{-N^2 \max_i |\lambda_i - \mu_i|}.$$

It follows that

$$\left| \det_{i,j=1,\ldots,N} \left[ c_{\lambda_i - i - \mu_j + j} \right] \frac{s_\lambda(1,\ldots,q^{1-N})}{s_\mu(1,\ldots,q^{1-N})} \right|$$
$$< const_N \sum_{\sigma \in S(N)} q^{-N^2 \max_i |\lambda_i - \mu_i|} \prod_{i=1}^{N} c_{\lambda_i - i - \mu_{\sigma(i)} + \sigma(i)}.$$

If $g(x) = \exp(\gamma x^{\pm 1})$ then $c_k r^k \to 0$ for any $r > 0$ as $k \to \infty$. Note that $\max_i |\lambda_i - \mu_i| = m$ implies that for any permutation $\sigma$ there exists $i$ such that $|\lambda_i - \mu_{\sigma(i)}| \geq m$. For this $i$ the product

$$q^{-N^2 \max_i |\lambda_i - \mu_i|} c_{\lambda_i - i - \mu_{\sigma(i)} + \sigma(i)}$$

is exponentially small (in $m$). Therefore, each term

$$q^{-N^2 \max_i |\lambda_i - \mu_i|} \prod_{i=1}^{N} c_{\lambda_i - i - \mu_{\sigma(i)} + \sigma(i)}$$

tends to zero exponentially fast as $\max_i |\lambda_i - \mu_i| \to \infty$ and we are done.

Finally, if $g(x) = (1 - \alpha x^{-1})^{-1}$ then $P_N(\mu \to \lambda; g(x)) = 0$ unless $\mu_i \geq \lambda_i$ for $1 \leq i \leq N$. In the latter case,

$$\frac{s_\lambda(1,\ldots,q^{1-N})}{s_\mu(1,\ldots,q^{1-N})} < const.$$

Expanding the determinant

$$\det_{i,j=1,\ldots,N} \left[ c_{\lambda_i - i - \mu_j^n + j} \right] = \sum_{\sigma} \prod_{i=1}^{N} c_{\lambda_i - i - \mu_{\sigma(i)} + \sigma(i)},$$

by the same argument as above we see that each term in the sum tends to zero exponentially fast as $\max_i |\lambda_i - \mu_i| \to \infty$. Therefore,

$$\left( \prod_{i=1}^{N} \frac{1}{g(q^{1-i})} \right) \det_{i,j=1,\ldots,N} \left[ c_{\lambda_i - i - \mu_j + j} \right] \frac{s_\lambda(1,\ldots,q^{1-N})}{s_\mu(1,\ldots,q^{1-N})}$$

tends to zero exponentially fast. □

*Proof of Proposition 6.12.* Let $h \in C_0(\overline{\mathbb{GT}_N})$; we want to check that $P_N^*(g)(h) \in C_0(\overline{\mathbb{GT}_N})$. By the definition of admissible functions, it suffices to check this property for elementary admissible functions $g(x) = (1 + \beta x^{\pm 1})$, $g(x) = \exp(\gamma x^{\pm 1})$ and $g(x) = (1 - \alpha x^{-1})^{-1}$. Moreover, since $C_0(\overline{\mathbb{GT}_N})$ is closed and $P_N^*(g)$ is a contraction, it is enough to check this property on a set of functions $h$ whose linear span is dense.

We choose the following system of functions ($\lambda \in \mathbb{GT}_N$, $k \geq 1$):

$$a_\lambda(\mu) = \begin{cases} 1, & \mu = \lambda, \\ 0, & \text{otherwise;} \end{cases}$$

$$b_{\lambda,k}(\mu) = \begin{cases} 1, & \mu_1 \geq \lambda_1, \dots, \mu_k \geq \lambda_k,\ \mu_{k+1} = \lambda_{k+1}, \dots, \mu_N = \lambda_N, \\ 0, & \text{otherwise.} \end{cases}$$

Let us start from $h = a_\lambda(\mu)$. We have

$$\left(\overline{P_N^*}(g)(h)\right)(\mu) = \overline{P_N}(\mu \to \lambda; g).$$

By definition, $\overline{P_N}(\mu \to \lambda; g) = 0$ if $\mu \in \overline{\mathbb{GT}_N} \setminus \mathbb{GT}_N$. Note that for a sequence $\mu^n$ of elements of $\mathbb{GT}_N$, $\mu^n \to \infty$ (in topology of $\overline{\mathbb{GT}_N}$) means that $\mu_N^n \to -\infty$, while $\mu^n \to \mu \in \overline{\mathbb{GT}_N} \setminus \mathbb{GT}_N$ implies $\mu_1^n \to +\infty$. Thus, to show that $\overline{P_N^*}(g)(h) \in C_0(\overline{\mathbb{GT}_N})$ we should prove that if $\mu^n$ is a sequence of elements of $\mathbb{GT}_N$ such that either $\mu_N^n \to -\infty$ or $\mu_1^n \to +\infty$ as $n \to \infty$, then $\overline{P_N}(\mu^n \to \lambda; g) = P_N(\mu^n \to \lambda; g) \to 0$ as $n \to \infty$. But if $\mu_N^n \to -\infty$ then $|\mu_N^n - \lambda_N| \to \infty$ and we may use Lemma 6.13. If $\mu_1^n \to +\infty$, then $|\mu_1^n - \lambda_1| \to \infty$ and Lemma 6.13 also provides the required estimate.

Now let $h = b_{\lambda,k}(\mu)$. The fact that $\left(\overline{P_N^*}(g)(h)\right)(\mu) \to 0$ as $\mu \to \infty$ (in other words, as $\mu_N \to -\infty$) again follows from Lemma 6.13. However, we need an additional argument to prove that the function $\overline{P_N^*}(g)(h)$ is continuous on $\overline{\mathbb{GT}_N}$.

Let us prove that if $\mu^n$ is a sequence of elements of $\overline{\mathbb{GT}_N}$ converging to $\mu$, then $\left(\overline{P_N^*}(g)(h)\right)(\mu^n) \to \left(\overline{P_N^*}(g)(h)\right)(\mu)$. One readily sees that there are two principal cases (the result for all other cases is a simple corollary of the results in one of these two cases):

- $\mu^n \in \mathbb{GT}_N \subset \overline{\mathbb{GT}_N}$ and $\mu \in \mathbb{GT}_N \subset \overline{\mathbb{GT}_N}$;
- $\mu^n \in \mathbb{GT}_N \subset \overline{\mathbb{GT}_N}$ and $\mu \in \overline{\mathbb{GT}_N} \setminus \mathbb{GT}_N$, i.e. $\mu_1 = \mu_2 = \dots = \mu_m = \infty$ while $\mu_{m+1} < \infty$ for some $1 \leq m \leq N$.

In the former case the sequence $\mu^n$ stabilizes, i.e. $\mu^n = \mu$ for large enough $n$, therefore, $\left(\overline{P_N^*}(g)(h)\right)(\mu^n) = \left(\overline{P_N^*}(g)(h)\right)(\mu)$ for large enough $n$. In the rest of the proof we concentrate on the latter case. From the definition of convergence in $\overline{\mathbb{GT}_N}$ we conclude that $\mu_i^n \to +\infty$ for $i = 1, \dots m$, and for $n > n_0$ we have $\mu_i^n = \mu_i$ for $i = m+1, \dots, N$. Without loss of generality we may assume $n_0 = 0$.

If $m > k$ then by Lemma 6.13

$$\left(\overline{P_N^*}(g)(h)\right)(\mu^n) \to 0 = \left(\overline{P_N^*}(g)(h)\right)(\mu).$$

Now suppose that $m = k$. We have:

$$\left(\overline{P_N^*}(g)(h)\right)(\mu^n) = \sum_{\nu\in B_{\lambda,k}} P_N(\mu^n \to \nu; g), \tag{6.8}$$

where $B_{\lambda,k} = \{u \in \mathbb{GT}_N : b_{\lambda,k}(u) = 1\}$.

Note that in $\mu^n$ the first $k$ coordinates are large while the last $N-k$ coordinates are fixed, and choose an integral sequence $r^n$ such that $r^n \to +\infty$ and $\mu_k^n - r^n \to \infty$. Let

$$\mathcal{A}^n = \{\nu \in \mathbb{GT}_N \mid \nu_1 \geq r^n, \ldots, \nu_k \geq r^n\}.$$

For every $n$ we divide the set $B_{\lambda,k}$ in the sum (6.8) into two disjoint parts

$$\sum_{\nu\in B_{\lambda,k}} P_N(\mu^n \to \nu; g) = \sum_{\nu\in \mathcal{B}^n_{\lambda,k}} P_N(\mu^n \to \nu; g) + \sum_{\nu\in \mathbb{B}^n_{\lambda,k}} P_N(\mu^n \to \nu; g)$$

where

$$\mathcal{B}^n_{\lambda,k} = B_{\lambda,k} \cap \mathcal{A}^n, \quad \mathbb{B}^n_{\lambda,k} = B_{\lambda,k} \setminus \mathcal{A}^n.$$

Observe that, as follows from Lemma 6.13,

$$\sum_{\nu\in \mathbb{B}^n_{\lambda,k}} P_N(\mu^n \to \nu; g) \to 0.$$

In the remaining sum

$$\sum_{\nu\in \mathcal{B}^n_{\lambda,k}} P_N(\mu^n \to \nu; g), \tag{6.9}$$

we use Lemma 6.5 for every term, i.e. for

$$P_N(\mu^n \to \nu; g(x)) = \left(\prod_{i=1}^{N} \frac{1}{g(q^{1-i})}\right) \det_{i,j=1,\ldots,N}\left[c_{\nu_i - i - \mu_j^n + j}\right] \frac{s_\nu(1,\ldots,q^{1-N})}{s_{\mu^n}(1,\ldots,q^{1-N})}.$$

As we have shown in the proof of Lemma 6.13 for all elementary admissible functions, the coefficients $c_k$ decay rapidly as $k$ grows. Therefore, the determinant of the matrix $\left[c_{\mu_i^n - i - \nu_j + j}\right]$ factorizes. We conclude that

$$\begin{aligned}
P_N(\mu^n \to \nu; g(x)) &= (1+o(1)) \\
&\times \left(\prod_{i=1}^{N-k} \frac{1}{g(q^{1-i})}\right) \det_{i,j=k+1,\ldots,N}\left[c_{\nu_i - i - \mu_j^n + j}\right] \frac{s_{(\nu_{k+1},\ldots,\nu_N)}(1,\ldots,q^{1-N+k})}{s_{(\mu^n_{k+1},\ldots,\mu^n_N)}(1,\ldots,q^{1-N+k})} \\
\times &\left(\prod_{i=N-k+1}^{N} \frac{1}{g(q^{1-i})}\right) \det_{i,j=1,\ldots,k}\left[c_{\nu_i - i - \mu_j^n + j}\right] \frac{s_{(\nu_1+N-k,\ldots,\nu_k+N-k)}(q^{k-N},\ldots,q^{1-N})}{s_{(\mu_1^n+N-k,\ldots,\mu_k^n+N-k)}(q^{k-N},\ldots,q^{1-N})},
\end{aligned} \tag{6.10}$$

where the term $o(1)$ uniformly tends to zero as $n \to \infty$.

Note that the second line in (6.10) is the transition probability

$$P_N\Big(\mu \to (\infty,\ldots,\infty,\lambda_{k+1},\ldots,\lambda_N); g(x)\Big) = \big(\overline{P_N^*}(g)(h)\big)(\mu).$$

As for the third line, observe that Lemma 2.2 implies

$$\sum_{(\nu_1,\ldots,\nu_k)\in\mathbb{GT}_k}\left(\prod_{i=N-k+1}^{N}\frac{1}{g(q^{1-i})}\right)\det_{i,j=1,\ldots,k}\left[c_{\nu_i-i-\mu_j^n+j}\right] \\ \times\frac{s_{(\nu_1+N-k,\ldots,\nu_N+N-k)}(q^{k-N},\ldots,q^{1-N})}{s_{(\mu_1^n+N-k,\ldots,\mu_k^n+N-k)}(q^{k-N},\ldots,q^{1-N})}=1.$$

Arguing as in Lemma 6.13, we conclude that we may replace the summation set by $\mathcal{B}^n_{\lambda,k}$, i.e. as $n\to\infty$,

$$\sum_{\nu\in\mathcal{B}^n_{\lambda,k}}\left(\prod_{i=N-k+1}^{N}\frac{1}{g(q^{1-i})}\right)\det_{i,j=1,\ldots,k}\left[c_{\nu_i-i-\mu_j^n+j}\right] \\ \times\frac{s_{(\nu_1+N-k,\ldots,\nu_N+N-k)}(q^{k-N},\ldots,q^{1-N})}{s_{(\mu_1^n+N-k,\ldots,\mu_k^n+N-k)}(q^{k-N},\ldots,q^{1-N})}\to 1.$$

Summing up, we proved that

$$\begin{aligned}\big(\overline{P_N^*}(g)(h)\big)(\mu^n) &= \sum_{\nu\in\mathcal{B}^n_{\lambda,k}}P_N(\mu^n\to\nu;g)+\sum_{\nu\in\mathbb{B}^n_{\lambda,k}}P_N(\mu^n\to\nu;g)\\ &=\big(\overline{P_N^*}(g)(h)\big)(\mu)(1+o(1))+o(1)\to\big(\overline{P_N^*}(g)(h)\big)(\mu).\end{aligned}$$

It remains to consider the case $m<k$. The statement in this case essentially follows from the case $m=k$. Indeed, decompose $B_{\lambda,k}$ into a disjoint union

$$B_{\lambda,k}=\bigcup_{\theta\in\Theta}B_{\theta,m},$$

where the union is taken over the set $\Theta\subset\mathbb{GT}_N$ consisting of all $\theta$ such that $\theta_i=\lambda_i$ for $i>k$, $\theta_i\ge\lambda_i$ for $m<i\le k$ and $\theta_i=\max(\lambda_i,\theta_{m+1})$ for $i\le m$. Now choose large enough $s$ and denote

$$\Theta^s=\Theta\cap\{\nu\in\mathbb{GT}_N\mid\nu_1<s\}.$$

We write

$$\big(\overline{P_N^*}(g)(b_{\lambda,k}\big)(\mu^n)=\sum_{\theta\in\Theta^s}\big(\overline{P_N^*}(g)(b_{\theta,m})\big)(\mu^n)+\sum_{\theta\in\Theta\setminus\Theta^s}\big(\overline{P_N^*}(g)(b_{\theta,m})\big)(\mu^n),$$

Note that the set $\Theta^s$ is finite. Therefore, using the already proven case $m=k$ we conclude that

$$\sum_{\theta\in\Theta^s}\big(\overline{P_N^*}(g)(b_{\theta,m})\big)(\mu^n)\to\sum_{\theta\in\Theta^s}\big(\overline{P_N^*}(g)(b_{\theta,m})\big)(\mu).$$

To finish the proof it remains to note that as follows from Lemma 6.13,

$$\sum_{\theta\in\Theta\setminus\Theta^s}\big(\overline{P_N^*}(g)(b_{\theta,m})\big)(\mu^n)$$

uniformly (in $n$) tends to zero as $s\to\infty$. ∎

**Theorem 6.14.** *For an admissible $g(x)$, $\overline{P_\infty}(g)$ is a Feller kernel on $\overline{\mathcal{N}}$.*

*Proof.* Let $h \in C_0(\overline{\mathcal{N}})$. We need to check that $\overline{P^*_\infty}(g)(h) \in C_0(\overline{\mathcal{N}})$. Since $C_0(\overline{\mathcal{N}})$ is closed and $P^*_\infty(g)$ is a contraction, it is enough to check this property on a dense set of functions $h$. Let us find a good dense set of functions.

Let $P^*_{\infty\to N}$ be a contraction operator from $\mathcal{B}(\overline{\mathbb{GT}_N})$ to $\mathcal{B}(\overline{\mathcal{N}})$ given by:

$$(P^*_{\infty\to N} f)(\nu) = \sum_{v\in\overline{\mathbb{GT}_N}} \mathcal{E}^\nu_N(v),$$

where $\mathcal{E}^\nu$ is the extreme coherent system corresponding to the measure $\delta^\nu$ on $\overline{\mathcal{N}}$.

**Lemma 6.15.** *$P^*_{\infty\to N}$ maps $C_0(\overline{\mathbb{GT}_N})$ to $C_0(\overline{\mathcal{N}})$.*

*Proof.* Again it suffices to check the lemma on a set of continuous functions whose linear span is dense. We choose the familiar system of functions

$$a_\lambda(\mu) = \begin{cases} 1, & \mu = \lambda, \\ 0, & \text{otherwise}, \end{cases}$$

$$b_{\lambda,k}(\mu) = \begin{cases} 1, & \mu_1 \ge \lambda_1, \dots, \mu_k \ge \lambda_k,\ \mu_{k+1} = \lambda_{k+1}, \dots, \mu_N = \lambda_N, \\ 0, & \text{otherwise}, \end{cases}$$

and their finite linear combinations. Theorem 6.4 implies that $P^*_{\infty\to N}(a_\lambda)$ and $P^*_{\infty\to N}(b_{\lambda,k})$ are continuous functions on $\overline{\mathcal{N}}$. It remains to check that they vanish at the infinity. If $\nu^n$ is a sequence of elements of $\overline{\mathcal{N}}$ tending to infinity (i.e. escaping from every compact set), then $\nu^n_1 \to -\infty$. Thus, we should check that if $\nu^n_1 \to -\infty$ then

$$\big(P^*_{\infty\to N}(a_\lambda)\big)(\nu^n) = \mathcal{E}^{\nu^n}_N(\lambda) \to 0.$$

Assume the opposite. Then there exists a subsequence $\{n_\ell\}$ such that $\mathcal{E}^{\nu^n}_N(\lambda) > c > 0$ for $n = n_\ell$. Let $\psi^\ell = \nu^{n_\ell} - \nu^{n_\ell}_1$. Then $\{\psi^\ell\}$ is a sequence of elements of a compact set. Hence, $\{\psi^\ell\}$ has a converging subsequence. Without loss of generality assume that already $\psi^\ell$ is converging, $\psi^\ell \to \psi$. Since, $\mathcal{E}^{\psi^\ell}_N$ is a probability measure on $\mathbb{GT}_N$ and $\mathcal{E}^{\psi^\ell}_N \to \mathcal{E}^{\psi}_N$, we must have $\mathcal{E}^{\psi^\ell}_N(\lambda - \nu^{n_\ell}_1) \to 0$.

Now observe the following property of measures $\mathcal{E}^\nu$ which was proved in [G2, Proposition 5.12]. For $e \in \mathbb{Z}$ and $\nu \in \mathcal{N}$ let $\nu - e$ be a sequence with coordinates $(\nu - e)_i = \nu_i - e_i$. In the same way for $\lambda \in \mathbb{GT}_N$ set $(\lambda - e)_i = \lambda_i - e$. Then we have $\mathcal{E}^{\nu-e}_N(\lambda - e) = \mathcal{E}^\nu_N(\lambda)$.

We conclude that $\mathcal{E}^{\nu^{n_\ell}}(\lambda) = \mathcal{E}^{\psi^\ell}_N(\lambda - \nu^{n_\ell}_1) \to 0$. Contradiction.

The argument for the functions $b_{\lambda,k}$ is similar and we omit it. □

**Lemma 6.16.** *The union of the sets $P^*_{\infty\to N}(C_0(\overline{\mathbb{GT}_N}))$ over $N \ge 0$ is dense in $C_0(\overline{\mathcal{N}})$.*

*Proof.* $\mathcal{M}(\overline{\mathcal{N}})$ is a Banach dual to $C_0(\overline{\mathcal{N}})$, thus, it is enough to check that if $\pi \in \mathcal{M}(\overline{\mathcal{N}})$ is such that $\int f d\pi = 0$ for all $N$ and all $f \in P^*_{\infty\to N}(C_0(\overline{\mathbb{GT}_N}))$, then $\pi \equiv 0$. The latter property is equivalent (by Fubini's theorem) to the following one: For any $N$ and any $f \in C_0(\overline{\mathbb{GT}_N})$, $\int f dE_q(\pi)_N = 0$, where as before $E_q$ stands for the map from $\mathcal{M}(\overline{\mathcal{N}})$ to $\varprojlim \mathcal{M}(\overline{\mathbb{GT}_N})$ ($P^*_{\infty\to N}$ is dual to $(E_q)_N$). But then $E_q(\pi)_N \equiv 0$, therefore, $E_q(\pi) \equiv 0$ and $\pi \equiv 0$. □

Now we can finish the proof of Theorem 6.14.

Let $h = P^*_{\infty\to N}(f)$. Then by the definitions

$$\overline{P^*_\infty}(g)(h) = \overline{P^*_\infty}(g)(P^*_{\infty\to N}(f)) = P^*_{\infty\to N}(\overline{P^*_N}(g)(f)).$$

But $\overline{P^*_N}(g)(f) \in C_0(\overline{\mathbb{GT}_N})$ by Proposition 6.12. Thus, $\overline{P^*_\infty}(g)(h) \in C_0(\overline{\mathcal{N}})$. □

**Proposition 6.17.** *A Markov process on $\overline{\mathbb{GT}_N}$ with semigroup of transition probabilities $\overline{P_N}(\lambda \to \mu; exp(t(\gamma_+ x + \gamma_-/x)))$ and arbitrary initial distribution is Feller.*

*Proof.* The first property is contained in Proposition 6.12. As for the second property, it immediately follows from the fact that $\exp(t(\gamma_+ x + \gamma_-/x))) \to 1$ as $t \to 0$ and definitions. □

**Theorem 6.18.** *A Markov process on $\overline{\mathcal{N}}$ with semigroup of transition probabilities $\overline{P_\infty}(u, A; \exp(t(\gamma_+ x + \gamma_-/x)))$ and arbitrary initial distribution is Feller.*

*Proof.* The first property is contained in Theorem 6.14. As for the second one, since $\overline{P^*_\infty}(g(x))$ is a contraction, we may check this property on a dense subset. If $h = P^*_{\infty\to N}(f)$ and $f \in C_0(\overline{\mathbb{GT}_N})$, then

$$\overline{P^*_\infty}(\exp(t(\gamma_+ x + \gamma_-/x)))(h) = P^*_{\infty\to N}(\overline{P^*_N}(\exp(t(\gamma_+ x + \gamma_-/x))(f)).$$

And this is a continuous map as a composition of a continuous map (by Proposition 6.17) and a contraction. □

# 7 PushASEP with particle-dependent jump rates

Fix parameters $\zeta_1, \dots, \zeta_N > 0$, $a, b \geq 0$, and assume that at least one the numbers $a$ and $b$ does not vanish. Consider $N$ particles in $\mathbb{Z}$ located at different sites and enumerated from left to right. The particle number $n$, $1 \leq n \leq N$, has two exponential clocks - the "right clock" of rate $a\zeta_n$ and the "left clock" of rate $b/\zeta_n$. When the right clock of particle number $n$ rings, it checks whether the position to the right of it is empty. If yes, then the particle jumps to the right by 1, otherwise it stays put. When the left clock of particle number $n$ rings, it jumps to the left by 1 and pushes the (maybe empty) block of particles sitting next to it.

Note that if $\zeta_1 = q^{1-N}$, $\zeta_2 = q^{2-N}$, $\dots$, $\zeta_{N-1} = q^{-1}$, $\zeta_N = 1$, and the process is started from the initial configuration $1-N, 2-N, \dots, -1, 0$ then this dynamics describes the evolution of $N$ leftmost particles of the process $\Pi(\mathcal{Y}^S_{\gamma_+,\gamma_-}(t))$ introduced in Section 5.3.

Let $P_t(x_1, \dots, x_N \mid y_1 \dots, y_N)$ denote the transition probabilities of the above process. The probabilities depend on $\zeta_1, \dots, \zeta_N, a, b$, but we omit these dependencies from the notation. In this section we study the asymptotic behavior of particles as time $t$ goes to infinity. In other words, we are interested in the asymptotics of $P_t(x_1, \dots, x_N \mid y_1 \dots, y_N)$ as $t \to \infty$.

One easily checks that if for some $r \in \mathbb{R}_{>0}$ and $1 \leq k \leq N$ we have $\zeta_k < r$ and $\xi_{k+1} > r$, $\dots$, $\zeta_N > r$, then at large times the first $k$ and the last $N-k$ particles behave independently. Thus, it is enough to study the case $\xi_N \leq$

$\min(\zeta_1, \ldots, \zeta_{N-1})$. Moreover, without loss of generality we may also assume that $\zeta_N = 1$. Thus, it suffices to consider the situation when we have $h \leq N$ indices $n_1 < n_2 < \cdots < n_h = N$ such that

$$\zeta_{n_1} = \zeta_{n_2} = \cdots = \zeta_{n_h} = 1,$$

and $\zeta_k > 1$ for $k$ not belonging to $\{n_1, \ldots, n_h\}$. Set $D = \{n_i\}_{i=1,\ldots,h}$.

To state the result on asymptotic behavior we need to introduce a certain distribution from the random matrix theory first. Let $M_n$ be a random $n \times n$ Hermitian matrix from the Gaussian Unitary Ensemble, see e.g. [Meh], [For], [AGZ] for the definition. We use the normalization for which diagonal matrix elements are real Gaussian random variables with variance 1. Denote the eigenvalues of $M_n$ by $\lambda_1^n \leq \lambda_2^n \leq \cdots \leq \lambda_n^n$. Let $M_k$ be the top left $k \times k$ submatrix of $M_n$, denote the eigenvalues of $M_k$ by $\lambda_1^k \leq \cdots \leq \lambda_k^k$. Finally, denote by $GUE_1^n$ the joint distribution of the smallest eigenvalues of matrices $M_k$. In other words, it is the joint distribution of the vector $(\lambda_1^n \leq \lambda_1^{n-1} \leq \cdots \leq \lambda_1^1)$.

Denote by $\mathfrak{G}_n(z)$, $n \in \mathbb{Z}$, the integral

$$\mathfrak{G}_n(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_2} u^n \exp(u^2/2 + uz) du,$$

where the contour of integration $\mathcal{C}_2$ is shown in Figure 3. Note that $\mathfrak{G}_0(z)$ is

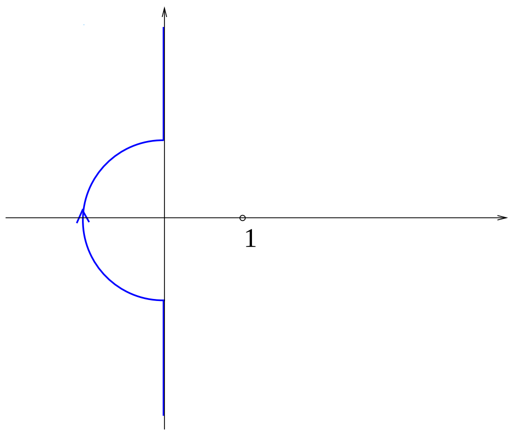


Figure 3: Contour of integration $\mathcal{C}_2$.

the density of the Gaussian distribution:

$$\frac{1}{2\pi i} \int_{\mathcal{C}_2} \exp(u^2/2 + uz) du = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2).$$

More generally, for $n \geq 1$ we have:

$$\mathfrak{G}_n(z) = \left(\frac{\partial}{\partial z}\right)^n \mathfrak{G}_0(z) = (-1)^n \frac{1}{\sqrt{\pi}\, 2^{(n+1)/2}} H_n\left(\frac{z}{\sqrt{2}}\right) \exp\left(-\frac{z^2}{2}\right),$$

where $H_n$ is the $n$th Hermite polynomial.

**Proposition 7.1.** *The probability distribution $GUE_1^n$ of the smallest eigenvalues of the GUE principal submatrices has density*

$$\rho(y_1, \dots, y_n) = \det[\mathfrak{G}_{k-l}(y_l)]_{k,l=1,\dots,n}, \quad y_1 \le y_2 \le \dots \le y_n.$$

*Proof.* We start from the formulas for the correlation functions of *all* eigenvalues of GUE minors found in [J4], [OR2], [JN1], [JN2]. Given a $n \times n$ GUE matrix, one constructs a point process on $\mathbb{N} \times \mathbb{R}$ which has a point $(m, y)$ if and only if $y$ is an eigenvalue of $m \times m$ top left submatrix. Let $\rho_n$ be the $n$th correlation function of this point process. Observe that the interlacing of the eigenvalues of the nested submatrices guarantees that

$$\rho_n(1, y_n; 2, y_{n-1}; \dots; n, y_1), \quad y_n \ge y_{n-1} \ge \dots \ge y_1,$$

is precisely the density of the distribution $GUE_1^n$. Then Theorem 2 in [OR2] and Theorem 1.3 of [JN1] yield that the density of $GUE_1^n$ is

$$\det_{i,j=1\dots n}[K(i, y_{n+1-i}; j, y_{n+1-j})],$$

where (we use a more convenient for us integral representation for the kernel, which can be found after the formula (6.17) in [JN1]; note that we use a different normalization for GUE)

$$\begin{aligned} K(r, \xi; s, \eta) &= \frac{\sqrt{2^{s-r} \exp(\frac{\eta^2 - \xi^2}{2})}}{2\sqrt{2}(\pi i)^2} \\ &\times \sum_{k=0}^{\infty} \int_{\mathcal{C}_1} du \exp(\sqrt{2}\xi u - u^2) u^{k-r} \int_{\mathcal{C}_2} dv v^{s-k-1} \exp(v^2 - \sqrt{2}\eta v), \end{aligned}$$

where $\mathcal{C}_1$ is anticlockwise oriented circle around the origin and $\mathcal{C}_2$ is the line $\Re v = 2$ from $-i\infty$ to $+i\infty$. Changing variables $z = -\sqrt{2}v$ and $w = -\sqrt{2}u$ we arrive at

$$(-1)^{s-r-1} \frac{\exp(\frac{\eta^2 - \xi^2}{2})}{(2\pi i)} \sum_{k=0}^{\infty} \int_{\mathcal{C}_3} dw \exp(-\xi w - w^2/2) w^{k-r} \mathfrak{G}_{s-k-1}(\eta),$$

where $\mathcal{C}_3$ is clockwise oriented circle around origin. Note that when $k \ge n \ge r$ the integral $\int_{\mathcal{C}_3} dw \exp(-\xi w - w^2/2) w^{k-r}$ vanishes. Thus, we may assume that the sum is over $k = 0, 1, \dots, n-1$.

Hence, $K(r, \xi; s, \eta)$ is the matrix element of the product of two matrices and

$$\det_{i,j=1\dots n}[K(i, y_{n+1-i}; j, y_{n+1-j})] = (-1)^n \det_{r,k=1,\dots,n}[A(r,k)] \det_{k,s=1,\dots,n}[\mathfrak{G}_{s-k}(y_{n+1-s})],$$

where

$$A(r,k) = \frac{1}{2\pi i} \int_{\mathcal{C}_3} dw \exp(y_{n+1-r} w - w^2/2) w^{k+1-r}.$$

Again note that $A(r,k) = 0$ unless $r > k$. Thus, the matrix $[A(r,k)]$ is triangular and

$$\det_{r,k=1,\dots,n}[A(r,k)] = \prod_{k=1}^{n} A(k,k)$$

with

$$A(k,k) = \frac{1}{2\pi i} \int_{\mathcal{C}_3} \frac{dw}{w} \exp(y_{n+1-r} w - w^2/2) = -1.$$

To finish the proof it remains to note that

$$\det_{k,s=1,\dots,n} [\mathfrak{G}_{s-k}(y_{n+1-s})] = \det_{k,s=1,\dots,n} [\mathfrak{G}_{k-s}(y_s)].$$

□

Now we are ready to state the main theorem of this section.

**Theorem 7.2.** *Fix parameters* $\big(\{\zeta_i\}_{i=1}^n, a, b\big)$ *as above and let* $y_1 < \dots < y_N$ *be arbitrary integers. Denote by* $(X_1(t), \dots, X_N(t))$ *the random vector distributed as*

$$P_t(x_1, \dots, x_N \mid y_1, \dots, y_N).$$

*The joint distribution of*

$$\Big( \frac{X_{n_1}(t) - (a-b)t}{\sqrt{(a+b)t}}, \frac{X_{n_2}(t) - (a-b)t}{\sqrt{(a+b)t}}, \dots, \frac{X_{n_h}(t) - (a-b)t}{\sqrt{(a+b)t}};$$

$$X_{n_1}(t) - X_{n_1-1}(t), \dots, X_2(t) - X_1(t);$$

$$X_{n_2}(t) - X_{n_2-1}(t), \dots, X_{n_1+2}(t) - X_{n_1+1}(t);$$

$$\dots$$

$$X_{n_h}(t) - X_{n_h-1}(t), \dots, X_{n_{h-1}+2}(t) - X_{n_{h-1}+1}(t) \Big)$$

*converges to*

$$GUE_1^h$$

$$\times \mathrm{Ge}(\zeta_{n_1-1}^{-1}) \times \dots \times \mathrm{Ge}(\zeta_1^{-1})$$

$$\dots$$

$$\times \mathrm{Ge}(\zeta_{n_h-1}^{-1}) \times \dots \times \mathrm{Ge}(\zeta_{n_{h-1}+1}^{-1}),$$

*where* $\mathrm{Ge}(p)$ *is a geometric distribution on* $\{1, 2, \dots\}$ *with parameter* $p$.

Theorem 7.2 follows from Proposition 7.1 and Proposition 7.3.

**Proposition 7.3.** *Set* $v = a - b$ *and denote*

$$x_1 = vt + \sqrt{(a+b)t} \cdot \widetilde{x}_1 + \hat{x}_1, \dots, x_{n_1} = vt + \sqrt{(a+b)t} \cdot \widetilde{x}_1 + \hat{x}_{n_1},$$

$$x_{n_1+1} = vt + \sqrt{(a+b)t} \cdot \widetilde{x}_2 + \hat{x}_{n_1+1}, \dots, x_{n_2} = vt + \sqrt{(a+b)t} \cdot \widetilde{x}_2 + \hat{x}_{n_2},$$

$$\dots$$

$$x_{n_{h-1}+1} = vt + \sqrt{(a+b)t} \cdot \widetilde{x}_h + \hat{x}_{n_{h-1}+1}, \dots, x_{n_h} = vt + \sqrt{(a+b)t} \cdot \widetilde{x}_h + \hat{x}_{n_h}.$$

*Then*

$$\lim_{t \to \infty} ((a+b)t)^{h/2} P_t(x_1, \dots, x_N \mid y_1, \dots, y_N)$$

$$= \prod_{i \in \{1, \dots, N\} \setminus D} (1 - \zeta_i^{-1}) \zeta_i^{1 - \hat{x}_{i+1} + \hat{x}_i} \cdot \det[\mathfrak{G}_{k-l}(\widetilde{x}_l)]_{k,l=1,\dots,h},$$

*and the convergence is uniform for* $\widetilde{x}_1, \dots, \widetilde{x}_h$ *belonging to compact sets.*

*Proof.* Our starting point is an explicit formula for $P_t(x_1,\dots,x_N \mid y_1,\dots,y_N)$ from [BF1]; it is a generalization of a similar formula for TASEP from [RS]. We have

$$P_t(x_1,\dots,x_N \mid y_1,\dots,y_N) \\ = \left(\prod_{i=1}^{N} \nu_i^{x_i-y_i} \exp(-at\zeta_i - bt/\zeta_i)\right) \det[F_{k,l}(x_l-y_k)], \quad (7.1)$$

where

$$F_{k,l}(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}_0} \frac{\prod_{i=1}^{k-1}(1-\zeta_i z)}{\prod_{j=1}^{l-1}(1-\zeta_j z)}\, z^{x-1}\exp(btz + atz^{-1})dz,$$

and the integration is over a positively oriented circle $\mathcal{C}_0$ of a small radius centered at 0.

Let us study the asymptotic behavior of $F_{k,l}(x)$ when $t\to\infty$ and $x = x(t) = (a-b)t + \sqrt{(a+b)t}\cdot\tilde{x} + \hat{x}$.

**Lemma 7.4.** *For $t\to\infty$ with $x = x(t) = (a-b)t + \sqrt{(a+b)t}\cdot\tilde{x}+\hat{x}$, we have*

$$\begin{aligned} F_{k,l}(x) &= \exp\left((a+b)t\right)\left((a+b)t\right)^{-1/2} t^{-n/2}(-1)^n \\ &\quad\times \frac{\prod_{i\in\{1,\dots,k-1\}\setminus D}(1-\zeta_i)}{\prod_{j\in\{1,\dots,l-1\}\setminus D}(1-\zeta_j)}\left(\mathfrak{G}_n(\tilde{x}) + o(1)\right) \\ &\quad + \sum_{m\in\{k\dots l-1\}\setminus D} C_{k,l,m}(t,x)\zeta_m^{1-x}\exp(bt\zeta_m^{-1} + at\zeta_m), \end{aligned} \quad (7.2)$$

*where the summation is only over indices $m$ corresponding to distinct $\zeta_m$,*

$$C_{k,l,m}(t,x) = Res_{z=\zeta_m^{-1}}\left(\frac{\prod_{i=1}^{k-1}(1-\zeta_i z)}{\prod_{j=1}^{l-1}(1-\zeta_j z)}(\zeta_m z)^{x-1}\exp(bt(z-\zeta_m^{-1}) + at(z^{-1}-\zeta_m))\right)$$

*(in particular, $C_{k,l,m}(t,x)$ has polynomial growth (or decay) in $t$ as $t\to\infty$) and*

$$n = |\{1,\dots,k-1\}\cap D| - |\{1,\dots,l-1\}\cap D|.$$

*The remainder $o(1)$ above is uniformly small for $\tilde{x}$ belonging to compact sets.*

**Remark.** Observe that in the limit regime of Lemma 7.4 we have

$$\begin{aligned} |\zeta_m^{-x}\exp(bt/\zeta_m + at\zeta_m)| &= \exp\left(t\left(-\frac{x}{t}\ln(\zeta_m) + b/\zeta_m + a\zeta_m\right)\right) \\ &\approx \exp\Big(t((b-a)\ln(\zeta_m) + b/\zeta_m + a\zeta_m)\Big). \end{aligned}$$

Since the function $(b-a)\ln(r) + b/r + ar$ of $r\in\mathbb{R}_+$ has a minimum at $r=1$, for any $\zeta_m\neq 1$ we have (for $t\gg 1$)

$$|\zeta_m^{1-x}\exp(bt/\zeta_m + ar)| \gg \exp((a+b)t).$$

Using the fact that $C_{k,l,m}(t,x)$ have polynomial growth (or decay) in $t$ as $t\to\infty$ we conclude that in the asymptotic decomposition (7.2) the first term is small comparing to the other ones.

*Proof of Lemma 7.4.* First, suppose that $b > 0$. Let us deform the integration contour so that it passes near the point 1. Since $\zeta_j \geq 1$, we need to add some residues:

$$F_{k,l}(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}_1} \frac{\prod_{i=1}^{k-1}(1-\zeta_i z)}{\prod_{j=1}^{l-1}(1-\zeta_j z)} z^{x-1} \exp(btz + atz^{-1}) dz$$
$$+ \sum_{m \in \{k,\dots,l-1\} \setminus D} Res_{z=\zeta_m^{-1}} \left( \frac{\prod_{i=1}^{k-1}(1-\zeta_i z)}{\prod_{j=1}^{l-1}(1-\zeta_j z)} z^{x-1} \exp(btz + atz^{-1}) \right), \quad (7.3)$$

where the contour $\mathcal{C}_1$ is shown in Figure 4. Here the summation is only over indices $m$ corresponding to distinct $\zeta_m$.

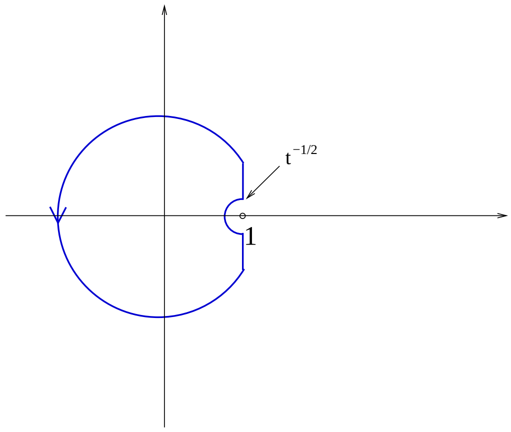


Figure 4: Contour of integration $\mathcal{C}_1$. The radius of the small arc is $t^{-1/2}$ and the radius of larger arc is a parameter $R$ that we choose later.

Observe that if $l \leq m < k$ and $\zeta_m > 1$, then

$$Res_{z=\zeta_m^{-1}} \left( \frac{\prod_{i=1}^{k-1}(1-\zeta_i z)}{\prod_{j=1}^{l-1}(1-\zeta_j z)} z^{x-1} \exp(btz + atz^{-1}) \right)$$
$$= C_{k,l,m}(t,x) \zeta_m^{1-x} \exp(bt\zeta_m^{-1} + at\zeta_m).$$

In particular, if the pole at $\zeta_m^{-1}$ is simple (which is true, for instance, if all $\zeta_m \neq 1$ are mutually distinct), then

$$Res_{z=\zeta_m^{-1}} \left( \frac{\prod_{i=1}^{k-1}(1-\zeta_i z)}{\prod_{j=1}^{l-1}(1-\zeta_j z)} z^{x-1} \exp(btz + atz^{-1}) \right)$$
$$= \frac{-\zeta_m^{-1}}{\prod_{j=\{k,\dots,l-1\} \setminus D}(1-\zeta_j/\zeta_m)} \zeta_m^{1-x} \exp(bt\zeta_m^{-1} + at\zeta_m).$$

Observe that

$$z^{x-1}\exp(btz+atz^{-1}) = \exp(t(\ln z(a-b+t^{-1/2}\tilde{x}\sqrt{a+b}+t^{-1}(\hat{x}-1))+bz+a/z))$$

and

$$\Re((a-b)\ln z + bz + a/z)$$

has a saddle point at $z=1$. We claim that for large $t$, only a small neighborhood of $z=1$ gives a non-negligible contribution the integral. Indeed, let $z=u+iv$. Then

$$\Re((a-b)\ln z + bz + a/z) = (a-b)\ln\sqrt{u^2+v^2} + bu + \frac{au}{\sqrt{u^2+v^2}}. \qquad (7.4)$$

Setting $u=1$ and differentiating with respect to $v$ we see that for a small enough $\varepsilon > 0$ the function

$$(a-b)\ln\sqrt{1+v^2} + b + \frac{a}{\sqrt{1+v^2}}$$

increases on $[-\varepsilon, 0]$, decreases on $[0,\varepsilon]$ and its maximum is $a+b$.

Now set the radius of the bigger arc $R$ to be equal to $\sqrt{1+\varepsilon^2}$ and note that along this arc

$$\begin{aligned}(a-b)\ln\sqrt{u^2+v^2} + bu + \frac{au}{\sqrt{u^2+v^2}} &= (a-b)\ln R + bu + \frac{au}{R}\\ < (a-b)\ln R + b + \frac{a}{R} &= (a-b)\ln\sqrt{1+\varepsilon^2} + b + \frac{a}{\sqrt{1+\varepsilon^2}}.\end{aligned}$$

Summing up, if we choose any $\delta > 0$, then everywhere outside the $\delta$-neighborhood of $z=1$ the function $f(z) = \Re((a-b)\ln z + bz + a/z)$ is smaller than $f(1+\delta)$ which, in turn, is smaller than $f(1) = a+b$. Therefore, the integral along our contour outside the $\delta$-neighborhood of $z=1$ can be bounded by $const \cdot \exp((a+b)t)\exp(t(f(1)-f(1+\delta)))$ and, thus, as $t \to \infty$ it becomes exponentially smaller than $\exp((a+b)t)$, which is the smallest term in our asymptotic expansion (7.2). Consequently, this integral contributes only to $o(1)$ term in (7.2) and can be neglected.

To calculate the integral in the small $\delta$-neighborhood of 1 we change the integration variable $z = 1 + t^{-1/2}u$ and arrive at the integral

$$\begin{aligned} t^{-1/2}\frac{1}{2\pi i}\int_{\mathcal{C}_2'} &\frac{\prod_{i=1}^{k-1}(1-\zeta_i(1+t^{-1/2}u))}{\prod_{j=1}^{l-1}(1-\zeta_j(1+t^{-1/2}u))}\\ &\times \exp\Big(t\big(\ln(1+t^{-1/2}u)(a-b+t^{-1/2}\tilde{x}\sqrt{a+b}+t^{-1}(\hat{x}-1))\\ &\qquad + b(1+t^{-1/2}u) + a/(1+t^{-1/2}u)\big)\Big)du\end{aligned}$$

with contour $\mathcal{C}_2'$ that is a part of contour $\mathcal{C}_2$ shown in Figure 3 between points $u = \pm i\delta t^{1/2}$

Simplifying the integrand we arrive at

$$\begin{aligned} t^{-1/2}\exp\big(t(a+b)\big)\frac{1}{2\pi i}\int_{\mathcal{C}_2'} &\frac{\prod_{i=1}^{k-1}(1-\zeta_i(1+t^{-1/2}u))}{\prod_{j=1}^{l-1}(1-\zeta_j(1+t^{-1/2}u))}\\ &\times\exp\big(u^2(a+b)/2 + u\tilde{x}\sqrt{a+b} + o(1)\big).\end{aligned}$$

Making the linear change of variables $v = u\sqrt{a+b}$ and sending $t$ to $\infty$ we obtain the required integral.

As for the case $b=0$, the argument is similar and we omit it. The only difference is that now contour of integration should have no vertical line part, i.e. it consists of the arcs of unit circle and $t^{-1/2}$-circle. $\square$

Now we continue the proof of Proposition 7.3.

The next step is to do certain elementary transformations to the matrix $[F_{k,l}(x_l-y_k)]$ in order to simplify its determinant. First, suppose that all $\zeta_m$ (except for those equal to 1) are distinct.

We take the $(N-1)$st row of matrix $[F_{k,l}(x_l-y_k)]_{k,l=1}^N$ (i.e. $k=N-1$), multiply it by

$$\zeta_m^{y_n-y_{N-1}} \frac{1}{\prod_{j=n}^{N-2}(1-\zeta_j/\zeta_{N-1})}, \tag{7.5}$$

and add to the $n$th row for $n=1,\dots,N-2$. As a result the term in (7.2) coming from the residue at $\zeta_{N-1}$ remains only in the $(N-1,N)$ matrix element.

Next, we take the $(N-2)$nd row of the matrix and add this row (again with coefficients) to rows $1,\dots,N-3$ so that the term in (7.2) coming from the residue at $\zeta_{N-2}$ remains only in the $(N-2,N-1)$ and $(N-2,N)$ matrix elements.

Repeating this procedure for every row $h$ such that $\zeta_h\neq 1$ we get a transformed matrix $[\widehat{F}_{k,l}(x_l-y_k)]_{k,l=1}^N$. In this new matrix, the term in (7.2) coming from the residue at $\zeta_h$ remains only in the matrix elements $\widehat{F}_{h-1,h}(x_h-y_h-1)$, $\dots$, $\widehat{F}_{h-1,N}(x_N-y_{h-1})$.

Let us now do some columns transformations. We take the second column ($l=2$) and add it with coefficients to the columns $3,\dots,N$ so that the terms coming from the residue at $\zeta_1$ vanish in columns $3,\dots,N$. Then we repeat this procedure for the third column and so on.

Finally, we get a matrix $[\widetilde{F}_{k,l}(x_l-y_k)]_{k,l=1}^N$ whose determinant coincides with that of $[F_{k,l}(x_l-y_k)]_{k,l=1}^N$, but this matrix has only $N-h$ elements with order of growth greater than $\exp((a+b)t)$ (i.e. elements with terms coming from the residues). These elements are $\widetilde{F}_{n-1,n}(x_n-y_{n-1})$ for $n\in\{2,\dots,N\}$, $n\notin D+1$,

$$\widetilde{F}_{n-1,n}(x_n-y_{n-1}) = -\zeta_{n-1}^{-1}\zeta_{n-1}^{1-x_n+y_{n-1}}\exp(bt\zeta_{n-1}^{-1}+at\zeta_{n-1})(1+o(1)).$$

It follows that asymptotically as $t\to\infty$ the determinant $\det[\widetilde{F}_{k,l}(x_l-y_k)]$ factorizes:

$$\begin{aligned}&\det[\widetilde{F}_{k,l}(x_l-y_k)]_{k,l=1,\dots,N}\\ &= \prod_{n\in\{2,\dots,N\}\setminus(D+1)} \zeta_{n-1}^{-x_n+y_{n-1}}\exp(bt\zeta_m^{-1}+at\zeta_m)\det[G_{k,l}]_{k,l=1,\dots,h}(1+o(1)),\end{aligned} \tag{7.6}$$

where $[G_{k,l}]_{k,l=1,\dots,h}$ is the submatrix of $[\widetilde{F}_{k,l}(x_l-y_k)]$ with rows $n_1,n_2,\dots,n_h$ and columns $1,n_1+2,n_2+1,\dots,n_{h-1}+1$. Looking back at row and column operations that we made with $[F_{k,l}(x_l-y_k)]$ to get $[\widetilde{F}_{k,l}(x_l-y_k)]$, we see that the determinant of $G_{k,l}$ coincides with the determinant of the Gaussian terms of the decomposition (7.2) of matrix elements $F_{k,l}(x_l-y_k)$ in rows $n_1,n_2,\dots,n_h$

and columns $1, n_1+1, n_2+1, \dots, n_{h-1}+1$. Therefore,

$$\det[G_{k,l}]_{k,l=1,\dots,h} = \exp(ht(a+b))(-1)^h((a+b)t)^{-h/2}\Bigg(o(1)$$

$$+\prod_{m=1}^{h}\prod_{i\in\{1,\dots,n_m-1\}\setminus D}(1-\zeta_i)\prod_{m=1}^{h-1}\prod_{j\in\{1,\dots,n_m\}\setminus D}\frac{1}{(1-\zeta_i)}\det[\mathfrak{G}_{k-l}(\widetilde{x}_l)]_{k,l=1,\dots,h}\Bigg)$$

$$= \exp(ht(a+b))(-1)^h((a+b)t)^{-h/2}\prod_{i\in\{1,\dots,N-1\}\setminus D}(1-\zeta_i)$$

$$\times\left(\det[\mathfrak{G}_{k-l}(\widetilde{x}_l)]_{k,l=1,\dots,h}+o(1)\right). \quad (7.7)$$

Combining (7.1) with asymptotic formulas (7.6) and (7.7) we conclude the proof.

Now suppose that some of $\zeta_i$ coincide. In this case the computation of the residues in the decomposition (7.2) becomes more complicated, since some of the poles are not simple. However, the scheme of the proof remains the same: First we transform the matrix $[F_{k,l}(x_l - y_k)]$ by means of elementary row transforms so that the terms in (7.2) coming from the residue at $\zeta_h$ remain only in the matrix elements $\widehat{F}_{h-1,h}(x_h - y_h - 1)$, $\dots$, $\widehat{F}_{h-1,N}(x_N - y_{h-1})$. Then we do elementary column transforms so that only $N-h$ elements with order of growth greater than $\exp((a+b)t)$ remain in the resulting matrix. After that the computation of the determinant repeats the case of distinct $\zeta_m$.

□

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