

# Quantum 3D Superstrings

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## ABSTRACT

The classical Green-Schwarz superstring action, with  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  spacetime supersymmetry, exists for spacetime dimensions  $D = 3, 4, 6, 10$ , but quantization in the light-cone gauge breaks Lorentz invariance unless *either*  $D = 10$ , which leads to critical superstring theory, *or*  $D = 3$ . We give details of results presented previously for the bosonic and  $\mathcal{N} = 1$  closed 3D (super)strings and extend them to the  $\mathcal{N} = 2$  3D superstring. In all cases, the spectrum is parity-invariant and contains anyons of irrational spin.

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# 1 Introduction

Quantization of a relativistic string in a  $D$ -dimensional Minkowski background spacetime is problematic unless  $D$  is the critical dimension ( $D = 26$  for the Nambu-Goto string and  $D = 10$  for superstrings). The difficulty is seen most clearly in the light-cone gauge, which removes all but the physical degrees of freedom that are present classically; this ensures unitarity but quantum anomalies then break Lorentz invariance in any (generic) non-critical dimension [1] (see also [2], and [3] for a recent detailed computation). A corollary is that Lorentz-covariant quantization in a (generic) non-critical dimension can lead to a unitary theory only if it introduces one or more additional variables that are not present classically. In fact, this option is available only in sub-critical dimensions and it has not yet proved useful for  $D > 2$  (see e.g. [4]).

These problems with non-critical string theories are well-known except for the qualification “generic”, which refers to an exception that we exploited in an earlier paper [5] to which the present paper is a sequel: light-cone gauge quantization preserves Lorentz invariance not only in the critical dimension but also for  $D = 3$  (3D), trivially for the Nambu-Goto string<sup>1</sup>. The light-cone gauge quantization of the 3D Nambu-Goto closed string was carried out in [5] and it was confirmed that Lorentz invariance is preserved in the quantum theory, without the need for any additional variables not present classically. Quantum consistency is not completely trivial, however, because the absence of tachyons imposes an upper bound on the intercept parameter, corresponding to a lower bound on the eigenvalues of the mass-squared operator. It was also noted in [5] that the low-lying states of non-zero spin appear in parity doublets. Here we prove that this was no coincidence: the quantum theory preserves parity as well as Lorentz invariance.

The 3D Nambu-Goto closed string is sufficiently simple that one can easily determine the Lorentz representations of the states in low-lying levels explicitly (rather than having to rely on implicit arguments based on matching degeneracies to dimensions of Lorentz representations). The spin of the states in levels 2 and 3 was found to depend on the intercept parameter, not surprisingly but there is no choice of this parameter for which the spins in both these levels are either integral or half-integral; in other words, the spectrum contains anyons<sup>2</sup>.

We also observed in [5] that the spectrum contains *irrational* spins for a generic allowed choice of the intercept parameter. Here we further show, by computation of the spectrum at level 4, that some states *necessarily* have irrational spin. This result is significant because it implies that the Lorentz group of the quantum 3D

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<sup>1</sup>This was pointed out at the May 2010 Solvay workshop on “Symmetries and Dualities in Gravitational Theories” in a talk by one of us based on a draft version of our subsequent paper, and also by T. Curtright in independent work on a related topic over the same period [6]. We have been led to understand that the exceptional status of the bosonic 3D string was already known to experts but we are not aware of any earlier reference. Some classical aspects of the light-cone gauge for 3D strings have been discussed previously by Siegel [7].

<sup>2</sup>By “anyon” we mean a particle with spin not equal to an integer or half-integer. This differs, in principle, from the definition in terms of statistics but spin and statistics are related by the 3D spin-statistics theorem; see e.g. [8].

string is neither  $SO(2,1)$  nor any finite multiple cover, such as the double cover  $SI(2; \mathbb{C})$ , but rather its universal cover  $\overline{SO}(2,1)$ . Irrational spin irreps of  $\overline{SO}(2,1)$  are infinite-dimensional [9,10], so an infinite component field is needed for any manifestly Lorentz-invariant field theoretic description of a particle of irrational spin. Since irrational spin particles appear in the 3D string spectrum, it should not be a surprise that the Lorentz invariance of 3D quantum strings cannot easily be seen using current methods of covariant quantization.

The Nambu-Goto string has a natural generalization to a spacetime supersymmetric Green-Schwarz (GS) superstring, which exists classically for spacetime dimension  $D = 3, 4, 6, 10$ , with either  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  supersymmetry [11]. The GS superstring action has a fermionic “ $\kappa$ -symmetry” gauge invariance, in addition to worldsheet reparametrization invariance, but there is an extension of the light-cone gauge that again eliminates all but the physical degrees of freedom that are present classically. Quantization in  $D = 10$  leads to standard critical superstring theory (after the inclusion of open strings in the  $\mathcal{N} = 1$  case). Light-cone gauge quantization of the 3D  $\mathcal{N} = 1$  GS superstring was carried out in [5]. Not only are there no Lorentz anomalies but there are also no super-Poincaré anomalies, and parity is also preserved. Moreover, the intercept parameter is now fixed by supersymmetry such that the ground state, which is doubly degenerate, is massless. This is entirely consistent with the possibility that there exists a 3D  $\mathcal{N} = 1$  superstring theory with an effective  $\mathcal{N} = 1$  3D supergravity action since only the dilaton and dilatino of the latter would propagate massless modes.

It is convenient to refer to the Nambu-Goto string with zero intercept parameter as the  $\mathcal{N} = 0$  string; this string has spin-3/2 states at level-1 and irrational spin anyons at level-2. The  $\mathcal{N} = 1$  string is a 3D heterotic string in the sense that its spectrum is a tensor product of Lorentz irreps arising from an  $\mathcal{N} = 0$  right-moving sector with supermultiplets from an  $\mathcal{N} = 1$  supersymmetric left-moving sector. The spectrum was computed through level-2 in [5]: it was found that there are semion states (spin  $1/4 + n/2$  for integer  $n$ ) at levels 1 and 2 (and irrational spins must occur at higher levels because they are present for  $\mathcal{N} = 0$ ). By tensoring two factors of the supersymmetric left-moving sector of the  $\mathcal{N} = 1$  string, one can deduce from the results of [5] that the  $\mathcal{N} = 2$  string has only bosons and fermions through level-2, and there is no obvious reason why anyons should appear in higher levels. This is one reason why a discussion of the  $\mathcal{N} = 2$  superstring was omitted (aside from a comment about zero-mass modes) from [5]: it was not clear that it exemplified our title “Anyons from Strings”.

The principal purpose of this paper is to extend the results of [5] to the 3D  $\mathcal{N} = 2$  GS superstring, but we also present details, omitted from the very brief account in [5], of the quantization of the 3D Nambu-Goto string and of the 3D  $\mathcal{N} = 1$  GS superstring. In all cases, we shall restrict our attention to closed strings, although it should become clear that many results for open 3D strings can be inferred from our closed string results. The main issue that we wish to address for  $\mathcal{N} = 2$  is whether the spectrum contains anyons. If not then we would need to explain why this quantum 3D string had not previously been found using Lorentz-covariant quantization methods.

It might have been necessary to invoke the usual difficulties with  $\kappa$ -symmetry, but a computation of the spectrum at level-3 suffices to show that irrational spins are also present in the spectrum of the  $\mathcal{N} = 2$  3D GS superstring.

We begin with a preliminary section that recalls pertinent features of 3D physics and introduces some of our notation. A novelty of this section is a ‘re-interpretation’ of the 3D supersymmetry algebra as the algebra of a model of supersymmetric quantum mechanics. This simplifies the analysis of the structure of massive 3D supermultiplets.

We then consider, in succession, the 3D closed Nambu-Goto string, the  $\mathcal{N} = 1$  GS superstring, and finally the  $\mathcal{N} = 2$  GS superstring. In each case we show how gauge invariances may be fixed so as to leave only the residual global gauge invariance under shifts of the string coordinate  $\sigma$ , which becomes the level-matching condition in the quantum theory. In this we follow the classic work of Goddard et al. [1] except that we start with the Hamiltonian form of the string action and thus obtain directly the Hamiltonian form of the light-cone gauge-fixed action; this simplifies the verification of (super)Poincaré invariance of the gauge-fixed quantum (super)string. Having established (super)Poincaré invariance, we then compute the spectrum at the first few levels, sufficient to show that the spectrum of each of the quantum (super)strings considered contains anyons of irrational spin. We conclude with a summary and some speculations on a possible unified framework for 3D superstrings.

## 2 3D Preliminaries

In cartesian coordinates  $\{\mathbb{X}^\mu; \mu = 0, 1, 2\}$ , we define the Minkowski metric  $\eta_{\mu\nu}$  and alternating pseudo-tensor  $\varepsilon^{\mu\nu\rho}$  such that

$$\eta = \text{diag}(-1, 1, 1), \quad \varepsilon^{012} = 1. \quad (2.1)$$

The ‘light-cone components’ are

$$X^\pm = \frac{1}{\sqrt{2}} (\mathbb{X}^1 \pm \mathbb{X}^0), \quad X = \mathbb{X}^2. \quad (2.2)$$

Similarly, the light-cone components of an arbitrary 3-vector  $\mathbb{U}$  are

$$U_\pm = \frac{1}{\sqrt{2}} (\mathbb{U}_1 \pm \mathbb{U}_0), \quad U = \mathbb{U}_2. \quad (2.3)$$

We also have

$$U^\pm = U_\mp = \frac{1}{\sqrt{2}} (\mathbb{U}_1 \mp \mathbb{U}_0) = \frac{1}{\sqrt{2}} (\mathbb{U}^1 \pm \mathbb{U}^0). \quad (2.4)$$

Note that

$$-U_0^2 + U_1^2 + U_2^2 \equiv \mathbb{U}^2 = 2U_+U_- + U^2. \quad (2.5)$$

We will make use of the following 3D vector algebra relations for arbitrary 3-vectors  $\mathbb{U}$  and  $\mathbb{V}$ :

$$\mathbb{U} \cdot \mathbb{V} = \mathbb{U}^\mu \mathbb{V}^\nu \eta_{\mu\nu}, \quad [\mathbb{U} \wedge \mathbb{V}]^\mu = \varepsilon^{\mu\nu\rho} \mathbb{U}_\nu \mathbb{V}_\rho, \quad U_\mu = \eta_{\mu\nu} U^\nu. \quad (2.6)$$

## 2.1 3D Dirac matrices and Majorana spinors

A convenient choice for the 3D Dirac matrices is

$$\Gamma^0 = i\sigma_2, \quad \Gamma^1 = \sigma_1, \quad \Gamma^2 = \sigma_3. \quad (2.7)$$

Observe that

$$\Gamma^{\mu\nu\rho} \equiv \Gamma^{[\mu}\Gamma^\nu\Gamma^{\rho]} = \varepsilon^{\mu\nu\rho} \mathbb{I}. \quad (2.8)$$

The Dirac matrices satisfy the identity

$$(\Gamma^\mu)^\alpha{}_{(\beta} (C\Gamma_\mu)_{\gamma\delta)} \equiv 0, \quad (2.9)$$

where  $C$  is the antisymmetric charge conjugation matrix satisfying  $C\Gamma_\mu C^{-1} = -\Gamma_\mu^T$ .

A Majorana spinor is a 2-component spinor such that

$$\bar{\psi} \equiv \psi^\dagger \Gamma^0 = \psi^T C. \quad (2.10)$$

For the above representation of the Dirac matrices we may choose

$$C = \Gamma^0, \quad (2.11)$$

in which case a Majorana spinor is a real  $Sl(2; \mathbb{R})$  doublet. For any *commuting* Majorana spinor  $\psi$ , the identity (2.9) implies that

$$\Gamma^\mu \psi (\bar{\psi} \Gamma_\mu \psi) \equiv 0. \quad (2.12)$$

The Dirac matrices in the light-cone basis are

$$\Gamma^\pm = \frac{1}{\sqrt{2}} (\Gamma^1 \pm \Gamma^0), \quad \Gamma \equiv \Gamma_2 = \sigma_3. \quad (2.13)$$

These satisfy

$$(\Gamma^\pm)^2 = 0, \quad \Gamma^\pm \Gamma^\mp = 1 \pm \sigma_3. \quad (2.14)$$

As for vectors,  $\Gamma^\pm = \Gamma_\mp$ .

## 2.2 Poincaré and super-Poincaré invariants

The 3D Poincaré group is generated by the 3-momentum  $\mathcal{P}_\mu$  and Lorentz 3-vector  $\mathcal{J}^\mu$  with non-zero commutators

$$[\mathcal{J}^\mu, \mathcal{J}^\nu] = i\varepsilon^{\mu\nu\rho} J_\rho, \quad [\mathcal{J}^\mu, \mathcal{P}^\nu] = i\varepsilon^{\mu\nu\rho} P_\rho. \quad (2.15)$$

In the light-cone basis this becomes

$$\begin{aligned} [\mathcal{J}^+, \mathcal{J}^-] &= i\mathcal{J}, & [\mathcal{J}, \mathcal{J}^\pm] &= \pm i\mathcal{J}^\pm, \\ [\mathcal{J}^\pm, \mathcal{P}_\mp] &= \pm i\mathcal{P}, & [\mathcal{J}, \mathcal{P}_\pm] &= \mp i\mathcal{P}_\pm. \end{aligned} \quad (2.16)$$

There are two Poincaré Casimirs:

$$M^2 \equiv -\mathcal{P}^2, \quad \Lambda = \mathcal{P}_\mu \mathcal{J}^\mu. \quad (2.17)$$

Unitary irreps of the Poincaré group are labelled by the values of these Casimirs [12]. In principle,  $M^2$  may be negative but only irreps with  $M^2 \geq 0$  are physical. We may therefore assume that  $M$  is real and non-negative. When  $M > 0$  we define the “relativistic helicity”, which we usually abbreviate to “helicity”, by

$$s = \Lambda/M. \quad (2.18)$$

This may take either sign, and parity flips the sign of  $s$ . We define  $|s|$  to be the spin. If the Lorentz group is  $SO(1, 2)$  then  $s$  is an integer. If the Lorentz group is  $Sl(2; \mathbb{R})$ , which is the double cover of  $SO(1, 2)$ , then  $s$  is an integer or half-integer. If the Lorentz group is the universal cover of  $SO(1, 2)$  then  $s$  can be any real number.

The  $\mathcal{N}$ -extended super-Poincaré algebra includes  $\mathcal{N}$  Majorana spinor generators  $\mathcal{Q}_a^\alpha$  ( $\alpha = 1, 2$ ) with the following commutation relations with the Poincaré generators:

$$[\mathcal{P}, \mathcal{Q}_a^\alpha] = 0, \quad [\mathcal{J}^\mu, \mathcal{Q}_a^\alpha] = -\frac{i}{2} (\Gamma^\mu)^\alpha{}_\beta \mathcal{Q}_a^\beta. \quad (2.19)$$

In addition they obey the following anticommutation relation

$$\{\mathcal{Q}_a^\alpha, \mathcal{Q}_b^\beta\} = \delta_{ab} (\Gamma^\mu C)^{\alpha\beta} \mathcal{P}_\mu. \quad (2.20)$$

The super-Poincaré Casimirs are (summation over  $a = 1, \dots, \mathcal{N}$ )

$$M^2 \equiv -\mathcal{P}^2, \quad \Omega \equiv \mathcal{P} \cdot \mathcal{J} + \frac{i}{4} \bar{\mathcal{Q}}_a \mathcal{Q}_a. \quad (2.21)$$

We shall call

$$\bar{s} = \Omega/M \quad (2.22)$$

the “(relativistic) superhelicity” of an  $\mathcal{N}$ -extended supermultiplet, and  $|\bar{s}|$  its super-spin.

### 2.3 3D Superspace and superforms

The extension of Minkowski spacetime to  $\mathcal{N}$ -extended superspace involves the introduction of  $\mathcal{N}$  *anticommuting* Majorana spinor coordinates  $\{\Theta_a; a = 1, \dots, \mathcal{N}\}$ . The supersymmetry transformations are

$$\delta_\epsilon \mathbb{X}^\mu = i \bar{\Theta}_a \Gamma^\mu \epsilon_a, \quad \delta_\epsilon \Theta_a = \epsilon_a, \quad (2.23)$$

where  $\epsilon_a$  are constant real anticommuting spinor parameters, and a sum over the index  $a$  is implicit. The factor of  $i$  in the expression for  $\delta_\epsilon \mathbb{X}$  is needed because we use the standard convention that the complex conjugate of a product of anticommuting factors reverses the order, which gives a minus sign for the complex conjugation of a fermion bilinear if the order is not changed.

A basis for the left-invariant differential 1-forms on superspace is provided by  $d\Theta_a$  and

$$\Pi^\mu = d\mathbb{X}^\mu + i\bar{\Theta}_a\Gamma^\mu d\Theta_a. \quad (2.24)$$

Allowing for non-constant  $\epsilon$ , one has

$$\delta_\epsilon\Pi^\mu = -2id\epsilon_a\Gamma^\mu\Theta_a, \quad (2.25)$$

which confirms the invariance for constant parameters  $\epsilon_a$ .

The WZ terms for the superstring can be constructed as follows [13]. Consider, for  $\mathcal{N} = 1$ , the following super-Poincaré invariant 3-form (the exterior product of forms is implicit):

$$h_3^{\mathcal{N}=1} = \Pi^\mu (d\bar{\Theta}\Gamma_\mu d\Theta). \quad (2.26)$$

The identity (2.9) implies that this 2-form is closed. It is also exact, in de Rham cohomology, because

$$h_3^{\mathcal{N}=1} = dh_2^{\mathcal{N}=1}, \quad h_2^{\mathcal{N}=1} = -d\mathbb{X}^\mu (\bar{\Theta}\Gamma_\mu d\Theta). \quad (2.27)$$

However,  $h_2$  is not super-Poincaré invariant, and cannot be made so by the addition of any exact 2-form, so  $h_3$  is non-trivial in Lie-superalgebra (Chevalley-Eilenberg) cohomology (see e.g [14]). Because  $h_3$  is super-Poincaré invariant the super-Poincaré variation of  $h_2$  is a closed 2-form, and this is sufficient for invariance of the integral of  $h_2$  over a string worldsheet. In fact, using the identity

$$2\Gamma^\mu d\Theta \bar{\Theta}\Gamma_\mu d\Theta \equiv \Gamma^\mu\Theta d\bar{\Theta}\Gamma_\mu d\Theta, \quad (2.28)$$

which is a consequence of (2.9), one finds that

$$\delta_\epsilon h_2^{\mathcal{N}=1} = d \left[ \bar{\epsilon}\Gamma_\mu\Theta \left( d\mathbb{X}^\mu + \frac{i}{3}\bar{\Theta}\Gamma^\mu\Theta \right) \right] - 2d\bar{\epsilon}\Gamma_\mu\Theta \left( d\mathbb{X}^\mu - \frac{i}{3}\Theta\Gamma^\mu d\Theta \right). \quad (2.29)$$

This is non-zero even when  $d\epsilon = 0$ , but it is then an exact 2-form.

There is a generalization to  $\mathcal{N} = 2$  with

$$h_3^{\mathcal{N}=2} = \Pi^\mu (d\bar{\Theta}_1\Gamma_\mu d\Theta_1 - d\bar{\Theta}_2\Gamma_\mu d\Theta_2). \quad (2.30)$$

The relative minus sign is required for closure of  $h_3$ , which can be written as  $dh_2$  with

$$h_2^{\mathcal{N}=2} = - \left( d\mathbb{X}^\mu + \frac{i}{2}\bar{\Theta}_a\Gamma^\mu d\Theta_a \right) (\bar{\Theta}_1\Gamma_\mu d\Theta_1 - \bar{\Theta}_2\Gamma_\mu d\Theta_2). \quad (2.31)$$

This 2-form is manifestly Poincaré invariant but its supersymmetry variation (allowing for non-constant parameters) is

$$\begin{aligned} \delta_\epsilon h_2^{\mathcal{N}=2} &= d \left[ \bar{\epsilon}_1\Gamma_\mu\Theta_1 \left( dX^\mu + \frac{i}{3}\bar{\Theta}_1\Gamma^\mu d\Theta_1 \right) - \bar{\epsilon}_2\Gamma_\mu\Theta_2 \left( dX^\mu + \frac{i}{3}\bar{\Theta}_2\Gamma^\mu d\Theta_2 \right) \right] \\ &\quad - 2d\bar{\epsilon}_1\Gamma_\mu\Theta_1 \left( dX^\mu - \frac{i}{3}\bar{\Theta}_1\Gamma^\mu d\Theta_1 + i\bar{\Theta}_2\Gamma^\mu d\Theta_2 \right) \\ &\quad + 2d\bar{\epsilon}_2\Gamma_\mu\Theta_2 \left( dX^\mu - \frac{i}{3}\bar{\Theta}_2\Gamma^\mu d\Theta_2 + i\bar{\Theta}_1\Gamma^\mu d\Theta_1 \right). \end{aligned} \quad (2.32)$$

This is an exact 2-form for constant  $\epsilon_a$ .

## 2.4 Parity

Parity is a  $\mathbb{Z}_2$  transformation  $\Pi$  that we may choose to have the following action on the coordinates of  $\mathcal{N} = 1$  superspace

$$\Pi : \mathbb{X}_2 \rightarrow -\mathbb{X}_2, \quad \Theta \rightarrow \Gamma_2 \Theta, \quad (2.33)$$

with all other coordinates being inert. For the extension to  $\mathcal{N} = 2$  (we will not need to consider  $\mathcal{N} > 2$ ) we choose to define parity as the  $\mathbb{Z}_2$  transformation

$$\Pi : \mathbb{X}_2 \rightarrow -\mathbb{X}_2, \quad \Theta_1 \rightarrow \Gamma_2 \Theta_1, \quad \Theta_2 \rightarrow -\Gamma_2 \Theta_2, \quad (2.34)$$

with all other coordinates being inert; these transformations imply the invariance of the 3-form  $h_3$ , and hence of the superstring WZ term. The  $\mathcal{N} = 2$  superstring model to be considered here is additionally invariant under the transformations  $\Theta_a \rightarrow -\Theta_a$ , separately for  $a = 1, 2$ , so we could choose to define parity without the relative sign for the  $\Theta_1$  and  $\Theta_2$  transformation. However, the relative minus sign is required for standard parity assignments within supermultiplets relevant to the  $\mathcal{N} = 2$  superstring spectrum, and for parity invariance of the massive  $\mathcal{N} = 2$  superparticle with a central charge [15].

Parity acts as the following outer automorphism of the  $\mathcal{N} = 1$  super-Poincaré algebra:

$$\Pi : \mathcal{P}_2 \rightarrow -\mathcal{P}_2, \quad \mathcal{J}^\pm \rightarrow -\mathcal{J}^\pm, \quad \mathcal{Q}_1 \rightarrow -\Gamma_2 \mathcal{Q}_1, \quad (2.35)$$

with all other basis generators being inert. Similarly for  $\mathcal{N} = 2$ , but with the relative sign difference discussed above:

$$\Pi : \mathcal{P}_2 \rightarrow -\mathcal{P}_2, \quad \mathcal{J}^\pm \rightarrow -\mathcal{J}^\pm, \quad \mathcal{Q}_1 \rightarrow -\Gamma_2 \mathcal{Q}_1, \quad \mathcal{Q}_2 \rightarrow \Gamma_2 \mathcal{Q}_2. \quad (2.36)$$

In both cases, it follows that

$$\Pi : \Lambda \rightarrow -\Lambda, \quad \Omega \rightarrow -\Omega. \quad (2.37)$$

## 2.5 3D Supermultiplets

In any hermitian operator realization of the super-Poincaré generators with non-vanishing  $\mathcal{P}_-$ , and positive  $M^2$ , we may define the new non-hermitian supercharges

$$\mathcal{S}_a = \frac{1}{\sqrt{\sqrt{2}\mathcal{P}_-}} \left[ \sqrt{2}\mathcal{P}_- \mathcal{Q}_a^1 - (\mathcal{P}_2 - iM) \mathcal{Q}_a^2 \right]. \quad (2.38)$$

These have the remarkably simple anticommutation relations

$$\{\mathcal{S}_a, \mathcal{S}_b\} = 0, \quad \{\mathcal{S}_a, \mathcal{S}_b^\dagger\} = 2\delta_{ab} M^2. \quad (2.39)$$

They also have simple commutation relations with the Poincaré invariant  $\Lambda$

$$[\Lambda, \mathcal{S}] = -\frac{1}{2}M\mathcal{S}, \quad [\Lambda, \mathcal{S}^\dagger] = \frac{1}{2}M\mathcal{S}^\dagger, \quad (2.40)$$



which shows that the action of any of  $\mathcal{S}_a$  on a helicity eigenstate lowers the helicity by  $1/2$ , whereas the action of any of  $\mathcal{S}_a^\dagger$  raises it by  $1/2$ . Of course,  $\mathcal{S}$  commutes with the super-invariant  $\Omega$ , which can be written for  $M \neq 0$  as

$$\Omega = \Lambda + \frac{1}{8M} \sum_a [\mathcal{S}_a, \mathcal{S}_a^\dagger]. \quad (2.41)$$

It follows immediately from this formula that the value of  $\Omega$  for a given supermultiplet is the average of the values of  $\Lambda$ , and hence that  $\bar{s}$  is the average of the helicities  $s$ .

Irreducible supermultiplets are built by the action of the operators  $\mathcal{S}_a^\dagger$  on a ‘‘Clifford vacuum’’ state  $|\rangle$  that is annihilated by the  $\mathcal{S}_a$ :

$$\mathcal{S}_a |\rangle = 0 \quad (a = 1, \dots, \mathcal{N}). \quad (2.42)$$

This gives a supermultiplet of states

$$\left( |\rangle, \mathcal{S}_a^\dagger |\rangle, \mathcal{S}_a^\dagger \mathcal{S}_b^\dagger |\rangle, \dots, \mathcal{S}_1^\dagger \dots \mathcal{S}_\mathcal{N}^\dagger |\rangle \right). \quad (2.43)$$

If the first of these states has relativistic helicity  $h$  then we get a supermultiplet of  $2^\mathcal{N}$  states with helicities ranging from  $h$  to  $h + \mathcal{N}/2$ , and ‘binomial’ multiplicities. As the superhelicity is the average of the helicities, the  $\bar{s} = h$  supermultiplet is the  $\bar{s} = 0$  supermultiplet with all component helicities shifted by  $h$ . For example, for  $\mathcal{N} = 1$ , the  $\bar{s} = h$  supermultiplet has helicities

$$s = \left( h - \frac{1}{4}, h + \frac{1}{4} \right). \quad (2.44)$$

This is an anyon supermultiplet when  $h - \frac{1}{4} \notin \mathbb{Z}$ . The special case of  $2h \in \mathbb{Z}$  yields semion supermultiplets; first studied for  $h = \frac{1}{2}$  in [16]. The  $h = 0$  case yields the spin- $\frac{1}{4}$  supermultiplet with  $s = (-\frac{1}{4}, \frac{1}{4})$ ; this has arisen in a number of distinct contexts [17–19], including the level-2 spectrum of the  $\mathcal{N} = 1$  3D string [5], because it is the unique parity-invariant irreducible  $\mathcal{N} = 1$  supermultiplet. The generic anyon supermultiplet has been studied in [20].

For  $\mathcal{N} = 2$ , the  $\bar{s} = h$  supermultiplet has helicities

$$s = \left( h - \frac{1}{2}, h, h, h + \frac{1}{2} \right). \quad (2.45)$$

Again, the superhelicity is the average of the helicities in the supermultiplet. For  $h = 0$  we get the parity-invariant 3D scalar supermultiplet with spin-0 and spin-1/2 states. *In the absence of a central charge*, this is the unique parity-invariant  $\mathcal{N} = 2$  supermultiplet.

When  $M = 0$  the  $\mathcal{N}$  charges  $\mathcal{S}_a$  are hermitian. These mutually-anticommuting hermitian charges also anticommute with the remaining  $\mathcal{N}$  linearly independent hermitian supercharges,  $Q_a$ , which we may choose such that  $\{Q_a, Q_b\} = 2\delta_{ab}$ . The charges  $\mathcal{S}_a$  annihilate the states of an irreducible representation of the super-Poincaré group, which are acted upon non-trivially only by the  $Q_a$ . For  $\mathcal{N} = 1$ , there is only

one charge  $Q$ , satisfying  $Q^2 = 1$ . In this exceptional case there is a trivial realization of  $Q$  by the identity, but if there exists an operator  $(-1)^F$  that anticommutes with  $Q$  (as is the case for the  $\mathcal{N} = 1$  superstring considered here) then the minimal realization is 2-dimensional: one bose state and one fermi state [19]. Although spin is not defined for massless particles, there are still two distinct unitary irreps of the Poincaré group corresponding to the distinction between bosons and fermions [12,21]. For  $\mathcal{N} = 2$  there are two charges  $Q_1$  and  $Q_2$  that are realized non-trivially and the minimal realization is again 2-dimensional but if there exists an operator  $(-1)^F$  that anticommutes with  $Q_1$  and  $Q_2$  (as is the case for the  $\mathcal{N} = 2$  superstring considered here) then the 2-dimensional realization is complex so there are two boson and two fermion states, which is also what one finds from quantization of the  $\mathcal{N} = 2$  massless 3D superparticle [19].

### 2.5.1 Central charges

The  $\mathcal{N}$ -extended super-Poincaré algebra admits central charges for  $\mathcal{N} \geq 2$ . For the  $\mathcal{N} = 2$  case, which is of potential relevance in light of comments that we make in the conclusions, the anticommutator (2.20) becomes

$$\left\{ \mathcal{Q}_a^\alpha, \mathcal{Q}_b^\beta \right\} = \delta_{ab} (\Gamma^\mu C)^{\alpha\beta} \mathcal{P}_\mu + \varepsilon_{ab} C^{\alpha\beta} Z, \quad (2.46)$$

where  $Z$  is a real central charge. This modification implies that (2.39) is modified to

$$\left\{ \mathcal{S}_a, \mathcal{S}_b \right\} = 0, \quad \left\{ \mathcal{S}_a, \mathcal{S}_b^\dagger \right\} = 2M (\delta_{ab} M - i\varepsilon_{ab} Z). \quad (2.47)$$

Unitarity requires that

$$M \geq |Z|. \quad (2.48)$$

An  $\mathcal{N} = 2$  massive parity-preserving superparticle model in which this bound is saturated was presented in [15]. In the quantum theory this describes a centrally-charged parity-invariant semion supermultiplet with helicity states  $s = (-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ . In the  $Z \rightarrow 0$  limit, both  $M$  and  $\Lambda$  go to zero, and the helicity  $\Lambda/M$  becomes ill-defined; the 4 massive states become the two massless bosonic and two massless fermionic states of a massless  $\mathcal{N} = 2$  supermultiplet.

## 2.6 The 3D massive particle

The Hamiltonian form of the time-reparametrization invariant action for a point particle of non-zero mass  $m$  and relativistic helicity  $s$  is

$$S[\mathbb{X}, \mathbb{P}] = \int d\tau \left\{ \dot{\mathbb{X}}^\mu \mathbb{P}_\mu - \frac{1}{2} \ell (\mathbb{P}^2 + m^2) \right\} - s S_{LWZ} \quad (2.49)$$

where  $\mathbb{P}^2 = \eta_{\mu\nu} \mathbb{P}^\mu \mathbb{P}^\nu$  and  $S_{LWZ}$  is the (parity violating) ‘‘Lorentz Wess-Zumino’’ (LWZ) term constructed from the Poincaré-invariant closed 2-form [22]

$$\frac{1}{2} (-\mathbb{P}^2)^{-\frac{3}{2}} \varepsilon^{\mu\nu\rho} \mathbb{P}_\mu d\mathbb{P}_\nu d\mathbb{P}_\rho. \quad (2.50)$$

By construction, the action is Poincaré invariant. The Noether charges are

$$\mathcal{P}_\mu = \mathbb{P}_\mu, \quad \mathcal{J}^\mu = [\mathbb{X} \wedge \mathbb{P}]^\mu - \frac{s}{m} \mathbb{P}^\mu. \quad (2.51)$$

The time reparametrization invariance is equivalent to gauge invariance under the infinitesimal “ $\alpha$ -symmetry” transformation

$$\delta_\alpha \mathbb{X}^\mu = \alpha \mathbb{P}_\mu, \quad \delta_\alpha \mathbb{P}_\mu = 0, \quad \delta_\alpha \ell = \dot{\alpha}, \quad (2.52)$$

with arbitrary parameter  $\alpha(\tau)$ . To quantize, we must deal with this gauge invariance. As our purpose here is to illustrate some features of the light-cone gauge fixing that we will use for strings, we proceed in this way by setting

$$X^+ = \tau. \quad (2.53)$$

This fixes the  $\alpha$ -gauge invariance of (2.52) provided that  $P_- \neq 0$ . With the same proviso, we may solve the mass-shell constraint for  $P_+$ , which is minus the Hamiltonian in the chosen gauge:

$$H = -P_+ = \frac{1}{2P_-} (P^2 + m^2). \quad (2.54)$$

The light-cone gauge action naturally depends on  $s$  but the  $s$ -dependence can be removed (following the procedure of [19]) by defining the new variable

$$Y^- = X^- - \frac{\Lambda P}{m^2 P_-}, \quad \Lambda = sm. \quad (2.55)$$

The light-cone gauge action then becomes

$$S[X, X^-; P, P_-] = \int d\tau \left\{ \dot{X} P + \dot{Y}^- P_- - H \right\}. \quad (2.56)$$

The Poincaré charges (2.51) in the light-cone gauge are

$$\begin{aligned} \mathcal{P} &= P, & \mathcal{P}_- &= P_-, & \mathcal{P}_+ &= -H, \\ \mathcal{J} &= Y^- P_- + \tau H, & \mathcal{J}^+ &= \tau P - X P_-, \\ \mathcal{J}^- &= -Y^- P - X H + \Lambda / P_-. \end{aligned} \quad (2.57)$$

The  $s$ -dependence is now entirely in  $\mathcal{J}^-$  and it is easily checked that  $\mathcal{P}_\mu \mathcal{J}^\mu = \Lambda$ , confirming that the particle has helicity  $s$ . The equations of motion imply that the Poincaré charges are time-independent; the explicit time-dependence is canceled by the implicit time-dependence due to the equations of motion.

Upon quantization we have the equal-time commutation relations (we set  $\hbar = 1$ )

$$[Y^-, P_-] = i, \quad [X, P] = i. \quad (2.58)$$

There are now operator ordering ambiguities in the expressions for  $\mathcal{J}$  and  $\mathcal{J}^-$ . These ambiguities are fixed by the twin requirements of hermiticity and closure of the Lorentz algebra. The quantum Lorentz generators are

$$\begin{aligned} \mathcal{J} &= \frac{1}{2} \{Y^-, P_-\} + \tau H, & \mathcal{J}^+ &= \tau P - Y P_-, \\ \mathcal{J}^- &= -Y^- P - \frac{1}{2} \{X, H\} + \Lambda / P_-. \end{aligned} \quad (2.59)$$

It should now be understood that the canonical variables in these expressions are operators, as is  $H$ . Again, the explicit time-dependence is cancelled by the implicit time-dependence of the operators. Using the equal-time commutation relations (2.58) one may verify that the commutation relations (2.16) are satisfied, and hence that the quantum theory preserves the Poincaré invariance of the classical theory. This was to be expected but virtually the same computation is what is needed to verify Poincaré invariance for the 3D string. The only difference is in the form of the Hamiltonian  $H$  and the Poincaré invariant  $\Lambda$ . As long as these operators commute, one finds that the commutation relations (2.16) are obeyed, so the proof of Lorentz invariance for the 3D string will reduce to checking that  $[H, \Lambda] = 0$ .

### 3 The 3D bosonic string

The Nambu-Goto action for the closed bosonic 3D string of tension  $T$  is

$$S[\mathbb{X}] = -T \int d\tau \oint \frac{d\sigma}{2\pi} \sqrt{\left(\dot{\mathbb{X}} \cdot \mathbb{X}'\right)^2 - \dot{\mathbb{X}}^2 (\mathbb{X}')^2}, \quad (3.1)$$

where an overdot indicates a derivative with respect to the arbitrary time parameter  $\tau$  and a prime indicates a derivative with respect to the arbitrary string coordinate  $\sigma$ , which we assume to be identified with  $\sigma + 2\pi$ . This action involves the background 3D Minkowski metric  $\eta$  through the scalar product. An equivalent action, with auxiliary 3-momentum  $\mathbb{P}$  conjugate to 3-vector  $\mathbb{X}$ , is

$$S[\mathbb{X}, \mathbb{P}; \ell, u] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbb{X}}^\mu \mathbb{P}_\mu - \frac{1}{2} \ell [\mathbb{P}^2 + (T\mathbb{X}')^2] - u \mathbb{X}'^\mu \mathbb{P}_\mu \right\}, \quad (3.2)$$

where  $\ell$  and  $u$  are Lagrange multipliers for the Hamiltonian and  $S^1$ -diffeomorphism constraints, respectively. Elimination of  $\mathbb{P}$ , followed by elimination of  $\ell$  and then  $u$ , leads back to (3.1). One may also verify that the equations of motion are equivalent to the Nambu-Goto equations.

The action (3.2) is invariant under the Poincaré transformations generated by the Noether charges

$$\mathcal{P}_\mu = \oint \frac{d\sigma}{2\pi} \mathbb{P}_\mu, \quad \mathcal{J}^\mu = \oint \frac{d\sigma}{2\pi} [\mathbb{X} \wedge \mathbb{P}]^\mu. \quad (3.3)$$

It is also invariant under the discrete parity transformation

$$\mathbb{X}_2 \rightarrow -\mathbb{X}_2, \quad \mathbb{P}_2 \rightarrow -\mathbb{P}_2, \quad (3.4)$$

with all other variables being inert. In addition to these rigid invariances, the action (3.2) is invariant under the following infinitesimal gauge transformations with parameters  $\alpha, \beta$ :

$$\begin{aligned} \delta \mathbb{X} &= \alpha \mathbb{P} + \beta \mathbb{X}', \\ \delta \mathbb{P} &= T^2 (\alpha \mathbb{X}')' + (\beta \mathbb{P})', \\ \delta \ell &= \dot{\alpha} + u' \alpha - u \alpha' + (\ell' \beta - \ell \beta') \\ \delta u &= \dot{\beta} + u' \beta - u \beta' + T^2 (\alpha \ell' - \ell \alpha'). \end{aligned} \quad (3.5)$$

Note that not only is the action gauge invariant but so also are the Noether charges.

### 3.1 Light-cone gauge

We now introduce the light-cone coordinates  $(X^+, X^-, X)$  and their conjugate momenta  $(P_+, P_-, P)$ . The light-cone gauge is defined by the choice

$$X^+ = \tau, \quad P_- = p_-(\tau), \quad (3.6)$$

where  $p_-(\tau)$  is a non-zero function of  $\tau$  only. This choice leaves only the residual global gauge invariance induced by a constant shift of  $\sigma$ :

$$\delta_{\beta_0} \mathbb{X} = \beta_0 \mathbb{X}', \quad \delta_{\beta_0} u = \dot{\beta}_0 + u' \beta_0, \quad \delta_{\beta_0} \ell = \beta_0 \ell', \quad (3.7)$$

where

$$\beta_0(\tau) = \oint \frac{d\sigma}{2\pi} \beta. \quad (3.8)$$

To obtain the action in light-cone gauge, we first define

$$\begin{aligned} x(\tau) &= \oint \frac{d\sigma}{2\pi} X, & x^-(\tau) &= \oint \frac{d\sigma}{2\pi} X^- \\ p(\tau) &= \oint \frac{d\sigma}{2\pi} P, & p_+(\tau) &= \oint \frac{d\sigma}{2\pi} P_+, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} \bar{X} &= X - x, & \bar{X}^- &= X^- - x^-, \\ \bar{P} &= P - p, & \bar{P}_+ &= P_+ - p_+, \end{aligned} \quad (3.10)$$

and also

$$u_0 = \oint \frac{d\sigma}{2\pi} u, \quad \bar{u} = u - u_0. \quad (3.11)$$

Using the gauge conditions (3.6), we now find that the string Lagrangian reduces to

$$\begin{aligned} L &= \dot{x}p + \dot{x}^- p_- + p_+ + \oint \frac{d\sigma}{2\pi} \dot{\bar{X}} \bar{P} - u_0 \oint \frac{d\sigma}{2\pi} \bar{X}' P - \oint \frac{d\sigma}{2\pi} \bar{u} \bar{X}' P \\ &+ p_- \oint \frac{d\sigma}{2\pi} \left\{ \bar{X}^- \bar{u}' - \ell \left( P_+ + \frac{1}{2p_-} [P^2 + (TX')^2] \right) \right\}. \end{aligned} \quad (3.12)$$

In this form of the action,  $\bar{X}^-$  is a Lagrange multiplier imposing the constraint  $\bar{u}' = 0$ , which implies  $\bar{u} = 0$ . The constraint imposed by the lapse function  $\ell$  is also easily solved:

$$P_+ = -\frac{1}{2p_-} [P^2 + (TX')^2]. \quad (3.13)$$

This leads to the Lagrangian density

$$L = \left\{ \dot{x}p + \dot{x}^- p_- + \oint \frac{d\sigma}{2\pi} \dot{\bar{X}} \bar{P} \right\} - H - u_0 \oint \frac{d\sigma}{2\pi} \bar{X}' \bar{P}, \quad (3.14)$$

where the Hamiltonian is

$$H \equiv -p_+ = \frac{1}{2p_-} (p^2 + \mathcal{M}^2) , \quad (3.15)$$

with

$$\mathcal{M}^2 = \oint \frac{d\sigma}{2\pi} [\bar{P}^2 + (T\bar{X}')^2] . \quad (3.16)$$

As expected, there is a residual global constraint imposed by  $u_0$ , corresponding to the residual global gauge invariance which is now just

$$\delta_{\beta_0} \phi = \beta_0 \phi' , \quad \delta_{\beta_0} u_0 = \dot{\beta}_0 . \quad (3.17)$$

The  $u_0$ -dependence of the Lagrangian (3.14) converts derivatives with respect to  $\tau$  into covariant  $\tau$  derivatives, defined for any dynamical variable  $\phi$  by

$$D_\tau \phi = \dot{\phi} - u_0 \phi' . \quad (3.18)$$

This transforms covariantly under (3.17):

$$\delta_{\beta_0} (D_\tau \phi) = \beta_0 (D_\tau \phi)' . \quad (3.19)$$

Using this notation, the light-cone-gauge action may now be written in a form that is manifestly invariant under this residual gauge invariance:

$$S = \int d\tau \left\{ \dot{x}p + \dot{x}^- p_- + \oint \frac{d\sigma}{2\pi} \bar{P} D_\tau \bar{X} - H \right\} . \quad (3.20)$$

This action is clearly still invariant under the parity transformation (3.4), which now reads

$$X \rightarrow -X , \quad P \rightarrow -P , \quad (3.21)$$

with all other variables being inert. It is also still Poincaré invariant, despite appearances: the infinitesimal transformations are easily found by working out the compensating gauge transformations needed to maintain the gauge choice when performing an infinitesimal Poincaré transformation, and these transformations can then be used to find the Noether charges. However, because the Noether charges are gauge-invariant, one finds the same result by simply substituting the gauge-fixing conditions into the expressions (3.3). This gives

$$\mathcal{P}_2 = p , \quad \mathcal{P}_- = p_- , \quad \mathcal{P}_+ = -H , \quad (3.22)$$

and

$$\mathcal{J} = x^- p_- + \tau H , \quad \mathcal{J}^+ = \tau p - x p_- , \quad \mathcal{J}^- = -x^- p - x H + \Lambda/p_- , \quad (3.23)$$

where

$$\Lambda = p_- \oint \frac{d\sigma}{2\pi} [\bar{X} \bar{P}_+ - \bar{X}^- \bar{P}] . \quad (3.24)$$

One may verify that all these charges are time-independent as a consequence of the equations of motion. The two Poincaré invariants are

$$\mathcal{P}^2 = -\mathcal{M}^2, \quad \mathcal{P} \cdot \mathcal{J} = \Lambda. \quad (3.25)$$

Observe that  $\Lambda$  depends on  $\bar{X}^-$  as well as the canonical variables of the final action, but the equation of motion of  $\bar{u}$  in (3.12) is

$$p_- (\bar{X}^-)' + p \bar{X}' = -\bar{X}' \bar{P} + \oint \frac{d\sigma}{2\pi} \bar{X}' \bar{P}, \quad (3.26)$$

which will allow us to express  $\bar{X}^-$  in terms of  $(p_-, p)$  and the Fourier coefficients of  $(\bar{X}, \bar{P})$ .

### 3.2 Fourier expansion

We see from (3.14) that the physical variables in the light-cone gauge are the canonical pairs  $(x, p)$ ,  $(x^-, p_-)$  and either  $(\bar{X}, \bar{P})$  or the coefficients in their Fourier expansions. As is standard, we actually choose to Fourier expand the combinations  $\bar{P} \pm T\bar{X}'$ :

$$\begin{aligned} \bar{P} - T\bar{X}' &= \sqrt{2T} \sum_{n=1}^{\infty} \frac{1}{n} [e^{in\sigma} \alpha_n + e^{-in\sigma} \alpha_n^*], \\ \bar{P} + T\bar{X}' &= \sqrt{2T} \sum_{n=1}^{\infty} \frac{1}{n} [e^{in\sigma} \tilde{\alpha}_n^* + e^{-in\sigma} \tilde{\alpha}_n]. \end{aligned} \quad (3.27)$$

This implies that

$$\begin{aligned} \bar{X} &= \frac{i}{\sqrt{2T}} \sum_{n=1}^{\infty} \frac{1}{n} [e^{in\sigma} (\alpha_n - \tilde{\alpha}_n^*) - e^{-in\sigma} (\alpha_n^* - \tilde{\alpha}_n)], \\ \bar{P} &= \sqrt{\frac{T}{2}} \sum_{n=1}^{\infty} [e^{in\sigma} (\alpha_n + \tilde{\alpha}_n^*) + e^{-in\sigma} (\alpha_n^* + \tilde{\alpha}_n)]. \end{aligned} \quad (3.28)$$

It follows from the first of these expressions that

$$\bar{X}' = -\sqrt{\frac{1}{2T}} \sum_{n=1}^{\infty} [e^{in\sigma} (\alpha_n - \tilde{\alpha}_n^*) + e^{-in\sigma} (\alpha_n^* - \tilde{\alpha}_n)], \quad (3.29)$$

and hence that the Lagrangian (3.14) may be written as

$$L = \left\{ \dot{x}p + \dot{x}^- p_- + i \sum_{n=1}^{\infty} \frac{1}{n} [\dot{\alpha}_n \alpha_n^* + \dot{\tilde{\alpha}}_n \tilde{\alpha}_n^*] \right\} - H + u_0 \sum_{n=1}^{\infty} [\alpha_n^* \alpha_n - \tilde{\alpha}_n^* \tilde{\alpha}_n], \quad (3.30)$$

where the Hamiltonian is as in (3.15) but now  $\mathcal{M}^2$  is expressed as a sum over Fourier modes

$$\mathcal{M}^2 = 2T \sum_{n=1}^{\infty} [\alpha_n^* \alpha_n + \tilde{\alpha}_n^* \tilde{\alpha}_n]. \quad (3.31)$$

Note that parity now acts as

$$x \rightarrow -x, \quad p \rightarrow -p, \quad \alpha_n \rightarrow -\alpha_n, \quad \tilde{\alpha}_n \rightarrow -\tilde{\alpha}_n. \quad (3.32)$$

To obtain expressions for the Lorentz generators (3.23) in terms of the same variables, we need an expression in terms of them for  $\bar{X}^-$ . To this end we use (3.26) to deduce that

$$\bar{X}^- = -\frac{1}{p_-} \left\{ p\bar{X} + \sum_{n=1}^{\infty} \frac{i}{n} \left[ e^{in\sigma} (\beta_n - \tilde{\beta}_n^*) - e^{-in\sigma} (\beta_n^* - \tilde{\beta}_n) \right] \right\}, \quad (3.33)$$

where

$$\beta_n = \frac{1}{2} \sum_{m=1}^{n-1} \alpha_m \alpha_{n-m} + \sum_{m>n} \alpha_m \alpha_{m-n}^*, \quad (3.34)$$

and similarly for  $\tilde{\beta}_n$ . The  $\beta_n$  and  $\tilde{\beta}_n$  coefficients also arise in the Fourier expansion of  $\bar{P}_+$ , as given in (3.13):

$$\bar{P}_+ = -\frac{1}{p_-} \left\{ p\bar{P} + T \sum_{n=1}^{\infty} \left[ e^{in\sigma} (\beta_n + \tilde{\beta}_n^*) + e^{-in\sigma} (\beta_n^* + \tilde{\beta}_n) \right] \right\}. \quad (3.35)$$

We now have Fourier expansions for each of the variables appearing in the integrand of the expression (3.24) for  $\Lambda$ . Using them, we deduce that

$$\Lambda = \Lambda_+ + \Lambda_-, \quad \Lambda_+ = \sqrt{2T} \lambda, \quad \Lambda_- = \sqrt{2T} \tilde{\lambda}, \quad (3.36)$$

where

$$\lambda = \sum_{n=1}^{\infty} \frac{i}{n} (\alpha_n^* \beta_n - \beta_n^* \alpha_n), \quad \tilde{\lambda} = \sum_{n=1}^{\infty} \frac{i}{n} (\tilde{\alpha}_n^* \tilde{\beta}_n - \tilde{\beta}_n^* \tilde{\alpha}_n). \quad (3.37)$$

### 3.2.1 Equations of motion

Before moving on to the quantum theory, we comment on the equations of motion. The Lagrangian (3.30) leads to the equations of motion

$$\dot{p} = \dot{p}_- = 0, \quad \dot{x} = p/p_-, \quad \dot{x}^- = -H, \quad (3.38)$$

and

$$D_\tau \alpha_n = -in\omega \alpha_n, \quad D_\tau \tilde{\alpha}_n = -in\omega \tilde{\alpha}_n, \quad (3.39)$$

where

$$\omega = T/p_-. \quad (3.40)$$

Using these equations, and the expression (3.35) for  $P_+$ , one can show that the expression (3.33) for  $\bar{X}^-$  implies that

$$p_- D_\tau \bar{X}^- = \bar{P}_+. \quad (3.41)$$



In the gauge  $u_0 = 0$ , the equations for  $(\alpha_n, \tilde{\alpha}_n)$  have the solution

$$\alpha_n(\tau) = \alpha_n(0)e^{-in\omega\tau}, \quad \tilde{\alpha}_n(\tau) = \tilde{\alpha}_n(0)e^{-in\omega\tau}, \quad (3.42)$$

which gives

$$\begin{aligned} \bar{P} - T\bar{X} &= \sqrt{2T} \sum_{n=1}^{\infty} \frac{1}{n} [e^{in[\sigma-\omega\tau]}\alpha_n(0) + c.c.] \\ \bar{P} + T\bar{X} &= \sqrt{2T} \sum_{n=1}^{\infty} \frac{1}{n} [e^{-in[\sigma+\omega\tau]}\tilde{\alpha}_n(0) + c.c.]. \end{aligned} \quad (3.43)$$

This confirms that the  $\alpha_n$  are the Fourier coefficients for right-moving modes and  $\tilde{\alpha}_n$  the Fourier coefficients for left-moving modes, but one might have expected to find that  $\omega = 1$  since waves on the string travel along it at the speed of light (which is  $c = 1$  in the units used here). However, the scale associated with the time variable  $\tau$  is arbitrary, and this is reflected in the arbitrariness of the angular frequency  $\omega = T/p_-$ . Note that  $p_-$  is set to a constant by the equations of motion. A natural choice is

$$p_- = T \quad (3.44)$$

since this implies that  $\omega = 1$ . However, it is important not to set  $p_- = T$  in the action; doing so would cause the  $\dot{x}^- p_-$  term to become an irrelevant total derivative and the action would no longer be Lorentz invariant. It is also important not to set  $p_- = T$  in the expressions for the Noether charges, at least before evaluation of Poisson brackets (classically) or commutators (quantum mechanically).

### 3.3 Quantum bosonic string

The non-zero Poisson brackets of the canonical variables in the action light-cone-gauge action (3.20) are

$$\{x, p\}_{pb} = 1, \quad \{x^-, p_-\}_{pb} = 1, \quad \{X(\sigma), P(\sigma')\}_{pb} = 2\pi \delta(\sigma - \sigma'). \quad (3.45)$$

In the quantum theory, these variables are promoted to operators with the commutation relations (we set  $\hbar = 1$ )

$$[x, p] = i, \quad [x^-, p_-] = i, \quad [X(\sigma), P(\sigma')] = 2\pi i \delta(\sigma - \sigma'). \quad (3.46)$$

The last of these can be achieved by promoting to operators the Fourier coefficients  $(\alpha_n, \tilde{\alpha}_n)$  so that the non-zero commutators are

$$[\alpha_n, \alpha_n^\dagger] = n, \quad [\tilde{\alpha}_n, \tilde{\alpha}_n^\dagger] = n, \quad \forall n \in \mathbb{Z}^+. \quad (3.47)$$

The quantum Hamiltonian is then

$$H = \frac{1}{2p_-} (p^2 + \mathcal{M}^2), \quad \mathcal{M}^2 = 2T (N + \tilde{N} - a), \quad (3.48)$$

where  $a$  is an arbitrary constant arising from operator ordering ambiguities, and the “level-number” operators  $N$  and  $\tilde{N}$  are

$$N = \sum_{n=1}^{\infty} \alpha_n^\dagger \alpha_n, \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_n^\dagger \tilde{\alpha}_n. \quad (3.49)$$

The constraint imposed by  $u_0$  is the level-matching condition, which must be imposed as a physical-state condition in the quantum theory: for physical state  $|\text{phys}\rangle$ ,

$$(N - \tilde{N}) |\text{phys}\rangle = 0. \quad (3.50)$$

The string ground state takes the tensor product form

$$|p, p_-\rangle \otimes |0\rangle_+ \otimes |0\rangle_-, \quad (3.51)$$

where  $|0\rangle_+$  is the ground state for the right-moving modes and  $|0\rangle_-$  is the ground state for the left-moving modes:

$$\alpha_n |0\rangle_+ = 0, \quad \tilde{\alpha}_n |0\rangle_- = 0, \quad \forall n \in \mathbb{Z}^+. \quad (3.52)$$

Excited string states are found by the action of oscillator creation operators on this ground state. Such states are eigenstates of the level operators  $N$  and  $\tilde{N}$ , with eigenvalues that we also call  $N$  and  $\tilde{N}$ . To be physical, these eigenstates must satisfy the level-matching condition  $N = \tilde{N}$ . We may therefore organize all physical states according to their level  $N$ . In addition,

$$\mathcal{M}^2 |_N = 2T (2N - a). \quad (3.53)$$

Because of the level-matching constraint, not only is the (level-0) oscillator ground state unique, for given  $p$  and  $p_-$ , but so also is the first (level-1) excited state,

$$\alpha_1^\dagger |0\rangle_+ \otimes \tilde{\alpha}_1^\dagger |0\rangle_- \equiv |1\rangle_+ \otimes |1\rangle_-. \quad (3.54)$$

There are four physical level-2 states, which are tensor products of

$$|1, 1\rangle_+ = \frac{1}{\sqrt{2}} (\alpha_1^\dagger)^2 |0\rangle_+, \quad |2\rangle_+ = \frac{1}{\sqrt{2}} \alpha_2^\dagger |0\rangle_+, \quad (3.55)$$

with the analogous two states built on  $|0\rangle_-$ . At level 3 we need to consider the three (orthonormal basis) states

$$\begin{aligned} |1, 1, 1\rangle_+ &= \frac{1}{\sqrt{6}} (\alpha_1^\dagger)^3 |0\rangle_+, & |1, 2\rangle_+ &= \frac{1}{\sqrt{2}} \alpha_1^\dagger \alpha_2^\dagger |0\rangle_+, \\ |3\rangle_+ &= \frac{1}{\sqrt{3}} \alpha_3^\dagger |0\rangle_+, \end{aligned} \quad (3.56)$$

and this leads to a total of nine physical states.

At level 4 we need to consider the five (orthonormal basis) states

$$\begin{aligned}
|1, 1, 1, 1\rangle_+ &= \frac{1}{\sqrt{4!}} (\alpha_1^\dagger)^4 |0\rangle_+, & |1, 1, 2\rangle_+ &= \frac{1}{2} (\alpha_1^\dagger)^2 \alpha_2^\dagger |0\rangle_+, \\
|1, 3\rangle_+ &= \frac{1}{\sqrt{3}} \alpha_1^\dagger \alpha_3^\dagger |0\rangle_+, & |2, 2\rangle_+ &= \frac{1}{2\sqrt{2}} (\alpha_1^\dagger)^2 \alpha_2^\dagger |0\rangle_+, \\
|4\rangle_+ &= \frac{1}{2} \alpha_4^\dagger |0\rangle_+, & &
\end{aligned} \tag{3.57}$$

and this leads to a total of twenty-five level-4 physical states.

### 3.3.1 Lorentz covariance and Parity

As the light-cone gauge renders the classical Lorentz invariance non-manifest, there is no guarantee that the quantum string will be Lorentz invariant. We must therefore check Lorentz invariance. The quantum translation generators are

$$\mathcal{P}_2 = p, \quad \mathcal{P}_- = p_-, \quad \mathcal{P}_+ = -H, \tag{3.58}$$

exactly as in (3.22) but now with the operator Hamiltonian of (3.48), and the quantum Lorentz generators are

$$\begin{aligned}
\mathcal{J} &= \frac{1}{2} \{x^-, p_-\} + \tau H, & \mathcal{J}^+ &= \tau p - x p_-, \\
\mathcal{J}^- &= -x^- p - \frac{1}{2} \{x, H\} + \Lambda/p_-, & &
\end{aligned} \tag{3.59}$$

Here,  $\Lambda = \Lambda_+ + \Lambda_-$ , with

$$\begin{aligned}
\frac{1}{\sqrt{2T}} \Lambda_+ &= \lambda \equiv \sum_{n=1}^{\infty} \frac{i}{n} (\alpha_n^\dagger \beta_n - \beta_n^\dagger \alpha_n), \\
\frac{1}{\sqrt{2T}} \Lambda_- &= \tilde{\lambda} \equiv \sum_{n=1}^{\infty} \frac{i}{n} (\tilde{\alpha}_n^\dagger \tilde{\beta}_n - \tilde{\beta}_n^\dagger \tilde{\alpha}_n),
\end{aligned} \tag{3.60}$$

where  $\beta_n$  and  $\tilde{\beta}_n$  are now the operators

$$\begin{aligned}
\beta_n &= \frac{1}{2} \sum_{m=1}^{n-1} \alpha_m \alpha_{n-m} + \sum_{m>n} \alpha_m \alpha_{m-n}^\dagger, \\
\tilde{\beta}_n &= \frac{1}{2} \sum_{m=1}^{n-1} \tilde{\alpha}_m \tilde{\alpha}_{n-m} + \sum_{m>n} \tilde{\alpha}_m \tilde{\alpha}_{m-n}^\dagger.
\end{aligned} \tag{3.61}$$

All operator ordering ambiguities in the quantum Lorentz generators are fixed by the requirements of hermiticity and closure of the algebra. In particular, there is no Lorentz anomaly, so the quantum theory is Lorentz invariant. This was to be expected because the ‘‘dangerous commutators’’ are antisymmetric in the  $(D - 2)$  ‘‘transverse space’’ indices and hence trivially absent for  $D = 3$ . As a result, the computation is

equivalent to the one that must be done for the massive particle except that one needs to check that  $[N, \lambda] = 0$ , which implies  $[H, \Lambda] = 0$ . For this step, it is convenient to first establish the commutation relations

$$[\alpha_n, \beta_m] = n\alpha_{n+m}, \quad [\alpha_n^\dagger, \beta_m] = \begin{cases} -n\alpha_{m-n} & n < m \\ 0 & n = m \\ -n\alpha_{n-m}^\dagger & n > m \end{cases}, \quad (3.62)$$

and then to use the identity

$$\sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \equiv \sum_{n=2}^{\infty} \sum_{m=1}^{n-1}. \quad (3.63)$$

It also remains true in the quantum theory that

$$\mathcal{P}^2 = -\mathcal{M}^2, \quad \mathcal{P} \cdot \mathcal{J} = \Lambda, \quad (3.64)$$

where the operators  $\mathcal{M}^2$  and  $\Lambda$  are given by (3.53) and (3.60) respectively. It is straightforward to verify that these two operators commute with each other and with all generators of the Poincaré algebra.

The parity operator of the quantum theory is<sup>3</sup>

$$II = II_0 \exp \left[ i\pi \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n^\dagger \alpha_n + \tilde{\alpha}^\dagger \tilde{\alpha}_n) \right], \quad (3.65)$$

where

$$II_0 = \int dp | -p \rangle \langle p|. \quad (3.66)$$

The operator  $II$  anticommutes with all the creation and annihilation operators. It therefore commutes with  $N$  and  $\tilde{N}$ , and hence with the Hamiltonian. Also, it *anti-commutes* with  $\Lambda$ . Parity is therefore preserved by the quantum theory, and all states of non-zero spin must appear in parity doublets of opposite-sign helicities. For the first few low-lying levels, this is verified by the explicit computations to follow.

### 3.3.2 Helicity spectrum

As  $\mathcal{M}^2$  and  $\Lambda$  commute, they are simultaneously diagonalizable. This means that  $\Lambda$  is block diagonal in a basis in which  $\mathcal{M}^2$  is diagonal, with blocks that may be labeled by the level number  $N$ . Since  $\lambda$  and  $\tilde{\lambda}$  commute, they too may be simultaneously diagonalized. It follows that

$$\lambda = \sum_{n=2}^{\infty} \lambda_n, \quad \tilde{\lambda} = \sum_{n=2}^{\infty} \tilde{\lambda}_n, \quad (3.67)$$

---

<sup>3</sup>Recall that the parity operator for the harmonic oscillator is  $\exp(i\pi\hat{N})$  where  $\hat{N}$  is the particle number operator.

where  $\lambda_n$  annihilates all states with  $N < n$  but not all those with  $N \geq n$ . The absence of  $n = 0$  and  $n = 1$  contributions to the sum is easily verified, and it implies that  $\Lambda$  annihilates the states at  $N = 0$  and  $N = 1$ ; this is expected because these levels each contain a single physical state which must be a parity singlet and hence a scalar. At level 2 we need consider only  $\lambda_2$  because  $\lambda_n$  for  $n \geq 3$  annihilates all states at levels  $N = 0, 1, 2$ . A computation shows that

$$\lambda_2 = \frac{3i}{4} \left[ \left( \alpha_1^\dagger \right)^2 \alpha_2 - \alpha_2^\dagger \alpha_1^2 \right]. \quad (3.68)$$

This reduces to the matrix  $(3/2)\sigma_2$  in the level-2 basis (3.55) so  $\lambda$  has eigenvalues  $\pm 3/2$ . The same is obviously true for  $\tilde{\lambda}$ , so the eigenvalues of  $\Lambda$  at level 2 are  $(0, 0, 3, -3)$  times  $\sqrt{2T}$ . We must divide by the level-2 mass  $\sqrt{2T(4-a)}$  to get the helicities, which are therefore

$$s_2 = \left( 0, 0, \pm \frac{3}{\sqrt{4-a}} \right). \quad (3.69)$$

As implied by parity, each non-zero spin occurs twice, once for each sign of the helicity.

At level 3 we need  $\lambda_3$  and a computation gives

$$\lambda_3 = \frac{7i}{6} \left( \alpha_1^\dagger \alpha_2^\dagger \alpha_3 - \alpha_3^\dagger \alpha_1 \alpha_2 \right). \quad (3.70)$$

We also need  $\lambda_2$  because, for example, it does not annihilate  $|1, 1, 1\rangle_+$ , but we do not need  $\lambda_4$  or higher terms. In the level-3 basis (3.56), one finds that  $\lambda$  reduces to the matrix

$$\frac{i}{2\sqrt{3}} \begin{pmatrix} 0 & -9 & 0 \\ 9 & 0 & -7\sqrt{2} \\ 0 & 7\sqrt{2} & 0 \end{pmatrix}, \quad (3.71)$$

which has eigenvalues  $(0, \pm\sqrt{179/12})$ . This leads to the level-3 helicity content

$$s_3 = \left( 0, 0, 0, \pm\sqrt{\frac{179}{12(6-a)}}, \pm\sqrt{\frac{179}{12(6-a)}}, \pm\sqrt{\frac{179}{3(6-a)}} \right). \quad (3.72)$$

Observe again that non-zero helicities appear in parity doublets of opposite helicity.

At level 4 we need  $\lambda_4$  and a computation gives

$$\lambda_4 = i \left\{ \frac{13}{12} \left( \alpha_3^\dagger \alpha_1^\dagger \alpha_4 - \alpha_4^\dagger \alpha_1 \alpha_3 \right) + \frac{3}{8} \left[ \left( \alpha_2^\dagger \right)^2 \alpha_4 - \alpha_4^\dagger (\alpha_2)^2 \right] \right\}. \quad (3.73)$$

We also need  $\lambda_3$  and  $\lambda_2$ . In the level-4 basis (3.57), one finds that  $\lambda$  reduces to the matrix

$$i \begin{pmatrix} 0 & -3\sqrt{6}/2 & 0 & 0 & 0 \\ 3\sqrt{6}/2 & 0 & -7\sqrt{3}/3 & -3\sqrt{2}/2 & 0 \\ 0 & 7\sqrt{3}/3 & 0 & 0 & -13\sqrt{3}/6 \\ 0 & 3\sqrt{2}/2 & 0 & 0 & -3\sqrt{2}/2 \\ 0 & 0 & 13\sqrt{3}/6 & 3\sqrt{2}/2 & 0 \end{pmatrix}, \quad (3.74)$$

which has eigenvalues

$$\left( 0, \pm \sqrt{\frac{1}{24} \left( 635 + \sqrt{258505} \right)}, \pm \sqrt{\frac{1}{24} \left( 635 - \sqrt{258505} \right)} \right). \quad (3.75)$$

Or, in plain numbers,  $\{0, \pm 6.9024\dots, \pm 2.29643\dots\}$ . The helicities are found by dividing these numbers by  $\sqrt{8-a}$ .

We are free to choose the intercept parameter  $a$  except that there are tachyons unless  $a \leq 0$ . The choice  $a = 0$  is natural because this makes the ground state a massless scalar. In this case the first excited state (level 1) is a massive scalar, there are then spins  $(0, 0, 3/2)$  at level 2 and some irrational spin anyons at level 3. We shall call this the  $\mathcal{N} = 0$  string since its spectrum is of direct relevance to the spectrum of the  $\mathcal{N} = 1$  superstring.

However, since the ground state of the critical bosonic string is a tachyon, the  $a > 0$  cases should perhaps be considered too. In particular, the choice  $a = 2$  leads to a tachyonic scalar ground state and a massless first-excited state, just like the critical bosonic string although the first excited state is a scalar in 3D. For  $a > 2$  this scalar excited state is a tachyon too but as long as  $a < 4$  there are no other tachyons. The limit  $a \rightarrow 4$  is singular because the level-2 mass goes to zero as  $a \rightarrow 4$  but the invariant  $\Lambda$  does not, so some helicities go to infinity. For  $a > 4$  there are non-scalar tachyonic excited states in addition to the scalar ground-state tachyon.

It was observed in [5] that there is no choice of  $a$  that avoids anyons in one of the levels 2 and 3, and that irrational spins occur for generic  $a$ . Taking into account the level 4 results, it becomes clear that the spectrum contains irrational spins for any choice of  $a$ . As we shall see, the analysis of this issue is simpler for the superstring because supersymmetry removes the ambiguity represented by the choice of the intercept parameter  $a$ .

## 4 The closed $\mathcal{N} = 1$ 3D Superstring

The action for the closed 3D  $\mathcal{N} = 1$  GS superstring of tension  $T$  is obtained from the Nambu-Goto string action in two steps. First, we replace  $d\mathbb{X}$  by the supersymmetry invariant 1-form  $\Pi$  of (2.24) (for  $\mathcal{N} = 1$ ). In other words

$$\dot{\mathbb{X}} \rightarrow \Pi_\tau = \dot{\mathbb{X}} + i\bar{\Theta}\Gamma\dot{\Theta}, \quad \mathbb{X}' \rightarrow \Pi_\sigma = \mathbb{X}' + i\bar{\Theta}\Gamma\dot{\Theta}'. \quad (4.1)$$

Next, we add to the resulting action a Wess-Zumino (WZ) term constructed from the closed, super-Poincaré invariant 3-form of (2.26). Applying this prescription to the Hamiltonian form of the 3D Nambu-Goto string action (3.2), we find the following ‘quasi-Hamiltonian’ form of the  $\mathcal{N} = 1$  3D superstring action:

$$S[\mathbb{X}, \mathbb{P}; \ell, u] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \Pi_\tau^\mu \mathbb{P}_\mu - \frac{1}{2} \ell [\mathbb{P}^2 + (T\Pi_\sigma)^2] - u \Pi_\sigma^\mu \mathbb{P}_\mu + iT \left( \dot{\mathbb{X}}^\mu \bar{\Theta} \Gamma_\mu \Theta' - \mathbb{X}'^\mu \bar{\Theta} \Gamma_\mu \dot{\Theta} \right) \right\}. \quad (4.2)$$

By construction, this action is invariant under worldsheet diffeomorphisms, and this is equivalent to invariance under “ $\alpha$ -symmetry” and “ $\beta$ -symmetry” gauge transformations that generalize (3.5). The gauge transformations of the Lagrange multiplier variables  $\ell$  and  $u$  are unchanged from those of (3.5) while the canonical variables have the gauge transformations

$$\begin{aligned}\delta\mathbb{X} &= \alpha \left[ \mathbb{P} - i\ell^{-1}\bar{\Theta}\Gamma \left( \dot{\Theta} - u\Theta' \right) \right] + \beta\mathbb{X}', \\ \delta\Theta &= \alpha\ell^{-1} \left( \dot{\Theta} - u\Theta' \right) + \beta\Theta', \\ \delta\mathbb{P} &= (T^2\alpha\Pi_\sigma + \beta\mathbb{P})' + 2i\alpha\ell^{-1}T \left( \bar{\Theta}'\Gamma\dot{\Theta} \right).\end{aligned}\quad (4.3)$$

The term linear in  $T$  in the action is the WZ term, and we have chosen its coefficient to ensure invariance of the action under the following fermionic gauge invariance (“ $\kappa$ -symmetry”) with anticommuting Majorana spinor parameter  $\kappa$ :

$$\begin{aligned}\delta_\kappa\Theta &= \Gamma_\mu(\mathbb{P}^\mu - T\Pi_\sigma^\mu)\kappa, & \delta_\kappa\mathbb{X}^\mu &= -i\bar{\Theta}\Gamma^\mu\delta_\kappa\Theta, & \delta_\kappa\mathbb{P}_\mu &= 2iT\bar{\Theta}'\Gamma_\mu\delta_\kappa\Theta, \\ \delta_\kappa\ell &= -4i\bar{\kappa} \left[ \dot{\Theta} + (\ell T - u)\Theta' \right], & \delta_\kappa u &= -T\delta_\kappa\ell.\end{aligned}\quad (4.4)$$

The action is  $\kappa$ -symmetric for either sign of  $T$  but we may choose  $T > 0$  and then allow for either sign of the WZ term. As the two models thus obtained are equivalent we may choose the sign as given. To verify the invariance, it is useful to use the fact that

$$\delta_\kappa h_3 = d\delta_\kappa h_2 = -2d \left[ \Pi^\mu (\delta_\kappa \bar{\Theta} \Gamma_\mu \Theta) \right], \quad (4.5)$$

which gives  $\delta_\kappa h_2$  up to the addition of an irrelevant closed form. Observe that

$$\det [\Gamma_\mu (\mathbb{P}^\mu - T\Pi_\sigma^\mu)] = -(\mathbb{P} - T\Pi_\sigma)^2 \approx 0, \quad (4.6)$$

where the symbol  $\approx$  stands for “weak equality” in the sense of Dirac. This implies that only one of the two independent components of  $\kappa$  has any effect, so that only one real component of  $\Theta$  can be gauged away.

The action (4.2) is both parity invariant (for reasons explained in subsection 2.4) and super-Poincaré invariant. The Poincaré Noether charges are

$$\begin{aligned}\mathcal{P}_\mu &= \oint \frac{d\sigma}{2\pi} \left\{ \mathbb{P}_\mu + iT\bar{\Theta}\Gamma_\mu\Theta' \right\}, \\ \mathcal{J}^\mu &= \oint \frac{d\sigma}{2\pi} \left\{ [\mathbb{X} \wedge (\mathbb{P} + iT\bar{\Theta}\Gamma\Theta')]^\mu + \frac{i}{2}\bar{\Theta}\Theta (\mathbb{P} - T\mathbb{X}')^\mu \right\}.\end{aligned}\quad (4.7)$$

The supersymmetry Noether charges are

$$\mathcal{Q}^\alpha = \sqrt{2} \oint \frac{d\sigma}{2\pi} \left\{ (\mathbb{P}^\mu - T\Pi_\sigma^\mu) (\Gamma_\mu\Theta)^\alpha - 2iT (\bar{\Theta}\Theta) (\Theta')^\alpha \right\}. \quad (4.8)$$

The  $\kappa$ -symmetry variation of all these charges is zero on the constraint surface, i.e. weakly zero.

## 4.1 Light-cone gauge

Light-cone gauge fixing proceeds as for the Nambu-Goto string but with the additional fixing of the kappa-symmetry by the relation [11]

$$\Gamma^+ \Theta = 0, \quad (4.9)$$

which implies that

$$\Theta = \sqrt{\frac{1}{2\sqrt{2} p_-}} \begin{pmatrix} \theta \\ 0 \end{pmatrix} \quad (4.10)$$

for some anticommuting worldsheet function  $\theta(\tau, \sigma)$ . We thus find that

$$\begin{aligned} \Pi_\tau^+ &= 1, & \Pi_\tau^- &= \dot{X}^- + \frac{i}{2p_-} \theta \dot{\theta}, & \Pi_\tau^2 &= \dot{X} \\ \Pi_\sigma^+ &= 0, & \Pi_\sigma^- &= (X^-)' + \frac{i}{2p_-} \theta \theta', & \Pi_\sigma^2 &= X'. \end{aligned} \quad (4.11)$$

As for the bosonic variables, it is convenient to define

$$\bar{\theta} = \theta - \vartheta, \quad \vartheta(\tau) = \oint \frac{d\sigma}{2\pi} \theta. \quad (4.12)$$

There should be no confusion with the notation for a conjugate spinor as  $\theta$  is not a 2-component spinor. In this notation, we find that the analog of (3.12) (but without the  $u = u_0 + \bar{u}$  split) is

$$\begin{aligned} L &= \dot{x}p + \dot{x}^- p_- + \frac{i}{2} \vartheta \dot{\vartheta} + \oint \frac{d\sigma}{2\pi} \left( \dot{X} \bar{P} + \frac{i}{2} \bar{\theta} \dot{\theta} \right) \\ &+ \frac{iT}{2p_-} \oint \frac{d\sigma}{2\pi} \bar{\theta} \theta' - \oint \frac{d\sigma}{2\pi} u \left( \bar{X}' P + \frac{i}{2} \theta \theta' \right) \\ &+ p_- \oint \frac{d\sigma}{2\pi} \left\{ \bar{X}^- u' - \ell \left( P_+ + \frac{1}{2p_-} [P^2 + (TX')^2] \right) \right\}. \end{aligned} \quad (4.13)$$

As before,  $\bar{X}^-$  is now a Lagrange multiplier for the constraint  $u' = 0$ , which we solve by writing  $u = u_0(\tau)$ . The constraint imposed by the Lagrange multiplier  $\ell$  is also exactly as before, and therefore has the same solution (3.13) for  $P_+$ . The resulting analog of the bosonic Lagrangian (3.14) is

$$\begin{aligned} L &= \left[ \dot{x}p + \dot{x}^- p_- + \frac{i}{2} \vartheta \dot{\vartheta} + \oint \frac{d\sigma}{2\pi} \left\{ \dot{X} \bar{P} + \frac{i}{2} \bar{\theta} \dot{\theta} \right\} \right] - H \\ &- u_0 \oint \frac{d\sigma}{2\pi} \left\{ \bar{X}' \bar{P} + \frac{i}{2} \bar{\theta} \theta' \right\}, \end{aligned} \quad (4.14)$$

where

$$\begin{aligned} H &= -p_+ - \frac{iT}{2} \oint \frac{d\sigma}{2\pi} \bar{\theta} \theta' \\ &= \frac{1}{2p_-} \left[ p^2 + \oint \frac{d\sigma}{2\pi} \left\{ \bar{P}^2 + (T\bar{X}')^2 - iT\bar{\theta} \theta' \right\} \right]. \end{aligned} \quad (4.15)$$



Notice that the Hamiltonian is no longer  $-p_+$  because of the fermionic contribution from the WZ term.

The Poincaré generators in the light-cone gauge are

$$\begin{aligned}\mathcal{P} &= p, & \mathcal{P}_- &= p_-, & \mathcal{P}_+ &= -H, \\ \mathcal{J} &= x^- p_- - \tau H, & \mathcal{J}^+ &= \tau p - x p_-, \\ \mathcal{J}^- &= -x^- p - x H + \Lambda/p_-, \end{aligned} \quad (4.16)$$

exactly as for the bosonic string, except that the Hamiltonian differs and now

$$\Lambda = p_- \oint \frac{d\sigma}{2\pi} [\bar{X} \bar{P}_+ - \bar{X}^- \bar{P}] + \frac{iT}{2} \left( \oint \frac{d\sigma}{2\pi} \bar{X} \bar{\theta} \bar{\theta}' + \vartheta \oint \frac{d\sigma}{2\pi} \bar{X} \bar{\theta}' \right). \quad (4.17)$$

Note the  $\vartheta$ -dependence in the last term of this expression. Note also the dependence on  $\bar{X}^-$  in the first integral; although this is not one of the canonical variables of the gauge-fixed action, its Fourier coefficients may again be expressed in terms of the Fourier coefficients of  $(\bar{X}, \bar{P})$ . However, in repeating this step one must now use the relation<sup>4</sup>

$$p_- (\bar{X}^-)' + p \bar{X}' + \frac{i}{2} \vartheta \bar{\theta}' = - \left( \bar{X}' \bar{P} + \frac{i}{2} \bar{\theta} \bar{\theta}' \right) + \oint \frac{d\sigma}{2\pi} \left( \bar{X}' \bar{P} + \frac{i}{2} \bar{\theta} \bar{\theta}' \right) \quad (4.18)$$

which replaces (3.26). The relation that replaces (3.41) is

$$p_- D_\tau \bar{X}^- = \bar{P}_+ + i \frac{T}{2p_-} \bar{\theta} \bar{\theta}' - i \frac{T}{2p_-} \oint \frac{d\sigma}{2\pi} \bar{\theta} \bar{\theta}'. \quad (4.19)$$

The supersymmetry charges in the light cone gauge are

$$\begin{aligned}\mathcal{Q}^1 &= \sqrt{\frac{1}{\sqrt{2} p_-}} \left[ p \vartheta + \oint \frac{d\sigma}{2\pi} (\bar{P} - T \bar{X}') \bar{\theta} \right], \\ \mathcal{Q}^2 &= \sqrt{\sqrt{2} p_-} \vartheta. \end{aligned} \quad (4.20)$$

Finally, the parity transformation (2.33) acts in the light-cone gauge via the transformation

$$X \rightarrow -X, \quad P \rightarrow -P, \quad (4.21)$$

with all other canonical variables, in particular  $\theta$ , being parity inert. It follows that the (classical) Hamiltonian  $H$  is invariant under parity, as expected.

## 4.2 Fourier expansion

We Fourier expand  $\bar{\theta}$  as

$$\bar{\theta} = \sum_{n=1}^{\infty} [e^{in\sigma} \xi_n + e^{-in\sigma} \xi_n^*]. \quad (4.22)$$

---

<sup>4</sup>This corrects a minus sign error in the corresponding relation of [5].

With the bosonic Fourier expansions as before, the Lagrangian (4.14) becomes

$$\begin{aligned}
L = & \dot{x}p - \dot{x}^-p_- + \frac{i}{2}\vartheta\dot{\vartheta} + i \sum_{n=1}^{\infty} \left[ \frac{1}{n} (\alpha_n^* \dot{\alpha}_n + \tilde{\alpha}_n^* \dot{\tilde{\alpha}}_n) + \xi_n^* \dot{\xi}_n \right] - H \\
& + u_0 \sum_{n=1}^{\infty} (\alpha_n^* \alpha_n - \tilde{\alpha}_n^* \tilde{\alpha}_n + n \xi_n^* \xi_n) .
\end{aligned} \tag{4.23}$$

The Hamiltonian again takes the form

$$H = \frac{1}{2p_-} (p^2 + \mathcal{M}^2) \tag{4.24}$$

but now with

$$\mathcal{M}^2 = 2T \sum_{n=1}^{\infty} (\alpha_n^* \alpha_n + \tilde{\alpha}_n^* \tilde{\alpha}_n + n \xi_n^* \xi_n) . \tag{4.25}$$

The Poincaré charges are as in (4.16) but now with the different expression (4.17) for the Poincaré invariant  $\Lambda$ . As this involves  $\bar{X}^-$ , we must first use (4.18) to express  $\bar{X}^-$  in terms of the canonical variables, or their Fourier coefficients. The result of this computation is

$$-p_- \bar{X}^- = p\bar{X} + \frac{i}{2}\vartheta\bar{\theta} + \sum_{n=1}^{\infty} \frac{i}{n} \left[ e^{in\sigma} (\beta_n + \gamma_n - \tilde{\beta}_n^*) - e^{-in\sigma} (\beta_n^* + \gamma_n^* - \tilde{\beta}_n) \right] \tag{4.26}$$

where  $\beta_n$  and  $\tilde{\beta}_n$  are as they were for the bosonic string, and

$$\gamma_n = \frac{1}{2} \sum_{m=1}^{n-1} (n-m) \xi_m \xi_{n-m} + \sum_{m>n} \left( m - \frac{n}{2} \right) \xi_{m-n}^* \xi_m . \tag{4.27}$$

One also needs the result that

$$\frac{i}{2} \oint \frac{d\sigma}{2\pi} \bar{X} \bar{\theta} \bar{\theta}' = \frac{1}{\sqrt{2T}} \sum_{n=1}^{\infty} \frac{i}{n} [\gamma_n (\alpha_n^* - \tilde{\alpha}_n) - \gamma_n^* (\alpha_n - \tilde{\alpha}_n^*)] \tag{4.28}$$

We then find that

$$\begin{aligned}
\Lambda_+ &= \sqrt{2T} \left[ \sum_{n=1}^{\infty} \frac{i}{n} \alpha_n^* (\beta_n + \gamma_n) + \frac{i}{2} \vartheta \sum_{n=1}^{\infty} \alpha_n^* \xi_n \right] + c.c. , \\
\Lambda_- &= \sqrt{2T} \sum_{n=1}^{\infty} \frac{i}{n} \tilde{\alpha}_n^* \tilde{\beta}_n + c.c.
\end{aligned} \tag{4.29}$$

The supersymmetry charges are now

$$\begin{aligned}
\mathcal{Q}^1 &= \sqrt{\frac{1}{\sqrt{2}p_-}} \left[ p\vartheta + \sqrt{2T} \sum_{n=1}^{\infty} (\alpha_n \xi_n^* + \alpha_n^* \xi_n) \right] , \\
\mathcal{Q}^2 &= \sqrt{\sqrt{2}p_-} \vartheta .
\end{aligned} \tag{4.30}$$

Using (4.29) and (4.30), we find that the super-Poincaré invariant  $\Omega$  of (2.21) takes the form

$$\Omega = \Omega_+ + \Omega_-, \quad (4.31)$$

where  $\Omega_- = \Lambda_-$  and

$$\Omega_+ = \sqrt{2T} \sum_{n=1}^{\infty} \frac{i}{n} [\alpha_n^* (\beta_n + \gamma_n) - (\beta_n + \gamma_n)^* \alpha_n] . \quad (4.32)$$

Note that the anticommuting zero mode  $\vartheta$ , present in  $\Lambda$ , cancels from  $\Omega$ .

### 4.3 Quantum $\mathcal{N} = 1$ 3D superstring

To quantize, we replace the bosonic variables by operators as before, and we promote the fermionic variables to operators satisfying the anti-commutation relations

$$\vartheta^2 = \frac{1}{2}, \quad \{\xi_n, \xi_n^\dagger\} = 1, \quad (4.33)$$

with all other anticommutators of these variables equal to zero. The quantum Hamiltonian has the form (4.24) with

$$\mathcal{M}^2 = 2T \left[ N + \tilde{N} + \nu \right], \quad \nu = \sum_{n=1}^{\infty} n \xi_n^\dagger \xi_n, \quad (4.34)$$

where the bosonic level number operators  $(N, \tilde{N})$  are as before. The level-matching constraint is now

$$\tilde{N} = N + \nu, \quad (4.35)$$

which implies that

$$\mathcal{M}^2 = 4T (N + \nu), \quad (4.36)$$

and hence that physical states of a given mass all appear at a particular level, given by  $N + \nu$ . The asymmetry in the level-matching condition is due to the fact that the fermionic operators  $\xi_n$  create right-moving modes on the string that are super-partners to the right-moving modes, whereas the left-moving bosonic modes have no super-partners. In effect, the  $\mathcal{N} = 1$  GS 3D closed superstring is a 3D heterotic string. Changing the sign of the WZ term in the action (4.2) would lead to super-partners for the left-moving bosonic modes instead of the right-moving ones, so there are two distinct  $\mathcal{N} = 1$  superstrings. Nevertheless, both of these potentially distinct (albeit equivalent) superstrings have exactly the same (parity preserving) 3D spectrum, so they are *identical* as quantum theories and we need not distinguish between them<sup>5</sup>.

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<sup>5</sup>There is nothing to prevent the strings under consideration here from self-intersecting, so we could consider a macroscopic figure-of-eight superstring in which the fermionic modes move clockwise in one loop of the “8” and anticlockwise in the other; this shows that the chiral nature of the worldsheet fermions does not imply a violation of 3D parity.

The operator versions of  $\Omega_{\pm}$  may be written as

$$\Omega_+ = \sqrt{2T} \left[ \lambda + \sum_{n=1}^{\infty} \frac{i}{n} (\alpha_n^\dagger \gamma_n - \alpha_n \gamma_n^\dagger) \right], \quad \Omega_- \equiv \Lambda_- = \sqrt{2T} \tilde{\lambda}, \quad (4.37)$$

where  $\lambda$  and  $\tilde{\lambda}$  are the operators of the bosonic string and  $\gamma_n$  is now the following operator:

$$\gamma_n = \frac{1}{2} \sum_{m=1}^{n-1} (n-m) \xi_m \xi_{n-m} + \sum_{m>n} \left( m - \frac{n}{2} \right) \xi_{m-n}^\dagger \xi_m. \quad (4.38)$$

The super-Poincaré invariant operator  $\Omega = \Omega_+ + \Omega_-$  is related to the Poincaré invariant operator  $\Lambda$  by

$$\Lambda = \Omega + \frac{1}{2\sqrt{2}} i \vartheta \Xi, \quad \Xi = \sqrt{4T} \sum_{n=1}^{\infty} (\alpha_n \xi_n^\dagger + \alpha_n^\dagger \xi_n). \quad (4.39)$$

The operator supercharges are

$$\mathcal{Q}^1 = \sqrt{\frac{1}{\sqrt{2} p_-}} \left[ p \vartheta + \frac{1}{\sqrt{2}} \Xi \right], \quad \mathcal{Q}^2 = \sqrt{\sqrt{2} p_-} \vartheta. \quad (4.40)$$

Using the fact that

$$\Xi^2 = \mathcal{M}^2, \quad (4.41)$$

for physical states satisfying the level-matching condition, it is straightforward to verify that the supercharges have the expected anticommutation relations. Although the above relation shows that the hermitian operator  $\Xi$  is a square root of  $\mathcal{M}^2$ , it has zero trace in the state space to be discussed below and so is not positive; it also anticommutes with  $\vartheta$ . However, the hermiticity of  $\Xi$  implies that  $\mathcal{M}^2$  is positive so there exists a positive square root hermitian operator  $\mathcal{M}$ ; it can be defined in a basis in which  $\mathcal{M}^2$  is diagonal by taking the positive square root of all diagonal entries. We may then introduce the new supercharge

$$\begin{aligned} \mathcal{S} &= \sqrt{\sqrt{2} p_-} \mathcal{Q}^1 - \frac{1}{\sqrt{\sqrt{2} p_-}} (p - i \mathcal{M}) \mathcal{Q}^2, \\ &= \frac{1}{\sqrt{2}} \left[ i \sqrt{2} \vartheta \mathcal{M} + \Xi \right]. \end{aligned} \quad (4.42)$$

For  $\mathcal{M} = 0$  this reduces to a factor times  $\Xi$ , which is (in this case) one real linear combination of the two hermitian supercharges  $\mathcal{Q}^\alpha$ . Otherwise,  $\mathcal{S}$  is non-hermitian, and we may trade the two hermitian supercharges  $\mathcal{Q}^\alpha$  for  $\mathcal{S}$  and its hermitian conjugate. Using the relation (4.41) one may verify that

$$\mathcal{S}^2 = 0, \quad \{ \mathcal{S}, \mathcal{S}^\dagger \} = 2 \mathcal{M}^2, \quad (4.43)$$

in accord with the discussion of subsection 2.5. These relations are valid only when the operators act on physical states because the validity of (4.41) requires the level-matching constraint.

Parity acts in the light-cone gauge of the  $\mathcal{N} = 1$  superstring in exactly the same way as it does in the bosonic theory. The parity operator  $\Pi$  is again given by (3.65); it has the property

$$\Pi \mathcal{S} = -\mathcal{S}^\dagger \Pi. \quad (4.44)$$

As  $\Pi$  commutes with both with  $N$  and  $\tilde{N}$ , as before, and trivially with  $\nu$ , it commutes with the Hamiltonian. Since it anticommutes with  $\Lambda$ , this means that massive states of non-zero spin must appear in degenerate parity doublets of opposite-sign helicity. However, two such degenerate states will not appear in the same supermultiplet unless this supermultiplet has zero superspin; this is because  $\Pi$  also anticommutes with  $\Omega$ , so massive supermultiplets of non-zero superspin must appear in degenerate pairs of opposite sign superhelicity.

### 4.3.1 Absence of anomalies

The Poincaré charges of the  $\mathcal{N} = 1$  superstring are exactly as given in (3.58) and (3.59) for the bosonic string but with the Hamiltonian  $H$  and Poincaré invariant  $\Lambda$  of the superstring. The absence of anomalies in the Lorentz algebra is again a direct consequence of the fact that  $H$  and  $\Lambda$  commute, but this is now an immediate consequence of the fact that

$$[\Xi, \Omega] = 0. \quad (4.45)$$

Moreover, this relation is now the fundamental one to check because it also implies that there is no anomaly in the commutation relation of the Lorentz charges with the supercharges. Most of the latter are just as for the superparticle; the only potentially problematic commutators are those which involve  $\mathcal{J}^-$ . We should find, for operators acting on physical states satisfying the level-matching constraint, that

$$[\mathcal{J}^-, \mathcal{Q}^1] = 0, \quad [\mathcal{J}^-, \mathcal{Q}^2] = -\frac{i}{\sqrt{2}} \mathcal{Q}^2. \quad (4.46)$$

This can be checked directly but it is essentially equivalent to a check of the commutation relations

$$[\Lambda, \mathcal{S}] = -\frac{1}{2} \mathcal{M} \mathcal{S}, \quad [\Lambda, \mathcal{S}^\dagger] = \frac{1}{2} \mathcal{M} \mathcal{S}^\dagger, \quad (4.47)$$

which ensure that massive supermultiplets consist of two states differing by helicity  $1/2$ , and these follow directly from (4.45).

To verify (4.45) we need only show that  $[\Xi, \Omega_+] = 0$  since it is manifest that  $\Xi$  commutes with  $\Omega_- = \Lambda_-$ . As  $\Xi$  is linear and  $\Omega$  quadratic in ‘fermions’, this commutator contains, in principle, terms that are linear and cubic in ‘fermions’. The cubic term vanishes as a consequence of the identity

$$\sum_{n=1}^{\infty} (\xi_n^\dagger \gamma_n + \gamma_n^\dagger \xi_n) \equiv 0, \quad (4.48)$$

which one proves by using the obvious identity

$$\sum_{m=1}^{n-1} \xi_m \xi_{n-m} \equiv 0. \quad (4.49)$$

To check that the term linear in ‘fermions’ is also zero, it is useful to begin by establishing the following commutation relations, which supplement those of (3.62):

$$[\xi_n, \gamma_m] = \left(n + \frac{m}{2}\right) \xi_{n+m}, \quad [\xi_n^\dagger, \gamma_m] = \begin{cases} \left(\frac{m}{2} - n\right) \xi_{m-n} & n < m \\ 0 & n = m \\ \left(\frac{m}{2} - n\right) \xi_{n-m}^\dagger & n > m \end{cases}. \quad (4.50)$$

### 4.3.2 Realization

The anticommutation relations (4.33) can be partially realized by setting

$$\vartheta = \frac{1}{\sqrt{2}} \sigma_1 \otimes \mathbb{I}, \quad \xi_n = \sigma_2 \otimes \chi_n, \quad \xi_n^\dagger = \sigma_2 \otimes \chi_n^\dagger, \quad (4.51)$$

where  $(\chi_n, \chi_n^\dagger)$  are a set of fermionic annihilation and creation operators:

$$\{\chi_n, \chi_m^\dagger\} = \delta_{mn}. \quad (4.52)$$

The operator

$$(-1)^F = \sigma_3 \otimes \mathbb{I} \quad (4.53)$$

anticommutes with  $\vartheta$  and  $\xi_n$ , and hence with the supercharges<sup>6</sup>. Let  $|\varsigma\rangle_+$  be the pair of states ( $\varsigma = \pm$ ) such that

$$(-1)^F |\varsigma\rangle_+ = \varsigma |\varsigma\rangle_+, \quad \chi_n |\varsigma\rangle_+ = 0 \quad (n \in \mathbb{Z}^+). \quad (4.54)$$

Then the doubly-degenerate oscillator ground state (for both bosonic and fermionic operators) is

$$|\varsigma\rangle = |0, \varsigma\rangle_+ \otimes |0\rangle_-, \quad |0, \varsigma\rangle_+ = |0\rangle_+ \otimes |\varsigma\rangle_+, \quad (4.55)$$

where  $|0, \varsigma\rangle_+$  is the ground state for the right-movers and  $|0\rangle_-$  is the ground state for the left-movers. The states  $|\varsigma\rangle$  are annihilated by  $\Xi$  and hence have zero mass. The operator  $\Xi$  is a real linear combination of the two hermitian supercharges for zero mass. The two states of  $|\varsigma\rangle$  are permuted by any other linearly-independent combination, e.g.  $\vartheta$ , so they form the two states of a single massless supermultiplet. As expected, one is a boson and the other a fermion. Excited string states, which are

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<sup>6</sup>The fact that the supercharges are 3D fermions suggests an interpretation of  $(-1)^F$  as an operator that counts spacetime fermion number modulo two. This allows us to distinguish bosonic from fermionic *massless* states: recall that this distinction survives the massless limit even though spin is not defined for massless 3D particles. The interpretation of  $(-1)^F$  in its action on massive states is less clear since these need not be either bosons or fermions, but we pass over this point here because massive states are characterized by their relativistic helicity  $s$ , which we may compute directly.

all massive, are found by acting on the ground state  $|\varsigma\rangle$  with creation operators, in such a way that the level-matching condition (4.35) is satisfied. We may therefore organize all physical states according to their level  $L$ , with mass  $M = \sqrt{4T}L$ . In the above realization, the operators  $\mathcal{M}$  and  $\Xi$  become

$$\mathcal{M} = \mathbb{I}_2 \otimes \mathcal{M}_{red}, \quad \Xi = \sigma_2 \otimes \Xi_{red} \quad (4.56)$$

where  $\mathcal{M}_{red}$  and  $\Xi_{red}$  are ‘reduced’ operators acting in the Fock space of the operators  $(\alpha_n, \alpha_n^\dagger)$  and  $(\chi_n, \chi_n^\dagger)$ . The non-hermitian supercharge  $\mathcal{S}$  is represented by

$$\mathcal{S} = \frac{1}{\sqrt{2}} [i\sigma_1 \otimes \mathcal{M}_{red} + \sigma_2 \otimes \Xi_{red}] \quad (4.57)$$

At a given mass level  $L > 0$ , for which  $\mathcal{M} = M = \sqrt{4T}L$ , we have

$$\mathcal{S}|_L = i\sqrt{2T} L [\sigma_1 \otimes I_L - i\sigma_2 \otimes \eta_L], \quad (4.58)$$

where  $\eta_L$  is an operator on the space of states at level  $L$  that squares to the identity but has zero trace. For a given eigenvalue of  $\eta_L$ , we get a supermultiplet by acting with  $\mathcal{S}^\dagger$  on a state annihilated by  $\mathcal{S}$  (as discussed in subsection 2.5) but since the eigenvalues of  $\eta_L$  come in  $\pm 1$  pairs, each massive level contains an even number of degenerate supermultiplets, half with  $\eta_L = 1$  and the other half with  $\eta_L = -1$ . All massive multiplets are therefore at least quadruply degenerate.

### 4.3.3 Low-lying excited states

The first excited states, at level-1, are

$$\begin{aligned} |1_B, \varsigma\rangle_+ \otimes |1_B\rangle_- &= \alpha_1^\dagger |0, \varsigma\rangle_+ \otimes \tilde{\alpha}_1^\dagger |0\rangle_-, \\ |1_F, \varsigma\rangle_+ \otimes |1_B\rangle_- &= \chi_1^\dagger |0, \varsigma\rangle_+ \otimes \tilde{\alpha}_1^\dagger |0\rangle_-, \end{aligned} \quad (4.59)$$

which gives us a total of four states at this level, and hence two  $\mathcal{N} = 1$  supermultiplets. The level-2 oscillator states, are constructed from tensor products of the ‘right-moving’ orthonormal states

$$\begin{aligned} |1_B, 1_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} (\alpha_1^\dagger)^2 |0, \varsigma\rangle_+, & |2_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} \alpha_2^\dagger |0, \varsigma\rangle_+ \\ |1_B, 1_F, \varsigma\rangle_+ &= \alpha_1^\dagger \chi_1^\dagger |0, \varsigma\rangle_+, & |2_F, \varsigma\rangle_+ &= \chi_2^\dagger |0, \varsigma\rangle_+, \end{aligned} \quad (4.60)$$

with the ‘left-moving’ level-2 states of the bosonic string

$$|1, 1\rangle_- = \frac{1}{\sqrt{2}} (\tilde{\alpha}_1^\dagger)^2 |0\rangle_-, \quad |2\rangle_- = \frac{1}{\sqrt{2}} \tilde{\alpha}_2^\dagger |0\rangle_-. \quad (4.61)$$

This gives us a total of 16 states, which must arrange themselves into eight  $\mathcal{N} = 1$  supermultiplets.

At level 3 we need to consider the following eight doubly-degenerate ‘right-moving’ (orthonormal basis) states

$$\begin{aligned}
|1_B, 1_B, 1_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{6}} \left(\alpha_1^\dagger\right)^3 |0, \varsigma\rangle_+, & |1_B, 2_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} \alpha_1^\dagger \alpha_2^\dagger |0, \varsigma\rangle_+, \\
|3_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{3}} \alpha_3^\dagger |0, \varsigma\rangle_+, & |1_F, 2_F, \varsigma\rangle_+ &= \chi_1^\dagger \chi_2^\dagger |0, \varsigma\rangle_+, \\
|1_B, 2_F, \varsigma\rangle_+ &= \alpha_1^\dagger \chi_2^\dagger |0, \varsigma\rangle_+, & |1_B, 1_B, 1_F, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} \left(\alpha_1^\dagger\right)^2 \chi_1^\dagger |0, \varsigma\rangle_+, \\
|2_B, 1_F, \varsigma\rangle_+ &= \frac{1}{\sqrt{3}} \alpha_2^\dagger \chi_1^\dagger |0, \varsigma\rangle_+, & |3_F, \varsigma\rangle_+ &= \chi_3^\dagger |0, \varsigma\rangle_+.
\end{aligned} \tag{4.62}$$

These must be tensored with the three ‘left-moving’ level-2 states of the bosonic  $\mathcal{N} = 0$  string

$$\begin{aligned}
|1, 1, 1\rangle_- &= \frac{1}{\sqrt{6}} \left(\tilde{\alpha}_1^\dagger\right)^3 |0\rangle_-, & |1, 2\rangle_- &= \frac{1}{\sqrt{2}} \tilde{\alpha}_1^\dagger \tilde{\alpha}_2^\dagger |0\rangle_-, \\
|3\rangle_- &= \frac{1}{\sqrt{3}} \tilde{\alpha}_3^\dagger |0\rangle_-.
\end{aligned} \tag{4.63}$$

This gives us a total of 48 states at level 3, and hence 24  $\mathcal{N} = 1$  supermultiplets.

#### 4.3.4 Superhelicities

In order to determine the spectrum of superhelicities, we must find the eigenvalues of the operator  $\Omega$ , which we can do by finding those of  $\Omega_+$  since the eigenvalues of  $\Omega_- = \Lambda_-$  have already been computed (to the level considered here). There will be a double degeneracy in the eigenvalues of  $\Omega$  due to the independence of this operator on the fermion zero mode  $\vartheta$ ; this is the degeneracy implied by  $\mathcal{N} = 1$  supersymmetry. As discussed in the previous subsection, there is a further double-degeneracy in massive levels, so the number of potentially distinct eigenvalues of  $\Omega$  at a given mass level is only a quarter of the number of states at that level. A further simplifying feature is that  $\Omega_+$  is a Grassmann even operator that does not mix states of different Grassmann parity (as determined by the Grassman parities of the operators used to construct the states, ignoring the Grassmann parity of the ground states).

The operator  $\Omega$  annihilates the two massless ground states, which form a massless supermultiplet of  $\mathcal{N} = 1$  3D supersymmetry comprising one boson and one fermion (recall that spin is not defined for massless particles). Potentially, these could be identified as a dilaton and dilatino. The operator  $\Omega$  also annihilates all level-1 states, which implies that the four states at this level yield two copies of the  $\mathcal{N} = 1$  semion supermultiplet with helicities  $(\frac{1}{4}, -\frac{1}{4})$ ; this is the supermultiplet one gets by quantizing the  $\mathcal{N} = 1$  massive superparticle [19]. At level 2, there are 16 states and so eight eigenvalues of  $\Omega$ . To compute them, we need to consider the  $4 \times 4$  matrix that results from the action of  $\lambda_2$  in the space spanned by the basis states (4.60). We find that

$$\lambda_2 = \frac{3i}{4} \left[ \left(\alpha_1^\dagger\right)^2 \alpha_2 - \alpha_2^\dagger \alpha_1^2 \right] + \frac{3i}{2} \left[ \alpha_1^\dagger \xi_1^\dagger \xi_2 - \xi_2^\dagger \xi_1 \alpha_1 \right]. \tag{4.64}$$



and then that the  $4 \times 4$  matrix is block-diagonal, with  $2 \times 2$  blocks. One block gets contributions only from the first term of  $\lambda_2$ ; this is just the the  $\mathcal{N} = 0$  string contribution of (3.68) and we have already seen that this gives the matrix  $(3/2)\sigma_2$ . The additional contribution to  $\lambda_2$  contributes only to the second  $2 \times 2$  block, which is also  $(3/2)\sigma_2$ . It follows that  $\Omega_+$  has eigenvalues  $\sqrt{2T}(\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{3}{2})$  whereas  $\Omega_- = \Lambda_-$  has eigenvalues  $\sqrt{2T}(\frac{3}{2}, -\frac{3}{2})$ . The eight eigenvalues of  $\Omega$  are therefore  $\sqrt{2T}(0, 0, 0, 0, 3, 3, -3, -3)$ . To get the superhelicities  $s_1$ , we have to divide this by the level-2 mass, which is  $2\sqrt{2T}$ ; this gives

$$s_1 = \left( -\frac{3}{2}, -\frac{3}{2}, 0, 0, 0, 0, \frac{3}{2}, \frac{3}{2} \right). \quad (4.65)$$

The supermultiplets with superhelicity  $s_1 = \pm\frac{3}{2}$  have helicities  $s = (\pm\frac{7}{4}, \pm\frac{5}{4})$ . These level-1 and level-2 results, which show that semions are present in the spectrum of the 3D  $\mathcal{N} = 1$  superstring, were announced in [5]. Here we continue the analysis to the next level.

At level 3 we find that

$$\begin{aligned} \lambda_3 = & \frac{7i}{6} \left( \alpha_1^\dagger \alpha_2^\dagger \alpha_3 - \alpha_3^\dagger \alpha_1 \alpha_2 \right) \\ & + i \left[ \frac{5}{2} \left( \alpha_1^\dagger \xi_2^\dagger \xi_3 - \xi_3^\dagger \xi_2 \alpha_1 \right) + \left( \alpha_2^\dagger \xi_1^\dagger \xi_3 - \xi_3^\dagger \xi_1 \alpha_2 \right) - \frac{1}{6} \left( \alpha_3^\dagger \xi_1 \xi_2 - \xi_2^\dagger \xi_1^\dagger \alpha_3 \right) \right]. \end{aligned} \quad (4.66)$$

In the level-3 basis (4.62) we have an  $8 \times 8$  matrix but it is block diagonal, with the following two  $4 \times 4$  blocks:

$$i \begin{pmatrix} 0 & -\frac{3\sqrt{3}}{2} & 0 & 0 \\ \frac{3\sqrt{3}}{2} & 0 & -\frac{7}{\sqrt{6}} & 0 \\ 0 & \frac{7}{\sqrt{6}} & 0 & -\frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \end{pmatrix}, \quad i \begin{pmatrix} 0 & +\frac{3}{\sqrt{2}} & 0 & -\frac{5}{2} \\ -\frac{3}{\sqrt{2}} & 0 & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & 0 & -\sqrt{2} \\ \frac{5}{2} & 0 & \sqrt{2} & 0 \end{pmatrix}. \quad (4.67)$$

The eigenvalues of these matrices are, respectively,

$$\pm \sqrt{\frac{15}{2} \pm \frac{9\sqrt{11}}{4}} \quad \text{and} \quad \pm \sqrt{3 \pm \frac{3\sqrt{7}}{4}}, \quad (4.68)$$

which shows that at the level 3 there are supermultiplets of irrational superhelicity.

## 5 The closed $\mathcal{N} = 2$ 3D Superstring

As for  $\mathcal{N} = 1$ , the  $\mathcal{N} = 2$  GS superstring action can be obtained from the bosonic string action in two steps. First, we make the replacement

$$\Pi_\tau \rightarrow \Pi_\tau = \dot{\mathbb{X}} + i\bar{\Theta}_a \Gamma \dot{\Theta}_a, \quad \Pi_\sigma \rightarrow \Pi_\sigma = \mathbb{X}' + i\bar{\Theta}_a \Gamma \Theta'_a. \quad (5.1)$$

Next, we add to the resulting action a Wess-Zumino (WZ) term constructed from the closed, super-Poincaré invariant 3-form  $h_3$  given in (2.30). This can be written

as  $h_3 = dh_2$  for  $h_2$  as given in (2.31), and the integral of  $h_2$  gives us the required WZ term. These considerations lead to the following quasi-Hamiltonian form of the  $\mathcal{N} = 2$  superstring action:

$$\begin{aligned}
S[\mathbb{X}, \mathbb{P}, \Theta_a; \ell, u] &= \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \Pi_\tau^\mu \mathbb{P}_\mu - \frac{1}{2} \ell [\mathbb{P}^2 + (T\Pi_\sigma)^2] - u \Pi_\sigma^\mu \mathbb{P}_\mu \right. \\
&\quad + iT \left[ \left( \dot{\mathbb{X}}^\mu + \frac{i}{2} \bar{\Theta}_a \Gamma^\mu \dot{\Theta}_a \right) (\bar{\Theta}_1 \Gamma_\mu \Theta'_1 - \bar{\Theta}_2 \Gamma_\mu \Theta'_2) \right. \\
&\quad \left. \left. - \left( \mathbb{X}'^\mu + \frac{i}{2} \bar{\Theta}_a \Gamma^\mu \Theta'_a \right) (\bar{\Theta}_1 \Gamma_\mu \dot{\Theta}_1 - \bar{\Theta}_2 \Gamma_\mu \dot{\Theta}_2) \right] \right\}. \tag{5.2}
\end{aligned}$$

This action has  $\alpha$ -symmetry and  $\beta$ -symmetry gauge invariances that generalize those of the  $\mathcal{N} = 1$  3D superstring. The transformations of the Lagrange multiplier variables are unchanged while those of the canonical variables are

$$\begin{aligned}
\delta \mathbb{X} &= \alpha \left[ \mathbb{P} - i\ell^{-1} \bar{\Theta}_a \Gamma (\dot{\Theta}_a - u\Theta'_a) \right] + \beta \mathbb{X}', \\
\delta \Theta_a &= \alpha \ell^{-1} (\dot{\Theta}_a - u\Theta'_a) + \beta \Theta'_a, \\
\delta \mathbb{P} &= (T^2 \alpha \Pi_\sigma + \beta \mathbb{P})' + 2i\alpha \ell^{-1} T (\bar{\Theta}'_1 \Gamma \dot{\Theta}_1 - \bar{\Theta}'_2 \Gamma \dot{\Theta}_2). \tag{5.3}
\end{aligned}$$

The term linear in  $T$  in the action (5.2) is the WZ term, and we have chosen its coefficient to ensure invariance under the following “ $\kappa$ -symmetry” gauge transformation with anticommuting Majorana spinor parameters  $\kappa_a$ :

$$\begin{aligned}
\delta_\kappa \mathbb{X}^\mu &= -i \bar{\Theta}_a \Gamma^\mu \delta_\kappa \Theta_a, & \delta_\kappa \mathbb{P}_\mu &= 2iT (\bar{\Theta}'_1 \Gamma_\mu \delta_\kappa \Theta_1 - \bar{\Theta}'_2 \Gamma_\mu \delta_\kappa \Theta_2), \\
\delta_\kappa \Theta_1 &= \Gamma_\mu (\mathbb{P}^\mu - T\Pi_\sigma^\mu) \kappa_1, & \delta_\kappa \Theta_2 &= \Gamma_\mu (\mathbb{P}^\mu + T\Pi_\sigma^\mu) \kappa_2, \\
\delta_\kappa \ell &= -4i\bar{\kappa}_1 \left[ \dot{\Theta}_1 + (\ell T - u) \Theta'_1 \right] - 4i\bar{\kappa}_2 \left[ \dot{\Theta}_2 + (-\ell T - u) \Theta'_2 \right], \\
\delta_\kappa u &= -T (\delta_{\kappa_1} \ell - \delta_{\kappa_2} \ell). \tag{5.4}
\end{aligned}$$

Because of the relative minus sign in the WZ term, its overall sign can be changed by the field redefinition  $\Theta_1 \leftrightarrow \Theta_2$ , so we may choose  $T > 0$ . To verify the  $\kappa$ -symmetry, it is useful to use the fact that

$$\delta_\kappa h_3 = d\delta_\kappa h_2 = -2d \left[ \Pi^\mu (\delta_\kappa \bar{\Theta}_1 \Gamma_\mu \Theta_1 - \delta_\kappa \bar{\Theta}_2 \Gamma_\mu \Theta_2) \right], \tag{5.5}$$

which gives  $\delta_\kappa h_2$  up to the addition of an irrelevant closed form. Observe that

$$\det [\Gamma_\mu (\mathbb{P}^\mu \mp T\Pi_\sigma^\mu)] = -(\mathbb{P} \mp T\Pi_\sigma)^2 \approx 0. \tag{5.6}$$

As for the  $N = 1$  superstring, this implies that only one of the two independent components of each  $\kappa_a$  has any effect, so that only one real component of each  $\Theta_a$  can be gauged away.

As for rigid symmetries, the action (5.2) is invariant under the parity transformation of (2.34). It is also super-Poincaré invariant, by construction. The Poincaré

Noether charges are

$$\begin{aligned}
\mathcal{P}_\mu &= \oint \frac{d\sigma}{2\pi} \{ \mathbb{P}_\mu + iT [ \bar{\Theta}_1 \Gamma_\mu \Theta'_1 - \bar{\Theta}_2 \Gamma_\mu \Theta'_2 ] \} , \\
\mathcal{J}^\mu &= \oint \frac{d\sigma}{2\pi} \{ [ \mathbb{X} \wedge ( \mathbb{P} + iT ( \bar{\Theta}_1 \Gamma \Theta'_1 - \bar{\Theta}_2 \Gamma \Theta'_2 ) ) ]^\mu \\
&\quad + \frac{i}{2} \bar{\Theta}_1 \Theta_1 ( \mathbb{P} - T \mathbb{X}' )^\mu + \frac{i}{2} \bar{\Theta}_2 \Theta_2 ( \mathbb{P} + T \mathbb{X}' )^\mu \\
&\quad + (iT/2) ( \bar{\Theta}_2 \Gamma^\mu \Theta'_2 \bar{\Theta}_1 \Theta_1 - \bar{\Theta}_1 \Gamma^\mu \Theta'_1 \bar{\Theta}_2 \Theta_2 ) \} . \tag{5.7}
\end{aligned}$$

The supersymmetry Noether charges are

$$\begin{aligned}
\mathcal{Q}_1^\alpha &= \sqrt{2} \oint \frac{d\sigma}{2\pi} \{ ( \mathbb{P}^\mu - T \Pi_\sigma ) ( \Gamma^\mu \Theta_1 )^\alpha - 2iT ( \bar{\Theta}_1 \Theta_1 ) \Theta'_1 \} , \\
\mathcal{Q}_2^\alpha &= \sqrt{2} \oint \frac{d\sigma}{2\pi} \{ ( \mathbb{P}^\mu + T \Pi_\sigma ) ( \Gamma^\mu \Theta_2 )^\alpha + 2iT ( \bar{\Theta}_2 \Theta_2 ) \Theta'_2 \} . \tag{5.8}
\end{aligned}$$

## 5.1 Light-cone gauge

The light-cone gauge fixing proceeds as for the  $\mathcal{N} = 1$  superstring but with the additional fixing of the larger kappa-symmetry gauge invariance by the condition

$$\Gamma^+ \Theta_a = 0, \quad a = 1, 2. \tag{5.9}$$

In this gauge,

$$\Theta_a = \sqrt{\frac{1}{2\sqrt{2} p_-}} \begin{pmatrix} \theta_a \\ 0 \end{pmatrix} \tag{5.10}$$

for some anticommuting worldsheet functions  $\theta_a(\tau, \sigma)$ . We thus find that

$$\begin{aligned}
\Pi_\tau^+ &= 1, & \Pi_\tau^- &= \dot{X}^- + \frac{i}{2p_-} \theta_a \dot{\theta}_a, & \Pi_\tau^2 &= \dot{X}, \\
\Pi_\sigma^+ &= 0, & \Pi_\sigma^- &= (X^-)' + \frac{i}{2p_-} \theta_a \theta'_a, & \Pi_\sigma^2 &= X'. \tag{5.11}
\end{aligned}$$

Again it is convenient, it is convenient to define

$$\bar{\theta}_a = \theta_a - \vartheta_a, \quad \vartheta_a(\tau) = \oint \frac{d\sigma}{2\pi} \theta_a . \tag{5.12}$$

and should again be no confusion with the notation for a conjugate spinor as  $\theta_a$  are not 2-component spinors. In this notation, we find that the analog of (3.12) (but without the  $u = \bar{u} + u_0$  split) is

$$\begin{aligned}
L &= \dot{x} p + \dot{x}^- p_- + \frac{i}{2} \vartheta_a \dot{\vartheta}_a + \oint \frac{d\sigma}{2\pi} \left\{ \dot{X} \bar{P} + \frac{i}{2} \bar{\theta}_a \dot{\theta}_a \right\} \\
&\quad + \frac{iT}{2p_-} \oint \frac{d\sigma}{2\pi} ( \bar{\theta}_1 \theta'_1 - \bar{\theta}_2 \theta'_2 ) - \oint \frac{d\sigma}{2\pi} u \left[ \bar{X}' P + \frac{i}{2} \theta_a \bar{\theta}'_a \right] \\
&\quad + p_- \oint \frac{d\sigma}{2\pi} \left\{ \bar{X}^- u' - \ell \left( P_+ + \frac{1}{2p_-} [ P^2 + (TX')^2 ] \right) \right\} . \tag{5.13}
\end{aligned}$$

As before,  $\bar{X}^-$  is now a Lagrange multiplier for the constraint  $u' = 0$ , which we solve by writing  $u = u_0(\tau)$ . The constraint imposed by the Lagrange multiplier  $\ell$  is also exactly as before, and therefore has the same solution (3.13) for  $P_+$ . The resulting analog of the bosonic Lagrangian (3.14) is

$$L = \left[ \dot{x}p + \dot{x}^-p_- + \frac{i}{2}\vartheta_a\dot{\vartheta}_a + \oint \frac{d\sigma}{2\pi} \left\{ \dot{\bar{X}}\bar{P} + \frac{i}{2}\bar{\theta}_a\dot{\theta}'_a \right\} \right] - H - u_0 \oint \frac{d\sigma}{2\pi} \left\{ \bar{X}'\bar{P} + \frac{i}{2}\bar{\theta}_a\bar{\theta}'_a \right\}, \quad (5.14)$$

where

$$H = -p_+ - \frac{iT}{2} \oint \frac{d\sigma}{2\pi} (\bar{\theta}_1\bar{\theta}'_1 - \bar{\theta}_2\bar{\theta}'_2) = \frac{1}{2p_-} \left[ p^2 + \oint \frac{d\sigma}{2\pi} \left\{ \bar{P}^2 + (T\bar{X}')^2 - iT(\bar{\theta}_1\bar{\theta}'_1 - \bar{\theta}_2\bar{\theta}'_2) \right\} \right]. \quad (5.15)$$

As for  $\mathcal{N} = 1$ , the Hamiltonian is not equal to  $-p_+$  because it gets a fermionic contribution from the WZ term.

The Poincaré generators in the light-cone gauge are

$$\begin{aligned} \mathcal{P} &= p, & \mathcal{P}_- &= p_-, & \mathcal{P}_+ &= -H, \\ \mathcal{J} &= x^-p_- + \tau H, & \mathcal{J}^+ &= \tau p - xp_-, \\ \mathcal{J}^- &= -x^-p - xH + \Lambda/p_-, \end{aligned} \quad (5.16)$$

exactly as for the  $\mathcal{N} = 1$  superstring except that the Hamiltonian differs and now now

$$\begin{aligned} \Lambda &= p_- \oint \frac{d\sigma}{2\pi} (\bar{X}\bar{P}_+ - \bar{X}^-\bar{P}) + \frac{iT}{2} \oint \frac{d\sigma}{2\pi} \bar{X} (\bar{\theta}_1\bar{\theta}'_1 - \bar{\theta}_2\bar{\theta}'_2) \\ &\quad + \frac{iT}{2} \left( \vartheta_1 \oint \frac{d\sigma}{2\pi} \bar{X}\bar{\theta}'_1 - \vartheta_2 \oint \frac{d\sigma}{2\pi} \bar{X}\bar{\theta}'_2 \right). \end{aligned} \quad (5.17)$$

Note the  $\vartheta_a$ -dependence of this expression. The Fourier coefficients of  $\bar{X}^-$  may again be expressed in terms of the Fourier coefficients of  $(\bar{X}, \bar{P})$ , but in repeating this step we should now use the  $\mathcal{N} = 2$  relation

$$p_- (\bar{X}^-)' + p\bar{X}' + \frac{i}{2}\vartheta_a\bar{\theta}'_a = - \left( \bar{X}'\bar{P} + \frac{i}{2}\bar{\theta}_a\bar{\theta}'_a \right) + \oint \frac{d\sigma}{2\pi} \left( \bar{X}'\bar{P} + \frac{i}{2}\bar{\theta}_a\bar{\theta}'_a \right), \quad (5.18)$$

which replaces (4.18). The relation that replaces (4.19) is

$$p_- D_\tau \bar{X}^- = \bar{P}_+ + i\frac{T}{2p_-} (\bar{\theta}_1\bar{\theta}'_1 - \bar{\theta}_2\bar{\theta}'_2) - i\frac{T}{2p_-} \oint \frac{d\sigma}{2\pi} (\bar{\theta}_1\bar{\theta}'_1 - \bar{\theta}_2\bar{\theta}'_2). \quad (5.19)$$

The supersymmetry charges in the light cone gauge are

$$\begin{aligned} \mathcal{Q}_1^1 &= \sqrt{\frac{1}{\sqrt{2}p_-}} \left[ p\vartheta_1 + \oint \frac{d\sigma}{2\pi} (\bar{P} - T\bar{X}') \bar{\theta}_1 \right], \\ \mathcal{Q}_1^2 &= \sqrt{\sqrt{2}p_-} \vartheta_1 \end{aligned} \quad (5.20)$$

and

$$\begin{aligned}\mathcal{Q}_2^1 &= \sqrt{\frac{1}{\sqrt{2}p_-}} \left[ p\vartheta_2 + \oint \frac{d\sigma}{2\pi} (\bar{P} + T\bar{X}') \bar{\theta}_2 \right], \\ \mathcal{Q}_2^2 &= \sqrt{\sqrt{2}p_-} \vartheta_2.\end{aligned}\tag{5.21}$$

Finally, parity acts in the light-cone gauge via the transformation

$$X \rightarrow -X, \quad P \rightarrow -P, \quad \theta_2 \rightarrow -\theta_2\tag{5.22}$$

with all other canonical variables being parity inert. The light-cone gauge Hamiltonian (5.15) is parity invariant, as expected.

## 5.2 Fourier expansion

We Fourier expand the  $\bar{\theta}_a$  as

$$\bar{\theta}_1 = \sum_{n=1}^{\infty} [e^{in\sigma} \xi_n + e^{-in\sigma} \xi_n^*], \quad \bar{\theta}_2 = \sum_{n=1}^{\infty} [e^{in\sigma} \tilde{\xi}_n^* + e^{-in\sigma} \tilde{\xi}_n].\tag{5.23}$$

With the bosonic Fourier expansions as before, the Lagrangian (5.14) becomes

$$\begin{aligned}L &= \dot{x}p - \dot{x}^- p_- + \frac{i}{2} \vartheta_a \dot{\vartheta}_a + i \sum_{n=1}^{\infty} \left[ \frac{1}{n} (\alpha_n^* \dot{\alpha}_n + \tilde{\alpha}_n^* \dot{\tilde{\alpha}}_n) + \xi_n^* \dot{\xi}_n + \tilde{\xi}_n^* \dot{\tilde{\xi}}_n \right] - H \\ &+ u_0 \sum_{n=1}^{\infty} \left[ \alpha_n^* \alpha_n - \tilde{\alpha}_n^* \tilde{\alpha}_n + n (\xi_n^* \xi_n - \tilde{\xi}_n^* \tilde{\xi}_n) \right].\end{aligned}\tag{5.24}$$

The Hamiltonian again takes the form

$$H = \frac{1}{2p_-} (p^2 + \mathcal{M}^2)\tag{5.25}$$

but now with

$$\mathcal{M}^2 = 2T \sum_{n=1}^{\infty} \left[ \alpha_n^* \alpha_n + \tilde{\alpha}_n^* \tilde{\alpha}_n + n (\xi_n^* \xi_n + \tilde{\xi}_n^* \tilde{\xi}_n) \right].\tag{5.26}$$

Similarly, the Poincaré charges are as in (5.16) with  $\Lambda = \Lambda_+ + \Lambda_-$ , with

$$\begin{aligned}\Lambda_+ &= \Omega_+ + \sqrt{\frac{T}{2}} i\vartheta_1 \sum_{n=1}^{\infty} (\alpha_n^* \xi_n + \alpha_n \xi_n^*), \\ \Lambda_- &= \Omega_- + \sqrt{\frac{T}{2}} i\vartheta_2 \sum_{n=1}^{\infty} (\tilde{\alpha}_n^* \tilde{\xi}_n + \tilde{\alpha}_n \tilde{\xi}_n^*),\end{aligned}\tag{5.27}$$

where  $\Omega_{\pm}$ , which sum to the  $\mathcal{N} = 2$  super-Poincaé invariant  $\Omega$ , are given by

$$\begin{aligned}\Omega_+ &= \sqrt{2T} \left[ \lambda + \sum_{n=1}^{\infty} \frac{i}{n} (\alpha_n^* \gamma_n - \alpha_n \gamma_n^*) \right], \\ \Omega_- &= \sqrt{2T} \left[ \tilde{\lambda} + \sum_{n=1}^{\infty} \frac{i}{n} (\tilde{\alpha}_n^* \tilde{\gamma}_n - \tilde{\alpha}_n \tilde{\gamma}_n^*) \right].\end{aligned}\quad (5.28)$$

In these expressions, the quantities  $\lambda$  and  $\tilde{\lambda}$  are as given in (3.37) for the bosonic string and  $\gamma_n$  is as given in (4.27) for the  $\mathcal{N} = 1$  string, with a formally identical expression for  $\tilde{\gamma}_n$  in terms of the ‘left-moving’ canonical variables. Note that the fermionic zero modes  $\vartheta_a$  cancel from  $\Omega_{\pm}$ .

The supersymmetry charges are

$$\begin{aligned}\mathcal{Q}_1^1 &= \sqrt{\frac{1}{\sqrt{2} p_-}} \left[ p \vartheta_1 + \sqrt{2T} \sum_{n=1}^{\infty} \{ \alpha_n \xi_n^* + \alpha_n^* \xi_n \} \right], \\ \mathcal{Q}_1^2 &= \sqrt{\sqrt{2} p_-} \vartheta_1,\end{aligned}\quad (5.29)$$

and

$$\begin{aligned}\mathcal{Q}_2^1 &= \sqrt{\frac{1}{\sqrt{2} p_-}} \left[ p \vartheta_2 + \sqrt{2T} \sum_{n=1}^{\infty} \{ \tilde{\alpha}_n \tilde{\xi}_n^* + \tilde{\alpha}_n^* \tilde{\xi}_n \} \right], \\ \mathcal{Q}_2^2 &= \sqrt{\sqrt{2} p_-} \vartheta_2.\end{aligned}\quad (5.30)$$

The upper number is the value of the spinor index  $\alpha$ , and the lower number is the value of the supersymmetry-number index  $a$ .

Parity now acts via the transformations

$$\begin{aligned}x &\rightarrow -x, & p &\rightarrow -p, & \vartheta_2 &\rightarrow -\vartheta_2, \\ \alpha_n &\rightarrow -\alpha_n, & \tilde{\alpha}_n &\rightarrow -\tilde{\alpha}_n, & \tilde{\xi}_n &\rightarrow -\tilde{\xi}_n.\end{aligned}\quad (5.31)$$

The asymmetry in the action on the fermi modes originates in the relative minus sign in the  $\Theta_a$  transformation of (2.34).

### 5.3 Quantum $\mathcal{N} = 2$ 3D superstring

To quantize, we replace the bosonic variables by operators as before, and we promote the fermionic variables to operators satisfying the anti-commutation relations

$$\{ \vartheta_a, \vartheta_b \} = \delta_{ab}, \quad \{ \xi_n, \xi_n^\dagger \} = 1, \quad \{ \tilde{\xi}_n, \tilde{\xi}_n^\dagger \} = 1, \quad (5.32)$$

with all other anticommutators of these variables equal to zero. The quantum Hamiltonian has the form (5.25) with

$$\mathcal{M}^2 = 2T \left[ N + \tilde{N} + \nu + \tilde{\nu} \right], \quad \nu = \sum_{n=1}^{\infty} n \xi_n^\dagger \xi_n, \quad \tilde{\nu} = \sum_{n=1}^{\infty} n \tilde{\xi}_n^\dagger \tilde{\xi}_n, \quad (5.33)$$

where the bosonic level number operators  $(N, \tilde{N})$  are as before. The level-matching constraint, on the eigenvalues of these operators, is now

$$\tilde{N} + \tilde{\nu} = N + \nu, \quad (5.34)$$

and we may use this to rewrite the mass-squared at level  $L = N + \nu$  as

$$\mathcal{M}^2|_L = 4TL, \quad L = N + \nu. \quad (5.35)$$

The quantum supersymmetry charges are obtained from the classical charges (5.36) and (5.36) in the usual way. The result is

$$\begin{aligned} \mathcal{Q}_1^1 &= \sqrt{\frac{1}{\sqrt{2}p_-}} \left[ p\vartheta_1 + \frac{1}{\sqrt{2}}\Xi \right], & \mathcal{Q}_1^2 &= \sqrt{\sqrt{2}p_-} \vartheta_1, \\ \mathcal{Q}_2^1 &= \sqrt{\frac{1}{\sqrt{2}p_-}} \left[ p\vartheta_2 + \frac{1}{\sqrt{2}}\tilde{\Xi} \right], & \mathcal{Q}_2^2 &= \sqrt{\sqrt{2}p_-} \vartheta_2, \end{aligned} \quad (5.36)$$

where

$$\Xi = \sqrt{4T} \sum_{n=1}^{\infty} (\alpha_n \xi_n^\dagger + \alpha_n^\dagger \xi_n), \quad \tilde{\Xi} = \sqrt{4T} \sum_{n=1}^{\infty} (\tilde{\alpha}_n \tilde{\xi}_n^\dagger + \tilde{\alpha}_n^\dagger \tilde{\xi}_n), \quad (5.37)$$

The operators  $\Xi$  and  $\tilde{\Xi}$  also appear in the relation between the quantum operators  $\Omega_\pm$  and  $\Lambda_\pm$ :

$$\Lambda_+ = \Omega_+ + \frac{i}{2\sqrt{2}}\vartheta_1\Xi, \quad \Lambda_- = \Omega_- + \frac{i}{2\sqrt{2}}\vartheta_2\tilde{\Xi}. \quad (5.38)$$

When these operators act on physical states satisfying the level-matching condition (5.34), they satisfy

$$\Xi^2 = \mathcal{M}^2 = \tilde{\Xi}^2, \quad \{\Xi, \tilde{\Xi}\} = 0. \quad (5.39)$$

For the reasons explained earlier for the  $\mathcal{N} = 1$  superstring, the absence of super-Poincaré anomalies is a consequence of the fact

$$[\Xi, \Omega_+] = 0, \quad [\tilde{\Xi}, \Omega_-] = 0. \quad (5.40)$$

The calculations needed to verify these commutation relations are also the same as those sketched earlier for the  $\mathcal{N} = 1$  superstring.

As for the  $\mathcal{N} = 1$  superstring, it is convenient to consider the supercharges

$$\mathcal{S}_a = \sqrt{\sqrt{2}p_-} \mathcal{Q}_a^1 - \frac{1}{\sqrt{\sqrt{2}p_-}} (p - i\mathcal{M}) \mathcal{Q}_a^2, \quad (5.41)$$

where  $\mathcal{M}$  is the positive square root of  $\mathcal{M}$ . This gives

$$\mathcal{S}_1 = i\mathcal{M}\vartheta_1 + \frac{1}{\sqrt{2}}\Xi, \quad \mathcal{S}_2 = i\mathcal{M}\vartheta_2 + \frac{1}{\sqrt{2}}\tilde{\Xi}. \quad (5.42)$$

Using (5.39), it is straightforward to verify that

$$\{\mathcal{S}_a, \mathcal{S}_b\} = 0, \quad \left\{ \mathcal{S}_a, \mathcal{S}_b^\dagger \right\} = 2\delta_{ab} \mathcal{M}^2. \quad (5.43)$$

The operators  $\mathcal{S}_a$  again commute with the operator version of  $\Omega$ , but

$$[\Lambda, \mathcal{S}_a] = -\frac{1}{2} \mathcal{M} \mathcal{S}_a, \quad [\Lambda, \mathcal{S}_a^\dagger] = \frac{1}{2} \mathcal{M} \mathcal{S}_a^\dagger. \quad (5.44)$$

The parity operator in the light-cone gauge takes the form

$$\Pi_{\mathcal{N}=2} = \Pi(-1)^{F_L} \quad (5.45)$$

where  $\Pi$  is the parity operator (3.65) of the bosonic string and  $\mathcal{N} = 1$  superstring, and the operator  $(-1)^{F_L}$  anticommutes with  $\vartheta_2$  and all  $\xi_n$  but commutes with all other canonical variables. As this operator anticommutes with both  $\Lambda_\pm$  and  $\Omega_\pm$ , both helicity and superhelicity eigenstates must appear in parity doublets of opposite sign eigenvalues.

We may similarly define an operator  $(-1)^{F_R}$  that anticommutes with  $\vartheta_1$  and all  $\xi_n$  but commutes with all other canonical variables. The operator

$$(-1)^F = (-1)^{F_L} (-1)^{F_R} \quad (5.46)$$

anticommutes with all fermionic canonical variables but commutes with all the bosonic canonical variables. As a consequence it anticommutes with all components of the supercharges  $\mathcal{Q}_a$ , so the action of one of these charges on an eigenstate of  $(-1)^F$  yields another eigenstate of  $(-1)^F$  but with opposite sign eigenvalue.

### 5.3.1 Realization

The canonical anticommutation relations (5.32) can be partially realized by setting

$$\begin{aligned} \sqrt{2}\vartheta_1 &= (\sigma_1 \otimes \mathbb{I}_+) \otimes (\sigma_3 \otimes \mathbb{I}_-), & \sqrt{2}\vartheta_2 &= (\mathbb{I}_2 \otimes \mathbb{I}_+) \otimes (\sigma_1 \otimes \mathbb{I}_-), \\ \xi_n &= (\sigma_2 \otimes \chi_n) \otimes (\sigma_3 \otimes \mathbb{I}_-), & \tilde{\xi}_n &= (\mathbb{I}_2 \otimes \mathbb{I}_+) \otimes (\sigma_2 \otimes \tilde{\chi}_n), \end{aligned} \quad (5.47)$$

where  $(\chi_n, \chi_n^\dagger)$  and  $(\tilde{\chi}_n, \tilde{\chi}_n^\dagger)$  are two *mutually-commuting* sets of operators obeying the anticommutation relations

$$\{\chi_n, \chi_m^\dagger\} = \delta_{nm} \mathbb{I}_+, \quad \{\tilde{\chi}_n, \tilde{\chi}_m\} = \delta_{nm} \mathbb{I}_-. \quad (5.48)$$

In this realization,

$$(-1)^{F_R} = (\sigma_3 \otimes \mathbb{I}_+) \otimes (\mathbb{I}_2 \otimes \mathbb{I}_-), \quad (-1)^{F_L} = (\mathbb{I}_2 \otimes \mathbb{I}_+) \otimes (\sigma_3 \otimes \mathbb{I}_-), \quad (5.49)$$

and hence

$$(-1)^F = (\sigma_3 \otimes \mathbb{I}_+) \otimes (\sigma_3 \otimes \mathbb{I}_-). \quad (5.50)$$

The fermi oscillator ground state is quadruply degenerate; a basis is provided by the four states

$$|\varsigma\rangle_+ \otimes |\tilde{\varsigma}\rangle_- \quad (\varsigma = \pm, \quad \tilde{\varsigma} = \pm), \quad (5.51)$$



where

$$(-1)^{F_R} |\varsigma\rangle_+ = \varsigma |\varsigma\rangle_+, \quad (-1)^{F_L} |\tilde{\varsigma}\rangle_- = \tilde{\varsigma} |\tilde{\varsigma}\rangle_- . \quad (5.52)$$

and

$$(\mathbb{I}_2 \otimes \chi_n) |\varsigma\rangle_+ = 0, \quad (\mathbb{I}_2 \otimes \tilde{\chi}_n) |\tilde{\varsigma}\rangle_- = 0 . \quad (5.53)$$

This means that the Fock vacua for the right and left oscillators (bosonic and fermionic) can be chosen to be, respectively,

$$|0, \varsigma\rangle_+ = |0\rangle_+ \otimes |\varsigma\rangle_+, \quad |0, \tilde{\varsigma}\rangle_- = |0\rangle_- \otimes |\tilde{\varsigma}\rangle_-, \quad (5.54)$$

where  $|0\rangle_{\pm}$  are the Fock vacuum states for the bosonic oscillators, as in (3.51). The quadruply-degenerate oscillator ground state of the string then takes the tensor product form

$$|\varsigma, \tilde{\varsigma}\rangle = |0, \varsigma\rangle_+ \otimes |0, \tilde{\varsigma}\rangle_- . \quad (5.55)$$

At a given level  $L > 0$ , the non-hermitian supercharges  $\mathcal{S}_a$  become

$$\begin{aligned} \mathcal{S}_1 &= i\sqrt{2TL} [\sigma_1 \otimes \mathbb{I}_+ - i\sigma_2 \otimes \eta_L] \otimes (\sigma_3 \otimes \mathbb{I}_-) \\ \mathcal{S}_2 &= i\sqrt{2TL} (\mathbb{I}_2 \otimes \mathbb{I}_+) \otimes [\sigma_1 \otimes \mathbb{I}_- - i\sigma_2 \otimes \tilde{\eta}_L] , \end{aligned} \quad (5.56)$$

where both the operators  $\eta_L$  and  $\tilde{\eta}_L$ , acting in the space of physical states at level  $L$ , are traceless and square to the identity, and so have (simultaneous) eigenvalues  $\pm 1$ . There are four possible choices of the signs  $(\eta_L, \tilde{\eta}_L)$ , and for each choice we get a supermultiplet by the action of  $\mathcal{S}_a^\dagger$  on states annihilated by  $\mathcal{S}_a$ . Each such supermultiplet has four states, so there is a minimal 16-fold degeneracy at each non-zero level.

### 5.3.2 Low-level excited states

Excited string states are found, as eigenstates of the level operators  $N + \nu$  and  $\tilde{N} + \tilde{\nu}$  with eigenvalues that we also call  $N + \nu$  and  $\tilde{N} + \tilde{\nu}$ , by the action of the creation operators on the oscillator vacuum state such that the level-matching condition (5.34) is satisfied. We may therefore organize all physical states according to their level  $L = N + \nu$ , with the corresponding mass being given by (5.33). Because of the quadruple degeneracy of the ground state there is a minimal quadruple degeneracy at each level, as required by  $\mathcal{N} = 2$  supersymmetry. There are a total of 16 first excited states, i.e. level-1 states:

$$\begin{aligned} |1_B, \varsigma\rangle_+ \otimes |1_B, \tilde{\varsigma}\rangle_- &= \alpha_1^\dagger |0, \varsigma\rangle_+ \otimes \tilde{\alpha}_1^\dagger |0, \tilde{\varsigma}\rangle_-, \\ |1_F, \varsigma\rangle_+ \otimes |1_F, \tilde{\varsigma}\rangle_- &= \chi_1^\dagger |0, \varsigma\rangle_+ \otimes \tilde{\chi}_1^\dagger |0, \tilde{\varsigma}\rangle_-, \end{aligned} \quad (5.57)$$

$$\begin{aligned} |1_F, \varsigma\rangle_+ \otimes |1_B, \tilde{\varsigma}\rangle_- &= \chi_1^\dagger |0, \varsigma\rangle_+ \otimes \tilde{\alpha}_1^\dagger |0, \tilde{\varsigma}\rangle_-, \\ |1_B, \varsigma\rangle_+ \otimes |1_F, \tilde{\varsigma}\rangle_- &= \alpha_1^\dagger |0\rangle_+ \otimes \tilde{\chi}_1^\dagger |0\rangle_- . \end{aligned} \quad (5.58)$$

These form four  $\mathcal{N} = 2$  supermultiplets.

The level-2 excited states are tensor products of the orthonormal states

$$\begin{aligned}
|1_B, 1_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} \left(\alpha_1^\dagger\right)^2 |0, \varsigma\rangle_+, & |2_B\rangle_+ &= \frac{1}{\sqrt{2}} \alpha_2^\dagger |0, \varsigma\rangle_+, \\
|1_B, 1_F, \varsigma\rangle_+ &= \alpha_1^\dagger \chi_1^\dagger |0, \varsigma\rangle_+, & |2_F, \varsigma\rangle_+ &= \chi_2^\dagger |0, \varsigma\rangle_+,
\end{aligned} \tag{5.59}$$

with the analogous states built on  $|0, \tilde{\varsigma}\rangle_-$ . This gives us a total of 64 level-2 states and hence 16  $\mathcal{N} = 2$  supermultiplets.

At level 3 we need to consider the (orthonormal basis) states

$$\begin{aligned}
|1_B, 1_B, 1_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{6}} \left(\alpha_1^\dagger\right)^3 |0, \varsigma\rangle_+, & |1_B, 2_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} \alpha_1^\dagger \alpha_2^\dagger |0, \varsigma\rangle_+, \\
|3_B, \varsigma\rangle_+ &= \frac{1}{\sqrt{3}} \alpha_3^\dagger |0, \varsigma\rangle_+ & |1_F, 2_F, \varsigma\rangle_+ &= \chi_1^\dagger \chi_2^\dagger |0, \varsigma\rangle_+ \\
|1_B, 2_F, \varsigma\rangle_+ &= \alpha_1^\dagger \chi_2^\dagger |0, \varsigma\rangle_+, & |1_B, 1_B, 1_F, \varsigma\rangle_+ &= \frac{1}{\sqrt{2}} \left(\alpha_1^\dagger\right)^2 \chi_1^\dagger |0, \varsigma\rangle_+, \\
|2_B, 1_F, \varsigma\rangle_+ &= \frac{1}{\sqrt{3}} \alpha_2^\dagger \chi_1^\dagger |0, \varsigma\rangle_+, & |3_F, \varsigma\rangle_+ &= \chi_3^\dagger |0, \varsigma\rangle_+.
\end{aligned} \tag{5.60}$$

Taking tensor products with the corresponding states built on  $|0, \tilde{\varsigma}\rangle_-$  gives a total of 256 states, and hence 64  $\mathcal{N} = 2$  supermultiplets.

To compute the spectrum of superhelicities at these levels we need to compute the eigenvalues of the quantum operator  $\Omega$ . In fact, it is sufficient to compute the eigenvalues of the operator  $\Omega_+$  because these eigenvalues are identical to those of  $\Omega_-$ . As neither  $\Omega_+$  nor  $\Omega_-$  depends on the zero modes  $\vartheta_a$ , each eigenvalue of  $\Omega$  has at least a four-fold degeneracy, so the number of eigenvalues of  $\Omega$  at any given level (counting multiplicity) equals the number of supermultiplets at that level, as required by  $\mathcal{N} = 2$  supersymmetry.

The four ground states are annihilated by  $\Omega$ , as must be since (relativistic 3D) superhelicity is not defined for massless particles. These states correspond to massless particles that are potentially identifiable as a dilaton and axion, and their superpartners. As we already saw for  $\mathcal{N} = 0, 1$ , the operator  $\Omega$  also annihilates the level-1 states, so there are four degenerate copies of the  $\mathcal{N} = 2$  supermultiplet of zero superhelicity at this level. The helicity content of this supermultiplet is  $s = (-1/2, 0, 0, 1/2)$ , so we get four 3D  $\mathcal{N} = 2$  scalar supermultiplets at level-1.

Similar considerations apply to the higher levels: the  $\mathcal{N} = 2$  helicity content at each level can be deduced directly from the  $\mathcal{N} = 1$  results of the previous section. For example, we saw that  $\Omega_+$  has the four eigenvalues  $\sqrt{2T}(-\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$  at level 2, so  $\Omega/\sqrt{2T}$  for the  $\mathcal{N} = 2$  superstring has eigenvalues  $(-3, 0, 3)$  with multiplicities  $(4, 8, 4)$ , leading to 4  $\mathcal{N} = 2$  supermultiplets of superhelicity  $s_2 = 3/2$ , another 4 with superhelicity  $s_2 = -3/2$  and 8 with zero superhelicity (so 16 in total, as required for the 16 supermultiplets at this level). The  $s_2 = 3/2$  supermultiplet has helicities  $s = (2, 3/2, 3/2, 1)$ ; it is a massive spin-2  $\mathcal{N} = 2$  supermultiplet.

As these results show, the states of the  $\mathcal{N} = 2$  3D superstring through level 2 are just standard bosons and fermions, but this simple feature does not extend to level 3.

Again, the level-3 content can be deduced from the previous results for  $\mathcal{N} = 1$ . From the eight level-3 eigenvalues of  $\Omega_+$  given in (4.68) we get a total of 64 eigenvalues of  $\Omega$ , as required for the 64 supermultiplets at this level. Eight of them have zero superhelicity but the rest have irrational superhelicities. As all helicities in such a supermultiplet are also irrational, we conclude not only that there are anyons in the spectrum of the  $\mathcal{N} = 2$  3D superstring, but also that these anyons are ‘generic’ ones of irrational spin.

## 6 Summary and Outlook

The quantum theory of strings below their critical dimension is problematic and generically involves the introduction of a new degree of freedom, the Liouville mode, that is not present in the classical string action. We say “generically” because there is an exception: the usual quantum Lorentz anomaly in the light-cone gauge, in which the action involves only physical worldvolume variables, is trivially absent for the Nambu-Goto string in a Minkowski spacetime of three dimensions (3D) [5, 6], so no Liouville mode is needed to guarantee unitarity and Lorentz invariance, at least for a free 3D Nambu-Goto string.

The implication is that the quantum spectrum contains states of definite mass and spin, and this was verified explicitly in [5], with a rather surprising result: the spins are not generically integer or half-integer. This is possible because the rotation subgroup of the universal cover  $\overline{SO}(1, 2)$  of the 3D Lorentz group is  $\overline{SO}(2) \cong \mathbb{R}$ . In the context of a relativistic theory, this implies that the states in the string spectrum generically describe “anyons”. There is an ambiguity in the string spectrum due to an operator ordering ambiguity: the mass-squared of the string ground state is arbitrary, although it must be non-negative to avoid tachyons. This ambiguity affects the spins as well as the masses. Consideration of both the level-2 and level-3 excited states led to the conclusion that some states are necessarily anyons and that they generically have irrational spin.

A similar conclusion was arrived at for the  $\mathcal{N} = 1$  3D Green-Schwarz (GS) superstring, but in that case the doubly-degenerate ground state is required by supersymmetry to be massless, so the quantum ambiguity of the bosonic 3D string is eliminated. The level-1 and level-2 excited states were shown in [5] to contain “semions” (a particular case of anyons for which the spin is  $1/4$  modulo a half-integer). In this paper we have given details of the computations behind these results, and we have extended them in a number of ways.

Firstly, we have extended the computation of the spectrum of the quantum 3D Nambu-Goto string to level-3. This allows us to strengthen our earlier conclusion concerning anyons in the spectrum: some of these anyons necessarily have *irrational* spin. This tells us that the Lorentz group really is  $\overline{SO}(1, 2)$  and not some finite cover of  $SO(1, 2)$ .

Secondly, we have established the same result for the  $\mathcal{N} = 1$  superstring by showing that irrational spin anyons are present in the spectrum at level-3. We have also established the absence of super-Poincaré anomalies. Classically, there are ac-

tually two  $\mathcal{N} = 1$  superstring theories, interchanged by worldsheet parity, because the string fermions propagate in one direction around the string. However, these two equivalent, but distinct, classical theories are identical as quantum theories because they describe exactly the same 3D spectrum.

Thirdly, we have extended the analysis to include the  $\mathcal{N} = 2$  GS superstring. In this case, the spectrum through level 2 consists only of bosons and fermions (i.e. particles of integer and half-odd-integer spins) so it was not previously clear to us whether the spectrum would contain anyons. In fact, the level-3 spectrum contains particles of irrational spin, this being a consequence of the presence of such states in the  $\mathcal{N} = 1$  superstring.

The fact that irrational spins appear in the spectrum of all 3D (super)strings implies that the Lorentz group is the infinite universal cover of  $SO(1, 2)$ , not the double cover that might have been expected, or any finite multiple cover. We believe that this may explain why existing covariant quantization methods do not appear to allow for the possibility of 3D strings: covariant quantization of even a free 3D *particle* is not straightforward if it has irrational spin.

We have made no attempt to explore whether the free 3D strings discussed here admit interactions. Again, this is already a difficult problem for particles of irrational spin. If interactions are possible then one would expect there to exist effective supersymmetric field theories describing the massless modes of the  $\mathcal{N} = 1$  and  $\mathcal{N} = 2$  3D superstring theories. Our results are consistent with this possibility even if the effective field theories are supposed to be supergravity theories because neither the metric nor the antisymmetric tensor fields that couple naturally to a string propagate massless modes in 3D.

For the  $\mathcal{N} = 2$  superstring, there are four massless states: a scalar and a pseudo-scalar, and their superpartners. The scalar might be interpretable as a dilaton. As a massless pseudo-scalar is dual to a massless vector in 3D, it would be natural to suppose that any effective field theory is some generally covariant theory involving an  $\mathcal{N} = 2$  vector multiplet. The vector potential of this supermultiplet could couple to particles carrying the central charge permitted by the  $\mathcal{N} = 2$  superalgebra. Although there are no such particles in the spectrum of a free  $\mathcal{N} = 2$  superstring, they might be non-perturbative excitations of an interacting  $\mathcal{N} = 2$  3D superstring, analogous to the D0-branes of critical superstring theory. If so, they might show up in an analysis of  $\mathcal{N} = 2$  open strings with Dirichlet boundary conditions, although it is difficult to see how this could be consistent with the fact that such particles are semions of spin  $1/4$  [15].

Finally, we recall that the  $\mathcal{N} = 2$  3D GS superstring is, classically, the double-dimensional reduction of the 4D supermembrane. In the context of a 4D spacetime that is a product of 3D Minkowski spacetime with a circle, the supermembrane can be wrapped on the circle to give a string. The  $\mathcal{N} = 2$  3D superstring is then found by ignoring the momentum modes in the extra dimension, but it would be interesting to see what effect these modes have on the string spectrum, and whether 3D “D0-branes” also emerge from this 4D perspective.

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