TWO-SCALE CONVERGENCE OF ELLIPTIC SPECTRAL PROBLEMS WITH INDEFINITE DENSITY FUNCTION IN PERFORATED DOMAINS

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Abstract

Spectral asymptotics of linear periodic elliptic operators with indefinite (sign-changing) density function is investigated in perforated domains with the two-scale convergence method. The limiting behavior of positive and negative eigencouples depends crucially on whether the average of the weight over the solid part is positive, negative or equal to zero. We prove concise homogenization results in all three cases.

1 Introduction

Many nonlinear problems lead, after linearization, to elliptic eigenvalue problems with an indefinite density function (see e.g., the survey paper by de Figueiredo[10] and the work of Hess and Kato[12, 13]). A vast literature in engineering, physics and applied mathematics deals with such problems arising, for instance, in the study of transport theory, reaction-diffusion equations and fluid dynamics. In 1904, Holmgren[15] considered the Dirichlet problem $\Delta u + \rho r(x,y)u = 0$, on a fixed bounded open set $\Omega \subset \mathbb{R}^2$ when ρ is continuous and changes sign; he proved the existence of a double sequence of real eigenvalues of finite multiplicity (one nonnegative and converging to $+\infty$, the other one negative and tending to $-\infty$) which can be characterized by the minimax principle. This result has been extended to higher dimensions, noncontinuous weight and coefficients in many papers including for example [3, 4, 21]. Asymptotic analysis of the eigenvalues has been visited by many mathematicians and is still a hot topic in mathematical analysis. Generally speaking, spectral asymptotics is a two folded research area. On the one hand it deals with asymptotic formulas (estimates) and asymptotic distribution of the eigenvalues. On the other hand it is concerned with homogenization of eigenvalues of oscillating operators on possibly varying

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domains such as perforated ones. This paper falls within the second framework, homogenization theory.

Let Ω be a bounded domain in $\mathbb{R}_x^N$ (the numerical space of variables $x = (x_1, ..., x_N)$, with integer $N \geq 2$) with C^1 boundary $\partial\Omega$. We define the perforated domain Ω^ε as follows. Let $T \subset Y = (0,1)^N$ be a compact subset in $\mathbb{R}_y^N$ with C^1 boundary ∂T and nonempty interior. For $\varepsilon > 0$, we define

$$t^\varepsilon = \{k \in \mathbb{Z}^N : \varepsilon(k+T) \subset \Omega\}$$

$$T^\varepsilon = \bigcup_{k \in t^\varepsilon} \varepsilon(k+T)$$

and

$$\Omega^\varepsilon = \Omega \setminus T^\varepsilon.$$

In this setup, T is the reference hole whereas $\varepsilon(k+T)$ is a hole of size ε and T^ε is the collection of the holes of the perforated domain Ω^ε. The family T^ε is made up with a finite number of holes since Ω is bounded. In the sequel, Y^* stands for $Y \setminus T$ and $n = (n_i)$ denotes the outer unit normal vector to ∂T with respect to Y^*.

We are interested in the spectral asymptotics (as $\varepsilon \to 0$) of the linear elliptic eigenvalue problem

$$\begin{cases} -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_i}\right) = \rho(\frac{x}{\varepsilon})\lambda_\varepsilon u_\varepsilon \text{ in } \Omega^\varepsilon \\ \\ \sum_{i,j=1}^{N} a_{ij}(\frac{x}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}n_i(\frac{x}{\varepsilon}) = 0 \text{ on } \partial T^\varepsilon \\ \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a_{ij} \in L^\infty(\mathbb{R}_y^N)$ ($1 \leq i, j \leq N$), with the symmetry condition $a_{ji} = a_{ij}$, the Y-periodicity hypothesis: for every $k \in \mathbb{Z}^N$ one has $a_{ij}(y+k) = a_{ij}(y)$ almost everywhere in $y \in \mathbb{R}_y^N$, and finally the (uniform) ellipticity condition: there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(y)\xi_j\xi_i \geq \alpha|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^N$ and for almost all $y \in \mathbb{R}_y^N$, where $|\xi|^2 = |\xi_1|^2 + \cdots + |\xi_N|^2$. The density function $\rho \in L^\infty(\mathbb{R}_y^N)$ is Y-periodic and changes sign on Y^*, that is, both the set $\{y \in Y^*, \rho(y) < 0\}$ and $\{y \in Y^*, \rho(y) > 0\}$ are of positive Lebesgue measure. This hypothesis makes the problem under consideration nonstandard. As stated above, it is well known (see [15, 21]) that under the preceding hypotheses, for each $\varepsilon > 0$ the spectrum of (1.1) is discrete and consists of two infinite sequences

$$0 < \lambda_\varepsilon^{1,+} \leq \lambda_\varepsilon^{2,+} \leq \cdots \leq \lambda_\varepsilon^{n,+} \leq \ldots, \quad \lim_{n \to +\infty} \lambda_\varepsilon^{n,+} = +\infty$$

and

$$0 > \lambda_\varepsilon^{1,-} \geq \lambda_\varepsilon^{2,-} \geq \cdots \geq \lambda_\varepsilon^{n,-} \geq \ldots, \quad \lim_{n \to +\infty} \lambda_\varepsilon^{n,-} = -\infty.$$

The asymptotic behavior of the eigencouples depends crucially on whether the average of ρ over Y^*, $M_{Y^*}(\rho) = \int_{Y^*} \rho(y)dy$, is positive, negative or equal to zero. All three cases are carefully investigated in this paper.

The homogenization of spectral problems has been widely explored. In a fixed domain, homogenization of spectral problems with point-wise positive density function goes back to Kesavan [17, 18]. In perforated domains, spectral asymptotics was first considered by Rauch and Taylor[27, 28] but the first homogenization result in that direction pertains to Vanninathan[30]. Since then a lot has been written on spectral asymptotics in perforated media, we mention the works [16, 26, 29] and the references therein to cite a few. Homogenization of elliptic operators with sing-changing density function in a fixed domain has been investigated by Nazarov et al. [20, 21, 22] via a combination of formal asymptotic expansion and Tartar's energy method. Recently, the Two-scale convergence method has been utilized to handle the homogenization process for some eigenvalue problems ([8, 9]) with constant density function.

In this paper we investigate in periodically perforated domains the spectral asymptotics of periodic elliptic linear differential operators of order two in divergence form with a sing-changing density function. We obtain accurate and concise homogenization results in all three cases: $M_{Y^*}(\rho) > 0$ (Theorem 3.1 and Theorem 3.3), $M_{Y^*}(\rho) = 0$ (Theorem 3.5) and $M_{Y^*}(\rho) < 0$ (Theorem 3.1 and Theorem 3.3), by using the two-scale convergence method[1, 19, 23, 31] introduced by Nguetseng[23] and further developed by Allaire[1]. Namely, if $M_{Y^*}(\rho) > 0$ then the positive eigencouples behave like in the case of point-wise positive density function, i.e., for $k \geq 1$, $\lambda_\varepsilon^{k,+}$ converges as $\varepsilon \to 0$ to the kth eigenvalue of the limit spectral problem on Ω, corresponding extended eigenfunctions converge along subsequences. As regards the "negative" eigencouples, $\lambda_\varepsilon^{k,-}$ converges to $-\infty$ at the rate $\frac{1}{\varepsilon^2}$ and the corresponding eigenfunctions oscillate rapidly. We use a factorization technique ([22, 30]) to prove convergence of $\{\lambda_\varepsilon^{k,-} - \frac{1}{\varepsilon^2}\lambda_1^-\}$ - where $(\lambda_1^-, \theta_1^-)$ is the first negative eigencouple to a local spectral problem - to the kth eigenvalue of a limit spectral problem which is different from that obtained for positive eigenvalues. As regards eigenfunctions, extensions of $\{\frac{u_\varepsilon^{k,-}}{(\theta_1^-)^\varepsilon}\}_{\varepsilon \in E}$ - where $(\theta_1^-)^\varepsilon(x) = \theta_1^-(\frac{x}{\varepsilon})$ - converge along subsequences to the kth eigenfunctions of the limit problem. In the case when $M_{Y^*}(\rho) = 0$, $\lambda_\varepsilon^{k,\pm}$ converges to $\pm\infty$ at the rate $\frac{1}{\varepsilon}$ and the limit spectral problem generates a quadratic operator pencil. We prove that $\varepsilon\lambda_\varepsilon^{k,\pm}$ converges to the $(k, \pm)^{th}$ eigenvalue of the limit operator, extended eigenfunctions converge along subsequences as well. The case when $M_{Y^*}(\rho) < 0$ is equivalent to that when $M_{Y^*}(\rho) > 0$, just replace ρ with $-\rho$. The reader may consider the reiteration procedure in multiscale periodically perforated domains to have some fun.

Unless otherwise specified, vector spaces throughout are considered over $\mathbb{R}$, and scalar functions are assumed to take real values. We will make use of the following notations. Let $F(\mathbb{R}^N)$ be a given function space. We denote by $F_{per}(Y)$ the space of functions in $F_{loc}(\mathbb{R}^N)$ that are Y-periodic, and by $F_{\#}(Y)$ the space of those functions $u \in F_{per}(Y)$ with $\int_Y u(y)dy = 0$. Finally, the letter E denotes throughout a family of strictly positive real numbers $(0 < \varepsilon < 1)$ admitting 0 as accumulation point. The numerical space $\mathbb{R}^N$ and its open sets are provided with the Lebesgue measure denoted by $dx = dx_1...dx_N$. The usual gradient operator will be denoted by D. The rest of the paper is organized as follows. Section 2 deals with some preliminary results while homogenization processes are considered

in Section 3.

2 Preliminaries

We first recall the definition and the main compactness theorems of the two-scale convergence method. Let Ω be an open bounded set in $\mathbb{R}_x^N$ (integer $N \geq 2$) and $Y = (0,1)^N$, the unit cube.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^2(\Omega)$ is said to two-scale converge in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega \times Y)$ if as $E \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\phi(x, \frac{x}{\varepsilon})dx \to \iint_{\Omega \times Y} u_0(x,y)\phi(x,y)dxdy \tag{2.1}$$

for all $\phi \in L^2(\Omega; \mathcal{C}_{per}(Y))$.

Notation. We express this by writing $u_\varepsilon \xrightarrow{2s} u_0$ in $L^2(\Omega)$.

The following compactness theorems (see [1, 23, 25]) are cornerstones of the two-scale convergence method.

Theorem 2.2. *Let $(u_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $L^2(\Omega)$. Then a subsequence E' can be extracted from E such that as $E' \ni \varepsilon \to 0$, the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ two-scale converges in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega \times Y)$.*

Theorem 2.3. *Let $(u_\varepsilon)_{\varepsilon \in E}$ be a bounded sequence in $H^1(\Omega)$. Then a subsequence E' can be extracted from E such that as $E' \ni \varepsilon \to 0$*

$$u_\varepsilon \quad \to \quad u_0 \quad in \ H^1(\Omega)\text{-}weak \tag{2.2}$$

$$u_\varepsilon \quad \to \quad u_0 \quad\quad in \ L^2(\Omega) \tag{2.3}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \quad \xrightarrow{2s} \quad \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \quad in \ L^2(\Omega) \ (1 \leq j \leq N) \tag{2.4}$$

where $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\#(Y))$. Moreover, as $E' \ni \varepsilon \to 0$ we have

$$\int_\Omega \frac{u_\varepsilon(x)}{\varepsilon} \psi(x, \frac{x}{\varepsilon})dx \to \iint_{\Omega \times Y} u_1(x,y)\psi(x,y)dxdy \tag{2.5}$$

for $\psi \in \mathcal{D}(\Omega) \otimes L^2_\#(Y)$.

Proof. The first part (2.2)-(2.4) is classical (see [1, 23]). The second part, (2.5), was proved in [25] in the general framework of deterministic homogenization but as it is of great importance in this paper and for the sake of completeness, we provide its proof in the periodic setting. Let $\psi = (\varphi, \theta) \in \mathcal{D}(\Omega) \times L^2_\#(Y)$. By the mean value zero condition over Y for θ we conclude that there exists a unique solution $\vartheta \in H^1_\#(Y)$ to

$$\begin{cases} \Delta_y \vartheta = \theta \quad in \ Y \\ \vartheta \in H^1_\#(Y). \end{cases}$$

Put $\phi = D_y \vartheta$. We get

$$\int_\Omega \frac{u_\varepsilon(x)}{\varepsilon} \psi(x, \frac{x}{\varepsilon}) dx = \int_\Omega \frac{u_\varepsilon(x)}{\varepsilon} \varphi(x) \theta(\frac{x}{\varepsilon}) dx =$$

$$\int_\Omega u_\varepsilon(x) \varphi(x) \operatorname{div}_x \phi(\frac{x}{\varepsilon}) dx = -\int_\Omega D_x(u_\varepsilon(x)\varphi(x)) \cdot \phi(\frac{x}{\varepsilon}) dx$$

A limite passage $(\varepsilon \to 0)$ using (2.4) yields

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{u_\varepsilon(x)}{\varepsilon} \psi(x, \frac{x}{\varepsilon}) dx = -\iint_{\Omega \times Y} [D_x u_0(x) + D_y u_1(x,y)] \varphi(x) \cdot \phi(y) dy dx$$

$$= -\iint_{\Omega \times Y} D_y u_1(x,y) \varphi(x) \cdot \phi(y) dy dx$$

$$= \iint_{\Omega \times Y} u_1(x,y) \varphi(x) \operatorname{div}_y \phi(y) dy dx$$

$$= \iint_{\Omega \times Y} u_1(x,y) \psi(x,y) dy dx.$$

This completes the proof. $\qquad\square$

We now gather some preliminary results we will need in our homogenization processes. We introduce the characteristic function χ_G of

$$G = \mathbb{R}_y^N \setminus \Theta$$

with

$$\Theta = \bigcup_{k \in \mathbb{Z}^N} (k + T).$$

It follows from the closeness of T that Θ is closed in $\mathbb{R}_y^N$ so that G is an open subset of $\mathbb{R}_y^N$. Next, let $\varepsilon \in E$ be arbitrarily fixed and define

$$V_\varepsilon = \{u \in H^1(\Omega^\varepsilon) : u = 0 \text{ on } \partial\Omega\}.$$

We equip V_ε with the $H^1(\Omega^\varepsilon)$-norm which makes it a Hilbert space. We recall the following classical extension result [7].

Proposition 2.4. *For each $\varepsilon \in E$ there exists an operator P_ε of V_ε into $H_0^1(\Omega)$ with the following properties:*

- *P_ε sends continuously and linearly V_ε into $H_0^1(\Omega)$.*

- *$(P_\varepsilon v)|_{\Omega^\varepsilon} = v$ for all $v \in V_\varepsilon$.*

- *$\|D(P_\varepsilon v)\|_{L^2(\Omega)^N} \le c\|Dv\|_{L^2(\Omega^\varepsilon)^N}$ for all $v \in V_\varepsilon$, where c is a constant independent of ε.*

Now, let $Q^\varepsilon = \Omega \setminus (\varepsilon\Theta)$. This is an open set in $\mathbb{R}^N$ and $\Omega^\varepsilon \setminus Q^\varepsilon$ is the intersection of Ω with the collection of the holes crossing the boundary $\partial\Omega$. We have the following result which implies that the holes crossing the boundary $\partial\Omega$ are of no effects as regards the homogenization processes since they are in arbitrary narrow stripe along the boundary.

Lemma 2.5. *[24] Let $K \subset \Omega$ be a compact set independent of ε. There is some $\varepsilon_0 > 0$ such that $\Omega^\varepsilon \setminus Q^\varepsilon \subset \Omega \setminus K$ for any $0 < \varepsilon \leq \varepsilon_0$.*

Next, we introduce the space

$$\mathbb{F}_0^1 = H_0^1(\Omega) \times L^2\left(\Omega; H_\#^1(Y)\right).$$

Endowed with the following norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^1} = \|D_x v_0 + D_y v_1\|_{L^2(\Omega \times Y)} \quad (\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1),$$

$\mathbb{F}_0^1$ is an Hilbert space admitting $F_0^\infty = \mathcal{D}(\Omega) \times [\mathcal{D}(\Omega) \otimes C_\#^\infty(Y)]$ as a dense subspace. This being so, for $(\mathbf{u}, \mathbf{v}) \in \mathbb{F}_0^1 \times \mathbb{F}_0^1$, let

$$a_\Omega(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^N \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) \left(\frac{\partial v_0}{\partial x_i} + \frac{\partial v_1}{\partial y_i}\right) dxdy.$$

This define a symmetric, continuous bilinear form on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$. We will need the following results whose proof can be found in [9].

Lemma 2.6. *Fix $\Phi = (\psi_0, \psi_1) \in F_0^\infty$ and define $\Phi_\varepsilon : \Omega \to \mathbb{R}$ ($\varepsilon > 0$) by*

$$\Phi_\varepsilon(x) = \psi_0(x) + \varepsilon \psi_1(x, \frac{x}{\varepsilon}) \quad (x \in \Omega).$$

If $(u_\varepsilon)_{\varepsilon \in E} \subset H_0^1(\Omega)$ is such that

$$\frac{\partial u_\varepsilon}{\partial x_i} \xrightarrow{2s} \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \quad in \quad L^2(\Omega) \ (1 \leq i \leq N)$$

as $E \ni \varepsilon \to 0$ for some $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$, then

$$a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) \to a_\Omega(\mathbf{u}, \Phi)$$

as $E \ni \varepsilon \to 0$, where

$$a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) = \sum_{i,j=1}^N \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial \Phi_\varepsilon}{\partial x_i} dx.$$

We now construct and point out the main properties of the so-called homogenized coefficients. Let $0 \leq j \leq N$ and put

$$a(u, v) = \sum_{i,j=1}^N \int_{Y^*} a_{ij}(y) \frac{\partial u}{\partial y_j} \frac{\partial v}{\partial y_i} dy,$$

$$l_j(v) = \sum_{k=1}^N \int_{Y^*} a_{kj}(y) \frac{\partial v}{\partial y_k} dy$$

and

$$l_0(v) = \int_{Y^*} \rho(y) v(y) dy$$

for $u, v \in H^1_\#(Y)$. Equipped with the seminorm

$$N(u) = \|D_y u\|_{L^2(Y^*)^N} \quad (u \in H^1_\#(Y)), \tag{2.6}$$

$H^1_\#(Y)$ is a pre-Hilbert space that is nonseparate and noncomplete. Let $H^1_\#(Y^*)$ be its separated completion with respect to the seminorm $N(\cdot)$ and $\mathbf{i}$ the canonical mapping of $H^1_\#(Y)$ into $H^1_\#(Y^*)$. we recall that

(i) $H^1_\#(Y^*)$ is a Hilbert space,

(ii) $\mathbf{i}$ is linear,

(iii) $\mathbf{i}(H^1_\#(Y))$ is dense in $H^1_\#(Y^*)$,

(iv) $\|\mathbf{i}(u)\|_{H^1_\#(Y^*)} = N(u)$ for every u in $H^1_\#(Y)$,

(v) If F is a Banach space and l a continuous linear mapping of $H^1_\#(Y)$ into F, then there exists a unique continuous linear mapping $L : H^1_\#(Y^*) \to F$ such that $l = L \circ \mathbf{i}$.

Proposition 2.7. *Let* $1 \le j \le N$. *The noncoercive local variational problems*

$$u \in H^1_\#(Y) \text{ and } a(u,v) = l_j(v) \text{ for all } v \in H^1_\#(Y) \tag{2.7}$$

and

$$u \in H^1_\#(Y) \text{ and } a(u,v) = l_0(v) \text{ for all } v \in H^1_\#(Y) \tag{2.8}$$

admit each at least one solution. Moreover, if χ^j *and* θ^j *(resp.* χ *and* θ*) are two solutions to (2.7) (resp. (2.8)), then*

$$D_y \chi^j = D_y \theta^j \quad (resp.\ D_y \chi = D_y \theta) \text{ a.e., in } Y^*. \tag{2.9}$$

Proof. We prove the result for (2.7). Proceeding as in the proof of [24, Lemma 2.5] we get a unique symmetric, coercive, continuous bilinear form $A(\cdot, \cdot)$ on $H^1_\#(Y^*) \times H^1_\#(Y^*)$ such that $A(\mathbf{i}(u), \mathbf{i}(v)) = a(u,v)$ for all $u, v \in H^1_\#(Y)$. Based on (v) above, we consider the continuous linear form $\mathbf{l}_j(\cdot)$ on $H^1_\#(Y^*)$ such that $\mathbf{l}_j(\mathbf{i}(u)) = l_j(u)$ for any $u \in H^1_\#(Y)$. Then $\chi^j \in H^1_\#(Y)$ satisfies (2.7) if and only if $\mathbf{i}(\chi^j)$ satisfies

$$\mathbf{i}(\chi^j) \in H^1_\#(Y^*) \text{ and } A(\mathbf{i}(\chi^j), V) = \mathbf{l}_j(V) \text{ for all } V \in H^1_\#(Y^*). \tag{2.10}$$

But $\mathbf{i}(\chi^j)$ is uniquely determined by (2.10). We deduce that (2.7) admits at least one solution and if χ^j and θ^j are two solutions, then $\mathbf{i}(\chi^j) = \mathbf{i}(\theta^j)$, which means χ^j and θ^j have the same neighborhoods in $H^1_\#(Y)$ or equivalently $N(\chi^j - \theta^j) = 0$. Hence (2.9). $\qquad \square$

Corollary 2.8. *Let* $1 \le i, j \le N$ *and* $\chi^j \in H^1_\#(Y)$ *be a solution to (2.7). The following homogenized coefficients*

$$q_{ij} = \int_{Y^*} a_{ij}(y)dy - \sum_{l=1}^{N} \int_{Y^*} a_{il}(y) \frac{\partial \chi^j}{\partial y_l}(y)dy \tag{2.11}$$

are well defined in the sense that they do not depend on the solution to (2.7).

Lemma 2.9. *The following assertions are true:* $q_{ji} = q_{ij}$ $(1 \leq i, j \leq N)$ *and there exists a constant* $\alpha_0 > 0$ *such that*

$$\sum_{i,j=1}^{N} q_{ij}\xi_j\xi_i \geq \alpha_0 |\xi|^2$$

for all $\xi \in \mathbb{R}^N$.

Proof. See e.g., [2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We now say a few words on the existence result for (1.1). The weak formulation of (1.1) reads: Find $(\lambda_\varepsilon, u_\varepsilon) \in \mathbb{C} \times V_\varepsilon$, $(u_\varepsilon \neq 0)$ such that

$$a^\varepsilon(u_\varepsilon, v) = \lambda_\varepsilon(\rho^\varepsilon u_\varepsilon, v)_{\Omega^\varepsilon}, \quad v \in V_\varepsilon, \tag{2.12}$$

where

$$(\rho^\varepsilon u_\varepsilon, v)_{\Omega^\varepsilon} = \int_{\Omega^\varepsilon} \rho^\varepsilon u_\varepsilon v \, dx.$$

Since ρ^ε changes sign, the classical results on the spectrum of semi-bounded self-adjoint operators with compact resolvent do not apply. To handle this, we follow the ideas in [22]. The bilinear form $(\rho^\varepsilon u_\varepsilon, v)_{\Omega^\varepsilon}$ defines a bounded linear operator $K^\varepsilon : V_\varepsilon \to V_\varepsilon$ such that

$$(\rho^\varepsilon u, v)_{\Omega^\varepsilon} = a^\varepsilon(K^\varepsilon u, v) \quad (u, v \in V_\varepsilon).$$

The operator K^ε is symmetric and its domains $D(K^\varepsilon)$ coincides with the whole V_ε, thus it is self-adjoint. Recall that the gradient norm is equivalent to the $H^1(\Omega^\varepsilon)$-norm on V_ε. Looking at $K^\varepsilon u$ as a solution to the boundary value problem

$$\begin{cases} -div(a(\dfrac{x}{\varepsilon})D_x(K^\varepsilon u)) = \rho^\varepsilon u & \text{in } \Omega^\varepsilon \\ a(\dfrac{x}{\varepsilon})D_x K^\varepsilon u \cdot n(\dfrac{x}{\varepsilon}) = 0 & \text{on } \partial T^\varepsilon \\ K^\varepsilon u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

we get a constant $C > 0$ such that $\|K^\varepsilon u\|_{V^\varepsilon} \leq C\|u\|_{L^2(\Omega^\varepsilon)}$. As V^ε is compactly embedded in $L^2(\Omega^\varepsilon)$ (indeed, $H^1(\Omega^\varepsilon) \hookrightarrow L^2(\Omega^\varepsilon)$ is compact as $\partial\Omega^\varepsilon$ is $\mathcal{C}^1$), the operator K^ε is compact. We can rewrite (2.12) as follows

$$K^\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon, \quad \mu_\varepsilon = \frac{1}{\lambda_\varepsilon}.$$

Notice that (see e.g., [5]) in the case $\rho \geq 0$ in Y, the operator K^ε is positive and its spectrum $\sigma(K^\varepsilon)$ lives in $[0, \|K^\varepsilon\|]$ and $\mu_\varepsilon = 0$ belongs to the essential spectrum $\sigma_e(K^\varepsilon)$. The essential spectrum of a self-adjoint operator L is by definition $\sigma_e(L) = \sigma_p^\infty(L) \cup \sigma_c(L)$, where $\sigma_p^\infty(L)$ is the set of eigenvalues of infinite multiplicity and $\sigma_c(L)$ is the continuous spectrum. The spectrum of K^ε is described by the following proposition whose proof is omitted since similar to that of [22, Lemma 1].

Lemma 2.10. *Let $\rho \in L^\infty_{per}(Y)$ be such that the sets $\{y \in Y^* : \rho(y) < 0\}$ and $\{y \in Y^* : \rho(y) > 0\}$ are both of positive Lebesgue measure. Then for any $\varepsilon > 0$, we have $\sigma(K^\varepsilon) \subset [-\|K^\varepsilon\|, \|K^\varepsilon\|]$ and $\mu = 0$ is the only element of the essential spectrum $\sigma_e(K^\varepsilon)$. Moreover, the discrete spectrum of K^ε consists of two infinite sequences*

$$\mu_\varepsilon^{1,+} \geq \mu_\varepsilon^{2,+} \geq \cdots \geq \mu_\varepsilon^{k,+} \geq \cdots \to 0^+,$$
$$\mu_\varepsilon^{1,-} \leq \mu_\varepsilon^{2,-} \leq \cdots \leq \mu_\varepsilon^{k,-} \leq \cdots \to 0^-.$$

Corollary 2.11. *The hypotheses are those of Lemma 2.10. Problem (1.1) has a discrete set of eigenvalues consisting of two sequences*

$$0 < \lambda_\varepsilon^{1,+} \leq \lambda_\varepsilon^{2,+} \leq \cdots \leq \lambda_\varepsilon^{k,+} \leq \cdots \to +\infty,$$
$$0 > \lambda_\varepsilon^{1,+} \geq \lambda_\varepsilon^{2,-} \geq \cdots \geq \lambda_\varepsilon^{k,-} \geq \cdots \to -\infty.$$

We are now in a position to state the main results of this paper.

3 Homogenization results

In this section we state and prove homogenization results for both cases $M_{Y^*}(\rho) > 0$ and $M_{Y^*}(\rho) = 0$. The homogenization results in the case when $M_{Y^*}(\rho) < 0$ can be deduced from the case $M_{Y^*}(\rho) > 0$ by replacing ρ with $-\rho$. We start with the less technical case.

3.1 The case $M_{Y^*}(\rho) > 0$

We start with the homogenization result for the positive part of the spectrum $(\lambda_\varepsilon^{k,+}, u_\varepsilon^{k,+})_{\varepsilon \in E}$.

3.1.1 Positive part of the spectrum

We assume (this is not a restriction) that the corresponding eigenfunctions are orthonormalized as follows

$$\int_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,+} u_\varepsilon^{l,+} dx = \delta_{k,l} \quad k,l = 1,2,\cdots \tag{3.1}$$

The homogenization results states as

Theorem 3.1. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda_\varepsilon^{k,+}, u_\varepsilon^{k,+})$ be the kth positive eigencouple to (1.1) with $M_{Y^*}(\rho) > 0$ and (3.1). Then, there exists a subsequence E' of E such that*

$$\lambda_\varepsilon^{k,+} \quad \to \quad \lambda_0^k \quad \text{in } \mathbb{R} \text{ as } E \ni \varepsilon \to 0 \tag{3.2}$$
$$P_\varepsilon u_\varepsilon^{k,+} \quad \to \quad u_0^k \quad \text{in } H_0^1(\Omega)\text{-weak as } E' \ni \varepsilon \to 0 \tag{3.3}$$
$$P_\varepsilon u_\varepsilon^{k,+} \quad \to \quad u_0^k \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \tag{3.4}$$
$$\frac{\partial P_\varepsilon u_\varepsilon^{k,+}}{\partial x_j} \quad \xrightarrow{2s} \quad \frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j} \text{ in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \ (1 \leq j \leq N) \tag{3.5}$$

where $(\lambda_0^k, u_0^k) \in \mathbb{R} \times H_0^1(\Omega)$ is the kth eigencouple to the spectral problem

$$
\begin{cases}
-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{1}{M_{Y^*}(\rho)} q_{ij} \frac{\partial u_0}{\partial x_j}\right) = \lambda_0 u_0 & in\ \Omega \\
\\
\qquad\qquad\qquad\qquad u_0 = 0 & on\ \partial\Omega \\
\\
\qquad\qquad \int_{\Omega} |u_0|^2 dx = \frac{1}{M_{Y^*}(\rho)},
\end{cases}
\tag{3.6}
$$

and where $u_1^k \in L^2(\Omega; H_\#^1(Y))$. Moreover, for almost every $x \in \Omega$ the following hold true:
(i) $u_1^k(x)$ is a solution to the noncoercive variational problem

$$
\begin{cases}
u_1^k(x) \in H_\#^1(Y) \\
\\
a(u_1^k(x), v) = -\sum_{i,j=1}^{N} \frac{\partial u_0^k}{\partial x_j} \int_{Y^*} a_{ij}(y) \frac{\partial v}{\partial y_i} dy \\
\\
\forall v \in H_\#^1(Y);
\end{cases}
\tag{3.7}
$$

(ii) We have

$$
\mathbf{i}(u_1^k(x)) = -\sum_{j=1}^{N} \frac{\partial u_0^k}{\partial x_j}(x) \mathbf{i}(\chi^j)
\tag{3.8}
$$

where χ^j is any function in $H_\#^1(Y)$ defined by the cell problem (2.7).

Proof. We present only the outlines since this proof is similar but less technical to that of the case $M_{Y^*}(\rho) = 0$.

Fix $k \geq 1$. By means of the minimax principle, as in [30], one easily proves the existence of a constant C independent of ε such that $\lambda_\varepsilon^{k,+} < C$. Clearly, for fixed $E \ni \varepsilon > 0$, $u_\varepsilon^{k,+}$ lies in V_ε, and

$$
\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon^{k,+}}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \lambda_\varepsilon^{k,+} \int_{\Omega^\varepsilon} \rho\left(\frac{x}{\varepsilon}\right) u_\varepsilon^{k,+} v\, dx
\tag{3.9}
$$

for any $v \in V_\varepsilon$. Bear in mind that $\int_{\Omega^\varepsilon} \rho\left(\frac{x}{\varepsilon}\right)(u_\varepsilon^{k,+})^2 dx = 1$ and choose $v = u_\varepsilon^k$ in (3.9). The boundedness of the sequence $(\lambda_\varepsilon^{k,+})_{\varepsilon \in E}$ and the ellipticity assumption (1.2) imply at once by means of Proposition 2.4 that the sequence $(P_\varepsilon u_\varepsilon^{k,+})_{\varepsilon \in E}$ is bounded in $H_0^1(\Omega)$. Theorem 2.3 applies and gives us $\mathbf{u}^k = (u_0^k, u_1^k) \in \mathbb{F}_0^1$ such that for some $\lambda_0^k \in \mathbb{R}$ and some subsequence $E' \subset E$ we have (3.2)-(3.5), where (3.4) is a direct consequence of (3.3) by the Rellich-Kondrachov theorem. For fixed $\varepsilon \in E'$, let Φ_ε be as in Lemma 2.6. Multiplying both sides of the first equality in (1.1) by Φ_ε and integrating over Ω leads us to the variational ε-problem

$$
\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial P_\varepsilon u_\varepsilon^{k,+}}{\partial x_j} \frac{\partial \Phi_\varepsilon}{\partial x_i} dx = \lambda_\varepsilon^{k,+} \int_{\Omega^\varepsilon} (P_\varepsilon u_\varepsilon^{k,+}) \rho\left(\frac{x}{\varepsilon}\right) \Phi_\varepsilon dx.
\tag{3.10}
$$

Sending $\varepsilon \in E'$ to 0, keeping (3.2)-(3.5) and Lemma 2.6 in mind, we obtain

$$
\sum_{i,j=1}^{N} \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j}\right) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\partial \psi_1}{\partial y_i}\right) dx dy = \lambda_0^k \iint_{\Omega \times Y^*} u_0^k \psi_0(x) \rho(y) dx dy.
$$

Therefore, $(\lambda_0^k, \mathbf{u}^k) \in \mathbb{R} \times \mathbb{F}_0^1$ solves the following *global homogenized spectral problem*:

$$\begin{cases} \text{Find } (\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{F}_0^1 \text{ such that} \\ \sum_{i,j=1}^N \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \right) \left(\frac{\partial \psi_0}{\partial x_i} + \frac{\partial \psi_1}{\partial y_i} \right) dx dy = \lambda M_{Y^*}(\rho) \int_\Omega u_0 \psi_0 \, dx \quad (3.11) \\ \text{for all } \Phi \in \mathbb{F}_0^1, \end{cases}$$

which leads to the macroscopic and microscopic problems (3.6)-(3.7) without any major difficulty.

As regards the normalization condition in (3.6), we use the decomposition $\Omega^\varepsilon = Q^\varepsilon \cup (\Omega^\varepsilon \setminus Q^\varepsilon)$ and the equality $Q^\varepsilon = \Omega \cap \varepsilon G$. On the one hand, when $E' \ni \varepsilon \to 0$,

$$\int_{Q^\varepsilon} \rho(\frac{x}{\varepsilon})(P_\varepsilon u_\varepsilon^{k,+})(P_\varepsilon u_\varepsilon^{l,+}) dx \to M_{Y^*}(\rho) \int_\Omega u_0^k u_0^l dx, \quad k,l = 1,2,\cdots$$

since

$$\int_{Q^\varepsilon} \rho(\frac{x}{\varepsilon})(P_\varepsilon u_\varepsilon^{k,+})(P_\varepsilon u_\varepsilon^{l,+}) \, dx = \int_\Omega \chi_G(\frac{x}{\varepsilon})\rho(\frac{x}{\varepsilon})(P_\varepsilon u_\varepsilon^{k,+})(P_\varepsilon u_\varepsilon^{l,+}) \, dx$$

and $(P_\varepsilon u_\varepsilon^{k,+})\chi_G^\varepsilon \rho^\varepsilon \rightharpoonup M_{Y^*}(\rho) u_0^k$ in $L^2(\Omega)$-weak and $P_\varepsilon u_\varepsilon^{l,+} \to u_0^l$ in $L^2(\Omega)$-strong as $E' \ni \varepsilon \to 0$. On the other hand, the same line of reasoning as in the proof of [9, Proposition 3.6] leads to

$$\lim_{E' \ni \varepsilon \to 0} \int_{\Omega^\varepsilon \setminus Q^\varepsilon} \rho(\frac{x}{\varepsilon})(P_\varepsilon u_\varepsilon^{k,+})(P_\varepsilon u_\varepsilon^{l,+}) \, dx = 0 \qquad (3.12)$$

The normalization condition in (3.6) follows thereby. In fact, we have just proved that $\{u_0^{k,+}\}_{k=1}^\infty$ is an orthogonal basis in $L^2(\Omega)$.   □

Remark 3.2.   • The eigenfunctions $\{u_0^k\}_{k=1}^\infty$ are orthonormalized by

$$\int_\Omega u_0^k u_0^l dx = \frac{\delta_{k,l}}{M_{Y^*}(\rho)} \quad k,l = 1,2,3,\cdots$$

• If λ_0^k is simple (this is the case for λ_0^1), then by Theorem 3.1, $\lambda_\varepsilon^{k,+}$ is also simple, for small ε, and we can choose the eigenfunctions $u_\varepsilon^{k,+}$ such that the convergence results (3.3)-(3.5) hold for the whole sequence E.

• Replacing ρ with $-\rho$ in (1.1), Theorem 3.1 also applies to the negative part of the spectrum in the case $M_{Y^*}(\rho) < 0$.

3.1.2   Negative part of the spectrum

We now investigate the negative part of the spectrum $(\lambda_\varepsilon^{k,-}, u_\varepsilon^{k,-})_{\varepsilon \in E}$. Before we can do this we need a few preliminaries and stronger regularity hypotheses on T, ρ and the coefficients $(a_{ij})_{i,j=1}^N$. We assume in this subsection that ∂T is $C^{2,\delta}$ and ρ and the coefficients $(a_{ij})_{i,j=1}^N$ are δ-Hölder continuous $(0 < \delta < 1)$.

Let $H^1_{per}(Y^*)$ denotes the space of functions in $H^1(Y^*)$ assuming same values on the opposite faces of Y. The following spectral problem is well posed

$$
\begin{cases}
\text{Find } (\lambda, \theta) \in \mathbb{C} \times H^1_{per}(Y^*) \\
-\sum_{i,j=1}^{N} \frac{\partial}{\partial y_j}\left(a_{ij}(y)\frac{\partial \theta}{\partial y_i}\right) = \lambda \rho(y)\theta \text{ in } Y^* \\
\sum_{i,j=1}^{N} a_{ij}(y)\frac{\partial \theta}{\partial y_j}n_i = 0 \text{ on } \partial T
\end{cases}
\tag{3.13}
$$

and possesses a spectrum with similar properties to that of (1.1), two infinite (positive and negative) sequences. We recall that (3.13) admits a unique nontrivial eigenvalue having an eigenfunction with definite sign, the first negative one, since we have $M_{Y^*}(\rho) > 0$ (see e.g., [6, 14]). In the sequel we will only make use of $(\lambda_1^-, \theta_1^-)$, the first negative eigencouple to (3.13). After proper sign choice we assume that

$$
\theta_1^-(y) > 0 \quad \text{in} \quad y \in Y^*.
\tag{3.14}
$$

We also recall that θ_1^- is δ-Hölder continuous(see e.g., [11]), hence can be extended to a Y-periodic function living in $L^\infty(\mathbb{R}^N_y)$ still denoted by θ_1^-. Notice that we have

$$
\int_{Y^*} \rho(y)(\theta_1^-(y))^2 dy < 0,
\tag{3.15}
$$

as is easily seen from the variational equality (keep the ellipticity hypothesis (1.2) in mind)

$$
\sum_{i,j=1}^{N} \int_{Y^*} a_{ij}\frac{\partial \theta_1^-}{\partial y_j}\frac{\partial \theta_1^-}{\partial y_i}dy = \lambda_1^- \int_{Y^*} \rho(y)(\theta_1^-(y))^2 dy.
$$

Bear in mind that problem (3.13) induces by a scaling argument the following equalities:

$$
\begin{cases}
-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(\frac{x}{\varepsilon})\frac{\partial \theta^\varepsilon}{\partial x_i}\right) = \frac{1}{\varepsilon^2}\lambda \rho(\frac{x}{\varepsilon})\theta(\frac{x}{\varepsilon}) \text{ in } Q^\varepsilon \\
\sum_{i,j=1}^{N} a_{ij}(\frac{x}{\varepsilon})\frac{\partial \theta^\varepsilon}{\partial x_j}n_i(\frac{x}{\varepsilon}) = 0 \text{ on } \partial Q^\varepsilon,
\end{cases}
\tag{3.16}
$$

where $\theta^\varepsilon(x) = \theta(\frac{x}{\varepsilon})$. However, θ^ε is not zero on $\partial\Omega$. We now introduce the following spectral problem (with an indefinite density function)

$$
\begin{cases}
\text{Find } (\xi_\varepsilon, v_\varepsilon) \in \mathbb{C} \times V_\varepsilon \\
-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(\widetilde{a}_{ij}(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon}{\partial x_i}\right) = \xi_\varepsilon \widetilde{\rho}(\frac{x}{\varepsilon})v_\varepsilon(x) \text{ in } \Omega^\varepsilon \\
\sum_{i,j=1}^{N} \widetilde{a}_{ij}(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon(x)}{\partial x_j}n_i(\frac{x}{\varepsilon}) = 0 \text{ on } \partial T^\varepsilon \\
\qquad\qquad\qquad v_\varepsilon(x) = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{3.17}
$$

with new spectral eigencouple $(\xi_\varepsilon, v_\varepsilon) \in \mathbb{C} \times V_\varepsilon$, where $\widetilde{a}_{ij}(y) = (\theta_1^-)^2(y) a_{ij}(y)$ and $\widetilde{\rho}(y) = (\theta_1^-)^2(y)\rho(y)$. Notice that $\widetilde{a}_{ij} \in L^\infty_{per}(Y)$ and $\widetilde{\rho} \in L^\infty_{per}(Y)$. As $0 < c_- \leq \theta_1^-(y) \leq c^+ < +\infty$ $(c_-, c^+ \in \mathbb{R})$, the operator on the left hand side of (3.17) is uniformly elliptic and Theorem 3.1 applies to the negative part of the spectrum of (3.17) (see (3.15) and Remark 3.2). The effective spectral problem for (3.17) reads

$$
\begin{cases}
-\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(\widetilde{q}_{ij}\frac{\partial v_0}{\partial x_i}\right) = \xi_0 M_{Y^*}(\widetilde{\rho}) v_0 & \text{in } \Omega \\[2ex]
v_0 = 0 \quad \text{on } \partial\Omega \\[2ex]
\displaystyle\int_\Omega |v_0|^2 dx = \frac{-1}{M_{Y^*}(\widetilde{\rho})}.
\end{cases}
\tag{3.18}
$$

The effective coefficients $\{\widetilde{q}_{ij}\}_{1\leq i,j\leq N}$ being defined as expected, i.e.,

$$
\widetilde{q}_{ij} = \int_{Y^*} \widetilde{a}_{ij}(y)dy - \sum_{l=1}^{N}\int_{Y^*} \widetilde{a}_{il}(y)\frac{\partial \widetilde{\chi}_1^j}{\partial y_l}(y)dy,
\tag{3.19}
$$

with $\widetilde{\chi}_1^l \in H_\#^1(Y^*)$ $(l = 1,...,N)$ being a solution to the following local problem

$$
\begin{cases}
\widetilde{\chi}_1^l \in H_\#^1(Y^*) \\[1ex]
\displaystyle\sum_{i,j=1}^{N}\int_{Y^*} \widetilde{a}_{ij}(y)\frac{\partial \widetilde{\chi}_1^l}{\partial y_j}\frac{\partial v}{\partial y_i}dy = \sum_{i=1}^{N}\int_{Y^*} \widetilde{a}_{il}(y)\frac{\partial v}{\partial y_i}dy \\[1ex]
\text{for all } v \in H_\#^1(Y^*).
\end{cases}
\tag{3.20}
$$

Notice that the spectrum of (3.18) is as follows

$$
0 > \xi_0^1 > \xi_0^2 \geq \xi_0^3 \geq \cdots \geq \xi_0^j \geq \cdots \to -\infty \text{ as } j \to \infty.
$$

Making use of (3.16), the same line of reasoning as in [30, Lemma 6.1] shows that the negative spectral parameters of problems (1.1) and (3.17) verify:

$$
u_\varepsilon^{k,-} = (\theta_1^-)^\varepsilon v_\varepsilon^{k,-} \quad (\varepsilon \in E,\ k = 1,2\cdots)
$$

and

$$
\lambda_\varepsilon^{k,-} = \frac{1}{\varepsilon^2}\lambda_1^- + \xi_\varepsilon^{k,-} + o(1), \quad (\varepsilon \in E,\ k = 1,2\cdots).
$$

The presence of the term $o(1)$ is due to integrals over $\Omega^\varepsilon \setminus Q^\varepsilon$, like the one in (3.12), which converge to zero with ε, remember that (3.16) holds in Q^ε but not Ω^ε. As will be seen below, the sequence $(\xi_\varepsilon^{k,-})_{\varepsilon \in E}$ is bounded in $\mathbb{R}$. In another words, $\lambda_\varepsilon^{k,-}$ is of order $1/\varepsilon^2$ and tends to $-\infty$ as ε goes to zero. It is now clear why the limiting behavior of negative eigencouples is not straightforward as that of positive ones.

The suitable orthonormalization condition for (3.17) is the one the reader is expecting:

$$
\int_{\Omega^\varepsilon} \widetilde{\rho}(\frac{x}{\varepsilon}) v_\varepsilon^{k,-} v_\varepsilon^{l,-} dx = -\delta_{k,l} \quad k,l = 1,2,\cdots
\tag{3.21}
$$

We now state the homogenization theorem for the negative part of the spectrum of (1.1).

Theorem 3.3. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda_\varepsilon^{k,-}, u_\varepsilon^{k,-})$ be the kth negative eigencouple to (1.1) with $M_{Y^*}(\rho) > 0$ and (3.21). Then, there exists a subsequence E' of E such that*

$$\lambda_\varepsilon^{k,-} - \frac{\lambda_1^-}{\varepsilon^2} \quad \rightarrow \quad \xi_0^k \quad \text{in } \mathbb{R} \text{ as } E \ni \varepsilon \to 0 \tag{3.22}$$

$$P_\varepsilon v_\varepsilon^{k,-} \quad \rightarrow \quad v_0^k \quad \text{in } H_0^1(\Omega)\text{-weak as } E' \ni \varepsilon \to 0 \tag{3.23}$$

$$P_\varepsilon v_\varepsilon^{k,-} \quad \rightarrow \quad v_0^k \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \tag{3.24}$$

$$\frac{\partial P_\varepsilon v_\varepsilon^{k,-}}{\partial x_j} \quad \xrightarrow{2s} \quad \frac{\partial v_0^k}{\partial x_j} + \frac{\partial v_1^k}{\partial y_j} \text{ in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \ (1 \leq j \leq N) \tag{3.25}$$

where $(\xi_0^k, v_0^k) \in \mathbb{R} \times H_0^1(\Omega)$ is the kth eigencouple to the spectral problem

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(\frac{1}{M_{Y^*}(\widetilde{\rho})}\widetilde{q}_{ij}\frac{\partial v_0}{\partial x_j}\right) = \xi_0 v_0 \quad \text{in } \Omega \\[2mm] \qquad\qquad\qquad\qquad v_0 = 0 \quad \text{on } \partial\Omega \\[2mm] \qquad\qquad\qquad \int_\Omega |v_0|^2 dx = \frac{-1}{M_{Y^*}(\widetilde{\rho})}, \end{cases} \tag{3.26}$$

and where $v_1^k \in L^2(\Omega; H_\#^1(Y))$. Moreover, for almost every $x \in \Omega$ the following hold true:
(i) $v_1^k(x)$ is a solution to the noncoercive variational problem

$$\begin{cases} v_1^k(x) \in H_\#^1(Y) \\[2mm] \widetilde{a}(v_1^k(x), u) = -\sum_{i,j=1}^N \frac{\partial v_0^k}{\partial x_j}\int_{Y^*}\widetilde{a}_{ij}(y)\frac{\partial u}{\partial y_i}dy \\[2mm] \forall u \in H_\#^1(Y); \end{cases} \tag{3.27}$$

(ii) We have

$$\mathbf{i}(v_1^k(x)) = -\sum_{j=1}^N \frac{\partial v_0^k}{\partial x_j}(x)\mathbf{i}(\widetilde{\chi}^j) \tag{3.28}$$

where $\widetilde{\chi}^j$ is any function in $H_\#^1(Y)$ defined by the cell problem (3.20).

Remark 3.4.　　• The eigenfunctions $\{v_0^k\}_{k=1}^\infty$ are orthonormalized by

$$\int_\Omega v_0^k v_0^l dx = \frac{-\delta_{k,l}}{M_{Y^*}(\widetilde{\rho})} \quad k,l = 1,2,3,\cdots$$

• Replacing ρ with $-\rho$ in (1.1), Theorem 3.3 adapts to the positive part of the spectrum in the case $M_{Y^*}(\rho) < 0$.

3.2　The case $M_{Y^*}(\rho) = 0$

We prove a homogenization result for both the positive part and the negative part of the spectrum simultaneously. As will be clear in the proof of Theorem 3.5 below, we assume in this case that the eigenfunctions are orthonormalized as follows

$$\int_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon})u_\varepsilon^{k,\pm} u_\varepsilon^{l,\pm} dx = \pm\varepsilon\delta_{k,l} \quad k,l = 1,2,\cdots \tag{3.29}$$

Let χ^0 be a solution to (2.8) and put

$$v^2 = \sum_{i,j=1}^{N} \int_{Y^*} a_{ij}(y)\frac{\partial \chi^0}{\partial y_j}\frac{\partial \chi^0}{\partial y_i}dy. \tag{3.30}$$

Indeed, the right hand side of (3.30) is positive and does not depend on a particular solution to (2.8). We now recall that the following spectral problem for a quadratic operator pencil with respect to v,

$$\begin{cases} -\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(q_{ij}\frac{\partial u_0}{\partial x_i}\right) = \lambda_0^2 v^2 u_0 \text{ in } \Omega \\ \\ u_0 = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.31}$$

has a spectrum consisting of two infinite sequences

$$0 < \lambda_0^{1,+} < \lambda_0^{2,+} \leq \cdots \leq \lambda_0^{k,+} \leq \ldots, \quad \lim_{n \to +\infty} \lambda_0^{k,+} = +\infty$$

and

$$0 > \lambda_0^{1,-} > \lambda_0^{2,-} \geq \cdots \geq \lambda_0^{k,-} \geq \ldots, \quad \lim_{n \to +\infty} \lambda_0^{k,-} = -\infty.$$

with $\lambda_0^{k,+} = -\lambda_0^{k,-}$ $k = 1,2,\cdots$ and with the corresponding eigenfunctions $u_0^{k,+} = u_0^{k,-}$. We note by passing that $\lambda_0^{1,+}$ and $\lambda_0^{1,-}$ are simple. We are now in a position to state the homogenization result in the present case.

Theorem 3.5. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda_\varepsilon^{k,\pm}, u_\varepsilon^{k,\pm})$ be the $(k,\pm)^{th}$ eigencouple to (1.1) with $M_{Y^*}(\rho) = 0$ and (3.29). Then, there exists a subsequence E' of E such that*

$$\begin{align} \varepsilon \lambda_\varepsilon^{k,\pm} &\to \lambda_0^{k,\pm} \quad \text{in } \mathbb{R} \text{ as } E \ni \varepsilon \to 0 \tag{3.32}\\ P_\varepsilon u_\varepsilon^{k,\pm} &\to u_0^{k,\pm} \quad \text{in } H_0^1(\Omega)\text{-weak as } E' \ni \varepsilon \to 0 \tag{3.33}\\ P_\varepsilon u_\varepsilon^{k,\pm} &\to u_0^{k,\pm} \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \tag{3.34}\\ \frac{\partial P_\varepsilon u_\varepsilon^{k,\pm}}{\partial x_j} &\xrightarrow{2s} \frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \text{ in } L^2(\Omega) \text{ as } E' \ni \varepsilon \to 0 \,(1 \leq j \leq N) \tag{3.35} \end{align}$$

where $(\lambda_0^{k,\pm}, u_0^{k,\pm}) \in \mathbb{R} \times H_0^1(\Omega)$ is the $(k,\pm)^{th}$ eigencouple to the following spectral problem for a quadratic operator pencil with respect to v,

$$\begin{cases} -\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(q_{ij}\frac{\partial u_0}{\partial x_j}\right) = \lambda_0^2 v^2 u_0 \text{ in } \Omega \\ \\ u_0 = 0 \text{ on } \partial\Omega \end{cases} \tag{3.36}$$

and where $u_1^{k,\pm} \in L^2(\Omega; H_\#^1(Y))$. We have the following normalization condition

$$\int_\Omega |u_0^{k,\pm}|^2\,dx = \frac{\pm 1}{\lambda_0^{k,\pm} v^2} \qquad k = 1,2,\cdots \tag{3.37}$$

Moreover, for almost every $x \in \Omega$ the following hold true:
(i) $u_1^{k,\pm}(x)$ is a solution to the noncoercive variational problem

$$
\begin{cases}
u_1^{k,\pm}(x) \in H_\#^1(Y) \\
a(u_1^{k,\pm}(x),v) = \lambda_0^{k,\pm} u_0^k(x) \int_{Y^*} \rho(y) v(y) dy - \sum_{i,j=1}^N \frac{\partial u_0^{k,+}}{\partial x_j}(x) \int_{Y^*} a_{ij}(y) \frac{\partial v}{\partial y_i} dy \qquad (3.38) \\
\forall v \in H_\#^1(Y);
\end{cases}
$$

(ii) We have

$$
\mathbf{i}(u_1^{k,\pm}(x)) = \lambda_0^{k,\pm} u_0^k(x) \mathbf{i}(\chi^0) - \sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) \mathbf{i}(\chi^j) \qquad (3.39)
$$

where χ^j $(1 \le j \le N)$ and χ^0 are functions in $H_\#^1(Y)$ defined by the cell problems (2.7) and (2.8), respectively.

Proof. Fix $k \ge 1$, using the minimax principle, as in [30], we get a constant C independent of ε such that $|\varepsilon \lambda_\varepsilon^{k,\pm}| < C$. We have $u_\varepsilon^{k,\pm} \in V_\varepsilon$ and

$$
\sum_{i,j=1}^N \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u_\varepsilon^{k,\pm}}{\partial x_j} \frac{\partial v}{\partial x_i} dx = (\varepsilon \lambda_\varepsilon^{k,\pm}) \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon}) u_\varepsilon^{k,\pm} v dx \qquad (3.40)
$$

for any $v \in V_\varepsilon$. Bear in mind that $\int_{\Omega^\varepsilon} \rho(\frac{x}{\varepsilon})(u_\varepsilon^{k,\pm})^2 dx = \pm \varepsilon$ and choose $v = u_\varepsilon^k$ in (3.40). The boundedness of the sequence $(\varepsilon \lambda_\varepsilon^{k,\pm})_{\varepsilon \in E}$ and the ellipticity assumption (1.2) imply at once by means of Proposition 2.4 that the sequence $(P_\varepsilon u_\varepsilon^{k,\pm})_{\varepsilon \in E}$ is bounded in $H_0^1(\Omega)$. Theorem 2.3 applies and gives us $\mathbf{u}^{k,\pm} = (u_0^{k,\pm}, u_1^{k,\pm}) \in \mathbb{F}_0^1$ such that for some $\lambda_0^{k,\pm} \in \mathbb{R}$ and some subsequence $E' \subset E$ we have (3.32)-(3.35), where (3.34) is a direct consequence of (3.33) by the Rellich-Kondrachov theorem. For fixed $\varepsilon \in E'$, let Φ_ε be as in Lemma 2.6. Multiplying both sides of the first equality in (1.1) by Φ_ε and integrating over Ω leads us to the variational ε-problem

$$
\sum_{i,j=1}^N \int_{\Omega^\varepsilon} a_{ij}(\frac{x}{\varepsilon}) \frac{\partial P_\varepsilon u_\varepsilon^{k,\pm}}{\partial x_j} \frac{\partial \Phi_\varepsilon}{\partial x_i} dx = (\varepsilon \lambda_\varepsilon^{k,\pm}) \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \rho(\frac{x}{\varepsilon}) \Phi_\varepsilon dx.
$$

Sending $\varepsilon \in E'$ to 0, keeping (3.32)-(3.35) and Lemma 2.6 in mind, we obtain

$$
a_\Omega(\mathbf{u}^{k,\pm}, \Phi) = \lambda_0^{k,\pm} \iint_{\Omega \times Y^*} \left(u_1^{k,\pm}(x,y) \psi_0(x) \rho(y) + u_0^{k,\pm} \psi_1(x,y) \rho(y) \right) dx dy \qquad (3.41)
$$

The right-hand side follows as explained below. Using the decomposition $\Omega^\varepsilon = Q^\varepsilon \cup (\Omega^\varepsilon \setminus Q^\varepsilon)$ and the equality $Q^\varepsilon = \Omega \cap \varepsilon G$ we arrive at

$$
\begin{aligned}
\frac{1}{\varepsilon} \int_{\Omega^\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \rho(\frac{x}{\varepsilon}) \Phi_\varepsilon dx &= \frac{1}{\varepsilon} \int_\Omega (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_0(x) \rho(\frac{x}{\varepsilon}) \chi_G(\frac{x}{\varepsilon}) dx \\
&+ \int_\Omega (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_1(x,\frac{x}{\varepsilon}) \rho(\frac{x}{\varepsilon}) \chi_G(\frac{x}{\varepsilon}) dx + o(1).
\end{aligned}
$$

On the one hand we have

$$\lim_{E' \ni \varepsilon \to 0} \int_\Omega (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_1(x, \tfrac{x}{\varepsilon}) \rho(\tfrac{x}{\varepsilon}) \chi_G(\tfrac{x}{\varepsilon}) \, dx = \iint_{\Omega \times Y} u_0^{k,\pm} \psi_1(x, y) \rho(y) \chi_G(y) \, dx dy.$$

On the other hand, owing to (2.5) of Theorem 2.3, the following holds:

$$\lim_{E' \ni \varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega (P_\varepsilon u_\varepsilon^{k,\pm}) \psi_0(x) \rho(\tfrac{x}{\varepsilon}) \chi_G(\tfrac{x}{\varepsilon}) \, dx = \iint_{\Omega \times Y} u_1^{k,\pm}(x, y) \psi_0(x) \rho(y) \chi_G(y) \, dx dy.$$

Indeed $\rho \chi_G \in L^2_\#(Y)$ as we clearly have $\int_Y \rho(y) \chi_G(y) dy = \int_{Y^*} \rho(y) dy = 0$. We have just proved that $(\lambda_0^{k,+}, \mathbf{u}^{k,\pm}) \in \mathbb{R} \times \mathbb{F}_0^1$ solves the following *global homogenized spectral problem*:

$$\begin{cases} \text{Find } (\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{F}_0^1 \text{ such that} \\ a_\Omega(\mathbf{u}, \Phi) = \lambda \iint_{\Omega \times Y^*} (u_1(x, y) \psi_0(x) \rho(y) + u_0 \psi_1(x, y) \rho(y)) \, dx dy \qquad (3.42) \\ \text{for all } \Phi \in \mathbb{F}_0^1. \end{cases}$$

To prove (i), choose $\Phi = (\psi_0, \psi_1)$ in (3.41) such that $\psi_0 = 0$ and $\psi_1 = \varphi \otimes v_1$, where $\varphi \in \mathcal{D}(\Omega)$ and $v_1 \in H^1_\#(Y)$ to get

$$\int_\Omega \varphi(x) \left[\sum_{i,j=1}^N \int_{Y^*} a_{ij}(y) \left(\frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \right) \frac{\partial v_1}{\partial y_i} dy \right] dx = \int_\Omega \varphi(x) \left[\lambda_0^{k,\pm} u_0^{k,\pm}(x) \int_{Y^*} v_1(y) \rho(y) dy \right] dx$$

Hence by the arbitrariness of φ, we have a.e. in Ω

$$\sum_{i,j=1}^N \int_{Y^*} a_{ij}(y) \left(\frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \right) \frac{\partial v_1}{\partial y_i} dy = \lambda_0^{k,\pm} u_0^{k,\pm}(x) \int_{Y^*} v_1(y) \rho(y) dy$$

for any v_1 in $H^1_\#(Y)$, which is nothing but (3.38).

Fix $x \in \overline{\Omega}$, multiply both sides of (2.7) by $-\frac{\partial u_0^{k,\pm}}{\partial x_j}(x)$ and sum over $1 \le j \le N$. Adding side by side to the resulting equality that obtained after multiplying both sides of (2.8) by $\lambda_0^{k,\pm} u_0^{k,\pm}(x)$, we realize that $z(x) = -\sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) \chi^j(y) + \lambda_0^{k,\pm} u_0^{k,\pm}(x) \chi^0(y)$ solves (3.38). Hence $\mathbf{i}(z(x)) = \mathbf{i}(u_1^{k,\pm}(x))$ by uniqueness of the solution to the coercive variational problem in $H^1_\#(Y^*)$ corresponding to the non-coercive variational problem (3.38) (see the proof of Proposition 2.7). Thus (3.39) since $\mathbf{i}$ is linear.

This being so, we recall that (3.39) precisely means that almost everywhere in $x \in \Omega$,

$$D_y u_1^{k,\pm}(x) = \lambda_0^{k,\pm} u_0^{k,\pm}(x) D_y \chi^0 - \sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) D_y \chi^j \quad \text{a.e. in } Y^*, \qquad (3.43)$$

so that there is some $c \in L^2(\Omega)$ with

$$u_1^{k,\pm}(x, y) = \lambda_0^{k,\pm} u_0^{k,\pm}(x) \chi^0(y) - \sum_{j=1}^N \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) \chi^j(y) + c(x) \quad \text{a.e. in } Y^*. \qquad (3.44)$$

Considering now $\Phi = (\psi_0, \psi_1)$ in (3.41) such that $\psi_0 \in \mathcal{D}(\Omega)$ and $\psi_1 = 0$ we get

$$\sum_{i,j=1}^{N} \iint_{\Omega \times Y^*} a_{ij}(y) \left(\frac{\partial u_0^{k,\pm}}{\partial x_j} + \frac{\partial u_1^{k,\pm}}{\partial y_j} \right) \frac{\partial \psi_0}{\partial x_i} dxdy = \lambda_0^{k,\pm} \iint_{\Omega \times Y^*} u_1^{k,\pm}(x,y) \rho(y) \psi_0(x) dxdy,$$

which by means of (3.43) and (3.44) leads to

$$\sum_{i,j=1}^{N} \int_{\Omega} q_{ij} \frac{\partial u_0^{k,\pm}}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dx + \lambda_0^{k,\pm} \sum_{i,j=1}^{N} \int_{\Omega} u_0^{k,\pm}(x) \frac{\partial \psi_0}{\partial x_i} dx \left(\int_{Y^*} a_{ij}(y) \frac{\partial \chi^0}{\partial y_j}(y) dy \right)$$

$$= -\lambda_0^{k,\pm} \sum_{j=1}^{N} \int_{\Omega} \frac{\partial u_0^{k,\pm}}{\partial x_j} \psi_0(x) dx \left(\int_{Y^*} \rho(y) \chi^j(y) dy \right) \tag{3.45}$$

$$+ (\lambda_0^{k,\pm})^2 \int_{\Omega} u_0^{k,\pm}(x) \psi_0(x) dx \left(\int_{Y^*} \rho(y) \chi^0(y) dy \right).$$

The term with $c(x)$ vanishes because of $M_{Y^*}(\rho) = 0$. Choosing χ^l $(1 \le l \le N)$ as test function in (2.8) and χ^0 as test function in (2.7) we observe that

$$\sum_{j=1}^{N} \int_{Y^*} a_{lj}(y) \frac{\partial \chi^0}{\partial y_j}(y) dy = \int_{Y^*} \rho(y) \chi^l(y) dy = a(\chi^l, \chi^0) \quad (l = 1, \cdots N).$$

Thus, in (3.45), the second term in the left hand side is equal to the first one in the right hand side. This leaves us with

$$\int_{\Omega} q_{ij} \frac{\partial u_0^{k,\pm}}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dx = (\lambda_0^{k,\pm})^2 \int_{\Omega} u_0^{k,\pm}(x) \psi_0(x) dx \left(\int_{Y^*} \rho(y) \chi^0(y) dy \right). \tag{3.46}$$

Choosing χ^0 as test function in (2.8) reveals that

$$\int_{Y^*} \rho(y) \chi^0(y) dy = a(\chi^0, \chi^0) = v^2.$$

Hence

$$\sum_{i,j=1}^{N} \int_{\Omega} q_{ij} \frac{\partial u_0^{k,\pm}}{\partial x_j} \frac{\partial \psi_0}{\partial x_i} dx = (\lambda_0^{k,\pm})^2 v^2 \int_{\Omega} u_0^{k,\pm}(x) \psi_0(x) dx,$$

and

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(q_{ij} \frac{\partial u_0^{k,\pm}}{\partial x_j}(x) \right) = (\lambda_0^{k,\pm})^2 v^2 u_0^{k,\pm}(x) \text{ in } \Omega.$$

Thus the convergence (3.32) holds for the whole sequence E. As regards (3.37), we notice that for fixed $k \ge 1$ and any $\phi \in \mathcal{D}(\Omega)$ one has (keep (2.5) in mind)

$$\lim_{E' \ni \varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega} (P_\varepsilon u_\varepsilon^{k,\pm}) \phi(x) \rho(\frac{x}{\varepsilon}) \chi_G(\frac{x}{\varepsilon}) dx = \iint_{\Omega \times Y^*} u_1^{k,\pm}(x,y) \phi(x) \rho(y) \, dxdy.$$

Hence, as $E' \ni \varepsilon \to 0$

$$\frac{1}{\varepsilon} (P_\varepsilon u_\varepsilon^{k,\pm}) \rho^\varepsilon \chi_G^\varepsilon \rightharpoonup \int_{Y^*} u_1^{k,\pm}(\cdot,y) \rho(y) \, dy \quad \text{in } L^2(\Omega) - \text{weak}.$$

Using once again the decomposition $\Omega^{\varepsilon} = Q^{\varepsilon} \cup (\Omega^{\varepsilon} \setminus Q^{\varepsilon})$ and the equality $Q^{\varepsilon} = \Omega \cap \varepsilon G$, we get as $E' \ni \varepsilon \to 0$

$$\frac{1}{\varepsilon} \int_{\Omega^{\varepsilon}} (P_{\varepsilon} u_{\varepsilon}^{k,\pm})(P_{\varepsilon} u_{\varepsilon}^{l,\pm}) \rho(\frac{x}{\varepsilon}) \, dx \to \iint_{\Omega \times Y^*} u_1^{k,\pm}(x,y) u_0^{l,\pm}(x) \rho(y) \, dxdy,$$

for fixed $l \geq 1$. This together with (3.29) and (3.44) yields

$$\lambda_0^{k,\pm} \nu^2 \int_{\Omega} u_0^{l,\pm} u_0^{k,\pm} \, dx - \sum_{j=1}^{N} a(\chi^j, \chi^0) \int_{\Omega} \frac{\partial u_0^{k,\pm}}{\partial x_j} u_0^{l,\pm} \, dx = \pm \delta_{k,l}, \quad k,l = 1,2,\cdots \qquad (3.47)$$

If $k = l$, then by Green's formula the sum on the left hand side vanishes and (3.47) reduces to the desired result. This concludes the proof. $\qquad \square$

Remark 3.6.   • The eigenfunctions $\{u_0^{k,\pm}\}_{k=1}^{\infty}$ are orthonormalized by

$$\iint_{\Omega \times Y^*} u_1^{l,\pm}(x,y) u_0^{k,\pm}(x) \rho(y) \, dxdy = \iint_{\Omega \times Y^*} u_1^{k,\pm}(x,y) u_0^{l,\pm}(x) \rho(y) \, dxdy = \pm \delta_{k,l}$$

$k,l = 1,2,\cdots$

• If $\lambda_0^{k,\pm}$ is simple (this is the case for $\lambda_0^{1,\pm}$), then by Theorem 3.5, $\lambda_{\varepsilon}^{k,\pm}$ is also simple, for small ε, and we can choose the eigenfunctions $u_{\varepsilon}^{k,\pm}$ such that the convergence results (3.3)-(3.5) hold for the whole sequence E.

Acknowledgments

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