Homogenization of Steklov Spectral Problems with Indefinite Density Function in Perforated Domains

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Abstract

The asymptotic behavior of second order self-adjoint elliptic Steklov eigenvalue problems with periodic rapidly oscillating coefficients and with indefinite (sign-changing) density function is investigated in periodically perforated domains. We prove that the spectrum of this problem is discrete and consists of two sequences, one tending to $-\infty$ and another to $+\infty$. The limiting behavior of positive and negative eigencouples depends crucially on whether the average of the weight over the surface of the reference hole is positive, negative or equal to zero. By means of the two-scale convergence method, we investigate all three cases.

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1 Introduction

In 1902, with a motivation coming from Physics, Steklov[27] introduced the following problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = \rho\lambda u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a scalar and ρ is a density function. The function u represents the steady state temperature on Ω such that the flux on the boundary $\partial\Omega$ is proportional to the temperature. In two dimensions, assuming $\rho = 1$, problem (1.1) can also be interpreted as a membrane with whole mass concentrated on the boundary. This problem has been later referred to as Steklov eigenvalue problem (Steklov is often transliterated as ”Stekloff”). Moreover, eigenvalue problems also arise from many nonlinear problems after linearization (see e.g., the work of Hess and Kato[11, 12] and that of de Figueiredo[9]). This paper deals with the

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limiting behavior of a sequence of second order elliptic Steklov eigenvalue problems with indefinite(sign-changing) density function in perforated domains.

Let Ω be a bounded domain in $\mathbb{R}^N_x$(the numerical space of variables $x=(x_1,...,x_N)$), with $\mathcal{C}^1$ boundary $\partial\Omega$ and with integer $N\geq 2$. We define the perforated domain Ω^ε as follows. Let $T\subset Y=(0,1)^N$ be a compact subset in $\mathbb{R}^N_y$ with $\mathcal{C}^1$ boundary ∂T $(\equiv S)$ and nonempty interior. For $\varepsilon>0$, we define

$$t^\varepsilon=\{k\in\mathbb{Z}^N:\varepsilon(k+T)\subset\Omega\}$$

$$T^\varepsilon=\bigcup_{k\in t^\varepsilon}\varepsilon(k+T)$$

and

$$\Omega^\varepsilon=\Omega\setminus T^\varepsilon.$$

In this setup, T is the reference hole whereas $\varepsilon(k+T)$ is a hole of size ε and T^ε is the collection of the holes of the perforated domain Ω^ε. The family T^ε is made up with a finite number of holes since Ω is bounded. In the sequel, Y^* stands for $Y\setminus T$ and $n=(n_i)_{i=1}^N$ denotes the outer unit normal vector to S with respect to Y^*.

We are interested in the spectral asymptotics (as $\varepsilon\to 0$) of the following Steklov eigenvalue problem

$$\begin{cases} -\sum_{i,j=1}^N\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}\right)=0 \text{ in } \Omega^\varepsilon\\ \sum_{i,j=1}^N a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}n_i(\frac{x}{\varepsilon})=\rho(\frac{x}{\varepsilon})\lambda_\varepsilon u_\varepsilon \text{ on } \partial T^\varepsilon\\ u_\varepsilon=0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where $a_{ij}\in L^\infty(\mathbb{R}^N_y)$ $(1\leq i,j\leq N)$, with the symmetry condition $a_{ji}=a_{ij}$, the Y-periodicity hypothesis: for every $k\in\mathbb{Z}^N$ one has $a_{ij}(y+k)=a_{ij}(y)$ almost everywhere in $y\in\mathbb{R}^N_y$, and finally the (uniform) ellipticity condition: there exists $\alpha>0$ such that

$$\sum_{i,j=1}^N a_{ij}(y)\xi_j\xi_i\geq\alpha|\xi|^2 \tag{1.3}$$

for all $\xi\in\mathbb{R}^N$ and for almost all $y\in\mathbb{R}^N_y$, where $|\xi|^2=|\xi_1|^2+\cdots+|\xi_N|^2$. The density function $\rho\in\mathcal{C}_{per}(Y)$ changes sign on S, that is, both the set $\{y\in S,\rho(y)<0\}$ and $\{y\in S,\rho(y)>0\}$ are of positive $N-1$ dimensional Hausdorf measure (the so-called surface measure). This hypothesis makes the problem under consideration nonstandard. We will see (Corollary 2.15) that under the preceding hypotheses, for each $\varepsilon>0$ the spectrum of (1.2) is discrete and consists of two infinite sequences

$$0<\lambda_\varepsilon^{1,+}\leq\lambda_\varepsilon^{2,+}\leq\cdots\leq\lambda_\varepsilon^{n,+}\leq\dots,\quad \lim_{n\to+\infty}\lambda_\varepsilon^{n,+}=+\infty$$

and

$$0>\lambda_\varepsilon^{1,-}\geq\lambda_\varepsilon^{2,-}\geq\cdots\geq\lambda_\varepsilon^{n,-}\geq\dots,\quad \lim_{n\to+\infty}\lambda_\varepsilon^{n,-}=-\infty.$$

The asymptotic behavior of the eigencouples depends crucially on whether the average of the density ρ over S, $M_S(\rho) = \int_S \rho(y)\, d\sigma(y)$, is positive, negative or equal to zero. All three cases are carefully investigated in this paper.

The homogenization of spectral problems has been widely explored. In a fixed domain, homogenization of spectral problems with point-wise positive density function goes back to Kesavan [14, 15]. Spectral asymptotics in perforated domains was studied by Vanninathan[29] an later in many other papers, including [7, 8, 13, 23, 24, 26] and the references therein. Homogenization of elliptic operators with sing-changing density function in a fixed domain with Dirichlet boundary conditions has been investigated by Nazarov et al. [17, 18, 19] via a combination of formal asymptotic expansion with Tartar's energy method. In porous media, spectral asymptotics of elliptic operator with sign changing density function is studied in [6] with the two scale convergence method.

The asymptotics of Steklov eigenvalue problems in periodically perforated domains was studied in [29] for the laplace operator and constant density ($\rho = 1$) using asymptotic expansion and Tartar's test function method. The same problem for a second order periodic elliptic operator has been studied in [24] (with $\mathcal{C}^\infty$ coefficients) and in [8] (with L^∞ coefficient) but still with constant density ($\rho = 1$). All the just-cited works deal only with one sequence of positive eigenvalues.

In this paper we take it to the general tricky step. We investigate in periodically perforated domains the asymptotic behavior of Steklov eigenvalue problems for periodic elliptic linear differential operators of order two in divergence form with L^∞ coefficients and a sing-changing density function. We obtain accurate and concise homogenization results in all three cases: $M_S(\rho) > 0$ (Theorem 3.1 and Theorem 3.3), $M_S(\rho) = 0$ (Theorem 3.5), $M_S(\rho) < 0$ (Theorem 3.1 and Theorem 3.3), by using the two-scale convergence method[1, 16, 20, 30] introduced by Nguetseng[20] and further developed by Allaire[1]. In short;

i) If $M_S(\rho) > 0$, then the positive eigencouples behave like in the case of point-wise positive density function, i.e., for $k \geq 1$, $\lambda_\varepsilon^{k,+}$ is of order ε and $\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+}$ converges as $\varepsilon \to 0$ to the k^{th} eigenvalue of the limit Dirichlet spectral problem, corresponding extended eigenfunctions converge along subsequences.

As regards the "negative" eigencouples, $\lambda_\varepsilon^{k,-}$ converges to $-\infty$ at the rate $\frac{1}{\varepsilon}$ and the corresponding eigenfunctions oscillate rapidly. We use a factorization technique ([19, 29]) to prove that

$$\lambda_\varepsilon^{k,-} = \frac{1}{\varepsilon}\lambda_1^- + \xi_\varepsilon^{k,-} + o(1), \qquad k = 1, 2\cdots$$

where $(\lambda_1^-, \theta_1^-)$ is the first negative eigencouple to the following local Steklov spectral problem

$$\begin{cases} -div(a(y)D_y\theta) = 0 \quad \text{in } Y^* \\ a(y)D_y\theta \cdot n = \lambda\rho(y)\theta \quad \text{on } \ S \\ \theta \quad Y - periodic, \end{cases} \tag{1.4}$$

and $\{\xi_\varepsilon^{k,\pm}\}_{k=1}^\infty$ are eigenvalues of a Steklov eigenvalue problem similar to (1.2). We then prove that $\{\frac{\lambda_\varepsilon^{k,-}}{\varepsilon} - \frac{\lambda_1^-}{\varepsilon^2}\}$ converges to the k^{th} eigenvalue of a limit Dirichlet spectral problem which is different from that obtained for positive eigenvalues. As regards eigenfunctions, extensions of $\{\frac{u_\varepsilon^{k,-}}{(\theta_1^-)^\varepsilon}\}_{\varepsilon\in E}$ - where $(\theta_1^-)^\varepsilon(x) = \theta_1^-(\frac{x}{\varepsilon})$ - converge along subsequences to the k^{th} eigenfunctions of the limit problem.

ii) If $M_S(\rho) = 0$, then the limit spectral problem generates a quadratic operator pencil and $\lambda_\varepsilon^{k,\pm}$ converges to the $(k,\pm)^{th}$ eigenvalue of the limit operator, extended eigenfunctions converge along subsequences as well. This case requires a new convergence result as regards the two-scale convergence theory, Lemma 2.7.

iii) The case when $M_S(\rho) < 0$ is equivalent to that when $M_S(\rho) > 0$, just replace ρ with $-\rho$.

Unless otherwise specified, vector spaces throughout are considered over $\mathbb{R}$, and scalar functions are assumed to take real values. We will make use of the following notations. Let $F(\mathbb{R}^N)$ be a given function space. We denote by $F_{per}(Y)$ the space of functions in $F_{loc}(\mathbb{R}^N)$ that are Y-periodic, and by $F_\#(Y)$ the space of those functions $u \in F_{per}(Y)$ with $\int_Y u(y)dy = 0$. Finally, the letter E denotes throughout a family of strictly positive real numbers ($0 < \varepsilon < 1$) admitting 0 as accumulation point. The numerical space $\mathbb{R}^N$ and its open sets are provided with the Lebesgue measure denoted by $dx = dx_1...dx_N$. The usual gradient operator will be denoted by D. For the sake of simple notations we hide trace operators. The rest of the paper is organized as follows. Section 2 deals with some preliminary results while homogenization processes are considered in Section 3.

2 Preliminaries

We first recall the definition and the main compactness theorems of the two-scale convergence method. Let Ω be an open bounded set in $\mathbb{R}_x^N$ (integer $N \geq 2$) and $Y = (0,1)^N$, the unit cube.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^2(\Omega)$ is said to two-scale converge in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega\times Y)$ if as $E \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\phi(x,\frac{x}{\varepsilon})dx \to \iint_{\Omega\times Y} u_0(x,y)\phi(x,y)dxdy \tag{2.1}$$

for all $\phi \in L^2(\Omega; \mathcal{C}_{per}(Y))$.

Notation. We express this by writing $u_\varepsilon \xrightarrow{2s} u_0$ in $L^2(\Omega)$.

The following compactness theorems [1, 20, 22] are cornerstones of the two-scale convergence method.

Theorem 2.2. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^2(\Omega)$. Then a subsequence E' can be extracted from E such that as $E' \ni \varepsilon \to 0$, the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ two-scale converges in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega\times Y)$.*

Theorem 2.3. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $H^1(\Omega)$. Then a subsequence E' can be extracted from E such that as $E' \ni \varepsilon \to 0$*

$$u_\varepsilon \to u_0 \quad \text{in } H^1(\Omega)\text{-weak} \tag{2.2}$$

$$u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega) \tag{2.3}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \xrightarrow{2s} \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \quad \text{in } L^2(\Omega) \ \ (1 \le j \le N) \tag{2.4}$$

where $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\#(Y))$. Moreover, as $E' \ni \varepsilon \to 0$ we have

$$\int_\Omega \frac{u_\varepsilon(x)}{\varepsilon}\psi(x, \frac{x}{\varepsilon})dx \to \iint_{\Omega\times Y} u_1(x,y)\psi(x,y)dxdy \tag{2.5}$$

for $\psi \in \mathcal{D}(\Omega)\otimes L^2_\#(Y)$.

In the sequel, S^ε stands for ∂T^ε and the surface measures on S and S^ε are denoted by $d\sigma(y)$ $(y \in Y)$, $d\sigma_\varepsilon(x)$ $(x \in \Omega, \varepsilon \in E)$, respectively. The space of squared integrable functions, with respect to the previous measures on S and S^ε are denoted by $L^2(S)$ and $L^2(S^\varepsilon)$ respectively. Since the volume of S^ε grows proportionally to $\frac{1}{\varepsilon}$ as $\varepsilon \to 0$, we endow $L^2(S^\varepsilon)$ with the scaled scalar product[25]

$$(u,v)_{L^2(S^\varepsilon)} = \varepsilon \int_{S^\varepsilon} u(x)v(x)d\sigma_\varepsilon(x) \quad \left(u, v \in L^2(S^\varepsilon)\right).$$

Definition 2.1 and theorem 2.2 then generalize as

Definition 2.4. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^2(S^\varepsilon)$ is said to two-scale converge to some $u_0 \in L^2(\Omega\times S)$ if as $E \ni \varepsilon \to 0$,

$$\varepsilon \int_{S^\varepsilon} u_\varepsilon(x)\phi(x, \frac{x}{\varepsilon})d\sigma_\varepsilon(x) \to \iint_{\Omega\times S} u_0(x,y)\phi(x,y)dxd\sigma(y)$$

for all $\phi \in \mathcal{C}(\overline{\Omega}; \mathcal{C}_{per}(Y))$.

Theorem 2.5. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^2(S^\varepsilon)$ such that*

$$\varepsilon \int_{S^\varepsilon} |u_\varepsilon(x)|^2 d\sigma_\varepsilon(x) \le C$$

where C is a positive constant independent of ε. There exists a subsequence E' of E such that $(u_\varepsilon)_{\varepsilon\in E'}$ two-scale converges to some $u_0 \in L^2(\Omega; L^2(S))$ in the sense of definition 2.4.

In the case when $(u_\varepsilon)_{\varepsilon\in E}$ is the sequence of traces on S^ε of functions in $H^1(\Omega)$, one can link its usual two-scale limit with its surface two-scale limits. The following proposition whose proof can be found in [2] clarifies this.

Proposition 2.6. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset H^1(\Omega)$ be such that*

$$\|u_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|Du_\varepsilon\|_{L^2(\Omega)^N} \le C,$$

where C is a positive constant independent of ε *and D denotes the usual gradient. The sequence of traces of* $(u_\varepsilon)_{\varepsilon\in E}$ *on* S^ε *satisfies*

$$\varepsilon\int_{S^\varepsilon}|u_\varepsilon(x)|^2d\sigma_\varepsilon(x)\leq C\quad(\varepsilon\in E)$$

and up to a subsequence E' *of* E, *it two-scale converges in the sense of Definition 2.4 to some* $u_0\in L^2(\Omega;L^2(S))$ *which is nothing but the trace on S of the usual two-scale limit, a function in* $L^2(\Omega;H^1_\#(Y))$. *More precisely, as* $E'\ni\varepsilon\to 0$

$$\begin{aligned}\varepsilon\int_{S^\varepsilon}u_\varepsilon(x)\phi(x,\frac{x}{\varepsilon})d\sigma_\varepsilon(x) &\to \iint_{\Omega\times S}u_0(x,y)\phi(x,y)dxd\sigma(y),\\ \int_{\Omega}u_\varepsilon(x)\phi(x,\frac{x}{\varepsilon})dxdy &\to \iint_{\Omega\times Y}u_0(x,y)\phi(x,y)dxdy,\end{aligned}$$

for all $\phi\in\mathcal{C}(\overline{\Omega};\mathcal{C}_{per}(Y))$.

In our homogenization process, we will need a generalization of (2.5) to periodic surfaces. Notice that (2.5) was proved for the first time in a deterministic setting by Nguetseng and Woukeng in [22] but to the best of our knowledge its generalization to periodic surface is not yet available in the literature. We prove it below and this is a non-negligible contribution to the theory of two-scale convergence.

Lemma 2.7. *Let* $(u_\varepsilon)_{\varepsilon\in E}\subset H^1(\Omega)$ *be such that as* $E\ni\varepsilon\to 0$

$$u_\varepsilon \xrightarrow{2s} u_0 \quad \text{in } L^2(\Omega) \tag{2.6}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \xrightarrow{2s} \frac{\partial u_0}{\partial x_j}+\frac{\partial u_1}{\partial y_j}\quad \text{in } L^2(\Omega)\ \ (1\leq j\leq N) \tag{2.7}$$

for some $u_0\in H^1(\Omega)$ *and* $u_1\in L^2(\Omega;H^1_\#(Y))$. *Then*

$$\lim_{\varepsilon\to 0}\int_{S^\varepsilon}u_\varepsilon(x)\varphi(x)\theta(\frac{x}{\varepsilon})d\sigma_\varepsilon(x)=\iint_{\Omega\times S}u_1(x,y)\varphi(x)\theta(y)dxd\sigma(y) \tag{2.8}$$

for all $\varphi\in\mathcal{D}(\Omega)$ *and* $\theta\in\mathcal{C}_\#(Y)$.

Proof. By the mean value zero condition over S for θ we conclude that there exists a solution $\vartheta\in H^1_\#(Y)$ to

$$\begin{cases}-\Delta_y\vartheta=0 \ \text{ in } Y^*\\ D_y\vartheta(y)\cdot n(y)=\theta(y)\ \text{ on } S,\end{cases}$$

where $n=(n_i)_{i=1}^N$ stands for the outward unit normal to S with respect to Y^*. Put $\phi=D_y\vartheta$. We get

$$\begin{aligned}\int_{\Omega^\varepsilon}D_xu_\varepsilon(x)\varphi(x)\cdot D_y\vartheta(y)dx &= \int_{S^\varepsilon}u_\varepsilon(x)\varphi(x)D_y\vartheta(\frac{x}{\varepsilon})\cdot n(\frac{x}{\varepsilon})d\sigma_\varepsilon(x)\\ &\quad -\int_{\Omega^\varepsilon}u_\varepsilon(x)D_x\varphi(x)\cdot D_y\vartheta(\frac{x}{\varepsilon})dx-\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}u_\varepsilon(x)\varphi(x)\Delta_y\vartheta(\frac{x}{\varepsilon})dx\\ &= \int_{S^\varepsilon}u_\varepsilon(x)\varphi(x)\theta(\frac{x}{\varepsilon})d\sigma_\varepsilon(x)-\int_{\Omega^\varepsilon}u_\varepsilon(x)D_x\varphi(x)\cdot\phi(\frac{x}{\varepsilon})dx.\end{aligned}$$

Sending ε to 0 yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{S^\varepsilon} u_\varepsilon(x)\varphi(x)\theta(\frac{x}{\varepsilon})\,d\sigma_\varepsilon(x) &= \iint_{\Omega\times Y^*}[D_x u_0(x)+D_y u_1(x,y)]\varphi(x)\cdot\phi(y)\,dxdy \\ &\quad + \iint_{\Omega\times Y^*} u_0(x) D_x\varphi(x)\cdot\phi(y)\,dxdy \\ &= \iint_{\Omega\times Y^*} D_y u_1(x,y)\varphi(x)\cdot\phi(y)\,dxdy.\end{aligned}$$

We finally have

$$\begin{aligned}\iint_{\Omega\times Y^*} D_y u_1(x,y)\varphi(x)\cdot\phi(y)\,dxdy &= -\iint_{\Omega\times Y^*} u_1(x,y)\varphi(x)\Delta_y\vartheta(y)\,dxdy \\ &\quad + \iint_{\Omega\times S} u_1(x,y)\varphi(x)\phi(y)\cdot n(y)\,dxd\sigma(y) \\ &= \iint_{\Omega\times S} u_1(x,y)\varphi(x)\theta(y)\,dxd\sigma(y).\end{aligned}$$

The proof is completed. □

We now gather some preliminary results. We introduce the characteristic function χ_G of

$$G = \mathbb{R}^N_y \setminus \Theta$$

with

$$\Theta = \bigcup_{k\in\mathbb{Z}^N}(k+T).$$

It is clear that G is an open subset of $\mathbb{R}^N_y$. Next, let $\varepsilon \in E$ be arbitrarily fixed and define

$$V_\varepsilon = \{u \in H^1(\Omega^\varepsilon) : u = 0 \text{ on } \partial\Omega\}.$$

We equip V_ε with the $H^1(\Omega^\varepsilon)$-norm which makes it a Hilbert space. We recall the following classical extension result [5].

Proposition 2.8. *For each $\varepsilon \in E$ there exists an operator P_ε of V_ε into $H^1_0(\Omega)$ with the following properties:*

- *P_ε sends continuously and linearly V_ε into $H^1_0(\Omega)$.*
- *$(P_\varepsilon v)|_{\Omega^\varepsilon} = v$ for all $v \in V_\varepsilon$.*
- *$\|D(P_\varepsilon v)\|_{L^2(\Omega)^N} \leq c\|Dv\|_{L^2(\Omega^\varepsilon)^N}$ for all $v \in V_\varepsilon$, where c is a constant independent of ε.*

Now, let $Q^\varepsilon = \Omega\setminus(\varepsilon\Theta)$. This defines an open set in $\mathbb{R}^N$ and $\Omega^\varepsilon\setminus Q^\varepsilon$ is the intersection of Ω with the collection of the holes crossing the boundary $\partial\Omega$. The following result implies that the holes crossing the boundary $\partial\Omega$ are of no effects as regards the homogenization process.

Lemma 2.9. *[21] Let $K \subset \Omega$ be a compact set independent of ε. There is some $\varepsilon_0 > 0$ such that $\Omega^\varepsilon \setminus Q^\varepsilon \subset \Omega\setminus K$ for any $0 < \varepsilon \leq \varepsilon_0$.*

We introduce the space

$$\mathbb{F}_0^1 = H_0^1(\Omega) \times L^2\left(\Omega; H_\#^1(Y)\right)$$

and endow it with the following norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^1} = \|D_x v_0 + D_y v_1\|_{L^2(\Omega\times Y)} \quad (\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1),$$

which makes it an Hilbert space admitting $F_0^\infty = \mathcal{D}(\Omega) \times [\mathcal{D}(\Omega) \otimes \mathcal{C}_\#^\infty(Y)]$ as a dense subspace. For $(\mathbf{u}, \mathbf{v}) \in \mathbb{F}_0^1 \times \mathbb{F}_0^1$, let

$$a_\Omega(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^N \iint_{\Omega\times Y^*} a_{ij}(y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right)\left(\frac{\partial v_0}{\partial x_i} + \frac{\partial v_1}{\partial y_i}\right) dxdy.$$

This define a symmetric, continuous bilinear form on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$. We will need the following results whose proof can be found in [8].

Lemma 2.10. *Fix $\Phi = (\psi_0, \psi_1) \in F_0^\infty$ and define $\Phi_\varepsilon : \Omega \to \mathbb{R}$ ($\varepsilon > 0$) by*

$$\Phi_\varepsilon(x) = \psi_0(x) + \varepsilon\psi_1(x, \frac{x}{\varepsilon}) \quad (x \in \Omega).$$

If $(u_\varepsilon)_{\varepsilon\in E} \subset H_0^1(\Omega)$ is such that

$$\frac{\partial u_\varepsilon}{\partial x_i} \xrightarrow{2s} \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \quad in \quad L^2(\Omega)\ (1 \leq i \leq N)$$

as $E \ni \varepsilon \to 0$ for some $\boldsymbol{u} = (u_0, u_1) \in \mathbb{F}_0^1$, then

$$a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) \to a_\Omega(\boldsymbol{u}, \Phi)$$

as $E \ni \varepsilon \to 0$, where

$$a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) = \sum_{i,j=1}^N \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial \Phi_\varepsilon}{\partial x_i} dx.$$

We now construct and point out the main properties of the so-called homogenized coefficients. Let $0 \leq j \leq N$ and put

$$a(u, v) = \sum_{i,j=1}^N \int_{Y^*} a_{ij}(y) \frac{\partial u}{\partial y_j}\frac{\partial v}{\partial y_i} dy,$$

$$l_j(v) = \sum_{k=1}^N \int_{Y^*} a_{kj}(y) \frac{\partial v}{\partial y_k} dy$$

and

$$l_0(v) = \int_S \rho(y) v(y) d\sigma(y)$$

for $u, v \in H_\#^1(Y)$. Equipped with the seminorm

$$N(u) = \|D_y u\|_{L^2(Y^*)^N} \quad (u \in H_\#^1(Y)), \tag{2.9}$$

$H_\#^1(Y)$ is a pre-Hilbert space that is nonseparate and noncomplete. Let $H_\#^1(Y^*)$ be its separated completion with respect to the seminorm $N(\cdot)$ and $\mathbf{i}$ the canonical mapping of $H_\#^1(Y)$ into $H_\#^1(Y^*)$. we recall that

(i) $H^1_\#(Y^*)$ is a Hilbert space,

(ii) $\mathbf{i}$ is linear,

(iii) $\mathbf{i}(H^1_\#(Y))$ is dense in $H^1_\#(Y^*)$,

(iv) $\|\mathbf{i}(u)\|_{H^1_\#(Y^*)} = N(u)$ for every u in $H^1_\#(Y)$,

(v) If F is a Banach space and l a continuous linear mapping of $H^1_\#(Y)$ into F, then there exists a unique continuous linear mapping $L : H^1_\#(Y^*) \to F$ such that $l = L \circ \mathbf{i}$.

Proposition 2.11. *Let $1 \le j \le N$. The noncoercive local variational problems*

$$u \in H^1_\#(Y) \text{ and } a(u,v) = l_j(v) \text{ for all } v \in H^1_\#(Y) \tag{2.10}$$

and

$$u \in H^1_\#(Y) \text{ and } a(u,v) = l_0(v) \text{ for all } v \in H^1_\#(Y) \tag{2.11}$$

admit each at least one solution. Moreover, if χ^j and θ^j (resp. χ and θ) are two solutions to (2.10) (resp. (2.11)), then

$$D_y\chi^j = D_y\theta^j \quad (resp.\ D_y\chi = D_y\theta) \text{ a.e., in } Y^*. \tag{2.12}$$

Proof. We prove the result for (2.10). Proceeding as in the proof of [21, Lemma 2.5] we prove that there exists a unique symmetric, coercive, continuous bilinear form $A(\cdot,\cdot)$ on $H^1_\#(Y^*) \times H^1_\#(Y^*)$ such that $A(\mathbf{i}(u), \mathbf{i}(v)) = a(u,v)$ for all $u, v \in H^1_\#(Y)$. Based on (v) above, we consider the linear form $\mathbf{l}_j(\cdot)$ on $H^1_\#(Y^*)$ such that $\mathbf{l}_j(\mathbf{i}(u)) = l_j(u)$ for any $u \in H^1_\#(Y)$. Then $\chi^j \in H^1_\#(Y)$ satisfies (2.10) if and only if $\mathbf{i}(\chi^j)$ satisfies

$$\mathbf{i}(\chi^j) \in H^1_\#(Y^*) \text{ and } A(\mathbf{i}(\chi^j), V) = \mathbf{l}_j(V) \text{ for all } V \in H^1_\#(Y^*). \tag{2.13}$$

But $\mathbf{i}(\chi^j)$ is uniquely determine by (2.13). We deduce that (2.10) admits at least one solution and if χ^j and θ^j are two solutions, then $\mathbf{i}(\chi^j) = \mathbf{i}(\theta^j)$, which means χ^j and θ^j have the same neighborhoods in $H^1_\#(Y)$ or equivalently $N(\chi^j - \theta^j) = 0$. Hence (2.12). □

Corollary 2.12. *Let $1 \le i, j \le N$ and $\chi^j \in H^1_\#(Y)$ be a solution to (2.10). The following homogenized coefficients*

$$q_{ij} = \int_{Y^*} a_{ij}(y)dy - \sum_{l=1}^{N} \int_{Y^*} a_{il}(y) \frac{\partial \chi^j}{\partial y_l}(y)dy \tag{2.14}$$

are well defined in the sense that they do not depend on the solution to (2.10).

Lemma 2.13. *The following assertions are true: $q_{ji} = q_{ij}$ $(1 \le i, j \le N)$ and there exists a constant $\alpha_0 > 0$ such that*

$$\sum_{i,j=1}^{N} q_{ij}\xi_j\xi_i \ge \alpha_0|\xi|^2$$

for all $\xi \in \mathbb{R}^N$.

Proof. See e.g., [3]. □

We now visit the existence result for (1.2). The weak formulation of (1.2) reads: Find $(\lambda_\varepsilon, u_\varepsilon) \in \mathbb{C} \times V_\varepsilon$, ($u_\varepsilon \neq 0$) such that

$$a^\varepsilon(u_\varepsilon, v) = \lambda_\varepsilon(\rho^\varepsilon u_\varepsilon, v)_{S^\varepsilon}, \quad v \in V_\varepsilon, \tag{2.15}$$

where

$$(\rho^\varepsilon u_\varepsilon, v)_{S^\varepsilon} = \int_{S^\varepsilon} \rho^\varepsilon u_\varepsilon v d\sigma_\varepsilon(x).$$

Since ρ^ε changes sign, the classical results on the spectrum of semi-bounded self-adjoint operators with compact resolvent do not apply. To handle this, we follow the ideas in [19]. The bilinear form $(\rho^\varepsilon u_\varepsilon, v)_{S^\varepsilon}$ defines a bounded linear operator $K^\varepsilon : V_\varepsilon \to V_\varepsilon$ such that

$$(\rho^\varepsilon u, v)_{S^\varepsilon} = a^\varepsilon(K^\varepsilon u, v) \quad (u, v \in V_\varepsilon).$$

The operator K^ε is symmetric and its domains $D(K^\varepsilon)$ coincides with the whole V_ε, thus it is self-adjoint. Recall the gradient norm is equivalent to the $H^1(\Omega^\varepsilon)$-norm on V_ε. Looking at $K^\varepsilon u$ as a solution to the boundary value problem

$$\begin{cases} -div(a(\frac{x}{\varepsilon})D_x(K^\varepsilon u)) = 0 & \text{in } \Omega^\varepsilon \\ a(\frac{x}{\varepsilon})D_x K^\varepsilon u \cdot n(\frac{x}{\varepsilon}) = \rho^\varepsilon u & \text{on } S^\varepsilon \\ K^\varepsilon u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

we get a constant $C > 0$ such that $\|K^\varepsilon u\|_{V^\varepsilon} \leq C\|u\|_{L^2(\Omega^\varepsilon)}$. As V^ε is compactly embedded in $L^2(\Omega^\varepsilon)$ (indeed, $H^1(\Omega^\varepsilon) \hookrightarrow L^2(\Omega^\varepsilon)$ is compact as $\partial\Omega^\varepsilon$ is $\mathcal{C}^1$), the operator K^ε is compact. We can rewrite (2.15) as follows

$$K^\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon, \quad \mu_\varepsilon = \frac{1}{\lambda_\varepsilon}.$$

We recall that (see e.g., [4]) in the case $\rho \geq 0$ on S, the operator K^ε is positive and its spectrum $\sigma(K^\varepsilon)$ lies in $[0, \|K^\varepsilon\|]$ and $\mu_\varepsilon = 0$ belongs to the essential spectrum $\sigma_e(K^\varepsilon)$. Let L be a self-adjoint operator and let $\sigma_p^\infty(L)$ and $\sigma_c(L)$ be its set of eigenvalues of infinite multiplicity and its continuous spectrum, respectively. We have $\sigma_e(L) = \sigma_p^\infty(L) \cup \sigma_c(L)$ by definition. The spectrum of K^ε is described by the following proposition whose proof is similar to that of [19, Lemma 1].

Lemma 2.14. *Let $\rho \in \mathcal{C}_{per}(Y)$ be such that the sets $\{y \in S : \rho(y) < 0\}$ and $\{y \in S : \rho(y) > 0\}$ are both of positive surface measure. Then for any $\varepsilon > 0$, we have $\sigma(K^\varepsilon) \subset [-\|K^\varepsilon\|, \|K^\varepsilon\|]$ and $\mu = 0$ is the only element of the essential spectrum $\sigma_e(K^\varepsilon)$. Moreover, the discrete spectrum of K^ε consists of two infinite sequences*

$$\mu_\varepsilon^{1,+} \geq \mu_\varepsilon^{2,+} \geq \cdots \geq \mu_\varepsilon^{k,+} \geq \cdots \to 0^+,$$
$$\mu_\varepsilon^{1,-} \leq \mu_\varepsilon^{2,-} \leq \cdots \leq \mu_\varepsilon^{k,-} \leq \cdots \to 0^-.$$

Corollary 2.15. *The hypotheses are those of Lemma 2.14. Problem (1.2) has a discrete set of eigenvalues consisting of two sequences*

$$0 < \lambda_\varepsilon^{1,+} \leq \lambda_\varepsilon^{2,+} \leq \cdots \leq \lambda_\varepsilon^{k,+} \leq \cdots \to +\infty,$$
$$0 > \lambda_\varepsilon^{1,+} \geq \lambda_\varepsilon^{2,-} \geq \cdots \geq \lambda_\varepsilon^{k,-} \geq \cdots \to -\infty.$$

We may now address the homogenization problem for (1.2).

3 Homogenization results

In this section we state and prove homogenization results for both cases $M_S(\rho) > 0$ and $M_S(\rho) = 0$. The homogenization results in the case when $M_S(\rho) < 0$ can be deduced from the case $M_S(\rho) > 0$ by replacing ρ with $-\rho$. We start with the less technical case.

3.1 The case $M_S(\rho) > 0$

We start with the homogenization result for the positive part of the spectrum $(\lambda_\varepsilon^{k,+}, u_\varepsilon^{k,+})_{\varepsilon\in E}$.

3.1.1 Positive part of the spectrum

We assume (this is not a restriction) that the corresponding eigenfunctions are orthonormalized as follows

$$\varepsilon \int_{S^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,+} u_\varepsilon^{l,+} d\sigma_\varepsilon(x) = \delta_{k,l} \quad k,l = 1,2,\cdots \tag{3.1}$$

and the homogenization results states as

Theorem 3.1. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda_\varepsilon^{k,+}, u_\varepsilon^{k,+})$ be the k^{th} positive eigencouple to (1.2) with $M_S(\rho) > 0$ and (3.1). Then, there exists a subsequence E' of E such that*

$$\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+} \quad \to \quad \lambda_0^k \quad \text{in } \mathbb{R} \text{ as } E \ni \varepsilon \to 0 \tag{3.2}$$

$$P_\varepsilon u_\varepsilon^{k,+} \quad \to \quad u_0^k \quad \text{in } H_0^1(\Omega)\text{-weak as } E' \ni \varepsilon \to 0 \tag{3.3}$$

$$P_\varepsilon u_\varepsilon^{k,+} \quad \to \quad u_0^k \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \tag{3.4}$$

$$\frac{\partial P_\varepsilon u_\varepsilon^{k,+}}{\partial x_j} \quad \xrightarrow{2s} \quad \frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j} \text{ in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \ (1 \leq j \leq N) \tag{3.5}$$

where $(\lambda_0^k, u_0^k) \in \mathbb{R} \times H_0^1(\Omega)$ is the k^{th} eigencouple to the spectral problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(\frac{1}{M_S(\rho)} q_{ij} \frac{\partial u_0}{\partial x_j}\right) = \lambda_0 u_0 \quad \text{in } \Omega \\ u_0 = 0 \quad \text{on } \partial\Omega \\ \displaystyle\int_\Omega |u_0|^2 dx = \frac{1}{M_S(\rho)}, \end{cases} \tag{3.6}$$

and where $u_1^k \in L^2(\Omega; H_\#^1(Y))$. Moreover, for almost every $x \in \Omega$ the following hold true:
(i) *$u_1^k(x)$ is a solution to the noncoercive variational problem*

$$\begin{cases} u_1^k(x) \in H_\#^1(Y) \\ a(u_1^k(x), v) = -\displaystyle\sum_{i,j=1}^N \frac{\partial u_0^k}{\partial x_j} \int_{Y^*} a_{ij}(y) \frac{\partial v}{\partial y_i} dy \\ \forall v \in H_\#^1(Y); \end{cases} \tag{3.7}$$

(ii) *We have*

$$\mathbf{i}(u_1^k(x)) = -\sum_{j=1}^{N} \frac{\partial u_0^k}{\partial x_j}(x)\mathbf{i}(\chi^j) \tag{3.8}$$

where χ^j is any function in $H^1_\#(Y)$ defined by the cell problem (2.10).

Proof. We present only the outlines since this proof is similar but less technical to that of the case $M_S(\rho) = 0$.

Fix $k \geq 1$. By means of the minimax principle, as in [29], one easily proves the existence of a constant C independent of ε such that $\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+} < C$. Clearly, for fixed $E \ni \varepsilon > 0$, $u_\varepsilon^{k,+}$ lies in V_ε, and

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon^{k,+}}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \left(\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+}\right) \varepsilon \int_{S^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,+} v \, d\sigma_\varepsilon(x) \tag{3.9}$$

for any $v \in V_\varepsilon$. Bear in mind that $\varepsilon \int_{S^\varepsilon} \rho(\frac{x}{\varepsilon})(u_\varepsilon^{k,+})^2 dx = 1$ and choose $v = u_\varepsilon^k$ in (3.9). The boundedness of the sequence $(\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+})_{\varepsilon\in E}$ and the ellipticity assumption (1.3) imply at once by means of Proposition 2.8 that the sequence $(P_\varepsilon u_\varepsilon^{k,+})_{\varepsilon \in E}$ is bounded in $H_0^1(\Omega)$. Theorem 2.3 and Proposition 2.6 apply simultaneously and gives us $\mathbf{u}^k = (u_0^k, u_1^k) \in \mathbb{F}_0^1$ such that for some $\lambda_0^k \in \mathbb{R}$ and some subsequence $E' \subset E$ we have (3.2)-(3.5), where (3.4) is a direct consequence of (3.3) by the Rellich-Kondrachov theorem. For fixed $\varepsilon \in E'$, let Φ_ε be as in Lemma 2.10. Multiplying both sides of the first equality in (1.2) by Φ_ε and integrating over Ω leads us to the variational ε-problem

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial P_\varepsilon u_\varepsilon^{k,+}}{\partial x_j} \frac{\partial \Phi_\varepsilon}{\partial x_i} dx = (\frac{1}{\varepsilon}\lambda_\varepsilon^{k,+}) \varepsilon \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,+}) \rho(\frac{x}{\varepsilon}) \Phi_\varepsilon \, d\sigma_\varepsilon(x). \tag{3.10}$$

Sending $\varepsilon \in E'$ to 0, keeping (3.2)-(3.5) and Lemma 2.10 in mind, we obtain

$$\sum_{i,j=1}^{N} \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j}\right) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\partial \psi_1}{\partial y_i}\right) dxdy = \lambda_0^k \iint_{\Omega \times S} u_0^k \psi_0(x) \rho(y) dx d\sigma(y).$$

Therefore, $(\lambda_0^k, \mathbf{u}^k) \in \mathbb{R} \times \mathbb{F}_0^1$ solves the following *global homogenized spectral problem*:

$$\begin{cases} \text{Find } (\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{F}_0^1 \text{ such that} \\ \displaystyle\sum_{i,j=1}^{N} \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\partial \psi_1}{\partial y_i}\right) dxdy = \lambda M_S(\rho) \int_\Omega u_0 \psi_0 \, dx \\ \text{for all } \Phi \in \mathbb{F}_0^1. \end{cases} \tag{3.11}$$

which leads to the macroscopic and microscopic problems (3.6)-(3.7) without any major difficulty. As regards the normalization condition in (3.6), we fix $k, l \geq 1$ and put

$$\langle \delta_\varepsilon^{k,+}, \varphi \rangle = \varepsilon \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,+}) \varphi(x) \rho(\frac{x}{\varepsilon}) d\sigma_\varepsilon(x) \qquad (\varepsilon \in E)$$

for $\varphi \in \mathcal{D}(\Omega)$. We have $P_\varepsilon u_\varepsilon^{l,+} \to u_0^{l,+}$ in $H^{-1}(\Omega)$-strong as $E' \ni \varepsilon \to 0$ by (3.4) and the Rellich-Kondrachov theorem. We also have

$$\delta_\varepsilon^{k,+} \rightharpoonup M_S(\rho) u_0^k \quad \text{in} \quad H^{-1}(\Omega)\text{-weak}$$

as $E' \ni \varepsilon \to 0$, since the following hold for any $\varphi \in \mathcal{D}(\Omega)$ (Proposition 2.6)

$$\lim_{E' \ni \varepsilon \to 0} \varepsilon \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,+}) \varphi(x) \rho(\frac{x}{\varepsilon}) d\sigma_\varepsilon(x) = \iint_{\Omega \times S} u_0^k(x) \varphi(x) \rho(y)\, dx d\sigma(y).$$

Hence,

$$\lim_{E' \ni \varepsilon \to 0} \varepsilon \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,+})(P_\varepsilon u_\varepsilon^{l,+}) \rho(\frac{x}{\varepsilon}) d\sigma_\varepsilon(x) = \iint_{\Omega \times S} u_0^k(x) u_0^l(x) \rho(y)\, dx d\sigma(y),$$

This concludes the proof. □

Remark 3.2.

- The eigenfunctions $\{u_0^k\}_{k=1}^\infty$ are in fact orthonormalized as follows

$$\int_\Omega u_0^k u_0^l dx = \frac{\delta_{k,l}}{M_S(\rho)} \quad k,l = 1,2,3,\cdots$$

- If λ_0^k is simple (this is the case for λ_0^1), then by Theorem 3.1, $\lambda_\varepsilon^{k,+}$ is also simple, for small ε, and we can choose the eigenfunctions $u_\varepsilon^{k,+}$ such that the convergence results (3.3)-(3.5) hold for the whole sequence E.

- Replacing ρ with $-\rho$ in (1.2), Theorem 3.1 also applies to the negative part of the spectrum in the case $M_S(\rho) < 0$.

3.1.2 Negative part of the spectrum

We now investigate the negative part of the spectrum $(\lambda_\varepsilon^{k,-}, u_\varepsilon^{k,-})_{\varepsilon \in E}$. Before we can do this we need a few preliminaries and stronger regularity hypotheses on T, ρ and the coefficients $(a_{ij})_{i,j=1}^N$. We assume in this subsection that ∂T is $C^{2,\delta}$ and ρ and the coefficients $(a_{ij})_{i,j=1}^N$ are δ-Hölder continuous ($0 < \delta < 1$).

Let $H^1_{per}(Y^*)$ denotes the space of functions in $H^1(Y^*)$ assuming same values on the opposite faces of Y. The following spectral problem is well posed

$$\begin{cases} \text{Find } (\lambda, \theta) \in \mathbb{C} \times H^1_{per}(Y^*) \\ -\displaystyle\sum_{i,j=1}^N \frac{\partial}{\partial y_j}\left(a_{ij}(y) \frac{\partial \theta}{\partial y_i}\right) = 0 \text{ in } \quad Y^* \\ \displaystyle\sum_{i,j=1}^N a_{ij}(y) \frac{\partial \theta}{\partial y_i} \nu_j = \lambda \rho(y) \theta(y) \text{ on } S \end{cases} \tag{3.12}$$

and possesses a spectrum with similar properties to that of (1.2), two infinite (one positive and another negative) sequences. We recall that since we have $M_S(\rho) > 0$, problem (3.12) admits a unique nontrivial eigenvalue having an eigenfunction with definite sign, the first negative one (see e.g., [28]). In the sequel we will only make use of $(\lambda_1^-, \theta_1^-)$, the first negative eigencouple to (3.12). After proper sign choice we assume that

$$\theta_1^-(y) > 0 \quad \text{in} \quad y \in Y^*. \tag{3.13}$$

We also recall that θ_1^- is δ-Hölder continuous(see e.g., [10]), hence can be extended to a function living in $\mathcal{C}_{per}(Y)$ still denoted by θ_1^-. Notice that we have

$$\int_S \rho(y)(\theta_1^-(y))^2\,d\sigma(y) < 0, \tag{3.14}$$

as is easily seen from the variational equality (keep the ellipticity hypothesis (1.3) in mind)

$$\sum_{i,j=1}^N \int_{Y^*} a_{ij}(y)\frac{\partial\theta_1^-}{\partial y_j}\frac{\partial\theta_1^-}{\partial y_i}\,dy = \lambda_1^- \int_S \rho(y)(\theta_1^-(y))^2\,d\sigma(y).$$

Bear in mind that problem (3.12) induces by a scaling argument the following equalities:

$$\begin{cases} -\displaystyle\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}(\frac{x}{\varepsilon})\frac{\partial\theta^\varepsilon}{\partial x_i}\right) = 0 \quad \text{ in } Q^\varepsilon \\ \displaystyle\sum_{i,j=1}^N a_{ij}(\frac{x}{\varepsilon})\frac{\partial\theta^\varepsilon}{\partial x_i}\nu_j(\frac{x}{\varepsilon}) = \frac{1}{\varepsilon}\lambda\rho(\frac{x}{\varepsilon})\theta(\frac{x}{\varepsilon}) \quad \text{ on } \partial Q^\varepsilon, \end{cases} \tag{3.15}$$

where $\theta^\varepsilon(x) = \theta(\frac{x}{\varepsilon})$. However, θ^ε is not zero on $\partial\Omega$. We now introduce the following Steklov spectral problem (with an indefinite density function)

$$\begin{cases} \text{Find } (\xi_\varepsilon, v_\varepsilon) \in \mathbb{C}\times V_\varepsilon \\ -\displaystyle\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(\widetilde{a}_{ij}(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon}{\partial x_i}\right) = 0 \quad \text{ in } \Omega^\varepsilon \\ \displaystyle\sum_{i,j=1}^N \widetilde{a}_{ij}(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon(x)}{\partial x_i}\nu_j(\frac{x}{\varepsilon}) = \xi_\varepsilon\widetilde{\rho}(\frac{x}{\varepsilon})v_\varepsilon(x) \text{ on } \partial T^\varepsilon \\ v_\varepsilon(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.16}$$

with new spectral parameters $(\xi_\varepsilon, v_\varepsilon) \in \mathbb{C}\times V_\varepsilon$, where $\widetilde{a}_{ij}(y) = (\theta_1^-)^2(y)a_{ij}(y)$ and $\widetilde{\rho}(y) = (\theta_1^-)^2(y)\rho(y)$. Notice that $\widetilde{a}_{ij}(y) \in L^\infty_{per}(Y)$ and $\widetilde{\rho}(y) \in \mathcal{C}_{per}(Y)$. As $0 < c_- \le \theta_1^-(y) \le c^+ < +\infty$ ($c_-, c^+ \in \mathbb{R}$), the operator on the left hand side of (3.16) is uniformly elliptic and Theorem 3.1 applies to the negative part of the spectrum of (3.16) (see (3.14) and Remark 3.2). The effective spectral problem for (3.16) reads

$$\begin{cases} -\displaystyle\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(\widetilde{q}_{ij}\frac{\partial v_0}{\partial x_i}\right) = \xi_0 M_S(\widetilde{\rho})v_0 \quad \text{ in } \Omega \\ v_0 = 0 \quad \text{ on } \partial\Omega \\ \displaystyle\int_\Omega |v_0|^2 dx = \frac{-1}{M_S(\widetilde{\rho})}. \end{cases} \tag{3.17}$$

The effective coefficients $\{\widetilde{q}_{ij}\}_{1\le i,j\le N}$ being defined as expected, i.e.,

$$\widetilde{q}_{ij} = \int_{Y^*} \widetilde{a}_{ij}(y)dy - \sum_{l=1}^N \int_{Y^*} \widetilde{a}_{il}(y)\frac{\partial\widetilde{\chi}_1^j}{\partial y_l}(y)dy, \tag{3.18}$$

with $\widetilde{\chi}_1^l \in H^1_\#(Y^*)$ $(l = 1,...,N)$ being a solution to the following local problem

$$\begin{cases} \widetilde{\chi}_1^l \in H^1_\#(Y^*) \\ \displaystyle\sum_{i,j=1}^N \int_{Y^*} \widetilde{a}_{ij}(y) \frac{\partial \widetilde{\chi}_1^l}{\partial y_j} \frac{\partial v}{\partial y_i} dy = \sum_{i=1}^N \int_{Y^*} \widetilde{a}_{il}(y) \frac{\partial v}{\partial y_i} dy \\ \text{for all } v \in H^1_\#(Y^*). \end{cases} \tag{3.19}$$

Notice that the spectrum of (3.17) is as follows

$$0 > \xi_0^1 > \xi_0^2 \geq \xi_0^3 \geq \cdots \geq \xi_0^j \geq \cdots \to -\infty \text{ as } j \to \infty.$$

Making use of (3.15) when following the same line of reasoning as in [29, Lemma 6.1], we obtain that the negative spectral parameters of problems (1.2) and (3.16) verify:

$$u_\varepsilon^{k,-} = (\theta_1^-)^\varepsilon v_\varepsilon^{k,-} \quad (\varepsilon \in E,\ k = 1,2\cdots) \tag{3.20}$$

and

$$\lambda_\varepsilon^{k,-} = \frac{1}{\varepsilon}\lambda_1^- + \xi_\varepsilon^{k,-} + o(1) \quad (\varepsilon \in E,\ k = 1,2\cdots). \tag{3.21}$$

The presence of the term $o(1)$ is due to integrals over $\Omega^\varepsilon \setminus Q^\varepsilon$, which converge to zero with ε, remember that (3.15) holds in Q^ε but not Ω^ε. As will be seen below, the sequence $(\xi_\varepsilon^{k,-})_{\varepsilon\in E}$ is bounded in $\mathbb{R}$. In another words, $\lambda_\varepsilon^{k,-}$ is of order $1/\varepsilon$ and tends to $-\infty$ as ε goes to zero. It is now clear why the limiting behavior of negative eigencouples is not straightforward as that of positive ones and requires further investigations, which have just been made.

Indeed, as the reader might be guessing now, the suitable orthonormalization condition for (3.16) is

$$\varepsilon \int_{S^\varepsilon} \widetilde{\rho}(\frac{x}{\varepsilon}) v_\varepsilon^{k,-} v_\varepsilon^{l,-}\, d\sigma_\varepsilon(x) = -\delta_{k,l} \quad k,l = 1,2,\cdots \tag{3.22}$$

which by means of (3.20) is equivalent to

$$\varepsilon \int_{S^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,-} u_\varepsilon^{l,-}\, d\sigma_\varepsilon(x) = -\delta_{k,l} \quad k,l = 1,2,\cdots \tag{3.23}$$

We may now state the homogenization theorem for the negative part of the spectrum of (1.2).

Theorem 3.3. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda_\varepsilon^{k,-}, u_\varepsilon^{k,-})$ be the k^{th} negative eigencouple to (1.2) with $M_S(\rho) > 0$ and (3.23). Then, there exists a subsequence E' of E such that*

$$\frac{\lambda_\varepsilon^{k,-}}{\varepsilon} - \frac{\lambda_1^-}{\varepsilon^2} \to \xi_0^k \quad \textit{in } \mathbb{R} \textit{ as } E \ni \varepsilon \to 0 \tag{3.24}$$

$$P_\varepsilon v_\varepsilon^{k,-} \to v_0^k \quad \textit{in } H_0^1(\Omega)\textit{-weak as } E' \ni \varepsilon \to 0 \tag{3.25}$$

$$P_\varepsilon v_\varepsilon^{k,-} \to v_0^k \quad \textit{in } L^2(\Omega) \textit{ as } E' \ni \varepsilon \to 0 \tag{3.26}$$

$$\frac{\partial P_\varepsilon v_\varepsilon^{k,-}}{\partial x_j} \xrightarrow{2s} \frac{\partial v_0^k}{\partial x_j} + \frac{\partial v_1^k}{\partial y_j} \textit{ in } L^2(\Omega) \textit{ as } E' \ni \varepsilon \to 0\ (1 \leq j \leq N) \tag{3.27}$$

where $(\xi_0^k, v_0^k) \in \mathbb{R} \times H_0^1(\Omega)$ *is the* k^{th} *eigencouple to the spectral problem*

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{1}{M_S(\widetilde{\rho})}\widetilde{q}_{ij}\frac{\partial v_0}{\partial x_j}\right) = \xi_0 v_0 \quad \text{in } \Omega \\ v_0 = 0 \quad \text{on } \partial\Omega \\ \int_\Omega |v_0|^2\, dx = \frac{-1}{M_S(\widetilde{\rho})}, \end{cases} \tag{3.28}$$

and where $v_1^k \in L^2(\Omega; H^1_\#(Y))$. *Moreover, for almost every* $x \in \Omega$ *the following hold true:*
(i) $v_1^k(x)$ *is a solution to the noncoercive variational problem*

$$\begin{cases} v_1^k(x) \in H^1_\#(Y) \\ \widetilde{a}(v_1^k(x), u) = -\sum_{i,j=1}^{N} \frac{\partial v_0^k}{\partial x_j}\int_{Y^*} \widetilde{a}_{ij}(y)\frac{\partial u}{\partial y_i}dy \\ \forall u \in H^1_\#(Y); \end{cases} \tag{3.29}$$

(ii) *We have*

$$\mathbf{i}(v_1^k(x)) = -\sum_{j=1}^{N} \frac{\partial v_0^k}{\partial x_j}(x)\mathbf{i}(\widetilde{\chi}^j) \tag{3.30}$$

where $\widetilde{\chi}^j$ *is any function in* $H^1_\#(Y)$ *defined by the cell problem (3.19).*

Remark 3.4.

- The eigenfunctions $\{v_0^k\}_{k=1}^\infty$ are orthonormalized by

$$\int_\Omega v_0^k v_0^l dx = \frac{-\delta_{k,l}}{M_S(\widetilde{\rho})} \quad k,l = 1,2,3,\cdots$$

- Replacing ρ with $-\rho$ in (1.2), Theorem 3.3 adapts to the positive part of the spectrum in the case $M_S(\rho) < 0$.

3.2 The case $M_S(\rho) = 0$

We prove an homogenization result for both the positive part and the negative part of the spectrum simultaneously. We assume in this case that the eigenfunctions are orthonormalized as follows

$$\int_{S^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,\pm} u_\varepsilon^{l,\pm}\, d\sigma_\varepsilon(x) = \pm\delta_{k,l} \quad k,l = 1,2,\cdots \tag{3.31}$$

Let χ^0 be a solution to (2.11) and put

$$\nu^2 = \sum_{i,j=1}^{N} \int_{Y^*} a_{ij}(y)\frac{\partial \chi^0}{\partial y_j}\frac{\partial \chi^0}{\partial y_i}dy. \tag{3.32}$$

Indeed, the right hand side of (3.32) is positive and does not depend on a particular solution to (2.11). We recall that the following spectral problem for a quadratic operator pencil with respect to ν,

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(q_{ij}\frac{\partial u_0}{\partial x_i}\right) = \lambda_0^2 \nu^2 u_0 \text{ in } \Omega \\ u_0 = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.33}$$

has a spectrum consisting of two infinite sequences

$$0 < \lambda_0^{1,+} < \lambda_0^{2,+} \leq \cdots \leq \lambda_0^{k,+} \leq \ldots, \quad \lim_{n\to+\infty} \lambda_0^{k,+} = +\infty$$

and

$$0 > \lambda_0^{1,-} > \lambda_0^{2,-} \geq \cdots \geq \lambda_0^{k,-} \geq \ldots, \quad \lim_{n\to+\infty} \lambda_0^{k,-} = -\infty.$$

with $\lambda_0^{k,+} = -\lambda_0^{k,-}$ $k = 1,2,\cdots$ and with the corresponding eigenfunctions $u_0^{k,+} = u_0^{k,-}$. We note by passing that $\lambda_0^{1,+}$ and $\lambda_0^{1,-}$ are simple. We are now in a position to state the homogenization result in the present case.

Theorem 3.5. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda_\varepsilon^{k,\pm}, u_\varepsilon^{k,\pm})$ be the $(k,\pm)^{th}$ eigencouple to (1.2) with $M_S(\rho) = 0$ and (3.31). Then, there exists a subsequence E' of E such that*

$$\lambda_\varepsilon^{k,\pm} \to \lambda_0^{k,\pm} \quad \text{in } \mathbb{R} \text{ as } E \ni \varepsilon \to 0 \tag{3.34}$$

$$P_\varepsilon u_\varepsilon^{k,\pm} \to u_0^{k,\pm} \quad \text{in } H_0^1(\Omega)\text{-weak as } E' \ni \varepsilon \to 0 \tag{3.35}$$

$$P_\varepsilon u_\varepsilon^{k,\pm} \to u_0^{k,\pm} \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \tag{3.36}$$

$$\frac{\partial P_\varepsilon u_\varepsilon^{k,\pm}}{\partial x_j} \xrightarrow{2s} \frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \text{ in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \ (1 \leq j \leq N) \tag{3.37}$$

where $(\lambda_0^{k,\pm}, u_0^{k,\pm}) \in \mathbb{R} \times H_0^1(\Omega)$ is the $(k,\pm)^{th}$ eigencouple to the following spectral problem for a quadratic operator pencil with respect to ν,

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(q_{ij}\frac{\partial u_0}{\partial x_j}\right) = \lambda_0^2 \nu^2 u_0 \text{ in } \Omega \\ u_0 = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.38}$$

and where $u_1^{k,\pm} \in L^2(\Omega; H_\#^1(Y))$. We have the following normalization condition

$$\int_\Omega |u_0^{k,\pm}|^2 \, dx = \frac{\pm 1}{\lambda_0^{k,\pm}\nu^2} \qquad k = 1,2,\cdots \tag{3.39}$$

Moreover, for almost every $x \in \Omega$ the following hold true:
(i) *$u_1^{k,\pm}(x)$ is a solution to the noncoercive variational problem*

$$\begin{cases} u_1^{k,\pm}(x) \in H_\#^1(Y) \\ a(u_1^{k,\pm}(x), v) = \lambda_0^{k,\pm} u_0^k(x) \displaystyle\int_S \rho(y) v(y)\, d\sigma(y) - \sum_{i,j=1}^{N} \frac{\partial u_0^{k,+}}{\partial x_j}(x) \int_{Y^*} a_{ij}(y) \frac{\partial v}{\partial y_i}\, dy \\ \forall v \in H_\#^1(Y); \end{cases} \tag{3.40}$$

(ii) *We have*

$$\mathbf{i}(u_1^{k,\pm}(x)) = \lambda_0^{k,\pm} u_0^k(x) \mathbf{i}(\chi^0) - \sum_{j=1}^{N} \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) \mathbf{i}(\chi^j) \tag{3.41}$$

where χ^j $(1 \leq j \leq N)$ and χ^0 are functions in $H_\#^1(Y)$ defined by the cell problems (2.10) and (2.11), respectively.

Proof. Fix $k \geq 1$, using the minimax principle, as in [29], we get a constant C independent of ε such that $|\lambda_\varepsilon^{k,\pm}| < C$. We have $u_\varepsilon^{k,\pm} \in V_\varepsilon$ and

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon^{k,\pm}}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \lambda_\varepsilon^{k,\pm} \int_{S^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,\pm} v \, d\sigma_\varepsilon(x) \tag{3.42}$$

for any $v \in V_\varepsilon$. Bear in mind that $\int_{S^\varepsilon} \rho(\frac{x}{\varepsilon})(u_\varepsilon^{k,\pm})^2 \, d\sigma_\varepsilon(x) = \pm 1$ and choose $v = u_\varepsilon^k$ in (3.42). The boundedness of the sequence $(\lambda_\varepsilon^{k,\pm})_{\varepsilon \in E}$ and the ellipticity assumption (1.3) imply at once by means of Proposition 2.8 that the sequence $(P_\varepsilon u_\varepsilon^{k,\pm})_{\varepsilon \in E}$ is bounded in $H_0^1(\Omega)$. Theorem 2.3 and Proposition 2.6 apply simultaneously and gives us $\mathbf{u}^{k,\pm} = (u_0^{k,\pm}, u_1^{k,\pm}) \in \mathbb{F}_0^1$ such that for some $\lambda_0^{k,\pm} \in \mathbb{R}$ and some subsequence $E' \subset E$ we have (3.34)-(3.37), where (3.36) is a direct consequence of (3.35) by the Rellich-Kondrachov theorem. For fixed $\varepsilon \in E'$, let Φ_ε be as in Lemma 2.10. Multiplying both sides of the first equality in (1.2) by Φ_ε and integrating over Ω leads us to the variational ε-problem

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial P_\varepsilon u_\varepsilon^{k,\pm}}{\partial x_j} \frac{\partial \Phi_\varepsilon}{\partial x_i} dx = \lambda_\varepsilon^{k,\pm} \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \rho(\frac{x}{\varepsilon}) \Phi_\varepsilon \, d\sigma_\varepsilon(x).$$

Sending $\varepsilon \in E'$ to 0, keeping (3.34)-(3.37) and Lemma 2.10 in mind, we obtain

$$a_\Omega(\mathbf{u}^{k,\pm}, \Phi) = \lambda_0^{k,\pm} \iint_{\Omega \times S} \left(u_1^{k,\pm}(x,y) \psi_0(x) \rho(y) + u_0^{k,\pm} \psi_1(x,y) \rho(y) \right) dx d\sigma(y) \tag{3.43}$$

The right-hand side follows as explained below. we have

$$\begin{aligned} \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \rho(\frac{x}{\varepsilon}) \Phi_\varepsilon \, d\sigma_\varepsilon(x) &= \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_0(x) \rho(\frac{x}{\varepsilon}) \, d\sigma_\varepsilon(x) \\ &+ \varepsilon \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_1(x, \frac{x}{\varepsilon}) \rho(\frac{x}{\varepsilon}) \, d\sigma_\varepsilon(x). \end{aligned}$$

On the one hand we have

$$\lim_{E' \ni \varepsilon \to 0} \varepsilon \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_1(x, \frac{x}{\varepsilon}) \rho(\frac{x}{\varepsilon}) \, dx = \iint_{\Omega \times S} u_0^{k,\pm} \psi_1(x,y) \rho(y) \, dx d\sigma(y).$$

On the other hand, owing to Lemma 2.7, the following holds:

$$\lim_{E' \ni \varepsilon \to 0} \int_{S^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_0(x) \rho(\frac{x}{\varepsilon}) \, d\sigma_\varepsilon(x) = \iint_{\Omega \times S} u_1^{k,\pm}(x,y) \psi_0(x) \rho(y) \, dx d\sigma(y).$$

We have just proved that $(\lambda_0^{k,+}, \mathbf{u}^{k,\pm}) \in \mathbb{R} \times \mathbb{F}_0^1$ solves the following *global homogenized spectral problem*:

$$\begin{cases} \text{Find } (\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{F}_0^1 \text{ such that} \\ a_\Omega(\mathbf{u}, \Phi) = \lambda \iint_{\Omega \times S} (u_1(x,y) \psi_0(x) + u_0(x) \psi_1(x,y)) \, \rho(y) \, dx d\sigma(y) \\ \text{for all } \Phi \in \mathbb{F}_0^1. \end{cases} \tag{3.44}$$

To prove (i), choose $\Phi = (\psi_0, \psi_1)$ in (3.43) such that $\psi_0 = 0$ and $\psi_1 = \varphi \otimes v_1$, where $\varphi \in \mathcal{D}(\Omega)$ and $v_1 \in H^1_\#(Y)$ to get

$$\int_\Omega \varphi(x) \left[\sum_{i,j=1}^N \int_{Y^*} a_{ij}(y) \left(\frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \right) \frac{\partial v_1}{\partial y_i} dy \right] dx = \int_\Omega \varphi(x) \left[\lambda_0^{k,\pm} u_0^{k,\pm}(x) \int_S v_1(y)\rho(y)\, d\sigma(y) \right] dx$$

Hence by the arbitrariness of φ, we have a.e. in Ω

$$\sum_{i,j=1}^N \int_{Y^*} a_{ij}(y) \left(\frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \right) \frac{\partial v_1}{\partial y_i} dy = \lambda_0^{k,\pm} u_0^{k,\pm}(x) \int_S v_1(y)\rho(y) d\sigma(y)$$

for any v_1 in $H^1_\#(Y)$, which is nothing but (3.40).

Fix $x \in \overline{\Omega}$, multiply both sides of (2.10) by $-\frac{\partial u_0^{k,\pm}}{\partial x_j}(x)$ and sum over $1 \le j \le N$. Adding side by side to the resulting equality that obtained after multiplying both sides of (2.11) by $\lambda_0^{k,\pm} u_0^{k,\pm}(x)$, we realize that $z(x) = -\sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x)\chi^j(y) + \lambda_0^{k,\pm} u_0^{k,\pm}(x)\chi^0(y)$ solves (3.40). Hence $\mathbf{i}(z(x)) = \mathbf{i}(u_1^{k,\pm}(x))$ by uniqueness of the solution to the coercive variational problem in $H^1_\#(Y^*)$ corresponding to the non-coercive variational problem (3.40) (see the proof of Proposition 2.11). Thus (3.41) since $\mathbf{i}$ is linear.

This being so, we recall that (3.41) precisely means that almost everywhere in $x \in \Omega$,

$$D_y u_1^{k,\pm}(x) = \lambda_0^{k,\pm} u_0^{k,\pm}(x) D_y \chi^0 - \sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) D_y \chi^j \quad \text{a.e. in } y \in Y^*. \tag{3.45}$$

Hence there is some $c \in L^2(\Omega)$ so that almost everywhere in $(x,y) \in \Omega \times Y^*$ we have

$$u_1^{k,\pm}(x,y) = \lambda_0^{k,\pm} u_0^{k,\pm}(x)\chi^0(y) - \sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x)\chi^j(y) + c(x). \tag{3.46}$$

But (3.46) still holds almost everywhere in $(x,y) \in \Omega \times S$ as S is of class $\mathcal{C}^1$. Considering now $\Phi = (\psi_0, \psi_1)$ in (3.43) such that $\psi_0 \in \mathcal{D}(\Omega)$ and $\psi_1 = 0$ we get

$$\sum_{i,j=1}^N \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \right) \frac{\partial \psi_0}{\partial x_i}\, dxdy = \lambda_0^{k,\pm} \iint_{\Omega \times S} u_1^{k,\pm}(x,y)\rho(y)\psi_0(x)\, dxd\sigma(y),$$

which by means of (3.45) and (3.46) leads to

$$\begin{aligned} &\sum_{i,j=1}^N \int_\Omega q_{ij} \frac{\partial u_0^{k,\pm}}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dx + \lambda_0^{k,\pm} \sum_{i,j=1}^N \int_\Omega u_0^{k,\pm}(x) \frac{\partial \psi_0}{\partial x_i} \left(\int_{Y^*} a_{ij}(y) \frac{\partial \chi^0}{\partial y_j}(y) dy \right) dx \\ &= -\lambda_0^{k,\pm} \sum_{j=1}^N \int_\Omega \frac{\partial u_0^{k,\pm}}{\partial x_j} \psi_0(x) \left(\int_S \rho(y)\chi^j(y)\, d\sigma(y) \right) dx \\ &+ (\lambda_0^{k,\pm})^2 \int_\Omega u_0^{k,\pm}(x)\psi_0(x) \left(\int_S \rho(y)\chi^0(y)\, d\sigma(y) \right) dx. \end{aligned} \tag{3.47}$$

The term with $c(x)$ vanishes because of $M_S(\rho)=0$. Choosing χ^l $(1\leq l\leq N)$ as test function in (2.11) and χ^0 as test function in (2.10) we observe that

$$\sum_{j=1}^{N}\int_{Y^*} a_{lj}(y)\frac{\partial\chi^0}{\partial y_j}(y)dy=\int_S \rho(y)\chi^l(y)\,d\sigma(y)=a(\chi^l,\chi^0)\quad (l=1,\cdots N).$$

Thus, in (3.47), the second term in the left hand side is equal to the first one in the right hand side. This leaves us with

$$\int_\Omega q_{ij}\frac{\partial u_0^{k,\pm}}{\partial x_j}\frac{\partial\psi_0}{\partial x_i}dx=(\lambda_0^{k,\pm})^2\int_\Omega u_0^{k,\pm}(x)\psi_0(x)dx\left(\int_S \rho(y)\chi^0(y)\,d\sigma(y)\right). \tag{3.48}$$

Choosing χ^0 as test function in (2.11) reveals that

$$\int_S \rho(y)\chi^0(y)\,d\sigma(y)=a(\chi^0,\chi^0)=\nu^2.$$

Hence

$$\sum_{i,j=1}^{N}\int_\Omega q_{ij}\frac{\partial u_0^{k,\pm}}{\partial x_j}\frac{\partial\psi_0}{\partial x_i}dx=(\lambda_0^{k,\pm})^2\nu^2\int_\Omega u_0^{k,\pm}(x)\psi_0(x)dx,$$

and

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(q_{ij}\frac{\partial u_0^{k,\pm}}{\partial x_j}(x)\right)=(\lambda_0^{k,\pm})^2\nu^2 u_0^{k,\pm}(x)\text{ in }\Omega.$$

Thus, the convergence (3.34) holds for the whole sequence E. As regards (3.39), we proceed as above. Fix $k,l\geq 1$ and put

$$\langle\delta_\varepsilon^{k,\pm},\varphi\rangle=\int_{S^\varepsilon}(P_\varepsilon u_\varepsilon^{k\pm})\varphi(x)\rho(\frac{x}{\varepsilon})d\sigma_\varepsilon(x)\qquad(\varepsilon\in E),$$

for $\varphi\in\mathcal{D}(\Omega)$. We have $P_\varepsilon u_\varepsilon^{l,\pm}\to u_0^{l,\pm}$ in $H^{-1}(\Omega)$-strong as $E'\ni\varepsilon\to 0$ by (3.35) and the Rellich-Kondrachov theorem. We also have

$$\delta_\varepsilon^{k,\pm}\rightharpoonup\int_S u_1^{k,\pm}(\cdot,y)\rho(y)d\sigma(y)\text{ in }H^{-1}(\Omega)\text{-weak}$$

as $E'\ni\varepsilon\to 0$, since (Lemma 2.7) for any $\varphi\in\mathcal{D}(\Omega)$, it holds that

$$\lim_{E'\ni\varepsilon\to 0}\int_{S^\varepsilon}(P_\varepsilon u_\varepsilon^{k,\pm})\varphi(x)\rho(\frac{x}{\varepsilon})d\sigma_\varepsilon(x)=\iint_{\Omega\times S}u_1^{k,\pm}(x,y)\varphi(x)\rho(y)\,dxd\sigma(y).$$

Hence,

$$\lim_{E'\ni\varepsilon\to 0}\int_{S^\varepsilon}(P_\varepsilon u_\varepsilon^{k,\pm})(P_\varepsilon u_\varepsilon^{l,\pm})\rho(\frac{x}{\varepsilon})d\sigma_\varepsilon(x)=\iint_{\Omega\times S}u_1^{k,\pm}(x,y)u_0^{l,\pm}\rho(y)\,dxd\sigma(y).$$

This together with (3.31) and (3.46) yields

$$\lambda_0^{k,\pm}\nu^2\int_\Omega u_0^{l,\pm}u_0^{k,\pm}\,dx-\sum_{j=1}^{N}a(\chi^j,\chi^0)\int_\Omega\frac{\partial u_0^{k,\pm}}{\partial x_j}u_0^{l,\pm}\,dx=\pm\delta_{k,l},\quad k,l=1,2,\cdots \tag{3.49}$$

If $k=l$, then by Green's formula the sum on the left hand side vanishes and (3.49) reduces to the desired result. This concludes the proof. □

Remark 3.6.

- The eigenfunctions $\{u_0^{k,\pm}\}_{k=1}^{\infty}$ are in fact orthonormalized as follows

$$\iint_{\Omega\times S} u_1^{l,\pm}(x,y)u_0^{k,\pm}(x)\rho(y)\,dxd\sigma(y) = \iint_{\Omega\times S} u_1^{k,\pm}(x,y)u_0^{l,\pm}(x)\rho(y)\,dxd\sigma(y) = \pm\delta_{k,l}$$

$k,l = 1,2,\cdots$

- If $\lambda_0^{k,\pm}$ is simple (this is the case for $\lambda_0^{1,\pm}$), then by Theorem 3.5, $\lambda_\varepsilon^{k,\pm}$ is also simple, for small ε, and we can choose the eigenfunctions $u_\varepsilon^{k,\pm}$ such that the convergence results (3.3)-(3.5) hold for the whole sequence E.

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References

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., **23** (1992), 1482-1518.

[2] G. Allaire, A. Damlamian and U. Hornung, *Two-scale convergence on periodic surfaces and applications*, In Proceedings of the International Conference on Mathematical Modelling of Flow through Porous Media (May 1995), A. Bourgeat et al. eds., pp.15-25, World Scientific Pub., Singapore (1996).

[3] A. Bensoussan, J.L. Lions and G. Papanicolaou, "Asymptotic analysis for periodic structures", North-Holland, Amsterdam, 1978.

[4] M.S. Birman, M.Z. Solomyack, "Spectral theory of self-adjoint operators in Hilbert spaces", D. Reidel Publishing Company, Dordrecht, 1987.

[5] D. Cioranescu and J. Saint Jean Paulin, *Homogenization in open sets with holes*, J. Math. Appl., **71** (1979), 590-607.

[6] H. Douanla, *Two-Scale convergence of elliptic spectral problems with indefinite density function in perforated domains*, arXiv:1106.3907v1 [math.AP].

[7] H. Douanla, *Two-Scale convergence of Stekloff eigenvalue problems in perforated domains*,, Boundary Value Problems, 2010 (2010), Article ID 853717, 15 pages.

[8] H. Douanla and N. Svanstedt, *Reiterated homogenization of linear eigenvalue problems in multiscale perforated domains beyond the periodic setting*, Commun. Math. Anal. **11** (2011), no. 1, 61-93.

[9] D. G. Figueiredo, *Positive solutions of semilinear elliptic problems*, Lecture Notes in Maths.,**954**, Springer-Verlag, Berlin (1982), 34-87.

[10] D. Gilbarg and N.S Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer, Berlin (2001), 34-87.

[11] P. Hess, *On bifurcation from infinity for positive solutions of secon order elliptic eigenvalue problems*, Nonlinear Phenomena in Mathematical Sciences., Academic Press, New York, (1982), 537-5

[12] P. Hess and T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial. Differential Equations., **5** (1980), 999-1030.

[13] S. Kaizu, *Homogenization of eigenvalue problems for the laplace operators with nonlinear terms in domains in many tiny holes*, Nonlin. Anal. TMA., **28** (1997), 377-391.

[14] S. Kesavan, *Homogenization of elliptic eigenvalue problems. I.*, Appl. Math. Optim., **5** (1979), 153-167.

[15] S. Kesavan, *Homogenization of elliptic eigenvalue problems. II.*, Appl. Math. Optim., **5** (1979), 197-216.

[16] D. Lukkassen, G. Nguetseng and P. Wall, *Two-scale convergence*, Int. J. Pure Appl. Math., **2** (2002), 35-86.

[17] S.A. Nazarov, *Asymptotics of negative eigenvalues of the Dirichlet problem with the density changing sign*, J. Math. Sci., **163** (2009), 151-175.

[18] S.A. Nazarov, A.L. Piatnitski, *Homogenization of the spectral Dirichlet problem for a system of differential equations with rapidly oscillating coefficients and changing sing density*, J. Math. Sci., **169** (2010), 212-248.

[19] S.A. Nazarov, I.L. Pankratova, A.L. Piatnitski, *Homogenization of the spectral problem for periodic elliptic operators with sign-changing density function*, Arch. Rational Mech. Anal., DOI: 10.1007/s00205-010-0370-2.

[20] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., **20**, (1989), 608-623.

[21] G. Nguetseng, *Homogenization in perforated domains beyond the periodic setting*, J. Math. Anal. Appl., **289** (2004), 608-628.

[22] G. Nguetseng, J.L. Woukeng, *Σ-convergence of nonlinear parabolic operators*, Nonlinear Anal., **66** (2007), 968-1004.

[23] O. A. Oleinik, G. A. Iosif'yan and A. S. Shamaev, *Mathematical problems of the theory of strongly nonhomogeneous elastic media [in Russian]*, Moscow, 1990.

[24] S. E. Pastukhova, *Averaging error for the Steklov problem in a perforated domain*, Differential Equations, **31** (1995), 975-986.

[25] M. Radu, *Homogenization techniques*, Diplomarbeit, University of Heidelberg: Faculty of Mathematics, July 1992.

[26] S. Roppongi, *Asymptotics of eigenvalues of the Laplacian with small spherical Robin boundary*, Osaka J. Math., **30** (1993), 783-811.

[27] M. V. Steklov, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecole normale Sup., **19** (1902), 455-490.

[28] O. Torne, *Steklov problem with an indefinite weight for the p-laplacian*, Elecron. J. Differ. Equ., **87** (2005), 1-8.

[29] M. Vanninathan, *Homogenization of eigenvalue problems in perforated domains*, Proc. Indian Acad. Sci. (Math. Sci.), **90** (1981), 239-271.

[30] V.V. Zhikov, On two-scale convergence. (Russian) *Tr. Semin. im. I. G. Petrovskogo* No. 23 (2003), 149–187, 410; translation in J. Math. Sci. (N. Y.) 120 (2004), no. 3, 1328–1352