ON THE VERLINDE FORMULAS FOR SO(3)-BUNDLES

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Abstract. This paper computes the quantization of the moduli space of flat SO(3)-bundles over an oriented surface with boundary, with prescribed holonomies around the boundary circles. The result agrees with the generalized Verlinde formula conjectured by Fuchs and Schweigert.

1. Introduction

Let G be a compact, connected Lie group, Σ a compact oriented surface of genus h with r boundary components. Given conjugacy classes $\mathcal{C}_1,\ldots,\mathcal{C}_r\subset G$, denote by

$$\mathcal{M}(\Sigma,\mathcal{C}_1,\ldots,\mathcal{C}_r) \tag{1}$$

the moduli space of flat G-bundles over Σ, with boundary holonomies in prescribed conjugacy classes $\mathcal{C}_j$. The choice of an invariant inner product on $\mathfrak{g}$ defines a symplectic structure on the moduli space. Under suitable integrality conditions the moduli space carries a pre-quantum line bundle L, and one can define the *quantization*

$$\mathcal{Q}(\mathcal{M}(\Sigma,\mathcal{C}_1,\ldots,\mathcal{C}_r))\in\mathbb{Z} \tag{2}$$

as the index of the Spin_c-Dirac operator with coefficients in L. (It may be necessary to use a partial desingularization as in [16].) Choosing a complex structure on Σ further defines a Kähler structure on the moduli space. If G is simply connected, Kodaira vanishing results [20] show that the above index coincides with the dimension of the space of holomorphic sections of L. It is given by the celebrated Verlinde formula [22, 21, 7, 19, 5]. For symplectic approaches to the Verlinde formulas, much in the spirit of the present paper, see [11, 10, 12, 6, 2].

Much less is known for non-simply connected groups. For surfaces without boundary ($r=0$), and taking $G=\mathrm{PU}(n)$, Verlinde-type formulas were obtained by Pantev [17] in the case $n=2$ and by Beauville [4] for n prime. For more general compact, semi-simple connected Lie groups, Fuchs and Schweigert [9] conjectured a generalization of the Verlinde formula, expressed in terms of *orbit Lie algebras*. Partial results on these conjectures were obtained in [2].

In this article, we will establish Fuchs-Schweigert formulas for the index (2) for the simplest case $G=\mathrm{SO}(3)$. We will use the recently developed quantization procedure [15, 14] for quasi-Hamiltonian actions with group-valued moment map [1]. In order to apply these techniques, we present the moduli spaces (1) as symplectic quotients of quasi-Hamiltonian $\tilde{G}$-spaces for the universal cover $\tilde{G}=\mathrm{SU}(2)$. In more detail, let $\mathcal{D}_i\subset\mathrm{SU}(2)$ be conjugacy classes, and consider the quasi-Hamiltonian SU(2)-space

$$\tilde{M}=\mathcal{D}_1\times\cdots\times\mathcal{D}_s\times\mathrm{SU}(2)^{2h}$$

with moment map the product of holonomies,

$$\tilde{\Phi}(d_1,\ldots,d_s,a_1,b_1,\ldots,a_h,b_h)=\prod_{i=1}^s d_i\prod_{j=1}^h(a_jb_ja_j^{-1}b_j^{-1}).$$

Put $M=\tilde{M}/\Gamma$, where $\Gamma\subset Z^{s+2h}$ is the subgroup preserving $\tilde{M}\subset \mathrm{SU}(2)^{s+2h}$ and $\tilde{\Phi}$. Then M is a quasi-Hamiltonian SU(2)-space, and all connected components of moduli spaces (2) are symplectic quotients $M/\!\!/\,\mathrm{SU}(2)$ for suitable choices of $\mathcal{D}_j$ (see Section 2.3). Our first main result gives necessary and sufficient conditions under which the space M admits a level k pre-quantization [13]. Using localization, we then compute the corresponding quantization $\mathcal{Q}(M)\in R_k(\mathrm{SU}(2))$, an element of the level k fusion ring (Verlinde ring). These results are summarized in Theorem 3.7. We reformulate the result as an equivariant version of the Fuchs-Schweigert formula (Theorem 4.1); the non-equivariant formula (see (16) in Section 4) is then obtained from a 'quantization commutes with reduction' principle.

Using the results of [14], it is also possible to compute quantizations of moduli spaces for non-simply connected groups of higher rank. However, the determination of the pre-quantization conditions and the evaluation of the fixed point contributions becomes more involved. We will return to these questions in a forthcoming paper; see also the author's abstracts in Oberwolfach Report No. 2011/09.

2. Preliminaries

The following notation, consistent with [15], will be used in this paper. For the Lie group SU(2) let T be the maximal torus given as the image of

$$j\colon\ \mathrm{U}(1)\to\mathrm{SU}(2),\quad j(z)=\left(\begin{smallmatrix} z & 0\\ 0 & \bar{z}\end{smallmatrix}\right).$$

Let $\Lambda=\ker\exp_T\subset\mathfrak{t}$ denote the integral lattice and $\Lambda^*\subset\mathfrak{t}^*$ its dual, the (real) weight lattice. Let $\rho\in\Lambda^*$ be the generator dual to the generator $\mathrm{d}j(2\pi i)\in\Lambda$. We will use the *basic inner product* on $\mathfrak{su}(2)$,

$$\xi\cdot\xi':=\frac{1}{4\pi^2}\mathrm{tr}(\xi^\dagger\xi'),\qquad \xi,\xi'\in\mathfrak{su}(2)$$

to identify $\mathfrak{su}(2)\cong\mathfrak{su}(2)^*$. Under this identification, $||\rho||^2=\frac{1}{2}$, and $\Lambda=2\Lambda^*$ with generator 2ρ. The following two elements of SU(2) will play a special role in this paper:

$$u_*=\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right),\quad t_*=\left(\begin{smallmatrix} i & 0\\ 0 & -i\end{smallmatrix}\right)$$

Observe that $t_*=\exp(\rho/2)$, with square $c=\exp\rho$ the non-trivial element in the center $Z:=Z(\mathrm{SU}(2))\cong\mathbb{Z}_2$. The element $u_*\in N(T)$ represents the non-trivial element of the Weyl group $W=N(T)/T\cong\mathbb{Z}_2$. Both u_*,t_* are contained in the conjugacy class $\mathcal{D}_*\subset\mathrm{SU}(2)$ of elements of trace 0. Note that $\mathcal{D}_*$ is the unique conjugacy class in SU(2) that is invariant under multiplication by Z. The quotient $\mathcal{C}_*=\mathcal{D}_*/Z\cong\mathbb{R}P(2)$ is the conjugacy class in SO(3) consisting of rotations by π.

2.1. The fusion ring $R_k(\mathrm{SU}(2))$.

We view the representation ring $R(\mathrm{SU}(2))$ as the subring of $C^\infty(\mathrm{SU}(2))$ generated by characters of SU(2)-representations. As a $\mathbb{Z}$-module, it is free with basis χ_0, χ_1, $\chi_2,\ldots$, where χ_m is the character of the irreducible SU(2)-representation on the m-th symmetric power $S^m(\mathbb{C}^2)$. The ring structure is determined by the formula

$$\chi_m\chi_{m'}=\chi_{m+m'}+\chi_{m+m'-2}+\cdots+\chi_{|m-m'|}.$$

For $k = 0, 1, 2, \ldots$ let $I_k(\mathrm{SU}(2))$ be the ideal generated by χ_{k+1} and let

$$R_k(\mathrm{SU}(2)) = R(\mathrm{SU}(2))/I_k(\mathrm{SU}(2))$$

be the *level k fusion ring* (or *Verlinde ring*). As a $\mathbb{Z}$-module, $R_k(\mathrm{SU}(2))$ is free, with basis $\tau_0, \tau_1, \ldots, \tau_k$ the images of $\chi_0, \chi_1, \ldots, \chi_k$ under the quotient homomorphism. Let $q = e^{\frac{i\pi}{k+2}}$ be the $2k+4$-th root of unity, and define *special points*

$$t_l = j(q^{l+1}), \quad l = 0, \ldots, k. \tag{3}$$

Then $I_k(\mathrm{SU}(2)) \subset R(\mathrm{SU}(2))$ has an alternative description as the ideal of characters vanishing at all special points (3). Hence, the evaluation of characters at the special points descends to evaluations $R_k(\mathrm{SU}(2)) \to \mathbb{C}, \ \tau \mapsto \tau(t_l)$.

The product in the complexified fusion ring $R_k(\mathrm{SU}(2)) \otimes_{\mathbb{Z}} \mathbb{C}$ can be diagonalized using the *S-matrix*, given by the Kac-Peterson formula

$$S_{m,l} = (\tfrac{k}{2} + 1)^{-\frac{1}{2}} \sin\big(\tfrac{\pi(l+1)(m+1)}{k+2}\big), \tag{4}$$

for $l, m = 0, 1, \ldots, k$. The S-matrix is orthogonal, and the alternative basis elements

$$\tilde{\tau}_l = \sum_m S_{0,l} S_{m,l} \tau_m$$

satisfy $\tilde{\tau}_m(t_l) = \delta_{m,l}$, hence

$$\tilde{\tau}_m \tilde{\tau}_{m'} = \delta_{m,m'} \tilde{\tau}_m.$$

The basis elements $\{\tau_0, \ldots, \tau_k\}$ are expressed in terms of the alternative basis as $\tau_m = \sum_l S_{0,l}^{-1} S_{m,l} \tilde{\tau}_l$.

2.2. **Quasi-Hamiltonian G-spaces.** We recall some basic definitions and facts from [1]. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, equipped with an invariant inner product, denoted by a dot $\cdot$. Let θ^L, θ^R denote the left-invariant, right-invariant Maurer-Cartan forms on G, and let $\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L]$ denote the Cartan 3-form on G. For a G-manifold M, and $\xi \in \mathfrak{g}$, let $\xi^\sharp$ denote the generating vector field, defined in terms of the action on functions $f \in C^\infty(M)$ by $(\xi^\sharp f)(x) = \frac{d}{dt}\Big|_{t=0} f(\exp(-t\xi).x)$. The Lie group G is itself viewed as a G-manifold for the conjugation action.

Definition 2.1. A quasi-Hamiltonian G-space is a triple (M, ω, Φ) consisting of a G-manifold M, a G-invariant 2-form ω on M, and an equivariant map $\Phi\colon M \to G$, called the moment map, satisfying:

(1) $d\omega + \Phi^*\eta = 0$,
(2) $\iota_{\xi^\sharp}\omega + \frac{1}{2}\Phi^*((\theta^L + \theta^R) \cdot \xi) = 0$ for all $\xi \in \mathfrak{g}$,
(3) at every point $x \in M$, $\ker \omega_x \cap \ker \mathrm{d}\Phi_x = \{0\}$.

The *fusion product* of two quasi-Hamiltonian G-spaces (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2) is the product $M_1 \times M_2$, with the diagonal G-action, 2-form

$$\omega = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2 + \tfrac{1}{2}\mathrm{pr}_1^*\Phi_1^*\theta^L \cdot \mathrm{pr}_2^*\Phi_2^*\theta^R, \tag{5}$$

and moment map $\Phi = \Phi_1\Phi_2$.

The *symplectic quotient* of a quasi-Hamiltonian G-space is the symplectic space $M/\!/G = \Phi^{-1}(e)/G$. Similar to the theory of Hamiltonian group actions, the group unit e is a regular value of Φ if and only if G acts locally freely on the level set $\Phi^{-1}(e)$, and in this case the pull-back of the 2-form to the level set descends to a symplectic 2-form on the orbifold $\Phi^{-1}(e)/G$. If e is a singular value, then $M/\!/G$ is a singular symplectic space as defined in [18].

The conjugacy classes $\mathcal{C} \subset G$ are basic examples of quasi-Hamiltonian G-spaces. The moment map is the inclusion into G, and the 2-form ω is given on generating vector fields by the formula

$$\omega_g(\zeta^\sharp(g), \xi^\sharp(g)) = \tfrac{1}{2}(\xi \cdot \mathrm{Ad}_g \zeta - \zeta \cdot \mathrm{Ad}_g \xi). \tag{6}$$

Together with the *double* $\mathbf{D}(G) = G \times G$, equipped with diagonal G-action and moment map $\Phi(g,h) = ghg^{-1}h^{-1}$, these are the building blocks of the main example appearing in this paper. As shown in [1], the moduli space of flat G-bundles over a compact, oriented surface Σ of genus h with s boundary components, with boundary holonomies in prescribed conjugacy classes $\mathcal{C}_j$, $j = 1, \ldots, s$, is a symplectic quotient of a fusion product:

$$M(\Sigma, \mathcal{C}_1, \ldots, \mathcal{C}_s) = \mathcal{C}_1 \times \cdots \times \mathcal{C}_s \times \mathbf{D}(G)^h /\!/ G. \tag{7}$$

If the group G is simply connected, then the fibers of the moment map for any compact, connected quasi-Hamiltonian G-space are connected. In particular, (7) is connected in that case. If G is non-simply connected, the space (7) may have several components.

To clarify the decomposition into components, we use the following construction. Suppose $p\colon \check{G} \to G$ is a homomorphism of compact, connected Lie groups, with finite kernel Z. Then Z is a subgroup of the center of $\check{G}$, and $G = \check{G}/Z$. For any quasi-Hamiltonian G-space (N, ω, Φ), let $\check{N}$ denote the fiber product defined by the pull-back square

$$\begin{array}{ccc} \check{N} & \xrightarrow{\check{\Phi}} & \check{G} \\ \big\downarrow{\scriptstyle p_N} & & \big\downarrow{\scriptstyle p} \\ N & \xrightarrow{\Phi} & G \end{array} \tag{8}$$

Then $(\check{N}, \check{\omega}, \check{\Phi})$ is a quasi-Hamiltonian $\check{G}$-space, for the diagonal $\check{G}$-action on $\check{N} \subset N \times \check{G}$, and with the 2-form $\check{\omega} = p_N^* \omega$. Simple properties of this construction are:

Proposition 2.2. *(i) We have a canonical identification of symplectic quotients*

$$\check{N}/\!/\check{G} \cong N/\!/G.$$

(ii) For a fusion product $N = N_1 \times \cdots \times N_r$ of quasi-Hamiltonian G-spaces, the space $\check{N}$ is a quotient of $\check{N}_1 \times \cdots \times \check{N}_r$ by the group $\{(c_1, \ldots, c_r) \in Z^r |\ \prod_{j=1}^r c_j = e\}$.

(iii) If $\Phi\colon N \to G$ lifts to a moment map $\Phi'\colon N \to \check{G}$, thus turning N into a quasi-Hamiltonian $\check{G}$-space (N, ω, Φ'), then

$$\check{N} = N \times Z$$

as a fusion product of quasi-Hamiltonian $\check{G}$-spaces. Here Z is viewed as a quasi-Hamiltonian $\check{G}$-space, with trivial action and with moment map the inclusion to $\check{G}$.

Proof. (i) By definition of $\check{N}$, the level sets $\check{\Phi}^{-1}(\check{e})$ and $\Phi^{-1}(e)$ are identified, and the pull-backs of the 2-forms to the level sets coincide. Since central elements in $\check{G}$ act trivially on $\check{N}$, the orbit spaces $\check{\Phi}^{-1}(\check{e})/\check{G}$ and $\Phi^{-1}(e)/G$ are identified as well.

(ii) Think of the spaces $\check{N}_i$ as submanifolds of $N_i \times \check{G}$. The canonical map

$$\check{N}_1 \times \cdots \times \check{N}_r \to \check{N},\ (x_1, g_1, x_2, g_2, \ldots, x_r, g_r) \mapsto (x_1, \ldots, x_r, g_1, \ldots, g_r)$$

is exactly the quotient map by $\{(c_1, \ldots, c_r) \in Z^r |\ \prod_{j=1}^r c_j = e\}$, and it preserves the $\check{G}$-actions and 2-forms.

(iii) The map $N \times Z \to \check{N},\ (x, c) \mapsto (x, \Phi'(x)c)$ is the desired diffeomorphism. □

2.3. **The moduli space example.** Our main interest is the moduli space of flat SO(3)-bundles with prescribed boundary holonomies, i.e. (7) with $G = \mathrm{SO}(3)$. In the notation of the previous Section, we will describe the quasi-Hamiltonian SU(2)-space $\check{N}$ associated to the quasi-Hamiltonian SO(3)-space

$$N = \mathcal{C}_1 \times \cdots \times \mathcal{C}_s \times \mathbf{D}(\mathrm{SO}(3))^h.$$

Choose conjugacy classes $\mathcal{D}_j \in \mathrm{SU}(2)$ with $p(\mathcal{D}_j) = \mathcal{C}_j$, and define a quasi-Hamiltonian SU(2)-space

$$\tilde{M} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathbf{D}(\mathrm{SU}(2))^h. \tag{9}$$

Put

$$M = \tilde{M}/\Gamma, \tag{10}$$

where $\Gamma \subset Z^{s+2h}$ consists of $\gamma = (\gamma_1, \ldots, \gamma_{s+2h})$ with the properties $\prod_{j=1}^s \gamma_j = e$ and $\gamma_j \mathcal{D}_j = \mathcal{D}_j$ for $j \le s$. (Equivalently, $\gamma_j = e$ for all $\mathcal{D}_j \ne \mathcal{D}_*$). The conditions guarantee that γ acts on $\tilde{M}$, preserving the 2-form and moment map which hence descend to $M = \tilde{M}/\Gamma$. Let $\mathcal{C}_* \cong \mathbb{R}P(2)$ be the SO(3)-conjugacy class consisting of rotations by π. It is the unique SO(3)-conjugacy class whose pre-image in SU(2) is connected. This pre-image is the SU(2)-conjugacy class $\mathcal{D}_* \cong S^2$ of matrices of trace 0.

Lemma 2.3. *With N as above, we have*

$$\check{N} \cong \begin{cases} M & \text{if} \quad \exists\, j\colon \mathcal{C}_j = \mathcal{C}_* \\ M \times Z & \text{if} \quad \forall\, j\colon \mathcal{C}_j \ne \mathcal{C}_*. \end{cases}$$

Proof. The moment map $\mathbf{D}(\mathrm{SU}(2)) \to \mathrm{SU}(2)$ (given by Lie group commutator) is invariant under the action of $Z \times Z$, hence it descends to a lift $\mathbf{D}(\mathrm{SO}(3)) \to \mathrm{SU}(2)$ of the commutator map for SO(3). Thus

$$\check{\mathbf{D}}(\mathrm{SO}(3)) = \mathbf{D}(\mathrm{SO}(3)) \times Z.$$

If $\mathcal{C}_j \ne \mathcal{C}_*$, the map $\mathcal{D}_j \to \mathcal{C}_j$ is a diffeomorphism, and defines a lift of the moment map $\mathcal{C}_j \hookrightarrow \mathrm{SO}(3)$. Hence

$$\check{\mathcal{C}}_j = \mathcal{D}_j \times Z$$

in that case. On the other hand, the conjugacy class $\mathcal{C}_*$ satisfies

$$\check{\mathcal{C}}_* = \mathcal{D}_*.$$

With these ingredients, the claim follows from Proposition 2.2. □

We may choose the labeling of the conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_s$ in such a way that $\mathcal{C}_j = \mathcal{C}_*$ for $j \le r$ and $\mathcal{C}_j \ne \mathcal{C}_*$ for $j > r$. The space (10) is then a fusion product

$$M = M' \times \mathcal{D}_{r+1} \times \cdots \times \mathcal{D}_s \times \mathbf{D}(\mathrm{SO}(3))^h, \tag{11}$$

where $\mathbf{D}(\mathrm{SO}(3))$ is viewed as a quasi-Hamiltonian $\mathrm{SU}(2)$-space (using the canonical lift of the $\mathrm{SO}(3)$ moment map, as in the proof of Lemma 2.3), and where

$$M' = (\mathcal{D}_* \times \cdots \times \mathcal{D}_*)/\Gamma'$$

with r factors, and with $\Gamma' = \{(\gamma_1, \ldots, \gamma_r) \in Z^r |\ \prod \gamma_j = e\}$. Let us describe the 2-form ω' of the space M', in terms of its pull-back $\tilde{\omega}'$ to the universal cover $\tilde{M} = \mathcal{D}_* \times \cdots \times \mathcal{D}_*$. Since the 2-form on $\mathcal{D}_*$ is just zero, only the fusion terms contribute. By iterative use of the formula (5) for the fusion product, one obtains

$$\tilde{\omega}' = \tfrac{1}{2} \sum_{i<j} g_i^* \theta^L \cdot \mathrm{Ad}_{g_{i+1} \cdots g_{j-1}} (g_j^* \theta^R), \tag{12}$$

where $g_i \colon \tilde{M} \to \mathcal{D}_* \subset \mathrm{SU}(2)$ denotes projection onto the i-th factor.

3. Quantization of the moduli space of flat SO(3)-bundles

In this section we use localization to compute the quantization of the space $M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathbf{D}(\mathrm{SU}(2))^h)/\Gamma$, as an element of the level k fusion ring $R_k(\mathrm{SU}(2))$.

3.1. Pre-quantization.

Recall that we fix the inner product $\cdot$ on $\mathfrak{su}(2)$ to be the *basic inner product*. Then $\eta \in \Omega^3(\mathrm{SU}(2))$ is integral, and represents a generator $x \in H^3(\mathrm{SU}(2); \mathbb{Z}) \cong \mathbb{Z}$. The condition $d\omega + \Phi^*\eta = 0$ from the definition of a quasi-Hamiltonian space says that the pair (ω, η) defines a relative cocycle in $\Omega^3(\Phi)$, the algebraic mapping cone of the pull-back map $\Phi^* \colon \Omega^*(G) \to \Omega^*(M)$. Let $k \in \mathbb{N}$.

Definition 3.1. [13, 15] A *level k pre-quantization* of a quasi-Hamiltonian $\mathrm{SU}(2)$-space (M, ω, Φ) is an integral lift $\alpha \in H^3(\Phi\,; \mathbb{Z})$ of the class $k[(\omega, \eta)] \in H^3(\Phi\,; \mathbb{R})$.

A necessary and sufficient condition for the existence of a level k pre-quantization is that for all smooth singular 2-cycles $\Sigma \in Z_2(M)$, and all smooth singular 3-chains $C \in C_3(G)$ such that $\partial C = \Phi(\Sigma)$,

$$k\Big(\int_\Sigma \omega + \int_C \eta\Big) \in \mathbb{Z}.$$

We list some basic properties and examples of level k pre-quantizations.

(a) The set of level k pre-quantizations is a torsor under the torsion group $\mathrm{Tor}(H^2(M, \mathbb{Z}))$ of isomorphism classes of flat line bundles.
(b) The level k pre-quantized conjugacy classes of $\mathrm{SU}(2)$ are exactly those of the elements $\exp(\frac{m}{k}\rho)$ with $m = 0, \ldots, k$ [15, Proposition 7.3].
(c) The double $\mathbf{D}(\mathrm{SO}(3))$ (viewed as a quasi-Hamiltonian $\mathrm{SU}(2)$-space) admits a level k pre-quantization if and only if k is even [15, Proposition 7.4].
(d) If M_1 and M_2 are pre-quantized quasi-Hamiltonian $\mathrm{SU}(2)$-spaces at level k, then their fusion product $M_1 \times M_2$ inherits a pre-quantization at level k. Conversely, a pre-quantization of the product induces pre-quantizations of the factors. See [13, Proposition 3.8].
(e) A level k pre-quantization of M induces a pre-quantization of the symplectic quotient $M/\!/\,\mathrm{SU}(2)$, equipped with the k-th multiple of the symplectic form.
(f) The long exact sequence in relative cohomology gives a necessary condition $k\Phi^*(x) = 0$ for the existence of a level k pre-quantization. If $H^2(M; \mathbb{R}) = 0$, this condition is also sufficient [13, Proposition 4.2].

(g) The existence of the canonical 'twisted Spin_c-structure' [15, Section 6] on quasi-Hamiltonian SU(2)-spaces (M,ω,Φ) implies that $2\Phi^*(x) = W^3(M)$, the third integral Stiefel-Whitney class. Since this is a 2-torsion class, $4\Phi^*(x) = 0$. In fact, there is a distinguished element $\beta \in H^3(\Phi\,;\mathbb{Z})$ whose image in $H^3(\mathrm{SU}(2)\,;\mathbb{Z})$ is $4x$. If $H^2(M,\mathbb{R}) = 0$, this element gives a distinguished level 4 pre-quantization.

Given a level k pre-quantization of a quasi-Hamiltonian SU(2)-space (M,ω,Φ) the construction from [15] produces a *quantization* $\mathcal{Q}(M) \in R_k(\mathrm{SU}(2))$, an element of the level k fusion ring. It is obtained as a push-forward in twisted equivariant K-homology, using the Freed-Hopkins-Teleman theorem [8] to identify $R_k(\mathrm{SU}(2))$ with the equivariant twisted K-homology of SU(2) at level $k+2$. This is the quasi-Hamiltonian counterpart of the Spin_c quantization of an ordinary compact Hamiltonian SU(2)-space, which produces an element of $R(\mathrm{SU}(2))$ as the equivariant index of a Spin_c-Dirac operator with coefficients in an equivariant pre-quantum line bundle. The quantization procedure for quasi-Hamiltonian G-spaces satisfies properties similar to its Hamiltonian analog. These include

(1) compatibility with products, $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$; and
(2) the 'quantization commutes with reduction' principle, $\mathcal{Q}(M/\!\!/G) = \mathcal{Q}(M)^G$.

Here $R_k(G) \to \mathbb{Z},\ \tau \mapsto \tau^G$ is the trace defined by $\tau_m^G = \delta_{m,0}$.

3.2. **Pre-quantization of M.** Let us now consider level k pre-quantizations of the quasi-Hamiltonian SU(2)-space

$$M = (\mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathbf{D}(\mathrm{SU}(2))^h)/\Gamma$$

from (10).

Theorem 3.2. *The quasi-Hamiltonian* SU(2)*-space M carries a level k pre-quantization if and only if the following conditions are satisfied:*

(i) The conjugacy classes $\mathcal{D}_j$ are of the form $\mathrm{SU}(2).\exp(\frac{m_j}{k}\rho)$ with $m_j \in \{0,\ldots,k\}$,
(ii) if $h \geq 1$, then $k \in 2\mathbb{N}$,
(iii) if the number of $\mathcal{D}_*$-factors is $r \geq 3$, then $k \in 4\mathbb{N}$.

Note that if at least one $\mathcal{D}_*$-factor appears, then the first condition requires that $k \in 2\mathbb{N}$ since $\mathcal{D}_* = \mathrm{SU}(2).\exp(\frac{1}{2}\rho)$.

Proof. Since a level k pre-quantization of M induces a level k pre-quantization of the universal cover $\tilde{M}$, it is a necessary condition that all $\mathcal{D}_j$ be pre-quantizable. That is, $\mathcal{D}_j = \mathrm{SU}(2).\exp(\frac{m_j}{k}\rho)$ with $m_j \in \{0,\ldots,k\}$.

Let us enumerate the conjugacy classes in such a way that $\mathcal{D}_1 = \ldots = \mathcal{D}_r = \mathcal{D}_*$. Using the decomposition (11) and the known pre-quantization conditions (b),(c) for the conjugacy classes $\mathcal{D}_j$ and the double $\mathbf{D}(\mathrm{SO}(3))$, together with the fusion property (d), the proof is reduced to the case $h = 0,\ s = r$. We may thus assume $M = (\mathcal{D}_* \times \cdots \times \mathcal{D}_*)/\Gamma$ with r factors. If $r = 1$ then $M = \mathcal{D}_*$, which is pre-quantized at level k if and only if k is even. Suppose $r > 1$. The non-trivial element $c \in Z$ acts on $H^2(\mathcal{D}_*;\mathbb{R}) \cong \mathbb{R}$ as multiplication by -1. Hence, Γ acts on $H^2(M;\mathbb{R}) \cong \mathbb{R}^r$ by componentwise sign changes. In particular, the Γ-invariant part is trivial. Since Γ acts freely, it follows that

$$H^2(M\,;\mathbb{R}) \cong H^2(\tilde{M}\,;\mathbb{R})^\Gamma = 0.$$

Hence, by Property (f), a level k pre-quantization exists if and only if $k\Phi^*(x) = 0$. If $r = 2$, so that $M = (\mathcal{D}_* \times \mathcal{D}_*)/\mathbb{Z}_2$, Poincaré duality gives that $H^3(M;\mathbb{Z}) \cong \mathbb{Z}_2$; therefore $2\Phi^*(x) = 0$. Hence the condition $k \in 2\mathbb{N}$ is also sufficient if $r = 2$.

It remains to consider the case $r \geq 3$. By Property (g), the condition $k \in 4\mathbb{N}$ is sufficient. Let us show that it is also necessary. Observe that the non-identity component of the normalizer, the circle $Tu_* = N(T) - T$, is a single conjugacy class inside $N(T)$. Since $u_* \in \mathcal{D}_*$, it follows that $Tu_* \subset \mathcal{D}_*$. Let $\tilde{X} \subset \tilde{M} = \mathcal{D}_* \times \cdots \times \mathcal{D}_*$ be the 2-torus given as the image of the map

$$T \times T \to \tilde{M}, \quad (h_1, h_2) \mapsto \big(h_1 u_*, h_2 u_*, h_1 h_2 u_*, u_*, \ldots, u_*\big),$$

and denote by X its image in M. Let $\tilde{\omega}_X, \omega_X$ be the pull-backs of the quasi-Hamiltonian 2-forms on $\tilde{X}$, X. Since $Tu_* = u_* T$, we have $\tilde{\Phi}(\tilde{X}) = \Phi(X) \subset T{u_*}^r$. Since the generator $x \in H^3(\mathrm{SU}(2), \mathbb{Z})$ pulls back to zero on this circle (for dimension reasons), the existence of a level k pre-quantization of M requires that $k \int_X \omega_X \in \mathbb{Z}$. Since the projection $\tilde{X} \to X$ is a 4-fold covering, $\int_X \omega_X = \frac{1}{4} \int_{\tilde{X}} \tilde{\omega}_X$. Hence it is necessary that $k \int_{\tilde{X}} \tilde{\omega}_X \in 4\mathbb{Z}$.

Let $\theta \in \Omega^1(T, \mathfrak{t})$ be the Maurer-Cartan form for T. From the general formula (12), and using $(hu_*)^*\theta^L = -h^*\theta,\ (hu_*)^*\theta^R = h^*\theta$, we obtain

$$\tilde{\omega}_X = \frac{1}{2}\big(- h_1^*\theta \wedge h_2^*\theta + h_1^*\theta \wedge (h_1 h_2)^*\theta - h_2^*\theta \wedge (h_1 h_2)^*\theta \big) = \frac{1}{2} h_1^*\theta \wedge h_2^*\theta.$$

Writing elements of T in the form $h = j(e^{2\pi i v})$, we may take $v \in [0,1]$ as the coordinate on $T \cong \mathbb{R}/\mathbb{Z}$. Since the lattice Λ is generated by 2ρ, we find $h_i^*\theta = 2dv_i \otimes \rho$, hence

$$\tilde{\omega}_X = 2||\rho||^2 \ dv_1 \wedge dv_2 = dv_1 \wedge dv_2$$

integrates to 1. This gives the condition $k \in 4\mathbb{N}$. □

3.3. **Fixed point components.** Suppose M is a level k pre-quantized quasi-Hamiltonian SU(2)-space, and let $\mathcal{Q}(M) \in R_k(\mathrm{SU}(2))$ be its quantization. By [15, Theorem 9.5], the numbers $\mathcal{Q}(M)(t)$ with $t = t_l,\ l = 0, \ldots, k$ are given as a sum of contributions from the fixed point manifolds of t:

$$\mathcal{Q}(M)(t) = \sum_{F \subset M^t} \int_F \frac{\widehat{A}(F)\ \mathrm{Ch}(\mathcal{L}_F,\, t)^{1/2}}{D_{\mathbb{R}}(\nu_F, t)}. \tag{13}$$

The ingredients of the right hand side will be described below, and explicitly computed in the context of our main example (10). The quantizations of SU(2)-conjugacy classes and of the double $\mathbf{D}(\mathrm{SO}(3))$ (viewed as a quasi-Hamiltonian SU(2)-space) were computed in [15].

For the remainder of this section, we therefore focus on the case $h = 0$, $s = r \geq 2$, i.e. $M = (\mathcal{D}_* \times \cdots \times \mathcal{D}_*)/\Gamma$.

3.3.1. *Fixed point sets of M.* We need to determine the components $F \subset M^t$ of the fixed point manifold for $t = t_l$, $l = 0, \ldots, k$, and describe various aspects of F and its normal bundle ν_F. Consider first a general regular element $t \in T^{\mathrm{reg}}$. Define the following two submanifolds of $\mathcal{D}_*$, labeled by the elements of the center $Z = \{e, c\}$ as follows:

$$Y^{(e)} = \mathcal{D}_* \cap T = \{t_*, {t_*}^{-1}\}, \quad Y^{(c)} = Tu_*.$$

Thus $Y^{(e)}$ is the fixed point set of $\mathrm{Ad}(t_*)$, while $Y^{(c)}$ consists of elements satisfying $\mathrm{Ad}(t_*)(g) = cg$. Note that both are Z-invariant. For $\gamma = (\gamma_1, \dots, \gamma_r) \in \Gamma$, consider the Γ-invariant submanifold

$$\tilde{F}^{(\gamma)} = Y^{(\gamma_1)} \times \cdots \times Y^{(\gamma_r)}.$$

and put $F^{(\gamma)} = \tilde{F}^{(\gamma)}/\Gamma$. Let $\mathsf{l}(\gamma)$ be the number of γ_i's that are equal to c. Then $\tilde{F}^{(\gamma)}$ is a disjoint union of $2^{r-\mathsf{l}(\gamma)}$ tori of dimension $\mathsf{l}(\gamma)$. Let $\varepsilon = (e, \dots, e)$ denote the group unit in Γ. If $\gamma \neq \varepsilon$, then Γ acts transitively on the set of components of $\tilde{F}^{(\gamma)}$. Hence $F^{(\gamma)}$ is a (connected) torus, and since $|\Gamma| = 2^{r-1}$, it follows that the projection restricts to a $2^{\mathsf{l}(\gamma)-1}$-fold covering on each component of $\tilde{F}^{(\gamma)}$. If $\gamma = \varepsilon$, $\tilde{F}^{(\varepsilon)}$ consists of 2^r points, and hence $F^{(\varepsilon)}$ consists of two points.

Proposition 3.3. *The fixed point set of $t \in T^{\mathrm{reg}}$ in M is*

$$M^t = \begin{cases} F^{(\varepsilon)} & \text{if} \quad t \notin \{t_*, {t_*}^{-1}\}, \\ \coprod_{\gamma \in \Gamma} F^{(\gamma)} & \text{if} \quad t \in \{t_*, {t_*}^{-1}\}. \end{cases}$$

Proof. An element $(g_1, \dots, g_r) \in \tilde{M}$ maps to a point in M^t if and only if there exists $\gamma = (\gamma_1, \dots, \gamma_r) \in \Gamma$ with $\mathrm{Ad}(t) g_i = g_i \gamma_i$, for $i = 1, \dots, r$. If $\gamma_i = e$, this condition gives $g_i \in T$, since t is regular. If $\gamma_i = c$, the condition says that $\mathrm{Ad}(g_i^{-1})(t) = \gamma_i t$. Since t is regular, this happens if and only if $t \in \{t_*,\, {t_*}^{-1}\}$, with $g_i \in N(T)$ representing the non-trivial Weyl group element. □

3.3.2. *The symplectic volume of the components of the fixed point set.* Each $F^{(\gamma)} \subset M^t$ is a quasi-Hamiltonian T-space, with moment map the restriction of Φ. (See e.g. [14, Proposition 3.1].) In particular, they are symplectic.

Lemma 3.4. *The symplectic volume of each component of $\tilde{F}^{(\gamma)}$ is equal to 1. Thus*

$$\mathrm{vol}(F^{(\gamma)}) = 2^{1-\mathsf{l}(\gamma)}.$$

Proof. The construction from [3] associates to any quasi-Hamiltonian G-space (with G compact, but possibly disconnected) a Liouville volume, in such a way that the volume of a fusion product is the product of the volumes. If $G = T$, so that the space is symplectic, the Liouville volume coincides with the symplectic volume. For a G-conjugacy class $\mathcal{C} \cong G/G_g$, the Liouville volume is given by the formula [3, Proposition 3.6]

$$\mathrm{vol}\, \mathcal{C} = |\mathrm{det}_{\mathfrak{g}_g^\perp}(1 - \mathrm{Ad}_g)|^{1/2} \frac{\mathrm{vol}(G)}{\mathrm{vol}(G_g)},$$

involving the Riemannian volumes of G and of the stabilizer group G_g. The spaces $Y^{(z)}$ for $z \in Z$ can be viewed as conjugacy classes for the group $N(T)$, of elements t_* if $z = e$ and u_* if $z = c$. Application of the formula gives

$$\mathrm{vol}(Y^{(z)}) = \begin{cases} 2 & \text{if } z = e \\ 1 & \text{if } z = c \end{cases}.$$

This is obvious for $z = e$, while for $z = c$ (so that $g = u_*,\ N(T)_g = \mathbb{Z}_4$) we have $|\det_{\mathfrak{t}}(1 - \mathrm{Ad}_{u_*})|^{1/2} = \sqrt{2}$ (since Ad_{u_*} acts as -1 on $\mathfrak{t}$), $\mathrm{vol}(N(T)) = 2\,\mathrm{vol}(T) = 2||\alpha|| = 2\sqrt{2}$, and $\mathrm{vol}(N(T)_g) = 4$. It follows that

$$\mathrm{vol}(\tilde{F}^{(\gamma)}) = \prod_{i=1}^{r} \mathrm{vol}(Y^{(\gamma_i)}) = 2^{r-\mathsf{l}(\gamma)}.$$

Since the moment map for the quasi-Hamiltonian $N(T)$-space $\tilde{F}^{(\gamma)}$ takes values in T, this coincides with the symplectic volume. Since $2^{r-\mathsf{l}(\gamma)}$ is also the number of components of $\tilde{F}^{(\gamma)}$, it follows that each component has volume 1. □

3.4. **Fixed point contributions.** In this Section, we assume that $M = (\mathcal{D}_* \times \cdots \times \mathcal{D}_*)/\Gamma$ carries a level k pre-quantization. Thus $k \in 2\mathbb{N}$ if $r = 2$ and $k \in 4\mathbb{N}$ if $r > 2$. Our aim is to compute the fixed point contributions to $\mathcal{Q}(M)(t)$, as described in formula (13), for $t = t_l$, $l = 0, \ldots, k$.

If $t \neq t_*$, Proposition 3.3 shows that $M^t = F^{(\varepsilon)}$ consists of just two points, covered by the set $\tilde{M}^t = \tilde{F}^{(\varepsilon)}$ (consisting of 2^r points). The fixed point contribution of $F^{(\varepsilon)}$ is just that for $\tilde{F}^{(\varepsilon)}$, divided by $|\Gamma| = 2^{r-1}$. Hence

$$\mathcal{Q}(M)(t) = 2^{1-r}\mathcal{Q}(\tilde{M}^t) = 2^{1-r}\mathcal{Q}(\mathcal{D}_*)^r(t),$$

with $\mathcal{Q}(\mathcal{D}_*) = \tau_{k/2}$ [15, Proposition 11.2] .

If $t = t_*$, $\mathcal{Q}(M)(t_*)$ is a sum over the contributions from all $F^{(\gamma)}$, $\gamma \in \Gamma$. The contribution from $F^{(\varepsilon)}$ is $2^{1-r}(\mathcal{Q}(\mathcal{D}_*)(t_*))^r$, as before. Calculation of the contributions from $F = F^{(\gamma)}$, $\gamma \neq \varepsilon$ requires more work:

Proposition 3.5. *The contribution of the fixed point manifold $F = F^{(\gamma)}$, $\gamma \neq \varepsilon$ to $\mathcal{Q}(M)(t_*)$ is*

$$\int_F \frac{\widehat{A}(F)\ \mathrm{Ch}(\mathcal{L}_F,\, t_*)^{1/2}}{D_{\mathbb{R}}(\nu_F, t_*)} = 2^{1-r}\left(\tfrac{k}{2}+1\right)^{\mathsf{l}(\gamma)/2}\varphi^{(\gamma)},$$

where the scalar $\varphi^{(\gamma)} = \mu_{F^{(\gamma)}}(t_*) \in \mathrm{U}(1)$ is the action of t_* on the pre-quantum line bundle over $F^{(\gamma)}$.

Proof. Since $F = F^{(\gamma)}$ is a torus, $\widehat{A}(F) = 1$. To compute the $D_{\mathbb{R}}$-class, note that the normal bundle of Tu_* in $\mathcal{D}_*$ is an orientable real line bundle, hence it is trivializable. Consequently, the normal bundle $\nu_{\tilde{F}^{(\gamma)}}$ to $\tilde{F}^{(\gamma)}$ in $\tilde{M}$ is trivializable, and thus the normal bundle $\nu_F = \nu_{\tilde{F}^{(\gamma)}}/\Gamma$ to F in M is a flat Euclidean vector bundle of rank $2r - \mathsf{l}(\gamma)$. The element t_* acts by multiplication by -1 on the fibers of ν_F, since $\mathrm{Ad}(t_*)$ has order 2 and cannot act trivially. By definition of the $D_{\mathbb{R}}$-class (see [2, Section 2.3] or [14, Section 5.3]), it follows that

$$D_{\mathbb{R}}(\nu_F, t_*) = i^{\mathrm{rank}(\nu_F)/2}\mathrm{det}_{\mathbb{R}}^{1/2}(1-(-1)) = (2i)^{r-\frac{\mathsf{l}(\gamma)}{2}}.$$

By [15, Proposition 9.3], the restriction $TM|_F$ inherits a distinguished Spin_c-structure (depending on the choice of level k pre-quantization), equivariant for the action of t_*. The line bundle $\mathcal{L}_F \to F$ is the Spin_c-line bundle associated to this Spin_c-structure, and

$$\mathrm{Ch}(\mathcal{L}_F,\, t_*)^{1/2} = \sigma(\mathcal{L}_F)(t_*)^{1/2}\exp(\tfrac{1}{2}c_1(\mathcal{L}_F))$$

is the square root of its equivariant Chern character, with $\sigma(\mathcal{L}_F)(t_*) \in \mathrm{U}(1)$ the action of t_* the Spin_c-line bundle. As discussed in [2, Section 2.3] (see also [14, Section 5.3]), the sign of the square root is determined as follows. Since Φ restricts to a surjective map $F \to T$, the fixed point set F meets $\Phi^{-1}(e)$. Pick any $x \in F \cap \Phi^{-1}(e)$. Observe that ω is non-degenerate at points of $\Phi^{-1}(e)$, and choose a t_*-invariant compatible complex structure to view T_xM as a Hermitian vector space. Let $A \in \mathrm{U}(T_xM)$ be the transformation defined by t_* and $A^{1/2}$ its unique square root for which all eigenvalues are of the form e^{iu} with $0 \leq u < \pi$. Then

$$\sigma(\mathcal{L}_F)(t_*)^{1/2} = \varphi^{(\gamma)}\mathrm{det}_{\mathbb{C}}(A^{1/2}).$$

Since t_* acts trivially on T_mF and as -1 on the normal bundle, the transformation $A^{1/2}$ acts trivially on T_xF and as i on the normal bundle. Thus $\det_{\mathbb{C}}(A^{1/2}) = i^{r-\mathsf{l}(\gamma)/2}$, which cancels a similar factor in the expression for the $D_{\mathbb{R}}$-class.

It remains to find the integral $\int_F \exp(\frac{1}{2}c_1(\mathcal{L}_F))$. To this end, we interpret $\mathcal{L}_F$ as a pre-quantum line bundle. By the same argument as in Property (g) of Section 3.1, (see also [15, Section 11.1]), the level k pre-quantization and the canonical twisted Spin_c-structure on M combine to give an element of $H^3(\Phi\,;\mathbb{Z})$ at level $2k+4$. Since $H^2(M\,;\mathbb{R}) = 0$, this element defines a pre-quantization at level $2k+4$. Pull-back to F defines a level $2k+4$ pre-quantization of F, with $\mathcal{L}_F$ as the pre-quantum line bundle. Hence $c_1(\mathcal{L}_F)$ is the $2k+4$-th multiple of the class of the symplectic form on F. It follows that

$$\int_F \exp(\tfrac{1}{2}c_1(\mathcal{L}_F)) = (k+2)^{\mathsf{l}(\gamma)}\operatorname{vol}(F) = 2^{1-\mathsf{l}(\gamma)/2}\left(\tfrac{k}{2}+1\right)^{\mathsf{l}(\gamma)/2}$$

where we have used Lemma 3.4. ☐

The phase factors $\varphi^{(\gamma)}$ depend on the choice of pre-quantization. Recall again that the set of pre-quantizations of a quasi-Hamiltonian SU(2)-space is a torsor under the group of isomorphism classes of flat line bundles. In our case this is the group

$$\mathrm{Tor}(H^2(M\,;\mathbb{Z})) \cong \mathrm{Hom}(\Gamma, \mathrm{U}(1)).$$

The homomorphism $\psi\colon \Gamma \to \mathrm{U}(1)$ defines the flat line bundle $\tilde{M} \times_\Gamma \mathbb{C}_\psi$, where $\mathbb{C}_\psi$ is the 1-dimensional Γ-representation defined by ψ. Changing the pre-quantization by such a flat line bundle changes $\varphi^{(\gamma)}$ for $F = F^{(\gamma)}$ to $\psi(\gamma)\varphi^{(\gamma)}$. By Property (g) of Section 3.1, and since $H^2(M;\mathbb{R}) = 0$, there is a *distinguished* pre-quantization at any level $k \in 4\mathbb{N}$. Hence, the inequivalent pre-quantizations at level $k \in 4\mathbb{N}$ are labeled by $\mathrm{Hom}(\Gamma, \mathrm{U}(1))$.

Lemma 3.6. *If $r \geq 3$ and $k \in 4\mathbb{N}$, the phase factor for the pre-quantization labeled by $\psi \in \mathrm{Hom}(\Gamma, \mathrm{U}(1))$ is given by*

$$\varphi^{(\gamma)} = (-1)^{\frac{k}{4}(r-\mathsf{l}(\gamma)/2)}\psi(\gamma).$$

Proof. The phase factor $\varphi^{(\gamma)}$ for the distinguished pre-quantization at level 4 is given by $\det_{\mathbb{C}}(A) = (-1)^{r-\mathsf{l}(\gamma)/2}$, in the notation from the proof of Proposition 3.5. For the distinguished pre-quantization at level $k \in 4\mathbb{N}$, we have to take the $\frac{k}{4}$-th power of this number, and changing the pre-quantization by ψ we have to multiply by $\psi(\gamma)$. ☐

If $r = 2$, there are $|\Gamma| = 2$ distinct pre-quantizations at all even levels $k \in 2\mathbb{N}$, related by elements $\psi \in \mathrm{Hom}(\Gamma, \mathrm{U}(1))$. Aside from the discrete fixed point set $F^{(\varepsilon)}$, there is a single non-discrete fixed point component $F^{(\gamma)}$ of t_*, given by $\gamma = (c, c)$. The non-trivial homomorphism $\psi \in \mathrm{Hom}(\Gamma, \mathrm{U}(1)) \cong \mathbb{Z}_2$ satisfies $\psi(c, c) = -1$, hence the weight $\varphi^{(\gamma)}$ is equal to 1 for one of the pre-quantizations and -1 for the other.

3.5. **Quantization of M.** We are now ready to summarize our computation of $\mathcal{Q}(M)$ for $M = (\mathcal{D}_* \times \cdots \times \mathcal{D}_*)/\Gamma$. Assuming that k is even, recall that $\mathcal{D}_*$ has a unique pre-quantization at level k, and $\mathcal{Q}(\mathcal{D}_*) = \tau_{k/2}$. Define an element

$$\chi = \tau_0 - \tau_2 + \tau_4 - \cdots + (-1)^{k/2}\tau_k \in R_k(\mathrm{SU}(2)).$$

By the orthogonality relations for $R_k(\mathrm{SU}(2))$, this element satisfies $\chi(t_*) = (\frac{k}{2}+1)$ and $\chi(t) = 0$ for $t = t_l$, $l \neq k/2$. Hence we may write the sum over the fixed point contributions as follows:

$$\mathcal{Q}(M)(t) = 2^{1-r}\Big(\tau_{k/2}(t)^r + \chi(t) \sum_{\gamma\in\Gamma\backslash\{\varepsilon\}} \big(\tfrac{k}{2}+1\big)^{\mathsf{l}(\gamma)/2-1}\,\varphi^{(\gamma)}\Big)$$

Theorem 3.7. *Consider the quasi-Hamiltonian* $\mathrm{SU}(2)$*-space* $M = (\mathcal{D}_*\times\cdots\times\mathcal{D}_*)/\Gamma$ *with* $r \geq 2$ *factors, where* $\Gamma \subset Z^r$ *consists of all* $\gamma = (\gamma_1, \ldots, \gamma_r)$ *with* $\prod_{i=1}^r \gamma_i = e$.

(1) *If* $r \geq 3$*, the space* M *is pre-quantized at level* k *if and only if* $k \in 4\mathbb{N}$*. The different pre-quantizations are indexed by the elements* $\psi \in \mathrm{Hom}(\Gamma, \mathrm{U}(1))$*, and the corresponding level* k *quantization is given by the formula,*

$$\mathcal{Q}_\psi(M) = 2^{1-r}\Big((\tau_{k/2})^r + \chi \sum_{\gamma\in\Gamma\backslash\{\varepsilon\}} \psi(\gamma)\big(\tfrac{k}{2}+1\big)^{\frac{\mathsf{l}(\gamma)}{2}-1}(-1)^{\frac{k}{4}(r-\frac{\mathsf{l}(\gamma)}{2})}\Big).$$

(2) *If* $r = 2$*, the space* M *is pre-quantized at level* k *if and only if* $k \in 2\mathbb{N}$*. At any such level, there are two distinct pre-quantizations indexed by the action* ± 1 *of* t_* *on the pre-quantum line bundle over* $F^{(\gamma)}$*, for* $\gamma = (c, c)$*. The corresponding level* k *quantizations of* M *are*

$$\mathcal{Q}_\pm(M) = \frac{1}{2}\left((\tau_{k/2})^2 \pm \chi\right).$$

3.6. **Multiplicity computations.** Being elements of $R_k(\mathrm{SU}(2))$, the coefficients of $\mathcal{Q}(M)$ in its decomposition with respect to the basis $\tau_0, \ldots, \tau_k$ must be integers. In this Section, we will compute these multiplicities for small r.

3.6.1. $r = 2$ *factors.* Assume $k \in 2\mathbb{N}$, and let $\mathcal{Q}_\pm(M)$ be the quantizations corresponding to the pre-quantizations labeled by ± 1. The multiplication rules for level k characters give

$$(\tau_{k/2})^2 = \tau_0 + \tau_2 + \ldots + \tau_k.$$

Hence, if $k \in 4\mathbb{N}$ we obtain

$$\mathcal{Q}_+(M) = \tau_0 + \tau_4 + \ldots + \tau_k,$$
$$\mathcal{Q}_-(M) = \tau_2 + \tau_6 + \ldots + \tau_{k-2},$$

while for $k \in 4\mathbb{N} - 2$,

$$\mathcal{Q}_+(M) = \tau_0 + \tau_4 + \ldots + \tau_{k-2},$$
$$\mathcal{Q}_-(M) = \tau_2 + \tau_6 + \ldots + \tau_k.$$

3.6.2. $r = 3$ *factors.* Let $\mathcal{Q}_\psi(M)$ denote the level $k \in 4\mathbb{N}$ pre-quantization indexed by $\psi \in \mathrm{Hom}(\Gamma, \mathrm{U}(1))$. Since $r = 3$, $\mathsf{l}(\gamma) = 2$ for any $\gamma \neq \varepsilon$ and the quantization formula simplifies to:

$$\mathcal{Q}_\psi(M) = \frac{1}{4}\Big(\tau_{2m}^3 + \chi \sum_{\gamma\neq\varepsilon} \psi(\gamma)\Big).$$

For the trivial homomorphism $\psi = 1$, we have $\sum_{\gamma\neq\varepsilon}\psi(\gamma) = 3$, while for a non-trivial homomorphism $\psi \neq 1$, $\sum_{\gamma\neq\varepsilon}\psi(\gamma) = -1$. We have,

$$(\tau_{k/2})^3 = \tau_0 + 3\tau_2 + \ldots + (\tfrac{k}{2}+1)\tau_{k/2} + \ldots + 3\tau_{k-2} + \tau_k.$$

We therefore obtain

$$\begin{aligned}\mathcal{Q}_\psi(M) &= (\tau_0 + 2\tau_4 + 3\tau_8 + \dots 3\tau_{k-8} + 2\tau_{k-4} + \tau_k) \\ &\quad + (\tau_6 + 2\tau_{10} + \dots + 2\tau_{k-10} + \tau_{k-6}) && \text{if } \psi = 1, \\ \mathcal{Q}_\psi(M) &= (\tau_0 + \tau_4 + 2\tau_8 + \dots + 3\tau_{k-8} + 2\tau_{k-4} + \tau_k) \\ &\quad + (\tau_2 + 2\tau_6 + 3\tau_{10} + \dots + 3\tau_{k-10} + 2\tau_{k-6} + \tau_{k-2}) && \text{if } \psi \neq 1.\end{aligned}$$

Note that the coefficients are symmetric about the midpoint $\frac{k}{2}$ of the interval $[0, k]$. In closed form, $\mathcal{Q}_\psi(M) = \sum_{j=0}^{k/2} a_{2j}\tau_{2j}$, where

$$a_{2j} = \begin{cases} \frac{1}{4}(2j + 1 + (4\delta_{\psi,1} - 1)(-1)^j) & : \ 2j \leq k/2, \\ \frac{1}{4}(k - 2j + 1 + (4\delta_{\psi,1} - 1)(-1)^j) & : \ 2j \geq k/2. \end{cases}$$

3.6.3. $r = 4$ *factors.* If $r = 4$ we have $|\Gamma| = 8$. There is a unique element $\gamma' \in \Gamma$ with $\mathsf{l}(\gamma') = 4$, and $\mathsf{l}(\gamma) = 2$ for $\gamma \neq \gamma', \varepsilon$. Hence we may write the quantization formula for levels $k \in 4\mathbb{N}$ as:

$$\mathcal{Q}_\psi(M) = \frac{1}{8}\Big(\tau_{k/2}^4 + \big(\psi(\gamma')\left(\tfrac{k}{2} + 1\right) + (-1)^{k/4} \sum_{\mathsf{l}(\gamma)=2} \psi(\gamma)\big)\chi\Big).$$

One finds that there are 4 homomorphisms ψ with $\sum_{\mathsf{l}(\gamma)=2} \psi(\gamma) = 0, \ \psi(\gamma') = -1$ and 3 homomorphisms with $\sum_{\mathsf{l}(\gamma)=2} \psi(\gamma) = -2, \ \psi(\gamma') = 1$. Of course, $\sum_{\mathsf{l}(\gamma)=2} \psi(\gamma) = 6, \ \psi(\gamma') = 1$ for $\psi = 1$. Therefore, we have

$$\mathcal{Q}_\psi(M) = \begin{cases} \frac{1}{8}\left(\tau_{k/2}^4 + \left(6(-1)^{k/4} + \left(\frac{k}{2} + 1\right)\right)\chi\right) & : \ \psi = 1 \\ \frac{1}{8}\left(\tau_{k/2}^4 - \left(\frac{k}{2} + 1\right)\chi\right) & : \ \sum_{\mathsf{l}(\gamma)=2} \psi(\gamma) = 0 \\ \frac{1}{8}\left(\tau_{k/2}^4 + \left(2(-1)^{k/4+1} + \left(\frac{k}{2} + 1\right)\right)\chi\right) & : \ \sum_{\mathsf{l}(\gamma)=2} \psi(\gamma) = -2 \end{cases}$$

with

$$(\tau_{k/2})^4 = \sum_{j=0}^{k/2} (\tfrac{k}{2} + 1 - 2j^2 + jk)\tau_{2j}.$$

One may verify that the multiplicities of τ_{2j} in $\mathcal{Q}_\psi(M)$ are integers, as required.

4. Fuchs-Schweigert

The formulas appearing in Theorem 3.7 may be rewritten in terms of the so-called S-matrix. For $z \in Z$, define $S_{m,l}^{(z)}$ by

$$S_{m,l}^{(z)} = \begin{cases} 1 & \text{if } z = c \\ \mathrm{S}_{m,l} & \text{if } z = e. \end{cases}$$

In the terminology of [9], $S_{m,l}^{(z)}$ is the S-matrix of the *orbit Lie algebra* associated to the central element z. (This interpretation may seem obscure for SU(2), but becomes natural for higher rank groups.) Consider once again the space $M = \tilde{M}/\Gamma$ from (10). Recall that Γ consists of elements $\gamma = (\gamma_1, \dots, \gamma_{s+2h}) \in Z^{s+2h}$ such that $\prod_{j=1}^{s} \gamma_j = e$, and $\gamma_j = e$ for all $j \leq s$ with $\mathcal{C}_j \neq \mathcal{C}_*$. In particular $|\Gamma| = 2^{2h+r-1}$ if $r \geq 1$, while $|\Gamma| = 2^{2h}$ if $r = 0$. To write the Fuchs-Schweigert formula, it is convenient to use the following notation. For $\gamma \in \Gamma$, let $\sum_l^{(\gamma)}$ denote the full sum $\sum_{l=0}^{k}$ if all $\gamma_i = e$, and consisting of the single term $l = \frac{k}{2}$ if at least one $\gamma_i \neq e$. (For higher rank groups, this becomes a sum over level k weights that are fixed under the action of all $\gamma_i \in Z$ on the set of level k weights.) We will prove the following equivariant analogue to the Fuchs-Schweigert formula:

Theorem 4.1. *Suppose the quasi-Hamiltonian* SU(2)*-space*

$$M = \big(\mathcal{D}_1 \times \cdots \times \mathcal{D}_s \times \mathbf{D}(\mathrm{SU}(2))^h\big)/\Gamma$$

is pre-quantized at level k. Then

$$\mathcal{Q}(M) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\varphi'(\gamma){\sum_l}^{(\gamma)} \frac{S^{(\gamma_1)}_{m_1,l}\cdots S^{(\gamma_s)}_{m_s,l}}{(S_{0,l})^{s+2h}}\ \tilde{\tau}_l, \tag{14}$$

where $\varphi'(\gamma) \in \mathrm{U}(1)$ are phase factors depending on the choice of pre-quantization, with $\varphi'(\varepsilon) = 1$.

An explicit description of the phase factors $\varphi'(\gamma)$ will be given during the course of the proof.

Proof of Theorem 4.1. The space M is a fusion product of the space $\widetilde{(\mathcal{C}_*)^r}$, conjugacy classes $\mathcal{D}_j \neq \mathcal{D}_*$, and h factors of $\mathbf{D}(\mathrm{SO}(3))$ (viewed as a quasi-Hamiltonian SU(2)-space). Since the fusion product in the basis $\tilde{\tau}_m$ is diagonalized, we may verify the formula separately for factors of these three types.

We begin with the case $h = 0, s = r$, with $r \geq 3$ (thus necessarily $k \in 4\mathbb{N}$). We re-write the right hand side of (14), separating the term $\gamma = \varepsilon$ from the sum over terms $\gamma \neq \varepsilon$. The right hand side of (14) becomes

$$\mathcal{Q}(M) = \frac{1}{|\Gamma|}\Big(\varphi'(\varepsilon)\sum_l \frac{(S_{k/2,l})^r}{(S_{0,l})^r}\tilde{\tau}_l + \sum_{\gamma\neq\varepsilon}\varphi'(\gamma)\frac{(S_{k/2,k/2})^{r-\mathsf{l}(\gamma)}}{(S_{0,k/2})^r}\ \tilde{\tau}_{k/2}\Big). \tag{15}$$

The sum over l is just $(\tau_{k/2})^r$. The element $\chi \in R_k(\mathrm{SU}(2))$ considered in Section 3.5 satisfies $\chi(t_l) = (\frac{k}{2}+1)\delta_{l,k/2}$ for $l = 0,\ldots,k$, hence

$$\tilde{\tau}_{\frac{k}{2}} = (\tfrac{k}{2}+1)^{-1}\chi.$$

Furthermore, by definition of the S-matrix,

$$S_{0,k/2} = (\tfrac{k}{2}+1)^{-\frac{1}{2}}, \qquad S_{k/2,k/2} = (\tfrac{k}{2}+1)^{-\frac{1}{2}}(-1)^{\frac{k}{4}}.$$

Equation (15) becomes

$$\mathcal{Q}(M) = \frac{1}{2^{r-1}}\Big(\varphi'(\varepsilon)(\tau_{k/2})^r + \sum_{\gamma\neq\varepsilon}\varphi'(\gamma)(-1)^{\frac{k}{4}(r-\mathsf{l}(\gamma))}(\tfrac{k}{2}+1)^{\frac{\mathsf{l}(\gamma)}{2}-1}\chi\Big)$$

which agrees with Theorem 3.7 for $\varphi'(\gamma) = \psi(\gamma)(-1)^{\frac{k\,\mathsf{l}(\gamma)}{8}}$.

The calculation is similar for the case $h = 0, s = r = 2,\ k \in 2\mathbb{N}$. Here, $|\Gamma| = 2$, and the generator $\gamma = (c,c) \in \Gamma$ has $\mathsf{l}(\gamma) = 2$. We hence obtain

$$\mathcal{Q}(M) = \frac{1}{2}\Big(\varphi'(e,e)(\tau_{k/2})^r + \varphi'(c,c)\chi\Big)$$

which agrees with Theorem 3.7 if we put $\varphi'(e,e) = 1$, and $\varphi'(c,c) = \pm 1$. If $h = 0$ and $s = r = 1,\ k \in 2\mathbb{N}$, then $\Gamma = \{e\}$, and the formula becomes $\mathcal{Q}(M) = \varphi'(e)\tau_{k/2}$, which is the correct expression for $\mathcal{Q}(\mathcal{D}_*)$ for $\varphi'(e) = 1$. Similarly, if $h = r = 0,\ s = 1$ so that M is a conjugacy class $\mathcal{D}_j \neq \mathcal{D}_*$, the formula reduces to $\mathcal{Q}(M) = \tau_{m_j} = \mathcal{Q}(\mathcal{D}_j)$.

Consider finally the case $h = 1,\ s = 0$ so that $M = \mathbf{D}(\mathrm{SO}(3))$. Pre-quantizability of this space requires $k \in 2\mathbb{N}$, and as shown in [15] the distinct pre-quantizations are indexed by $\varphi \in \mathrm{Hom}(\Gamma, \mathrm{U}(1))$, with $\Gamma = Z \times Z$. Separating off the term (e,e), (14) becomes

$$\mathcal{Q}(M) = \frac{1}{4}\Big(\varphi'(\epsilon)\sum_l \frac{1}{S_{0,l}^2}\tilde{\tau}_l + \sum_{\gamma\neq(e,e)} \varphi'(\gamma)\frac{1}{S_{0,k/2}^2}\tilde{\tau}_{k/2}\Big).$$

We have $\frac{1}{S_{0,k/2}^2}\tilde{\tau}_{k/2} = \chi$, and

$$\mathcal{Q}(\mathbf{D}(\mathrm{SU}(2))) = \sum_m \tau_m^2 = \sum_{l,m} \frac{S_{m,l}^2}{S_{0,l}^2}\tilde{\tau}_l = \sum_l \frac{1}{S_{0,l}^2}\tilde{\tau}_l,$$

where we use the symmetry and orthogonality of the S-matrix. Thus the formula may be re-written

$$\mathcal{Q}(M) = \frac{1}{4}\Big(\varphi'(\epsilon)\mathcal{Q}(\mathbf{D}(\mathrm{SU}(2))) + \sum_{\gamma\neq(e,e)} \varphi'(\gamma)\ \chi\Big).$$

This agrees with the formula for $\mathcal{Q}(\mathbf{D}(\mathrm{SO}(3))$ given in [15, Section 11.4] if one puts $\varphi'(e,e) = 1$ and $\varphi'(\gamma) = (-1)^{k/2}\varphi(\gamma)$ for $\gamma \neq (e,e)$. □

By combining this result with the 'quantization commutes with reduction' theorem for quasi-Hamiltonian spaces [15, Theorem 10.1], and since the coefficient of τ_0 in $\tilde{\tau}_l$ is $S_{0,l}^2$, we obtain the Fuchs-Schweigert formula [9] for the SO(3) moduli space $\mathcal{M}(\Sigma, \mathcal{C}_1, \ldots, \mathcal{C}_s)$, where Σ is of genus h with s boundary components. Recall that this moduli space has up two 2 connected components, of the form $M/\!\!/\,\mathrm{SU}(2)$ for suitable choice of lifts $\mathcal{D}_j$. We have,

$$\mathcal{Q}(M/\!\!/\,\mathrm{SU}(2)) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\varphi'(\gamma){\sum_l}^{(\gamma)} \frac{S_{m_1,l}^{(\gamma_1)}\cdots S_{m_s,l}^{(\gamma_s)}}{(S_{0,l})^{s+2h-2}}. \tag{16}$$

Remark 4.2. The above Fuchs-Schweigert type formula computes the quantization of the moduli space of SO(3)-bundles interpreted as the *index* of a pre-quantum line bundle, while the original conjecture in [9] concerns the dimension of the space of conformal blocks. It is expected that, just as in the case of simply-connected groups, the space of conformal blocks can be re-interpreted as the space of holomorphic sections, and that a Kodaira vanishing result can further identify its dimension with the index considered here. We are not aware of a reference addressing such questions in generality for non-simply connected groups.

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