CONTINUUM STATISTICS OF THE AIRY$_2$ PROCESS

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ABSTRACT. We develop an exact determinantal formula for the probability that the Airy$_2$ process is bounded by a function g on a finite interval. As an application, we provide a direct proof that $\sup(\mathcal{A}_2(x) - x^2)$ is distributed as a GOE random variable. Both the continuum formula and the GOE result have applications in the study of the end point of an unconstrained directed polymer in a disordered environment. We explain Johansson's [14] observation that the GOE result follows from this polymer interpretation and exact results within that field. In a companion paper [16] these continuum statistics are used to compute the distribution of the endpoint of directed polymers.

1. Introduction

The Airy$_2$ process $\mathcal{A}_2$ was introduced in Prähofer and Spohn [17] in the study of the scaling limit of a discrete polynuclear growth (PNG) model. It is expected to govern the asymptotic spatial fluctuations in a wide variety of random growth models on a one dimensional substrate with curved initial conditions, and the point-to-point free energies of directed random polymers in $1+1$ dimensions (the KPZ universality class). It also arises as the scaling limit of the top eigenvalue in Dyson's Brownian motion [7] for the Gaussian Unitary Ensemble (GUE) of random matrix theory (see [3] for more details).

$\mathcal{A}_2$ is defined through its finite-dimensional distributions, which are given by a determinantal formula: given $x_0, \dots, x_n \in \mathbb{R}$ and $t_0 < \cdots < t_n$ in $\mathbb{R}$,

$$\mathbb{P}(\mathcal{A}_2(t_0) \le x_0, \dots, \mathcal{A}_2(t_n) \le x_n) = \det(I - \mathrm{f}^{1/2} K_{\mathrm{ext}} \mathrm{f}^{1/2})_{L^2(\{t_0,\dots,t_n\}\times\mathbb{R})}, \tag{1.1}$$

where we have counting measure on $\{t_0, \dots, t_n\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{t_0, \dots, t_n\} \times \mathbb{R}$ by $\mathrm{f}(t_j, x) = \mathbf{1}_{x \in (x_j,\infty)}$, and the *extended Airy kernel* [17, 12, 15] is defined by

$$K_{\mathrm{ext}}(t, \xi; t', \xi') = \begin{cases} \int_0^\infty d\lambda\, e^{-\lambda(t-t')} \operatorname{Ai}(\xi+\lambda) \operatorname{Ai}(\xi'+\lambda), & \text{if } t \ge t' \\ \int_{-\infty}^0 d\lambda\, e^{-\lambda(t-t')} \operatorname{Ai}(\xi+\lambda) \operatorname{Ai}(\xi'+\lambda), & \text{if } t < t', \end{cases}$$

where $\operatorname{Ai}(\cdot)$ is the Airy function. In particular, the one point distribution of $\mathcal{A}_2$ is given by the Tracy-Widom largest eigenvalue distribution for GUE.

K. Johansson [14] proved the remarkable fact that

Theorem 1. *For every $m \in \mathbb{R}$,*

$$\mathbb{P}\Big(\sup_{t\in\mathbb{R}} \big(\mathcal{A}_2(t) - t^2\big) \le m\Big) = F_{\mathrm{GOE}}(4^{1/3} m).[1]$$

Here F_{GOE} denotes the Tracy-Widom largest eigenvalue distribution for the Gaussian Orthogonal Ensemble (GOE) [22]. It also arises as the one point distribution of the Airy$_1$ process, which governs the asymptotic spatial fluctuations in one dimensional random growth models with flat initial conditions, and the point-to-line free energies of directed random polymers in $1+1$ dimensions.

The proof of Theorem 1 in [14] is indirect, using a functional limit theorem for the convergence of the PNG model to the Airy$_2$ process, together with the connection between

[1] The factor $4^{1/3}$ corrects a minor mistake in Johansson's statement. See Section 2 for a discussion.

the PNG process and a certain last passage percolation model for which Baik and Rains [4] had proved the connection with GOE. In this article we develop a method to compute continuum probabilities for the Airy$_2$ process. This is then used to provide a direct proof of Theorem 1 starting only from determinantal formulas.

Theorem 1 reflects a universal behaviour seen in a large class of one dimensional systems (the KPZ universality class starting with flat initial conditions) and therefore has attracted quite a bit of interest at the physical level. Much of the recent work is on finite systems of N nonintersecting random walks, the so-called vicious walkers [11]. [9, 19, 20] obtain various expressions for the maximum and position of the maximum at the finite N level. [13] uses non-rigorous methods from gauge theory to obtain the GOE distribution in the large N limit, and furthermore connect the problem to Yang-Mills theory.

Our computation of continuum probabilities starts with the following (earlier) variant of (1.1) due to Prähofer and Spohn [17],

$$\begin{aligned}(1.2)\quad & \mathbb{P}(\mathcal{A}_2(t_0)\le x_0,\dots,\mathcal{A}_2(t_n)\le x_n)\\ &\qquad = \det\Big(I-K_{\mathrm{Ai}}+\bar P_{x_0}e^{(t_0-t_1)H}\bar P_{x_1}e^{(t_1-t_2)H}\cdots\bar P_{x_n}e^{(t_n-t_0)H}K_{\mathrm{Ai}}\Big),\end{aligned}$$

where K_{Ai} is the *Airy kernel*

$$K_{\mathrm{Ai}}(x,y)=\int_{-\infty}^{0}d\lambda\,\mathrm{Ai}(x-\lambda)\,\mathrm{Ai}(y-\lambda),$$

H is the *Airy Hamiltonian* $H=-\partial_x^2+x$ and $\bar P_a$ denotes the projection onto the interval $(-\infty,a]$. Here, and in everything that follows, the determinant means the Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$. The equivalence of (1.1) and (1.2) is proved in [17] and [18].

Fix an $L>0$. Given $g\in H^1([-L,L])$ (i.e. both g and its derivative are in $L^2([-L,L])$), define an operator Θ_L^g acting on $L^2(\mathbb{R})$ as follows: $\Theta_L^g f(\cdot)=u(L,\cdot)$, where $u(L,\cdot)$ is the solution at time L of the boundary value problem

$$\begin{aligned}\partial_t u+Hu&=0\quad\text{for } x<g(t),\ t\in(-L,L)\\ u(-L,x)&=f(x)\mathbf{1}_{x<g(-L)}\\ u(t,x)&=0\quad\text{for } x\ge g(t).\end{aligned}$$

The fact that this problem makes sense for $g\in H^1([-L,L])$ is easy to prove and can be seen from the proof of Proposition 3.2 below. By taking a fine mesh in t and using the cyclic property of determinants we obtain a continuum version of (1.2):

Theorem 2.

$$(1.3)\qquad \mathbb{P}(\mathcal{A}_2(t)\le g(t)\ \textit{for}\ t\in[-L,L])=\det\big(I-K_{\mathrm{Ai}}+e^{LH}K_{\mathrm{Ai}}\Theta_L^g e^{LH}K_{\mathrm{Ai}}\big).$$

An expression in terms of determinants of solution operators of boundary value problems may not seem very practical. But in fact one can give an explicit expression for the kernel of the operator Θ_L^g in terms of Brownian motion. Let $b(s)$ denote a Brownian motion with diffusion coefficient 2. By the Feynman-Kac formula,

$$u(L,x)=\mathbb{E}_{b(-L)=x}\Big(f(b(L))e^{-\int_{-L}^{L}b(s)ds}\mathbf{1}_{b(s)\le g(s)\text{ on }[-L,L]}\Big).$$

The potential x is removed by a parabolic shift,

$$\begin{aligned}\Theta_L^g f(x)&=\mathbb{E}_{b(-L)=x}\Big(f(b(L))e^{-\int_{-L}^{L}b(s)ds}\mathbf{1}_{b(s)\le g(s)\text{ on }[-L,L]}\Big)\\ &=\mathbb{E}_{b(-L)=x}\Big(f(b(L))e^{-L(b(L)+b(-L))+2L^3/3+\int_{-L}^{L}s\,db(s)-\int_{-L}^{L}s^2ds}\mathbf{1}_{b(s)\le g(s)\text{ on }[-L,L]}\Big)\\ &=\mathbb{E}_{b(-L)=x-L^2}\Big(f(b(L)+L^2)e^{-(b(L)+b(-L)+2L^2)L+2L^3/3}\mathbf{1}_{b(s)+s^2\le g(s)\text{ on }[-L,L]}\Big),\end{aligned}$$

where in the second equality we used integration by parts and added and subtracted $2L^3/3$, and in the third one we used the Cameron-Martin-Girsanov formula. This gives

Theorem 3. *Let $\Theta_L^g(x,y)$ denote the integral kernel of Θ_L^g. Then*

$$\Theta_L^g(x,y) = e^{-(x+y)L+2L^3/3}\frac{e^{-(x-y)^2/8L}}{\sqrt{8\pi L}}\mathbb{P}_{\hat{b}(-L)=x,\hat{b}(L)=y}\Big(\hat{b}(s) \leq g(s) - s^2 + L^2 \text{ on } [-L,L]\Big),$$

where the probability is computed with respect to a Brownian bridge $\hat{b}(s)$ from x at time $-L$ to y at time L and with diffusion coefficient 2.

This gives a formula which can be used in applications. The obvious one is the case $g(t) = t^2 + m$, in which the probability can easily be computed by the reflection principle (method of images). A second one is the computation of the joint distribution of the max and argmax of the Airy$_2$ process minus a parabola, which appears in a companion paper [16]. The simple result in the case $g(t) = t^2 + m$ is that

$$\Theta_L := \Theta_L^{g(t)=t^2+m} = \bar{P}_{m+L^2}e^{-2LH}\bar{P}_{m+L^2} - \bar{P}_{m+L^2}R_L\bar{P}_{m+L^2}, \tag{1.4}$$

where R_L is the reflection term

$$R_L(x,y) = \frac{1}{\sqrt{8\pi L}}e^{-(x+y-2m-2L^2)^2/8L-(x+y)L+2L^3/3}.$$

To obtain the $L\to\infty$ asymptotics, decompose Θ_L as

$$\Theta_L = e^{-2LH} - R_L - \Omega_L$$

where $\Omega_L = \big(\bar{P}_{m+L^2}R_L\bar{P}_{m+L^2} - R_L\big) + \big(\bar{P}_{m+L^2}e^{-2LH}\bar{P}_{m+L^2} - e^{-2LH}\big)$. In Section 5 we will show that

$$\widetilde{\Omega}_L := e^{LH}K_{\mathrm{Ai}}\Omega_L e^{LH}K_{\mathrm{Ai}} \xrightarrow[L\to\infty]{} 0 \tag{1.5}$$

in trace norm. Referring to (1.3), we have $e^{LH}K_{\mathrm{Ai}}e^{-2LH}e^{LH}K_{\mathrm{Ai}} = K_{\mathrm{Ai}}$, so the key point is the limiting behaviour in L of $e^{LH}K_{\mathrm{Ai}}R_Le^{LH}K_{\mathrm{Ai}}$. Remarkably, it does not depend on L and gives the kernel of F_{GOE}, thus providing a proof of Theorem 1.

Theorem 4.

$$e^{LH}K_{\mathrm{Ai}}R_Le^{LH}K_{\mathrm{Ai}} = A\bar{P}_0\hat{R}\bar{P}_0A^*,$$

where the A is the Airy transform, *$Af(x) := \int_{-\infty}^{\infty} dz\,\mathrm{Ai}(x-z)f(z)$, and*

$$\hat{R}(\lambda,\tilde{\lambda}) := 2^{-1/3}\,\mathrm{Ai}(2^{-1/3}(2m-\lambda-\tilde{\lambda})).$$

Furthermore,

$$\det\Big(I - A\bar{P}_0\hat{R}\bar{P}_0A^*\Big) = F_{\mathrm{GOE}}(4^{1/3}m). \tag{1.6}$$

The last equality is a version of the determinantal formula for F_{GOE} proved by Ferrari and Spohn [10]:

$$F_{\mathrm{GOE}}(m) = \det(I - P_0B_mP_0), \quad \text{where} \quad B_m(x,y) = \mathrm{Ai}(x+y+m).$$

This can be seen as follows. Using the cyclic property of the determinant and the *reflection operator* $\sigma f(x) = f(-x)$ we may rewrite the determinant in (1.6) as

$$\det\Big(I - A\sigma A\bar{P}_0\hat{R}\bar{P}_0A^*A\sigma\Big) = \det\Big(I - \sigma\bar{P}_0\hat{R}\bar{P}_0\sigma\Big) = \det\Big(I - P_0\sigma\hat{R}\sigma P_0\Big), \tag{1.7}$$

where we have used the identity $A\sigma A = \sigma$. On the other hand we have $\sigma\hat{R}\sigma(\lambda,\tilde{\lambda}) = 2^{-1/3}\,\mathrm{Ai}(2^{-1/3}(\lambda+\tilde{\lambda}+2m))$. Performing the change of variables $\lambda\mapsto 2^{1/3}\lambda$, $\tilde{\lambda}\mapsto 2^{1/3}\tilde{\lambda}$ in the Fredholm determinant shows that the determinants in (1.7) equal $\det(I - P_0B_{4^{1/3}m}P_0)$.

The rest of the paper is organized as follows. In Section 2 we give an overview of the approach of Johansson [14] explaining how Theorem 1 can be obtained indirectly using the connection of the Airy$_2$ process with last passage percolation. Sections 3 and 4 are devoted respectively to the proofs of Theorems 2 and 4. Finally, Section 5 provides the proof of (1.5).

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2. Indirect derivation of Theorem 1 through last passage percolation

As we mentioned in the introduction, Johansson [14] presented an indirect proof of the main result of this paper by way of the PNG model. His idea was entirely correct, but in the process of translating between the available results at the time, a factor of $4^{1/3}$ was lost. The purpose of this section is to explain Johansson's approach and explain where the missing $4^{1/3}$ went.

We consider the PNG model with two types of initial conditions (droplet and flat), and show that by coupling them to the same Poisson point environment we can represent the one-point distribution for the flat case as the maximum of the interface in the droplet case. Asymptotics of this relationship leads to the identity in Theorem 1.

Consider a space-time Poisson point process P of intensity 2. Define a height function above x at time t as

$$h_g(x,t) = \max_{\pi: g \to (x,t)} T(\pi)$$

where g represents a space-time curve $(g(x),x)_{x\in\mathbb{R}}$, π is a Lipschitz 1 function of time (i.e., $|\pi(s)-\pi(s')| \le |s-s'|$ for all s,s'), $\pi : g \to (x,t)$ means that π starts at a point of the form $(g(x),x)$ and ends at the point (x,t), and $T(\pi)$ represents the sum of the number of Poisson points that π touches. We will specialize this definition to two cases. In the *droplet* geometry (for which we write h^{droplet}) we take $f = \{|x|,x\}_{x\in\mathbb{R}}$, hence we only consider paths originating along a wedge. As a result the maximal path will always originate at the origin $(0,0)$. In the *flat* geometry (for which we write h^{flat}) we take $f = \{0,x\}_{x\in\mathbb{R}}$, hence we consider Lipschitz paths starting in any spatial location at time 0 and ending at x at time t.

Couple P to another Poisson point process $\tilde{P}$ via $\tilde{P}(A) = P(\tau_t A)$, where for any Borel set $A \in \mathbb{R}^2$, $(y,s) \in \tau_t A$ if and only if $(-y, t-s) \in A$ (one should think of this as a time-reversal of the Poisson point process where $s \mapsto t-s$ and $x \mapsto -x$). Let $\tilde{h}^{\text{flat}}$ represent the flat geometry height function built on the $\tilde{P}$ Poisson point process. Then the following relation holds

$$\tilde{h}^{\text{flat}}(t,0) = \max_{x\in\mathbb{R}} h^{\text{droplet}}(t,x).$$

Asymptotic fluctuation statistics have been derived for both the droplet and flat geometries and (up to justification of taking the limit inside the maximum, as done in [14] for a related model) the limiting statistics also respect the same relationship above. Specifically [17] (see also [5] for the specific choices of scaling used below) shows that

$$\lim_{t\to\infty} \frac{h^{\text{droplet}}(t, t^{2/3}x) - 2t}{t^{1/3}} = \mathcal{A}_2(x) - x^2.$$

This implies that (up to the justifications mentioned above)

$$\lim_{t\to\infty} \frac{\tilde{h}^{\text{flat}}(t,0)-2t}{t^{1/3}} = \max_{x\in\mathbb{R}} \left(\mathcal{A}_2(x)-x^2\right).$$

On the other hand, [6] shows that

$$\lim_{t\to\infty} \frac{\tilde{h}^{\text{flat}}(t,0)-2t}{t^{1/3}} = 2^{1/3}\mathcal{A}_1(0)$$

where A_1 is the Airy$_1$ process. Combining these two identities shows that

$$\mathbb{P}\big(\max_{x\in\mathbb{R}}(\mathcal{A}_2(x)-x^2)\le m\big) = \mathbb{P}\big(\mathcal{A}_1(0)\le 2^{-1/3}m\big) = F_{\mathrm{GOE}}(4^{1/3}m),$$

where the last equality follows from work of Ferrari and Spohn [10] which shows that $\mathbb{P}(\mathcal{A}_1(0)<m)=F_{\mathrm{GOE}}(2m)$.

3. Proof of Theorem 2

To prove this result we need first to recall some facts about Fredholm determinants, trace class operators and Hilbert-Schmidt operators (see Section 2.3 in [2] for more details, a complete treatment can be found in [21]). Consider a separable Hilbert space $\mathcal{H}$ and let A be a bounded linear operator acting on $\mathcal{H}$. Let $|A|=\sqrt{A^*A}$ be the unique positive square root of the operator A^*A. The *trace norm* of A is defined as $\|A\|_1=\sum_{n=1}^{\infty}\langle e_n,|A|e_n\rangle$, where $\{e_n\}_{n\ge1}$ is any orthormal basis of $\mathcal{H}$. We say that $A\in\mathcal{B}_1(\mathcal{H})$, the family of *trace class operators*, if $\|A\|_1<\infty$. For $A\in\mathcal{B}_1(\mathcal{H})$, one can define the trace $\operatorname{tr}(A)=\sum_{n=1}^{\infty}\langle e_n,Ae_n\rangle$ and then the *Hilbert-Schmidt norm* $\|A\|_2=\sqrt{\operatorname{tr}(|A|^2)}$. We say that $A\in\mathcal{B}_2(\mathcal{H})$, the family *Hilbert-Schmidt operators*, if $\|A\|_2<\infty$. The following lemma collects some results which we will need in the sequel, they can be found in Chapters 1-3 of [21]:

Lemma 3.1.

(a) *$A\mapsto\det(I+A)$ is a continuous function on $\mathcal{B}_1(\mathcal{H})$. Explicitly,*

$$|\det(I+A)-\det(I+B)|\le\|A-B\|_1\exp(\|A\|_1+\|B\|_1+1).$$

(b) *If $A\in\mathcal{B}_1(\mathcal{H})$ and $A=BC$ with $B,C\in\mathcal{B}_2(\mathcal{H})$, then $\|A\|_1\le\|B\|_2\|C\|_2$.*

(c) *If $\|A\|_{\mathrm{op}}$ denotes the operator norm of A in $\mathcal{H}$, then $\|A\|_{\mathrm{op}}\le\|A\|_2\le\|A\|_1$ and $\|AB\|_1\le\|A\|_{\mathrm{op}}\|B\|_1$.*

(d) *If $A\in\mathcal{B}_2(\mathcal{H})$, then $\|A^*\|_2=\|A\|_2$. If A has integral kernel $A(x,y)$, then*

$$\|A\|_2=\left(\int dx\,dy\,|A(x,y)|^2\right)^{1/2}.$$

Proposition 3.2. *Assume $g\in H^1([-L,L])$. The operators $e^{-2LH}-\Theta^g_{n,L}$ and $e^{-2LH}-\Theta^g_L$ are in $\mathcal{B}_2(L^2(\mathbb{R}))$, with $\|e^{-2LH}-\Theta^g_{n,L}\|_2$ bounded uniformly in n. Furthermore, for any fixed L we have, writing $n_k=2^k$,*

$$\lim_{k\to\infty}\|(e^{-2LH}-\Theta^g_{n_k,L})-(e^{-2LH}-\Theta^g_L)\|_2=0.$$

Proof. Assume first that $g(s)=s^2+m$ for some $m\in\mathbb{R}$ and recall the formula for $\Theta^g_L(x,y)$ given in Theorem 3:

$$\Theta^g_L(x,y)=\frac{e^{-(x-y)^2/8L-(x+y)L+2L^3/3}}{\sqrt{8\pi L}}\mathbb{P}_{\hat{b}(-L)=x,\hat{b}(L)=y}\Big(\hat{b}(s)\le m+L^2\text{ on }[-L,L]\Big). \tag{3.1}$$

Similarly, the kernel of e^{-2LH} equals the above one with the probability replaced by 1, and hence

$$\big(e^{-2LH}-\Theta^g_L\big)(x,y)=\frac{e^{-(x-y)^2/8L-(x+y)L+2L^3/3}}{\sqrt{8\pi L}} \cdot \mathbb{P}_{\hat b(-L)=x,\hat b(L)=y}\Big(\hat b(s)\geq m+L^2 \text{ for some } s\in[-L,L]\Big).$$

Therefore

$$\|e^{-2LH}-\Theta^g_L\|_2^2\leq\frac{1}{8\pi L}\int_{\mathbb{R}^2}dx\,dy\,\big[e^{-(x-y)^2/8L-(x+y)L+2L^3/3}\big]^2<\infty,$$

that is, $e^{-2LH}-\Theta^g_L\in\mathcal{B}_2(L^2(\mathbb{R}))$.

Next we observe that we can apply the Feynman-Kac and Cameron-Martin-Girsanov formulas directly on $\Theta^g_{n,L}$ (n times) exactly as we did for Θ^g_L, and it is not hard to check that we get a formula analogous to (3.1):

$$\Theta^g_{n,L}(x,y)=\frac{1}{\sqrt{8\pi L}}e^{-(x-y)^2/8L-(x+y)L+2L^3/3}\,\mathbb{P}_{\hat b^n(-L)=x,\hat b^n(L)=y}\Big(\hat b^n(s)\leq m+L^2 \text{ on } [-L,L]\Big),$$

where $\hat b^n$ is now a discrete time random walk with Gaussian jumps with mean 0 and variance $2L/n$, started at time $-L$ at x, conditioned to hit y at time L, and jumping at times $t^n_i=-L+2iL/n$, $i\geq 0$. We deduce then that

$$\big(e^{-2LH}-\Theta^g_{n,L}\big)(x,y)=\frac{1}{\sqrt{8\pi L}}e^{-(x-y)^2/8L-(x+y)L+2L^3/3} \cdot \mathbb{P}_{\hat b^n(-L)=x,\hat b^n(L)=y}\Big(\hat b^n(t^n_i)\geq m+L^2 \text{ for some } i\in\{0,\dots,n\}\Big).$$

Thus we obtain for $\|e^{-2LH}-\Theta^g_{n,L}\|_2$ the same bound as for $\|e^{-2LH}-\Theta^g_L\|_2$ which is, in particular, independent of n.

Finally, in order to deal with $(e^{-2LH}-\Theta^g_L)-(e^{-2LH}-\Theta^{n_k}_L)$, we couple the Brownian bridge $\hat b$ and the conditioned random walk $\hat b^{n_k}$ by simply letting $\hat b^{n_k}(t^{n_k}_i)=\hat b(t^{n_k}_i)$ for each $i=0,\dots,n_k$. Since the Brownian bridge hits the barrier at $m+L^2$ whenever the conditioned random walk does, it is clear that

$$\big|\big(e^{-2LH}-\Theta^g_L\big)(x,y)-\big(e^{-2LH}-\Theta^{n_k}_L\big)(x,y)\big|=\frac{1}{\sqrt{8\pi L}}e^{-(x-y)^2/8L-(x+y)L+2L^3/3}q_{n_k}(x,y),$$

where $q_{n_k}(x,y)$ is the probability that the Brownian bridge $\hat b(s)$ hits $m+L^2$ for $s\in[-L,L]$ but not for any $s\in\{t^{n_k}_0,\dots,t^{n_k}_{2n_k}\}$. Since every point is regular for one-dimensional Brownian motion, it is clear that $q_{n_k}(x,y)\searrow 0$ as $k\to\infty$ for every fixed x,y, and thus by the monotone convergence theorem we deduce that $\|(e^{-2LH}-\Theta^g_L)-(e^{-2LH}-\Theta^{n_k}_L)\|_2\to 0$ as $k\to\infty$.

To extend the result to $g\in H^1([-L,L])$ we note that everything in the above argument deals with almost sure properties of a Brownian motion $b(s)$ killed at the boundary $m+L^2$. In the general case we will have by Theorem 3 a Brownian motion $b(s)$ killed at the boundary $g(s)-s^2+L^2$ or, equivalently, a process $\tilde b(s)=b(s)-g(s)+s^2$ killed at the boundary L^2. Thus the extension follows from the Cameron-Martin-Girsanov Theorem. □

Proof of Theorem 2. For $n>0$ let $t_i=-L+2iL/n$, $i=0,\dots,n$. Given $g\in H^1([-L,L])$ we have by (1.2) that

$$\mathbb{P}(\mathcal{A}_2(t_0)\leq a(t_0),\dots,\mathcal{A}_2(t_n)\leq a(t_n))=\det\Big(I-K_{\rm Ai}+\Theta^g_{n,L}e^{2LH}K_{\rm Ai}\Big),$$

where

$$\Theta^g_{n,L}=\bar P_{g(t_0)}e^{(t_0-t_1)H}\bar P_{g(t_1)}e^{(t_1-t_2)H}\cdots e^{(t_{n-1}-t_n)H}\bar P_{g(t_n)}.$$

Since the Airy$_2$ process has a version with continuous paths (see Theorem 4.3 in [17] and also [14]), the probability above converges as $n \to \infty$ to $\mathbb{P}(\mathcal{A}_2(t) \leq g(t)$ for $t \in [-L, L])$. On the other hand we may rewrite $K_{\mathrm{Ai}} - \Theta^g_{n,L} e^{2LH} K_{\mathrm{Ai}}$ as $(e^{-2LH} - \Theta^g_{n,L}) e^{2LH} K_{\mathrm{Ai}}$ (and do the same with Θ^g_L instead) and then use Proposition 3.2, Lemma 3.1 and the fact that $e^{2LH} K_{\mathrm{Ai}} \in \mathcal{B}_2(L^2(\mathbb{R}))$ to deduce that

$$\lim_{n\to\infty} \det\bigl(I - K_{\mathrm{Ai}} + \Theta^g_{n,L} e^{2LH} K_{\mathrm{Ai}}\bigr) = \det\bigl(I - K_{\mathrm{Ai}} + \Theta^g_L e^{2LH} K_{\mathrm{Ai}}\bigr).$$

The result now follows by the cyclic property of determinants. □

4. Proof of Theorem 4

Using the facts that e^{LH} and K_{Ai} commute and that $K_{\mathrm{Ai}} = A\bar{P}_0 A^*$ we have

$$e^{LH} K_{\mathrm{Ai}} R_L e^{LH} K_{\mathrm{Ai}} = A\bar{P}_0 A^* e^{LH} R_L e^{LH} A\bar{P}_0 A^*.$$

Because $e^{rH} A(x, \lambda) = e^{\lambda r} \operatorname{Ai}(x - \lambda)$ this can be rewritten as $A\bar{P}_0 \hat{R}_L \bar{P}_0 A^*$ where

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{1}{\sqrt{8\pi L}} \int_{\mathbb{R}^2} d\tilde{z}\, dz\, e^{-(z+\tilde{z}-2m-2L^2)^2/8L-(z+\tilde{z})L+(\lambda+\tilde{\lambda})L+2L^3/3} \operatorname{Ai}(z - \lambda) \operatorname{Ai}(\tilde{z} - \tilde{\lambda}).$$

Applying the change of variables $2u = z + \tilde{z}$, $2v = z - \tilde{z}$, we get

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{1}{\sqrt{2\pi L}} \int_{\mathbb{R}^2} du dv\, e^{-\frac{(u-m-L^2)^2}{2L} - 2uL + (\lambda+\tilde{\lambda})L + \frac{2}{3}L^3} \operatorname{Ai}(u + v - \lambda) \operatorname{Ai}(u - v - \tilde{\lambda}).$$

Using the formula

$$\int_{-\infty}^{\infty} dx \operatorname{Ai}(a + x) \operatorname{Ai}(b - x) = 2^{-1/3} \operatorname{Ai}(2^{-1/3}(a + b))$$

(see, for example, (3.108) in [23]), the v integral equals $2^{-1/3} \operatorname{Ai}(2^{-1/3}(2u - \lambda - \tilde{\lambda}))$. Therefore

$$\begin{aligned}\hat{R}_L(\lambda, \tilde{\lambda}) &= \frac{2^{-1/3}}{\sqrt{2\pi L}} \int_{-\infty}^{\infty} du\, e^{-(u-m-L^2)^2/2L-2uL+(\lambda+\tilde{\lambda})L+2L^3/3} \operatorname{Ai}(2^{-1/3}(2u - \lambda - \tilde{\lambda}))\\ &= \frac{2^{-1/3}}{\sqrt{2\pi L}} \frac{1}{2\pi i} \int_\Gamma dt \int_{-\infty}^{\infty} du\, e^{-(u-m-L^2)^2/2L-2uL+(\lambda+\tilde{\lambda})L+2L^3/3+t^3/3-2^{-1/3}t(\lambda+\tilde{\lambda}-2u)},\end{aligned}$$

where in the second equality we have used the contour integral representation of the Airy function, $\operatorname{Ai}(x) = \frac{1}{2\pi i} \int_\Gamma dt\, e^{t^3/3 - tx}$ with $\Gamma = \{c + is \colon s \in \mathbb{R}\}$ and c any positive real number. The u integral is just a Gaussian integral, and computing it we get

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{2^{-1/3}}{2\pi i} \int_\Gamma dt\, e^{t^3/3+2^{1/3}Lt^2+[4^{1/3}L^2+2^{-1/3}(\lambda+\tilde{\lambda}-2m)]t+2L^3/3+(\lambda+\tilde{\lambda})L}.$$

Now we perform the change of variables $t = s - 2^{1/3}L$ to obtain

$$\hat{R}_L(\lambda, \tilde{\lambda}) = \frac{2^{-1/3}}{2\pi i} \int_{\Gamma'} ds\, e^{s^3/3-2^{-1/3}(2m-\lambda-\tilde{\lambda})s} = 2^{-1/3} \operatorname{Ai}(2^{-1/3}(2m - \lambda - \tilde{\lambda}))$$

(here the contour Γ' is simply Γ shifted by $2^{1/3}L$, so the integral still gives an Airy function). Note how all the terms involving L have canceled.

5. Proof of (1.5)

We need to state two results first. The first one is a version of Laplace's method, which we state without proof (see, for instance, [8]):

Lemma 5.1. *Let*

$$I(M)=\int_{\Omega}dx\,f(x)e^{\varphi(x)M},$$

where $\Omega\subseteq\mathbb{R}^n$ is a (possibly unbounded) open polygonal domain and f and φ are smooth functions defined on $\overline{\Omega}$. Assume that the local maxima of φ are attained at a finite subset $\{x_1,\dots,x_n\}$ of $\overline{\Omega}$. Then there is a constant $C>0$ such that

$$|I(M)|\leq C\sum_{k=1}^{n}M^{-\kappa_i}|f(x_i)|e^{\varphi(x_i)M}$$

for large enough M, where $\kappa_i=(n+1)/2$ if $x_i\in\partial\Omega$ and $\kappa_i=n/2$ if $x_i\in\Omega$.

The second result involves the norm of a certain integral operator.

Lemma 5.2. *Let $E_L=e^{LH/2}A\bar{P}_0$. Then*

$$\|E_L\|_{\mathrm{op}}=\|e^{LH/2}K_{\mathrm{Ai}}\|_{\mathrm{op}}\leq(2L)^{-1/2}.$$

Proof. The equality is a direct consequece of the Plancherel formula (5.1) for the Airy transform,

$$\int f^2=\int(Af)^2=\int(A^*f)^2, \tag{5.1}$$

and the fact that $K_{\mathrm{Ai}}=A\bar{P}_0A^*$. By Lemma 3.1 $\|e^{LH/2}K_{\mathrm{Ai}}\|_{\mathrm{op}}\leq\|e^{LH/2}K_{\mathrm{Ai}}\|_2$, for which we have

$$\begin{aligned}\|e^{LH/2}K_{\mathrm{Ai}}\|_2^2&=\int_{\mathbb{R}^2}dx\,dy\int_{(-\infty,0]^2}d\lambda\,d\tilde{\lambda}\,e^{(\lambda+\tilde{\lambda})L}\,\mathrm{Ai}(x-\lambda)\,\mathrm{Ai}(y-\lambda)\,\mathrm{Ai}(x-\tilde{\lambda})\,\mathrm{Ai}(y-\tilde{\lambda})\\&=\int_{(-\infty,0]^2}d\lambda\,d\tilde{\lambda}\,e^{(\lambda+\tilde{\lambda})L}\delta_{\lambda=\tilde{\lambda}}=(2L)^{-1}.\end{aligned}$$

□

We will also use the following well-known estimates for the Airy function (see (10.4.59-60) in [1]): for $x>0$,

$$|\mathrm{Ai}(x)|\leq Ce^{-2/3x^{3/2}}\quad\text{for }x>0,\qquad|\mathrm{Ai}(x)|\leq C\quad\text{for }x\leq0. \tag{5.2}$$

We write $\widetilde{\Omega}_L=\widetilde{\Omega}_L^1+\widetilde{\Omega}_L^2$ where

$$\begin{aligned}\widetilde{\Omega}_L^1&=e^{LH}K_{\mathrm{Ai}}\left(\bar{P}_{c+L^2}R_L\bar{P}_{c+L^2}-R_L\right)e^{LH}K_{\mathrm{Ai}},\\\widetilde{\Omega}_L^2&=e^{LH}K_{\mathrm{Ai}}\left(\bar{P}_{c+L^2}e^{-2LH}\bar{P}_{c+L^2}-e^{-2LH}\right)e^{LH}K_{\mathrm{Ai}},\end{aligned}$$

The proof of (1.5) is contained in the next two lemmas.

Lemma 5.3.

$$\|\widetilde{\Omega}_L^1\|_1\xrightarrow[L\to\infty]{}0.$$

Proof. From the definition of $\widetilde{\Omega}_L^1$ we can write

$$\widetilde{\Omega}_L^1(x,y)=\frac{1}{\sqrt{8\pi L}}\int_{(-\infty,0]^2}d\tilde{\lambda}\,d\lambda\int_{\widetilde{D}}d\tilde{z}\,dz\,e^{-(z+\tilde{z}-2m-2L^2)^2/8L-(z+\tilde{z})L+(\lambda+\tilde{\lambda})L+2L^3/3}\cdot\mathrm{Ai}(x-\lambda)\,\mathrm{Ai}(z-\lambda)\,\mathrm{Ai}(\tilde{z}-\tilde{\lambda})\,\mathrm{Ai}(y-\tilde{\lambda}),$$

where $\widetilde{D} = \mathbb{R}^2 \setminus [m+L^2,\infty)^2$. Since $K_{\mathrm{Ai}} = A\bar{P}_0A^*$ and $e^{LH}K_{\mathrm{Ai}} = e^{LH/2}K_{\mathrm{Ai}}e^{LH/2}$, we factorize this operator as

$$\widetilde{\Omega}^1_L = E_L\hat{\Omega}^1_L E_L^*,$$

where $E_L = e^{LH/2}A\bar{P}_0$ was defined in Lemma 5.2 and

$$\hat{\Omega}^1_L(\lambda,\tilde{\lambda}) = \frac{1}{\sqrt{8\pi L}}\int_D dw\,d\tilde{w}\,e^{-(w+\tilde{w})^2/8L+(w+\tilde{w})L-2mL-4L^3/3-(\lambda+\tilde{\lambda})L/2} \cdot \mathrm{Ai}(-\lambda-w+m+L^2)\,\mathrm{Ai}(-\tilde{\lambda}-\tilde{w}+m+L^2)\mathbf{1}_{\lambda,\tilde{\lambda}\le 0}, \tag{5.3}$$

where $D = \mathbb{R}^2\setminus[0,\infty)^2$ and we have performed the change of variables $w = -z+m+L^2$, $\tilde{w} = -\tilde{z}+m+L^2$. Using Lemmas 3.1 and 5.2 we get

$$\|\widetilde{\Omega}^1_L\|_1 \le \|E_L\|_{\mathrm{op}}\|\hat{\Omega}^1_L E_L^*\|_2 \le \|E_L\|^2_{\mathrm{op}}\|\hat{\Omega}^1_L\|_2 \le (2L)^{-2}\|\hat{\Omega}^1_L\|_2. \tag{5.4}$$

Therefore it will be enough to show that $\|\hat{\Omega}^1_L\|_2$ goes to zero as $L\to\infty$.

Applying the change of variables $w\mapsto L^2w$ and $\tilde{w}\mapsto L^2\tilde{w}$ in (5.3) the kernel becomes

$$\hat{\Omega}^1_L(\lambda,\tilde{\lambda}) = \frac{L^{7/2}e^{(\lambda+\tilde{\lambda})L/2-2mL}}{\sqrt{8\pi}}\int_D dw\,d\tilde{w}\,e^{L^3f(w,\tilde{w})}\,\mathrm{Ai}(L^2(1-w)+m-\lambda) \cdot \mathrm{Ai}(L^2(1-\tilde{w})+m-\tilde{\lambda})\mathbf{1}_{\lambda,\tilde{\lambda}\le 0}, \tag{5.5}$$

where $f(w,\tilde{w}) = \frac{-(w+\tilde{w})^2}{8} + (w+\tilde{w}) - \frac{4}{3}$.

We split the region D into the union of three disjoint regions of pairs $(w,\tilde{w})$: $D_1 = \{w\le 1, \tilde{w}\le 1\} - \{0\le w\le 1, 0\le\tilde{w}\le 1\}$, $D_2 = \{w\le 0, \tilde{w}\ge 1\}$ and $D_2' = \{w\ge 1, \tilde{w}\le 0\}$. By the triangle inequality we can bound $|\hat{\Omega}^1_L(\lambda,\tilde{\lambda})|$ by the sum of the absolute value of the integrals over the regions D_1, D_2 and D_2'. Notice that, due to the symmetry of our formula, we do not need to bound the integral on D_2', as the bound for D_2 immediately follows also for D_2'.

Let us focus first on the integral over region D_1. Here we have $1-w\ge 0$ as well as $1-\tilde{w}\ge 0$, so using (5.2) we can bound the integral over this region

$$\frac{L^{7/2}e^{(\lambda+\tilde{\lambda})L/2-2mL}}{\sqrt{8\pi}}\int_{D_1} dw\,d\tilde{w}\,\left|e^{L^3f_1(w,\tilde{w})}\right|\mathbf{1}_{\lambda,\tilde{\lambda}\le 0}$$

where

$$f_1(w,\tilde{w}) = \frac{-(w+\tilde{w})^2}{8} + (w+\tilde{w}) - \tfrac{4}{3} - \tfrac{2}{3}(1-w)^{3/2} - \tfrac{2}{3}(1-\tilde{w})^{3/2}.$$

This function f_1 achieves its maximum on D_1 at the points $(1,0)$ and $(0,1)$, where its value is $-9/8$. Lemma 5.1 then allows to conclude that the integral is bounded by

$$CL^{3/2}(L^3)^{-3/2}e^{-9/8L^3+(\lambda+\tilde{\lambda})L/2-2mL}\mathbf{1}_{\lambda,\tilde{\lambda}\le 0}.$$

Let us now turn to the bound the integral over region D_2 (and hence also D_2'). This bound is slightly harder owing to the fact that one of the Airy functions is oscillatory (rather than rapidly decaying) in this region. As readily derived from the contour integral representation of the Airy function by deforming the contour and performing a change of variables, $\mathrm{Ai}(\cdot)$ may alternatively be expressed as

$$\mathrm{Ai}(x) = \mathrm{Re}\left[\frac{\sqrt{-x}}{2\pi i}\int_\Gamma ds\,\exp(i(-x)^{3/2}(-s+s^3/3))\right],$$

where Γ is the contour $\{s = a+b(a)i : a>0\}$ with $b(a) = (a-1)\sqrt{\frac{a+2}{3a}}$. This contour is the steepest descent contour for $f(s) = i(-s+s^3/3)$ and has the property that $\mathrm{Im}\,f(s) = \mathrm{Im}\,f(s_0) = -2/3$, where $s_0 = 1$ is a critical point of f. Along Γ we can write $f(s) =$

$-2/3i+g(s)$ where $g(s)$ is real valued, $g(s_0)=0$ and $g(s)$ decays to $-\infty$ monotonically and quadratically with respect to $|s-s_0|$. Thus we may also write

$$\mathrm{Ai}(x)=\tfrac{1}{2}(G(-x)+G(-\bar{x})) \tag{5.6}$$

where

$$G(x)=\exp(-\tfrac{2}{3}x^{3/2}i)\frac{\sqrt{x}}{2\pi}\int_{\Gamma}ds\,\exp(x^{3/2}g(s)).$$

This expansion of the Airy function is the key to our oscillatory asymptotics.

By applying the change of variables $w=0+L^{-3/2}v$ and $\tilde{w}=4+L^{-3/2}\tilde{v}$ the integral we wish to bound is given by

(5.7)

$$\hat{\Omega}^1(\lambda,\tilde{\lambda})=\frac{L^{1/2}e^{-(\lambda+\tilde{\lambda})L/2}}{\sqrt{8\pi}}\int_{-\infty}^{0}dv\,h_L(v)\int_{-3L^{3/2}}^{\infty}d\tilde{v}\,e^{-\frac{(v+\tilde{v})^2}{8}}\,\mathrm{Ai}(-3L^2-L^{1/2}\tilde{v}+m-\tilde{\lambda})\mathbf{1}_{\lambda,\tilde{\lambda}\le 0},$$

where

$$h_L(v)=e^{\frac{2}{3}L^3}\,\mathrm{Ai}(L^2-L^{1/2}v+m-\lambda).$$

We will focus on the inner integral in $\tilde{v}$ and prove that it is bounded by e^{-CL^3} for some $C>0$. As the Airy function is bounded on the real axis, we readily find that due to the Gaussian term, we may cut our integral outside of a region $R_\delta=(-\delta L^{3/2},\delta L^{3/2})$ by introducing an error of order $e^{-C'L^3}$. Thus we may restrict attention to R_δ.

Using the expansion given by equation (5.6), the inner integral may be written as $\frac{1}{2}(I_L^1+I_L^2)$ where

$$I_L^1=\int_{-\delta L^{3/2}}^{\delta L^{3/2}}d\tilde{v}\,e^{-\frac{(v+\tilde{v})^2}{8}}G(3L^2+L^{1/2}\tilde{v}+\tilde{\lambda}),\quad I_L^2=\int_{-\delta L^{3/2}}^{\delta L^{3/2}}d\tilde{v}\,e^{-\frac{(v+\tilde{v})^2}{8}}G(3L^2+L^{1/2}\bar{\tilde{v}}+\tilde{\lambda}).$$

(note that in I_L^2 we take the complex conjugate of $\tilde{v}$ inside G). We wish to show that both I_L^1 and I_L^2 are bounded by e^{-CL^3} for possibly different positive constants C. Let us focus on I_L^1 first.

On the region of integration we may perform a change of variables from $\tilde{v}$ to r by setting

$$L^3r=(3L^2+L^{1/2}\tilde{v}+\tilde{\lambda})^{3/2}.$$

This shows that

$$I_L^1=\frac{2}{3}L^{3/2}\int_{3^{3/2}-\delta'}^{3^{3/2}+\delta'}dr\,r^{-1/3}e^{-\frac{L^3}{8}(v+[r^{2/3}-3-\tilde{\lambda}L^{-1/2}])^2}G((L^3r)^{2/3}).$$

Since we can consider an arbitrary δ before the change of variables, we can likewise consider an arbitrary $\delta'>0$ for which to bound I_L^1. Plugging in the expression for G we have

$$I_L^1=\frac{L^{5/2}}{3\pi}\int_{3^{3/2}-\delta'}^{3^{3/2}+\delta'}dr\,e^{-L^3\left[\frac{(vL^{-3/2}+[r^{2/3}-3-\tilde{\lambda}L^{-2}])^2}{8}-\frac{2}{3}ri\right]}\int_{\Gamma}ds\,\exp(L^3rg(s))$$

For simplicity we set $v=\tilde{\lambda}=0$, though the argument below does not rely on this assumption and applies equally well for all $\tilde{\lambda}\le 0$ and $v<0$ as necessary. Under this simplification we have

$$I_L^1=\frac{L^{5/2}}{3\pi}\int_{3^{3/2}-\delta'}^{3^{3/2}+\delta'}dr\,e^{-L^3\left[\frac{(r^{2/3}-3)^2}{8}-\frac{2}{3}ri\right]}\int_{\Gamma}\exp(L^3rg(s))ds.$$

Observe that this integrand is analytic in r. Thus by Cauchy's theorem, rather than integrating from $3^{3/2}-\delta'$ to $3^{3/2}+\delta'$ along the real axis, we may do so along any other curve between these points. Due to the properties of $g(s)$ along Γ, as long as $\mathrm{Re}(r)>0$ we have that

$$\left|\int_{\Gamma}ds\exp(L^3rg(s))\right|\le\int_{\Gamma}ds\exp(L^3\,\mathrm{Re}(r)g(s)),$$

which is certainly a bounded function of r for $\mathrm{Re}(r) > 0$. The decay of the integrand is thus controlled by

$$\mathrm{Re}\Big(-\Big[\tfrac{1}{8}(r^{2/3}-3)^2 - \tfrac{2}{3}ri\Big]\Big). \tag{5.8}$$

Informed by this we may deform the r integration contour to the contour $B = B_1 \cup B_2 \cup B_3$ where $B_1 = \{3^{3/2} - \delta' + iy : y \in [0,\eta]\}$, $B_2 = \{x + i\eta : x \in [3^{3/2} - \delta', 3^{3/2} + \delta']\}$ and $B_3 = \{3^{3/2} + \delta' + iy : y \in [0,\eta]\}$. It is an exercise in basic complex analysis to see that one can choose η in such a way that, along the contour B, (5.8) stays bounded below a constant $-C$ for $C > 0$. This implies that the exponential is bounded by e^{-CL^3} along that curve and hence the entire integral $|I_L^1| \leq e^{-CL^3}$ for some $C > 0$. The integral I_L^2 is likewise bounded by using the same argument but deforming to a contour $\bar{B}$ which is the reflection of B through the real axis.

Returning to (5.7), it now suffices to prove that $h_L(v)$ does not grow like e^{CL^3} (for $v < 0$). This follows readily from using (5.2), which imply that

$$\log(h_L(v)) \approx \tfrac{2}{3}L^3 - \tfrac{2}{3}(L^2 - L^{1/2}v + \lambda)^{2/3} \approx CL^{3/2}v, \tag{5.9}$$

for some fixed $C > 0$. Note how the L^3 terms perfectly cancel. This shows that the entire integral in (5.7) can be bounded like e^{-CL^3}.

As noted before, we may likewise develop a bound for the integral over region D_2' and thus we have established that

$$|\hat{\Omega}_L^1(\lambda,\tilde{\lambda})| < e^{-CL^3} e^{(\lambda+\tilde{\lambda})L/2 - 2cL}\mathbf{1}_{\lambda,\tilde{\lambda}\leq 0}$$

for some $C > 0$. The Hilbert-Schmidt norm of $\hat{\Omega}_L^1$ is then readily computed by integrating $|\hat{\Omega}_L^1(\lambda,\tilde{\lambda})|^2$ over $\lambda,\tilde{\lambda} \leq 0$. Doing this gives the upper bound

$$\|\hat{\Omega}_L^1\|_2^2 \leq e^{-CL^3} e^{-2cL}, \tag{5.10}$$

which goes to zero as $L \to \infty$. Using this and (5.4) the result follows. ☐

Lemma 5.4.

$$\|\widetilde{\Omega}_L^2\|_1 \xrightarrow[L\to\infty]{} 0.$$

Proof. The proof of this result is the same as that of the previous lemma. Using the definition of $\widetilde{\Omega}_L^2$ and factorizing as in the above proof we get

$$\widetilde{\Omega}_L^2 = E_L \hat{\Omega}_L^2 E_L^*$$

with

$$\hat{\Omega}_L^2(\lambda,\tilde{\lambda}) = \frac{e^{-2mL+(\lambda+\tilde{\lambda})L/2}}{\sqrt{8\pi L}} \int_D d\tilde{w}\, dw\, e^{-(w-\tilde{w})^2/8L + (w+\tilde{w})L - 4L^3/3} \cdot \mathrm{Ai}(w + L^2 - \lambda + m)\,\mathrm{Ai}(\tilde{w} + L^2 - \tilde{\lambda} + m)\mathbf{1}_{\lambda,\tilde{\lambda}\leq 0}.$$

Applying the change of variables $w \mapsto L^2 w$ and $\tilde{w} \mapsto L^2\tilde{w}$ the kernel becomes

$$\hat{\Omega}_L^2(\lambda,\tilde{\lambda}) = \frac{L^{7/2}e^{(\lambda+\tilde{\lambda})L/2 - 2mL}}{\sqrt{8\pi}} \int_D dw\, d\tilde{w}\, e^{L^3\tilde{f}(w,\tilde{w})}\,\mathrm{Ai}(L^2(1-w) + m - \lambda) \cdot \mathrm{Ai}(L^2(1-\tilde{w}) + m - \tilde{\lambda})\mathbf{1}_{\lambda,\tilde{\lambda}\leq 0},$$

where $\tilde{f}(w,\tilde{w}) = \frac{-(w-\tilde{w})^2}{8} + (w+\tilde{w}) - \frac{4}{3}$. Note the similarity with (5.5), the only difference being that in $\tilde{f}$ we have a term $-(w-\tilde{w})^2/8$ instead of $-(w+\tilde{w})^2/8$.

As in the above proof all we need is to bound $\|\hat{\Omega}_L^2\|_2$, and to that end we split D into the same three regions D_1, D_2 and D_2'. The integral over D_1 is easy to bound, exactly as before. On D_2 (and thus also on D_2') we can repeat the same argument as before. The only diference is that, when we apply the change of variables $w = 0 + L^{-3/2}v$ and $\tilde{w} = 4 + L^{-3/2}\tilde{v}$, the function $h_L(v)$ in the resulting integral in (5.7) is now multiplied by $e^{2vL^{3/2}}$, coming from the difference between $\tilde{f}$ and the function f defined after (5.5). This change does not affect the bound on the $\tilde{v}$ integral (I_L^1 and I_L^2 in the above proof). It is straightforward to check that the rest of the proof is not affected either (note in fact that the only place where the definition of $h_L(v)$ is used is (5.9), and the approximation there is still valid). Hence the bound given in (5.10) is also valid for $\|\hat{\Omega}_L^2\|_2$ and the result follows. □

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