A unified stability property in spin glasses.

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Abstract

Gibbs' measures in the Sherrington-Kirkpatrick type models satisfy two asymptotic stability properties, the Aizenman-Contucci stochastic stability and the Ghirlanda-Guerra identities, which play a fundamental role in our current understanding of these models. In this paper we show that one can combine these two properties very naturally into one unified stability property.

Key words: Gibbs measures, spin glass models, stability.

1 Introduction and main results.

Let us consider a random discrete probability measure G on the unit ball of a separable Hilbert space, $G = \sum_{l\geq 1} w_l \delta_{\xi_l}$. We assume that the weights are arranged in non-increasing order, $w_1 \geq w_2 \geq \ldots$, and denote by $Q = (\xi_l \cdot \xi_{l'})_{l,l'\geq 1}$ the matrix of scalar products of the points in the support of G. Let $(\sigma^l)_{l\geq 1}$ be an i.i.d. sequence of *configurations* from this measure and denote by $R_{l,l'} = \sigma^l \cdot \sigma^{l'}$ the scalar product, or *overlap*, of σ^l and $\sigma^{l'}$. For any $n \geq 1$ and a function $f = f(\sigma^1, \ldots, \sigma^n)$ of n configurations we will denote its average with respect to $G^{\otimes\infty}$ by

$$\langle f \rangle = \sum_{l_1,\ldots,l_n\geq 1} w_{l_1} \cdots w_{l_n} f(\xi_{l_1}, \ldots, \xi_{l_n}). \tag{1.1}$$

We will denote by $\mathbb{E}$ the expectation with respect to the randomness of G. Random measure G is said to satisfy the Ghirlanda-Guerra identities [6] if for any $n \geq 2$, any bounded measurable function f of the overlaps $(R_{l,l'})_{l,l'\leq n}$ and any integer $p \geq 1$ we have

$$\mathbb{E}\langle f R_{1,n+1}^p \rangle = \frac{1}{n} \mathbb{E}\langle f \rangle \mathbb{E}\langle R_{1,2}^p \rangle + \frac{1}{n} \sum_{l=2}^{n} \mathbb{E}\langle f R_{1,l}^p \rangle. \tag{1.2}$$

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Given integer $p \geq 1$, let $(g_p(\xi_l))_{l\geq 1}$ be a Gaussian sequence conditionally on G indexed by the points $(\xi_l)_{l\geq 1}$ with covariance

$$\mathrm{Cov}\big(g_p(\xi_l), g_p(\xi_{l'})\big) = (\xi_l \cdot \xi_{l'})^p. \tag{1.3}$$

Given $t \geq 0$, consider a new sequence of weights

$$w_l^t = \frac{w_l e^{t g_p(\xi_l)}}{\sum_{j\geq 1} w_j e^{t g_p(\xi_j)}} \tag{1.4}$$

defined by a random change of density proportional to $e^{t g_p(\xi_l)}$. Let (w_l^π) be the weights (w_l^t) arranged in the non-increasing order and let $\pi : \mathbb{N} \to \mathbb{N}$ be the permutation keeping track of where each index came from, $w_l^\pi = w_{\pi(l)}^t$. Let us define by

$$G^\pi = \sum_{l\geq 1} w_l^\pi \delta_{\xi_{\pi(l)}} \quad \text{and} \quad Q^\pi = \big(\xi_{\pi(l)} \cdot \xi_{\pi(l')}\big)_{l,l'\geq 1} \tag{1.5}$$

the probability measure G after the change of density proportional to $e^{t g_p(\xi_l)}$ and the matrix Q rearranged according to the reordering of weights. Measure G is said to satisfy the Aizenman-Contucci stochastic stability [1] if for any $p \geq 1$ and $t \geq 0$,

$$\big((w_l^\pi)_{l\geq 1}, Q^\pi\big) \stackrel{d}{=} \big((w_l)_{l\geq 1}, Q\big) \tag{1.6}$$

where equality in distribution is in the sense of finite dimensional distributions of these arrays.

Random measures satisfying (1.2) and (1.6) arise as the asymptotic analogues of Gibbs' measures in the Sherrington-Kirkpatrick type models and these two stability properties have been used extensively in proving structural results for such measures (see e.g. [2], [3], [4], [7], [8], [9], [10], [12], [14], [13], [15]). In this paper we will show that one can combine (1.2) and (1.6) into a joint stability property as follows. It is known (Theorem 2 in [7]) that if the measure G satisfies the Ghirlanda-Guerra identities and if q^* is the supremum of the support of the distribution of the overlap $R_{1,2}$ under $\mathbb{E}G^{\otimes 2}$ then with probability one G is concentrated on the sphere of radius $\sqrt{q^*}$. Let

$$b_p = (q^*)^p - \mathbb{E}\langle R_{1,2}^p \rangle. \tag{1.7}$$

Then the following holds.

Theorem 1 *Random measure G satisfies the Ghirlanda-Guerra identities (1.2) and the Aizenman-Contucci stochastic stability (1.6) if and only if it is concentrated on the sphere of constant radius $\sqrt{q^*}$ with probability one and for any $p \geq 1$ and $t \geq 0$,*

$$\Big(\big(w_l^\pi\big)_{l\geq 1}, \big(g_p(\xi_{\pi(l)}) - b_p t\big)_{l\geq 1}, Q^\pi\Big) \stackrel{d}{=} \Big(\big(w_l\big)_{l\geq 1}, \big(g_p(\xi_l)\big)_{l\geq 1}, Q\Big). \tag{1.8}$$

where equality in distribution is in the sense of finite dimensional distributions.

The stability property (1.8) is known for the Poisson-Dirichlet cascades (Ruelle probability cascades in the terminology of [3]), which is the result of Talagrand, Theorem 15.2.1 in [15]. Of course, the big open problem is whether it holds only for the Poisson-Dirichlet cascades which is equivalent to proving that any measure satisfying (1.8) is ultrametric.

The Ghirlanda-Guerra identities do not require the random measure G to be discrete and, in fact, the Aizenman-Contucci stochastic stability can be formulated not only for discrete measures as well. We will mention this more general formulation in the next section. However, we prefer to state our main result in the setting of discrete measures since it allows for a particularly attractive formulation (1.8) in the spirit of competing particle systems, as in [12] and [3]. Moreover, from the point of view of studying structural properties of such measures one can without loss of generality start with discrete measures since it is easy to show that sampling an i.i.d. sequence of points from the original measure and assigning them new independent weights from the Poisson-Dirichlet distribution creates a discrete measure which still satisfies both properties. On the other hand, almost any geometric property of the original measure will be encoded into a countable i.i.d. sample and, therefore, this new discrete measure.

2 Proof.

Let $(\rho^l)_{l\geq 1}$ be an i.i.d. sequence from measure G^π defined in (1.5) and denote by $S_{l,l'} = \rho^l \cdot \rho^{l'}$ the overlap of ρ^l and $\rho^{l'}$. Analogously to (1.1), for any $n \geq 1$ and a function $f = f(\rho^1, \ldots, \rho^n)$ of n configurations we will denote its average with respect to $(G^\pi)^{\otimes\infty}$ by

$$\langle f \rangle_\pi = \sum_{l_1,\ldots,l_n \geq 1} w^\pi_{l_1} \cdots w^\pi_{l_n} \, f(\xi_{\pi(l_1)}, \ldots, \xi_{\pi(l_n)}). \tag{2.1}$$

We now will denote by $\mathbb{E}$ the expectation with respect to the randomness of G and the Gaussian sequence (g_p). Let us first make a simple observation that equality of finite dimensional distributions in (1.6) and (1.8) implies equality of averages with respect to the random measures in the following sense.

Lemma 1 *If (1.8) holds then for any $k \geq 1$, any bounded measurable function f of the overlaps on k replicas and any integers $n_1, \ldots, n_k \geq 0$,*

$$\mathbb{E}\Big\langle \prod_{l\leq k} \big(g_p(\rho^l) - b_p t\big)^{n_l} f\big((S_{l,l'})_{l,l'\leq k}\big) \Big\rangle_\pi = \mathbb{E}\Big\langle \prod_{l\leq k} g_p(\sigma^l)^{n_l} f\big((R_{l,l'})_{l,l'\leq k}\big) \Big\rangle \tag{2.2}$$

Under (1.6), this holds with all $n_l = 0$.

Remark. One can consider (2.2) with all $n_l = 0$ as the definition of the Aizenman-Contucci stochastic stability for non-atomic measures in which case $(g_p(\xi))$ is the Gaussian field with covariance (1.3). Moreover, in this case (2.2) should be considered as the analogue of (1.8).

Proof. This is obvious by separating the sum in (1.1) and (2.1) into finitely many terms corresponding to the largest weights and the remaining small weights. For example,

$$\begin{aligned}\mathbb{E}\Big\langle\prod_{l\le k} g_p(\sigma^l)^{n_l} f\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle &= \mathbb{E}\sum_{j_1,\ldots,j_k\ge 1} w_{j_1}\cdots w_{j_n}\prod_{l\le k} g_p(\xi_{j_l})^{n_l} f\big((\xi_{j_l}\cdot\xi_{j_{l'}})_{l,l'\le k}\big)\\ &= \mathbb{E}\sum_{j_1,\ldots,j_k\le N} w_{j_1}\cdots w_{j_n}\prod_{l\le k} g_p(\xi_{j_l})^{n_l} f\big((\xi_{j_l}\cdot\xi_{j_{l'}})_{l,l'\le k}\big)+\mathcal{R}_N,\end{aligned}$$

where the remainder $\mathcal{R}_N$ consists of the terms with at least one index $j_1,\ldots,j_k > N$. The left hand side of (2.2) can be similarly broken into two sums. The finite sums are equal because they involve only finitely many elements of the arrays (1.8) which are equal in distribution by assumption. Thus, we only need to show that $\mathcal{R}_N$ becomes small for large N. Since f is bounded, taking expectation in Gaussian random variables $(g_p(\xi_l))$ first we get

$$|\mathcal{R}_N| \le L(f,n_1,\ldots,n_k)\,\mathbb{E}\sum_{(j_1,\ldots,j_k\le N)^c} w_{j_1}\cdots w_{j_n} \le Lk\,\mathbb{E}\sum_{j>N} w_j$$

which goes to zero as $N\to\infty$. The remainder for the left hand side of (2.2) is controlled similarly.

□

The "if" part of the Theorem 1 is easy since assuming (1.8) we only need to prove (1.2) and this follows from integration by parts of (2.2) with $n_1 = 1, n_2 = \ldots = n_k = 0$. In this case the right hand side is zero by averaging $g_p(\sigma^1)$ first and the left hand side is

$$\begin{aligned}\mathbb{E}\Big\langle\big(g_p(\rho^1)-b_p t\big) f\big((S_{l,l'})_{l,l'\le k}\big)\Big\rangle_\pi &= t\mathbb{E}\Big\langle\big(\sum_{l=1}^k S^p_{1,l} - b_p - kS^p_{1,k+1}\big) f\big((S_{l,l'})_{l,l'\le k}\big)\Big\rangle_\pi\\ &= t\mathbb{E}\Big\langle\big(\sum_{l=1}^k R^p_{1,l} - b_p - kR^p_{1,k+1}\big) f\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle\\ &= t\mathbb{E}\Big\langle\big(\sum_{l=2}^k R^p_{1,l} + \mathbb{E}\langle R^p_{1,2}\rangle - kR^p_{1,k+1}\big) f\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle\end{aligned}$$

where in the second line we used (1.6) part of (1.8) and Lemma 1, and in the third line we used (1.7) and the fact that $\xi_l\cdot\xi_l = q^*$. The fact that the last sum is zero is exactly (1.2).

To prove the "only if" part we need the following key lemma.

Lemma 2 *If (1.2) and (1.6) hold then (2.2) holds.*

Proof. The proof is by induction on $N = n_1+\ldots+n_k$. When $N=0$, (2.2) is the consequence of (1.6) by Lemma 1. Suppose (2.2) holds for all $k\ge 1$, all f and for all $N\le N_0$. Clearly, we only need to prove the case of powers $n_1+1, n_2,\ldots,n_k$. Writing

$$g_p(\sigma^1)^{n_1+1} = g_p(\sigma^1)\,g_p(\sigma^1)^{n_1}$$

and using Gaussian integration by parts for $g_p(\sigma^1)$ we can rewrite the right hand side of (2.2) with n_1+1 instead of n_1 as

$$\sum_{l\le k} n_l \mathbb{E}\Big\langle g_p(\sigma^1)^{n_1}\ldots g_p(\sigma^l)^{n_l-1}\ldots g_p(\sigma^k)^{n_k} R_{1,l}^p\, f\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle. \tag{2.3}$$

Again, writing

$$(g_p(\rho^1)-b_p t)^{n_1+1} = (g_p(\rho^1)-b_p t)(g_p(\rho^1)-b_p t)^{n_1}$$

and using Gaussian integration by parts for $g_p(\rho^1)$ we can rewrite the left hand side of (2.2) with n_1+1 instead of n_1 as $\mathrm{I}+\mathrm{II}$ where I is given by

$$\sum_{l\le k} n_l \mathbb{E}\Big\langle \big(g_p(\rho^1)-b_p t\big)^{n_1}\ldots \big(g_p(\rho^l)-b_p t\big)^{n_l-1}\ldots \big(g_p(\rho^k)-b_p t\big)^{n_k} S_{1,l}^p\, f\big((S_{l,l'})_{l,l'\le k}\big)\Big\rangle_\pi \tag{2.4}$$

and II is given by

$$t\,\mathbb{E}\Big\langle \prod_{l\le k}\big(g_p(\rho^l)-b_p t\big)^{n_l}\Big(\sum_{l\le k} S_{1,l}^p - b_p - k S_{1,k+1}^p\Big) f\big((S_{l,l'})_{l,l'\le k}\big)\Big\rangle_\pi. \tag{2.5}$$

By induction hypothesis, (2.4) is equal to (2.3) and (2.5) is equal to

$$t\,\mathbb{E}\Big\langle \prod_{l\le k} g_p(\sigma^l)^{n_l}\Big(\sum_{l\le k} R_{1,l}^p - b_p - k R_{1,k+1}^p\Big) f\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle. \tag{2.6}$$

Since $\langle\cdot\rangle$ does not depend on the Gaussian sequence $(g_p(\xi_l))$ we can take expectation $\mathbb{E}_g$ with respect to the randomness of this sequence conditionally on G first and notice that

$$\mathbb{E}_g \prod_{l\le k} g_p(\sigma^l)^{n_l} = f'\big((R_{l,l'})_{l,l'\le k}\big)$$

is the function f' of the overlaps of k configurations $\sigma^1,\ldots,\sigma^k$. Therefore, (2.6) is equal to

$$\begin{aligned} & t\,\mathbb{E}\Big\langle \Big(\sum_{l=1}^k R_{1,l}^p - b_p - k R_{1,k+1}^p\Big)(f f')\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle \\ &= t\,\mathbb{E}\Big\langle \Big(\sum_{l=2}^k R_{1,l}^p + \mathbb{E}\langle R_{1,2}^p\rangle - k R_{1,k+1}^p\Big)(f f')\big((R_{l,l'})_{l,l'\le k}\big)\Big\rangle = 0, \end{aligned} \tag{2.7}$$

where in the first equality we again used (1.7) and the fact that $\xi_l\cdot\xi_l = q^*$ and the second equality is by the Ghirlanda-Guerra identities (1.2). This finishes the proof.

□

The equality of joint moments (2.2) proved in Lemma 2 implies the following.

Lemma 3 *If (1.2) and (1.6) hold then*

$$\Big(\big(g_p(\rho^l)-b_p t\big)_{l\ge 1}, (S_{l,l'})_{l,l'\ge 1}\Big) \stackrel{d}{=} \Big(\big(g_p(\sigma^l)\big)_{l\ge 1}, (R_{l,l'})_{l,l'\ge 1}\Big) \tag{2.8}$$

where equality in distribution is in the sense of finite dimensional distributions.

Proof. By choosing f to be monomials, (2.2) gives the equality of joint moments of the corresponding elements of the two arrays in (2.8). In our case the joint moments uniquely determine joint distributions, for example, by the main result in [11] which states that we only need to ensure the uniqueness of one dimensional marginals and the fact that the one dimensional marginals are either bounded or Gaussian.

□

Finally, we will show that (2.8) implies (1.8). First of all, by the well-known result of Talagrand (Section 1.2 in [13]), the Ghirlanda-Guerra identities imply that the weights (w_l) must have a Poisson-Dirichlet distribution $PD(m)$ where m is determined by

$$\mathbb{E}\langle I(R_{1,2}=q^*)\rangle = \mathbb{E}\sum_{l\geq 1} w_l^2 = 1-m.$$

This means that if (u_l) is a Poisson point process on $(0,\infty)$ with intensity measure $x^{-m-1}dx$ then $w_l = u_l/\sum_{j\geq 1} u_j$. In particular, all the weights are different with probability one. This point is not crucial but it makes for an easier argument. The reason why (2.8) implies (1.8) is because one can easily reconstruct the arrays in (1.8) from the arrays (2.8) using that (σ^l) is an i.i.d. sample from (ξ_l) according to weights (w_l) and (ρ^l) is an i.i.d. sample from $(\xi_{\pi(l)})$ according to weights (w_l^π). The key observation here is that given arrays (2.8) we know exactly when $\sigma^l = \sigma^{l'}$ and $\rho^l = \rho^{l'}$ since this is equivalent to $R_{l,l'} = q^*$ and $S_{l,l'} = q^*$. Therefore, given $N \geq 1$ and

$$\big((g_p(\sigma^l))_{l\leq N}, (R_{l,l'})_{l,l'\leq N}\big)$$

we can partition the set $\{1,\ldots,N\}$ according to the equivalence relation $l \sim l'$ defined by $R_{l,l'} = q^*$, let the sequence of weights $(w_l^N)_{l\geq 0}$ be the proportions of the sets in this partition arranged in non-increasing order and extended by zeros and, given any integer j in the element of the partition corresponding to the weight w_l^N, define $\xi_l^N = \sigma^j$. We let $Q^N = (\xi_l^N \cdot \xi_{l'}^N)_{l,l'\geq 1}$. The elements of (ξ_l^N) and Q^N with indices corresponding to zero weights w_l^N can be set to some fixed values, and we break ties between w_l^N by any pre-determined rule. Similarly, given

$$\big((g_p(\rho^l) - b_p t)_{l\leq N}, (S_{l,l'})_{l,l'\leq N}\big)$$

we can construct sequences $(\tilde{w}_l^N), (\tilde{\xi}_l^N)$ and $\tilde{Q}^N = (\tilde{\xi}_l^N \cdot \tilde{\xi}_{l'}^N)$. Equation (2.8) implies that for any fixed $k \geq 1$,

$$\Big((\tilde{w}_l^N)_{l\leq k}, \big(g_p(\tilde{\xi}_l^N) - b_p t\big)_{l\leq k}, (\tilde{q}_{l,l'}^N)_{l,l'\leq k}\Big) \stackrel{d}{=} \Big((w_l^N)_{l\leq k}, \big(g_p(\xi_l^N)\big)_{l\leq k}, (q_{l,l'}^N)_{l,l'\leq k}\Big).$$

It remains to observe that the right hand side converges

$$\Big((w_l^N)_{l\leq k}, \big(g_p(\xi_l^N)\big)_{l\leq k}, (q_{l,l'}^N)_{l,l'\leq k}\Big) \to \Big((w_l)_{l\leq k}, \big(g_p(\xi_l)\big)_{l\leq k}, (q_{l,l'})_{l,l'\leq k}\Big) \tag{2.9}$$

almost surely and, similarly, the left hand side converges a.s. to the corresponding array from the left hand side of (1.8). To prove (2.9), we notice that by construction

$$G^N := \sum_{l\geq 1} w_l^N \delta_{\xi_l^N} = \frac{1}{N}\sum_{i\leq N} \delta_{\sigma_i}$$

is the empirical measure based on the sample $\sigma^1, \ldots, \sigma^N$ from the measure $G = \sum_{l\geq 1} w_l \delta_{\xi_l}$. By the strong law of large number for empirical measures (e.g. Theorem 11.4.1 in [5]), the laws $G^N \to G$ almost surely and since the Poisson-Dirichlet weights (w_l) are all different a.s., the largest k weights must converge $(w_l^N)_{l\leq k} \to (w_l)_{l\leq k}$ almost surely and for large enough N we must have $(\xi_l^N)_{l\leq k} = (\xi_l)_{l\leq k}$ and, thus, (2.9) holds.

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