Metric Compatible or Noncompatible Finsler–Ricci Flows

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Abstract

There were elaborated different models of Finsler geometry using the Cartan (metric compatible), or Berwald and Chern (metric non–compatible) connections, the Ricci flag curvature etc. In a series of works, we studied (non)commutative metric compatible Finsler and nonholonomic generalizations of the Ricci flow theory [see S. Vacaru, J. Math. Phys. **49** (2008) 043504; **50** (2009) 073503 and references therein]. The goal of this work is to prove that there are some models of Finsler gravity and geometric evolution theories with generalized Perelman's functionals, and correspondingly derived nonholonomic Hamilton evolution equations, when metric noncompatible Finsler connections are involved. Following such an approach, we have to consider distortion tensors, uniquely defined by the Finsler metric, from the Cartan and/or the canonical metric compatible connections. We conclude that, in general, it is not possible to elaborate self–consistent models of geometric evolution with arbitrary Finsler metric noncompatible connections.

Keywords: Finsler geometry, Ricci flows, geometric evolution.

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1 Motivation and Introduction

Geometric analysis and evolution equations are important topics of research in modern mathematics and physics, see original R. Hamilton's [1, 2] and G. Perelman's [3, 4, 5] works and reviews of results in [6, 7, 8]. In 2007, it was published a communication at a Conference in memory of M. Matsumoto (at Sapporo, in 2005), where D. Bao [9] mentioned that the idea to study such problems related to Finsler geometry came to S. -S. Chern in 2004. Unfortunately, the famous mathematician had not published his proposals/results on a Finsler–Ricci flow theory.[1]

[1] It was one–two years after famous Grisha Perelman's electronic preprints containing the proof of the Thurston/ Poincarè conjecture were put in arXiv.org. That induced a number of papers on geometric flows and applications related to various branches of mathematics, physics, optimization etc. I'm grateful to D. Bao and E. Peyghan for important correspondence, historical remarks about S. Chern original ideas, and discussions

In May-June, 2005, there were a series of lectures of N. Higson at Madrid, Spain, where the R. Hamilton and G. Perelman fundamental contributions in mathematics were discussed with respect to possible applications in modern gravity, cosmology and astrophysics. The author of this paper attended one of those lectures at CSIC, Madrid. At that time, he worked in some directions of nonholonomic mechanics and Finsler geometry and geometric methods of constructing exact solutions in Einstein gravity and modifications. He knew that Chern's connection in Finsler geometry is metric noncompatible which gives rise to a number of difficulties for applications related to standard theories of physics (see discussions in [10, 11, 12, 13]; we also mention here some most important monographs on Finsler geometry [14, 15, 16, 17, 18]). It is obvious that a general extension of the Hamilton–Perelman theory for metric noncompatible spaces, including Finsler models, is not possible. If $\mathbf{Dg} \neq 0$ for a metric $\mathbf{g}$ and a linear connection $\mathbf{D}$ (such geometric objects may be Finsler or other types), the evolution of geometric objects on a real parameter χ can not be determined only by a Ricci tensor (see relevant formulas on next page and rigorous definitions in sections 3 and 4).

In our works, we preferred to use the Cartan connection and metric compatible modifications and generalizations of Finsler geometry because the geometric constructions and proofs of the main results are quite similar to those for Riemannian spaces but for some special classes of Finsler connections. A series of results were developed for the theory of nonholonomic Ricci flows (with additional non–integrable constraints) for certain classes of Einstein, Finsler, Lagrange and other nonholonomic, noncommutative, nonsymmetric, fractional and stochastic spacetimes and geometries [19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

The problem of Ricci flows and Finsler geometry was considered again in a recent paper [29][2], where Finsler–Ricci flow type evolution equations are studied following D. Bao's heuristic proposals related to geometric flows and Finsler geometry. In such a case, even the Berwald connection (which is also metric noncompatible) is involved, the constructions may be associated to the Cartan metric compatible connection. A new definition/type of the Ricci tensor [30, 9, 18] which is symmetric and seem to provide an alternative approach to formulating Finsler like gravity and Ricci flow theories is considered. Such results are original and important. Nevertheless,

on Finsler–Ricci flows and almost Kähler models of Finsler geometry and generalizations.

[2] I thank E. Peyghan for sending two preliminary versions of their work before the authors would publish the results in a preprint or journal version

the geometric evolution equations with right side Ricci flag curvature postulated in the mentioned works (by D. Bao and A. Tayebi and E. Peyghan) were not derived from certain generalized Perelman's functionals. It was is not clear if such equations may describe an evolution gradient process (we shall prove this in the present paper, as a particular case). We also noted that it was not stated if, and when, the models of Finsler–Ricci flows with flag curvature may have certain limits to standard Laplacian operators and Levi–Civita configurations - this would be an important argument that such theories may describe well–defined evolution processes.

In this work we extend our former results on Finsler–Ricci flows for metric compatible connections in a more general context when metric noncompatible Finsler connections (like the Berwald and Chern ones) are used for nonholonomic deformations of Perelman's functionals. We shall analyze possible relations to former results on nonholonomic Ricci flows and Lagrange–Finsler evolution models via Cartan type (metric compatible) connections which positively describe geometric evolution processes in a self–consistent and similar manner to the Ricci flow theory on Riemannian manifolds.

R. S. Hamilton's evolution equations were postulated for real Riemannian manifolds [1, 2] following heuristic arguments,

$$\frac{\partial g_{ij}}{\partial \chi} = -2Ric_{ij}, \qquad g_{ij}\mid_{\chi=0} = {}^{\circ}g_{ij}(x^k).$$

In these equations, geometric flows of metrics $g_{ij}(\chi, x^k)$ are considered for a real parameter χ on a manifold M when local coordinates x^k are labeled by indices $i, j, ... = 1, 2, ..., n = dimM$. The Ricci tensor Ric_{ij} in defined by the Levi–Civita connection ∇ of g_{ij} (for our purposes, it is enough to work with non–normalized flows).

For a Finsler fundamental/generating function[3], $F(x^k, y^a)$, we can consider

$$^{v}\tilde{g}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1}$$

as a "vertical" (v) metric on typical fiber if $\det|\ ^{v}\tilde{g}_{ij}| \neq 0$.[4] Following a formal analogy to Hamilton's works, but for $^{v}\tilde{g}_{ij}$ on tangent bundle TM,

[3]see definitions and details in next section

[4]On TM, we can identify the horizontal, h, and v–indices, i.e. $i, j, ...$ and $a, b, ...$). In our work, left "up" and "low" indices are used as labels, for instance, "F" being associated to "Finsler" etc. We cite the monographs [14, 15, 16, 17, 18] on main Finsler geometry methods and comprehensive bibliography and our papers [10, 11], for critical remarks, principles and perspectives of applications in modern physics, cosmology and geometric mechanics. We suggest readers to consult such works for reviews of results and notation conventions.

we can postulate certain evolution equations of type

$$\frac{\partial\ ^{v}\tilde{g}_{ij}}{\partial\chi} \sim\ ^{F}Ric_{ij}, \qquad (2)$$

where $^{F}Ric_{ij}$ is a variant of Ricci tensor constructed for a model of Finsler geometry and flows/evolution of fundamental Finsler functions are parametrized by $F(\chi, x^k, y^a)$. A heuristic definition of $^{F}Ric_{ij}$ is related to an important question if a chosen Finsler type Ricci tensor would limit, or not, a Laplacian operator $^{F}\Delta$ derived in metric compatible form for a Finsler geometry model. The answer is affirmative for Laplacians determined by the Levi–Civita and/or Cartan connections but not for models of Finsler geometry when $^{F}Ric_{ij}$ is introduced in a "nonstandard" form, or using a general metric noncompatible Finsler connection. In [30, 9, 29], the problem if and how a Laplacian $^{F}\Delta$ may be associated naturally to the Ricci flag (and Akbar–Zadeh's) curvature and geometric flows was not analyzed.

The goal of this paper is to prove that (nonholonomically constrained) Finsler–Ricci flow evolution equations and corresponding $^{F}Ric_{ij}$ can be derived for some classes of metric noncompatible Finsler connections and/or Akbar–Zadeh's Ricci curvature. If such geometric objects are determined in unique forms (up to frame/coordinate transforms) by respective distortion tensors which, in their turns, are also completely defined by a Finsler fundamental function, we can formulate well defined Finsler evolution theories. In our approach, we use our former results and techniques elaborated for the models of geometric Finsler evolution with the Cartan connection and certain metric compatible generalizations [21, 22, 24]. Such constructions are very similar to those for Riemannian spaces but derived with respective Finsler connections and adapted frames. This allows us to define certain generalized Perelman functionals and associated entropy and thermodynamical type values and derive Hamilton type evolutions equations.

The paper is organized as follows. In Section 2 we survey the most important geometric constructions and the basic language on metric compatible Finsler spaces. In Section 3 there are defined the fundamental geometric objects for metric noncompatible Finsler spaces (using distortions from compatible ones) and defined the most important formulas for Einstein–Finsler spaces. The material outlined in the first three sections is oriented to non–experts on Finsler geometry but researchers on geometric analysis and mathematical physics. Perelman's functionals are defined for special classes of metric noncompatible Finsler spaces in Section 4. There are proven main theorems on Finsler–Ricci flows and nonholonomic, in general, metric noncompatible geometric evolution equations. We also speculate on statistical analogy and thermodynamics for Finsler–Ricci flows.

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2 Metric Compatible Finsler Geometries

In this section, we provide an introduction and analyze some common features and differences of (pseudo) Riemannian and metric compatible Finsler geometry models (proofs are omitted, see details in Refs. [10, 11, 13]). The fundamental geometric objects are presented in Figure 1, for a comparative study of Riemann and Finsler spaces. In section 3 and Figure 2, we shall analyze the most important formulas for metric compatible and noncompatible Finsler geometry models. We emphasize that in Ricci flow theories, it is convenient to work both with global and coordinate/index formulas and equations. Some historical remarks will be presented in order to explain the most important ideas and results in Finsler geometry and related evolution/gravity theories.

2.1 Finsler and Riemannian metrics

Let M be a real C^∞ manifold of dimension $\dim M = n$ and denote by TM its tangent bundle. Denoting by T_xM the tangent spaces at $x \in M$, we have $TM = \bigcup_{x\in M} T_xM$.

A Finsler fundamental/generating function (metric) is a function $F : TM \to [0,\infty)$ subjected to the conditions:

1. $F(x,y)$ is C^∞ on $\widetilde{TM} := TM\backslash\{0\}$, where $\{0\}$ denotes the set of zero sections of TM on M;

2. $F(x,\beta y) = \beta F(x,y)$, for any $\beta > 0$, i.e. it is a positive 1–homogeneous function on the fibers of TM;

3. $\forall y \in \widetilde{T_xM}$, the Hessian ${}^v\tilde{g}_{ij}$ (1) is nondegenerate and positive definite[5].

The term "metric" for F is used in Finsler geometry because it defines on TM a nonlinear quadratic element

$$ds^2 = F^2(x,dx) \tag{3}$$

[5] this condition should be relaxed for models of Finsler gravity with finite, in general, locally anisotropic speed of light [10, 11]

for $dx^i \sim y^i$. The well–known and very important example of (pseudo) Riemannian geometry, determined by a metric tensor $g_{ij}(x^k)$, is a particular case with quadratic form $F = \sqrt{|g_{ij}(x)y^i y^j|}$ when

$$ds^2 = g_{ij}(x)dx^i dx^j \qquad (4)$$

and the signature of g_{ij} is of type $(+,+,+,+)$, or $(+,+,+,-)$, for corresponding space like, or spacetime, manifolds. It should be noted that the condition (4) allows us to identify the fiber of TM with a flat (pseudo) Euclidean space, respectively, Minkowski spacetime, in any point $x \in M$. The tangent spaces T_xM are considered in (pseudo) Riemannian geometry on M in order to define geometrically tensors and forms by analogy to flat spaces. For instance, a vector $A = \{A_i(x)\} \in TM$ in any system of reference/coordinates, has coefficients $A_i(x)$ depending only on base coordinates x^k but not on y^a. The fundamental geometric objects (for instance, the Levi–Civita connection ∇ and respective curvature tensor and Ricci tensor) are completely and uniquely determined by a metric tensor ${}^h g = \{g_{ij}(x)\}$ following the condition of metric compatibility and zero torsion, see left blocks 2l, 3l, 4l and 5l in Figure 1. This is a result of the "quadratic" condition (4) when, in general, geometric and/or gravity theory models based on (pseudo) Riemannian geometry, and various Einstein/Riemann–Cartan or metric–affine generalizations, are for geometric/physical objects depending only on x–coordinates. Any given (pseudo) Riemannian metric structure naturally generates a unique chain

$${}^h g(x) \to \nabla(x) \to {}^{\nabla}\mathcal{R}(x) \to {}^{\nabla}Ric(x)$$

following well–defined geometric rules. The "standard" theory of Ricci flows [1, 2, 3, 4, 5] was formulated for (pseudo) Riemannian) metrics $g_{ij}(\chi, x)$ depending on a real flow parameter χ (for simplicity, we omit details on geometric flows of (almost) Kähler geometries).

If a Finsler metric F is generic nonlinear, the problem of constructing geometric models on TM became more sophisticate. Any relation of type (3) for a class of correspondingly defined functions F allows us to study various metric properties of T_xM and, in general, of TM, including fiber constructions, using ${}^v\tilde{g}_{ij}(x,y)$ (1) and its possible projections, conformal transforms etc. For instance, it is well known that B. Riemann in his famous thesis [31] considered the first example of Finsler metric with non–quadratic quadratic elements (see historical remarks and references in [14, 15, 16, 17, 18]; that why the term Riemann–Finsler geometry was introduced in modern literature) even he elaborated a complete geometric model only for Riemannian spaces. Nevertheless, to know the metric properties is not enough for constructing a complete geometric model on TM for a given F and respective ${}^{v}\tilde{g}_{ij}$. We need more assumptions, for instance, how we chose to define connections naturally determined by F because for generic Finsler metrics there is not a unique analog of the Levi–Civita connection.

2.2 Cartan–Finsler geometry

The first complete geometric model of Finsler geometry is due to E. Cartan [14]. Roughly speaking, the Cartan–Finsler geometry is a variant of the well known Riemann–Cartan one, with nonzero torsion, but constructed on TM in a form when all geometric objects are generated by F following the conditions of metric compatibility and vanishing of "pure" horizontal and vertical components of torsion. Here we note that the Cartan–Finsler torsion (see block 5r] in Figure 1 and block 4) in Figure 2) is different from that used, for instance, in Einstein–Cartan gravity when torsion is considered as an additional (to metric) tensor field for which additional (algebraic) field equations are introduced. For the Cartan–Finsler model, the torsion field is completely determined by metrics F and ${}^{v}\tilde{g}_{ij}[F]$, when (at least, in principle) a complete metric ${}^{F}\mathbf{g}$ can be constructed on total TM following certain well defined geometric principles.

2.2.1 The canonical N–connection, adapted frames and metrics

Nevertheless, the Cartan–Finsler space is not only a Riemann–Cartan geometry on TM with metric tensor and metric compatible connection with torsion (all induced by F). This is also an example of nonholonomic manifold/bundle space when the geometric objects are adapted to a non–integrable distribution on TM induced by F in such a form that canonical semi–spray and nonlinear connection (N–connection) structures are defined. In the mentioned first monograph on Finsler geometry [14], the concept of N–connection is considered in coordinate form (the first global definitions are due to Ehresmann [32] and A. Kawaguchi [33, 34], see details in [16] and, for the Einstein gravity and generalizations, in [10, 13]). Let us analyze, in brief, such constructions. A N–connection $\mathbf{N}$ can be defined as a non–integrable (there are used equivalent terms like nonholonomic and/or anholonomic) distribution

$$TTM = hTM \oplus vTM \qquad (5)$$

into conventional horizontal (h) and vertical (v) subspaces[6]. Locally, such a geometric object is determined by its coefficients $\{N_i^a\}$, when $\mathbf{N} = N_i^a(u)dx^i \otimes \partial/\partial y^a$, and characterized by its curvature (Neijenhuis tensor)

$$\mathbf{\Omega} = \frac{1}{2}\Omega_{ij}^a \ d^i \wedge d^j \otimes \partial_a,$$

with coefficients

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (6)$$

In Cartan–Finsler geometry, the N–connection is canonically determined by F following a geometric/variational principle: The value $L = F^2$ is considered as an effective regular Lagrangian on TM and action integral

$$S(\tau) = \int_0^1 L(x(\tau), y(\tau))d\tau, \text{ for } y^k(\tau) = dx^k(\tau)/d\tau,$$

for $x(\tau)$ parametrizing smooth curves on a manifold M with $\tau \in [0,1]$. The Euler–Lagrange equations $\frac{d}{d\tau}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0$ are equivalent to the "nonlinear geodesic" (equivalently, semi–spray) equations $\frac{d^2 x^k}{d\tau^2} + 2\tilde{G}^k(x,y) = 0$, where

$$\tilde{G}^k = \frac{1}{4}\tilde{g}^{kj}\left(y^i \frac{\partial^2 L}{\partial y^j \partial x^i} - \frac{\partial L}{\partial x^j}\right), \quad (7)$$

for $\tilde{g}^{kj}$ being inverse to ${}^v\tilde{g}_{ij} \equiv \tilde{g}_{ij}$ (1), defines the canonical N–connection

$$\tilde{N}_j^a := \frac{\partial \tilde{G}^a(x,y)}{\partial y^j}. \quad (8)$$

A fundamental Finsler function $F(x,y)$ induces naturally a N–adapted frame structure (defined linearly by $\tilde{N}_j^a$), $\tilde{\mathbf{e}}_\nu = (\tilde{\mathbf{e}}_i, e_a)$, where

$$\tilde{\mathbf{e}}_i = \frac{\partial}{\partial x^i} - \tilde{N}_i^a(u)\frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}, \quad (9)$$

and the dual frame (coframe) structure is $\tilde{\mathbf{e}}^\mu = (e^i, \tilde{\mathbf{e}}^a)$, where

$$e^i = dx^i \text{ and } \mathbf{e}^a = dy^a + \tilde{N}_i^a(u)dx^i. \quad (10)$$

[6] In our works, we use "boldface" symbols for spaces and geometric objects endowed/adapted to N–connection structure.

There are satisfied nontrivial nonholonomy relations

$$[\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\beta] = \tilde{\mathbf{e}}_\alpha \tilde{\mathbf{e}}_\beta - \tilde{\mathbf{e}}_\beta \tilde{\mathbf{e}}_\alpha = \tilde{W}^{\gamma}_{\alpha\beta} \tilde{\mathbf{e}}_\gamma \qquad (11)$$

with (antisymmetric) nontrivial anholonomy coefficients $\tilde{W}^b_{ia} = \partial_a \tilde{N}^b_i$ and $\tilde{W}^a_{ji} = \tilde{\Omega}^a_{ij}$. This is a reason to say that a Finsler geometry is a nonholonomic one when F defines a "preferred" frame structure on TM.[7] If a generating function F is of particular quadratic type (4), the values $\tilde{N}^a_j$, $\tilde{\mathbf{e}}_\alpha$ and $\tilde{W}^{\gamma}_{\alpha\beta}$ can be parametrized in some forms not depending explicitly on y^a. In such cases, $\tilde{\mathbf{e}}_\alpha$ can be arbitrary frames not depending on a "degenerate" Finsler, i.e. on a (pseudo) Riemannian metric $g_{ij}(x)$.

Using data $(\tilde{g}_{ij}, \tilde{\mathbf{e}}_\alpha)$, we can define a canonical (Sasaki type) metric structure on $\widetilde{TM}$,

$$\tilde{\mathbf{g}} = \tilde{g}_{ij}(x,y)\ e^i \otimes e^j + \tilde{g}_{ij}(x,y)\ \tilde{\mathbf{e}}^i \otimes\ \tilde{\mathbf{e}}^j. \qquad (12)$$

It is possible to use other geometric principles for "lifts and projections" with ${}^v\tilde{g}_{ij}$ on the typical fiber, when from a given F it is constructed a metric on total/horizontal spaces of TM. Nevertheless, for models of locally anisotropic/Finsler gravity on TM, with a generalized covariance principle, such constructions are equivalent up to certain frame/coordinate transforms $\tilde{\mathbf{e}}_\gamma \to \mathbf{e}_{\gamma'} = e^{\gamma}{}_{\gamma'} \tilde{\mathbf{e}}_\gamma$. In such cases, we can omit "tilde" on symbols and write, in general, $\mathbf{g} = \{\mathbf{g}_{\alpha\beta}\}$ and $\mathbf{N} = \{N^a_i = e^a{}_{a'} e_i{}^{i'} N^{a'}_{i'}\}$. There is a subclass of transforms preserving a prescribed splitting (5).

We note that in Finsler geometry and generalizations there are used terms like distinguished tensor/ metric/ connection etc (in brief, d–tensor, d–metric, d–connection) for geometric objects adapted to N–connection splitting when coefficients are computed with respect to frames of type (9) and (10). For instance, a d–vector $\mathbf{X} = ({}^hX,\ {}^vX) = X^i \tilde{\mathbf{e}}_i + X^a e_a$.

2.2.2 Torsion and curvature of d–connections

For any d–metric $\tilde{\mathbf{g}}$ (12), we may construct in standard form, on TM, its Levi–Civita connection $\tilde{\nabla}$. Nevertheless, such a linear connection is not used in Finsler geometry because it is not adapted to the N–connection structure $\mathbf{N}$. This motivates the definition of a new class of linear connections.

A distinguished connection (d–connection) $\mathbf{D}$ on TM is a linear connection conserving under parallelism the Whitney sum (5). For any $\mathbf{D}$, there is a decomposition into h– and v–covariant derivatives,

$$\mathbf{D}_{\mathbf{X}} \doteqdot \mathbf{X}\rfloor\mathbf{D} = {}^hX\rfloor\mathbf{D} + {}^vX\rfloor\mathbf{D} = D_{{}^hX} + D_{{}^vX} = {}^hD_X + {}^vD_X,$$

[7] Such a N–adapted frame system of reference does not prohibits us to consider arbitrary frame and coordinate transforms on TM.

where "$\rfloor$" denotes the interior product.

The torsion of a d–connection $\mathbf{D}$ is defined in standard from by d–tensor field

$$\mathcal{T}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{Y} - \mathbf{D}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}], \qquad (13)$$

for which a N–adapted h-v–decomposition is possible,

$$\mathcal{T}(\mathbf{X},\mathbf{Y}) = T(\ ^{h}X,\ ^{h}Y) + T(\ ^{h}X,^{v}Y) + T(\ ^{v}X,\ ^{h}Y) + T(\ ^{v}X,\ ^{v}Y).$$

The curvature of a d–connection $\mathbf{D}$ is

$$\mathcal{R}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_{\mathbf{X}}\mathbf{D}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\mathbf{D}_{\mathbf{X}} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]} \qquad (14)$$

for any d–vectors $\mathbf{X},\mathbf{Y}$, with a corresponding h–v–decomposition (for simplicity, we omit such formulas) .

The N–adapted components $\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}$ of a d–connection $\mathbf{D}_{\alpha} = (\mathbf{e}_{\alpha}\rfloor\mathbf{D})$ are computed following equations $\mathbf{D}_{\alpha}\mathbf{e}_{\beta} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\mathbf{e}_{\gamma}$, or $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}(u) = (\mathbf{D}_{\alpha}\mathbf{e}_{\beta})\rfloor\mathbf{e}^{\gamma}$. Respective splitting into h– and v–covariant derivatives are given by

$$h\mathbf{D} = \{\mathbf{D}_{k} = \left(L^{i}_{jk}, L^{a}_{bk}\right)\},\ \text{and}\ \ v\mathbf{D} = \{\mathbf{D}_{c} = \left(C^{i}_{jc}, C^{a}_{bc}\right)\},$$

where, by definition,

$$L^{i}_{jk} = (\mathbf{D}_{k}\mathbf{e}_{j})\rfloor e^{i},\quad L^{a}_{bk} = (\mathbf{D}_{k}e_{b})\rfloor\mathbf{e}^{a},\ C^{i}_{jc} = (\mathbf{D}_{c}\mathbf{e}_{j})\rfloor e^{i},\quad C^{a}_{bc} = (\mathbf{D}_{c}e_{b})\rfloor\mathbf{e}^{a}.$$

A set of coefficients $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc}\right)$ completely define a d–connection $\mathbf{D}$ on TM enabled with N–connection structure.

The simplest way to perform computations with d–connections is to use N–adapted differential forms. The d–connection 1–form is $\mathbf{\Gamma}^{\alpha}{}_{\beta} = \mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}\mathbf{e}^{\gamma}$. For instance, the h–v–coefficients $\mathbf{T}^{\alpha}{}_{\beta\gamma} = \{T^{i}{}_{jk}, T^{i}{}_{ja}, T^{a}{}_{ji}, T^{a}{}_{bi}, T^{a}{}_{bc}\}$ of torsion $\mathbf{T}$ (13) are computed using formulas

$$\mathcal{T}^{\alpha} := \mathbf{D}\mathbf{e}^{\alpha} = d\mathbf{e}^{\alpha} + \mathbf{\Gamma}^{\alpha}{}_{\beta}\wedge\mathbf{e}^{\beta}.$$

We obtain

$$\begin{aligned} T^{i}{}_{jk} &= L^{i}{}_{jk} - L^{i}{}_{kj},\ T^{i}{}_{ja} = -T^{i}{}_{aj} = C^{i}{}_{ja},\ T^{a}{}_{ji} = \Omega^{a}{}_{ji}, \\ T^{a}{}_{bi} &= \frac{\partial N^{a}_{i}}{\partial y^{b}} - L^{a}{}_{bi},\ T^{a}{}_{bc} = C^{a}{}_{bc} - C^{a}{}_{cb}. \end{aligned} \qquad (15)$$

Similarly, we can compute the N–adapted components $\mathbf{R}^{\alpha}{}_{\beta\gamma\delta}$ of the curvature $\mathbf{R}$ (14),

$$\mathcal{R}^{\alpha}{}_{\beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}{}_{\beta} = d\mathbf{\Gamma}^{\alpha}{}_{\beta} - \mathbf{\Gamma}^{\gamma}{}_{\beta}\wedge\mathbf{\Gamma}^{\alpha}{}_{\gamma} = \mathbf{R}^{\alpha}{}_{\beta\gamma\delta}\mathbf{e}^{\gamma}\wedge\mathbf{e}^{\delta}. \qquad (16)$$

For simplicity, we omit formulas for an explicit h–v–parametrization of $\mathbf{R}^{\alpha}{}_{\beta\gamma\delta}$, see details in Refs. [10, 13, 16].

There is a sub–class of d–connections $\mathbf{D}$ on $\mathbf{TM}$ which are metric compatible to a d–metric

$$\mathbf{g} = \ g_{ij}(x,y)\ e^{i}\otimes e^{j} + \ h_{ab}(x,y)\ \mathbf{e}^{a}\otimes \mathbf{e}^{b} \qquad (17)$$

with N–adapted decomposition $\mathbf{g} = hg \oplus_{\mathbf{N}} vg = [hg, vg]$.[8] The condition of compatibility $\mathbf{Dg}=\mathbf{0}$ split in respective conditions for h-v–components, $D_j g_{kl}=0, D_a g_{kl}=0, D_j h_{ab}=0, D_a h_{bc}=0.$

We can construct a canonical d–connection $\widehat{\mathbf{D}}$ completely defined by a d–metric $\mathbf{g}$ (17) in metric compatible form, $\widehat{\mathbf{D}}\mathbf{g}=0$, and with zero h- and v-torsions (with $\widehat{T}^{i}{}_{jk}=0$ and $\widehat{T}^{a}{}_{bc}=0$ but, in general, nonzero $\widehat{T}^{i}{}_{ja}, \widehat{T}^{a}{}_{ji}$ and $\widehat{T}^{a}{}_{bi}$, see (15)). The coefficients of $\widehat{\mathbf{D}}$, computed with respect to N–adapted frames are $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$ for

$$\begin{aligned}
\widehat{L}^{i}_{jk} &= \frac{1}{2} g^{ir}\left(\mathbf{e}_{k}g_{jr}+\mathbf{e}_{j}g_{kr}-\mathbf{e}_{r}g_{jk}\right), \\
\widehat{L}^{a}_{bk} &= e_{b}(N^{a}_{k})+\frac{1}{2}h^{ac}\left(e_{k}h_{bc}-h_{dc}\ e_{b}N^{d}_{k}-h_{db}\ e_{c}N^{d}_{k}\right), \\
\widehat{C}^{i}_{jc} &= \frac{1}{2}g^{ik}e_{c}g_{jk},\ \widehat{C}^{a}_{bc}=\frac{1}{2}h^{ad}\left(e_{c}h_{bd}+e_{c}h_{cd}-e_{d}h_{bc}\right).
\end{aligned} \qquad (18)$$

For any metric structure $\mathbf{g}$ on TM, we can compute also the Levi–Civita connection ∇ for which ${}^{\nabla}\mathcal{T}^{\alpha}=0$ and $\nabla\mathbf{g}=\mathbf{0}$. There is a canonical distortion relation

$$\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}} \qquad (19)$$

where both connections $\widehat{\mathbf{D}}, \nabla$ and $\widehat{\mathbf{Z}}$ (such a distortion tensor is an algebraic combination of nontrivial torsion coefficients $\widehat{T}^{i}{}_{ja}, \widehat{T}^{a}{}_{ji}$ and $\widehat{T}^{a}{}_{bi}$) are uniquely defined by the same metric structure $\mathbf{g}$. Taking $\mathbf{g}=\tilde{\mathbf{g}}$, such values $\tilde{\widehat{\mathbf{D}}}, \tilde{\nabla}$ and

[8] Any d–metric $\mathbf{g}=g_{\alpha\beta}du^{\alpha}du^{\beta}$ on TM, via corresponding frame/coordinate transforms can be parametrized in the form (17) and $\tilde{\mathbf{g}}$ (12) (in the last case, we have to prescribe a generating function F). This mean that on the total space of a tangent bundle endowed with metric structure $\mathbf{g}$ we can always introduce Finsler variables when $\mathbf{g}=\tilde{\mathbf{g}}$ and there is a h–v–splitting $\mathbf{N}=\tilde{\mathbf{N}}$. The constructions are performed equivalently but depend on the type of geometric structure chosen to be the fundamental one. If F is prescribed, then we construct data $\left(F:\tilde{\mathbf{N}},\tilde{\mathbf{g}}\right)$ which up to frame transforms [$\mathbf{e}_{\gamma'}=e^{\gamma}{}_{\gamma'}\tilde{\mathbf{e}}_{\gamma}$; the vielbeins $e^{\gamma}{}_{\gamma'}$ have to be defined as a solution of an algebraic quadratic equations $\mathbf{g}_{\alpha'\beta'}=e^{\alpha}{}_{\alpha'}e^{\beta}{}_{\beta'}\tilde{\mathbf{g}}_{\alpha\beta}$ for given $\mathbf{g}_{\alpha'\beta'}$ and $\tilde{\mathbf{g}}_{\alpha\beta}$] are equivalent to some data $(\mathbf{N},\mathbf{g})$. Inversely, we can fix any $(\mathbf{N},\mathbf{g})$ (in particular, $\mathbf{N}$ can be for any conventional h–v–splitting) and then chose any convenient F when via frame transforms $(\mathbf{N},\mathbf{g})\rightarrow(\tilde{\mathbf{N}},\tilde{\mathbf{g}})$.

$\widehat{\widehat{\mathbf{Z}}}$ can be derived from a Finsler metric F (for simplicity, we omit explicit coordinate formulas for ∇ and $\widehat{\mathbf{Z}}$, see details in [10, 13, 16]). This allows us to construct a complete model of Finsler space on TM. Such a canonical metric compatible geometry is determined by data $\left(F:\mathbf{g},\mathbf{N},\widehat{\mathbf{D}}\right)$.

2.2.3 The Cartan d–connection

Historically, E. Cartan [14] used another type of metric compatible d–connection $\tilde{\mathbf{D}}$ which via frame transforms and deformations can be related to $\widehat{\mathbf{D}}$ (18). If we consider that $\widehat{L}^{a}_{bk}\to\widehat{L}^{i}_{jk}$ and $\widehat{C}^{i}_{jc}\to\ \widehat{C}^{a}_{bc}$, by identifying respectively $a=n+i$ with i and $b=n+j$, we obtain the so–called normal d–connection $\ {}^{n}\mathbf{D}=(\widehat{L}^{i}{}_{jk},\widehat{C}^{i}_{jc})$ with N–adapted 1–form

$$\widehat{\mathbf{\Gamma}}^{i}{}_{j}=\widehat{\mathbf{\Gamma}}^{i}{}_{j\gamma}\mathbf{e}^{\gamma}=\widehat{L}^{i}{}_{jk}e^{k}+\widehat{C}^{i}_{jc}\mathbf{e}^{c},$$

where

$$\widehat{L}^{i}{}_{jk}=\frac{1}{2}g^{ih}(\mathbf{e}_{k}g_{jh}+\mathbf{e}_{j}g_{kh}-\mathbf{e}_{h}g_{jk}),\widehat{C}^{a}{}_{bc}=\frac{1}{2}g^{ae}(e_{b}h_{ec}+e_{c}h_{eb}-e_{e}h_{bc}). \quad (20)$$

Taking $\mathbf{g}=\tilde{\mathbf{g}}$, when $\tilde{h}_{ij}=\tilde{g}_{ij},$ and $\mathbf{N}=\tilde{\mathbf{N}}$ in (20), we define the Cartan d–connection $\ \tilde{\mathbf{D}}=(\tilde{L}^{i}{}_{jk},\tilde{C}^{i}_{jc})$.

For $\tilde{\mathbf{D}}$, the nontrival h– and v–components of torsion $\tilde{\mathbf{T}}^{\alpha}_{\beta\gamma}=\{\tilde{T}^{i}_{jc},\tilde{T}^{a}_{ij},\tilde{T}^{a}_{ib}\}$ and curvature $\tilde{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}=\{\tilde{R}^{i}{}_{hjk},\tilde{P}^{i}{}_{jka},\tilde{S}^{a}{}_{bcd}\}$ are respectively

$$\tilde{T}^{i}_{jc}=\tilde{C}^{i}{}_{jc},\tilde{T}^{a}_{ij}=\tilde{\Omega}^{a}_{ij},\tilde{T}^{a}_{ib}=e_{b}\left(\tilde{N}^{a}_{i}\right)-\tilde{L}^{a}{}_{bi}, \quad (21)$$

and

$$\begin{aligned}\tilde{R}^{i}{}_{hjk} &= \tilde{\mathbf{e}}_{k}\tilde{L}^{i}{}_{hj}-\tilde{\mathbf{e}}_{j}\tilde{L}^{i}{}_{hk}+\tilde{L}^{m}{}_{hj}\tilde{L}^{i}{}_{mk}-\tilde{L}^{m}{}_{hk}\tilde{L}^{i}{}_{mj}-\tilde{C}^{i}{}_{ha}\tilde{\Omega}^{a}{}_{kj}, \quad (22)\\ \tilde{P}^{i}{}_{jka} &= e_{a}\tilde{L}^{i}{}_{jk}-\tilde{\mathbf{D}}_{k}\tilde{C}^{i}{}_{ja},\ \tilde{S}^{a}{}_{bcd}=e_{d}\tilde{C}^{a}{}_{bc}-e_{c}\tilde{C}^{a}{}_{bd}+\tilde{C}^{e}{}_{bc}\tilde{C}^{a}{}_{ed}-\tilde{C}^{e}{}_{bd}\tilde{C}^{a}{}_{ec}.\end{aligned}$$

We note that h– and v–components of torsion are zero, $\tilde{T}^{i}_{jk}=0$ and $\tilde{T}^{a}_{bc}=0,$ even there are also nontrivial components $\tilde{T}^{a}_{ij}$ and $\tilde{T}^{a}_{ib}$.

The Cartan d–connection is characterized by a unique distortion relation

$$\tilde{\mathbf{D}}=\tilde{\nabla}+\tilde{\mathbf{Z}}, \quad (23)$$

where all values $\tilde{\mathbf{D}}$,$\tilde{\nabla}$ and $\tilde{\mathbf{Z}}$ are determined (up to frame/coordinate transforms) by F and $\tilde{\mathbf{g}}$. On TM, the data $\left(F;\tilde{\mathbf{g}},\tilde{\mathbf{N}},\tilde{\mathbf{D}}\right)$ define a model of Cartan–Finsler geometry.

2.2.4 The almost Kähler model of Cartan–Finsler geometry

There is a fundamental result by M. Matsumoto [15] which allows us to reformulate $\left(F;\tilde{\mathbf{g}},\tilde{\mathbf{N}},\tilde{\mathbf{D}}\right)$, equivalently, as an almost Kähler geometry. Let us consider a linear operator $\tilde{\mathbf{J}}$ acting on vectors on TM following formulas

$$\tilde{\mathbf{J}}(\tilde{\mathbf{e}}_i) = -e_i \text{ and } \tilde{\mathbf{J}}(e_i) = \tilde{\mathbf{e}}_i,$$

where the superposition $\tilde{\mathbf{J}} \circ \tilde{\mathbf{J}} = -\mathbf{I}$, for the unity matrix $\mathbf{I}$.

A Finsler fundamental function $F(x,y)$ and the corresponding Sasaki type metric $\tilde{\mathbf{g}}$ (12) induce, respectively, a canonical 1–form $\tilde{\omega} = F\frac{\partial F}{\partial y^i}e^i$ and a canonical 2–form

$$\tilde{\theta} = \ \tilde{g}_{ij}(x,y)\mathbf{e}^i \wedge e^j. \qquad (24)$$

Such objects are associated to $\mathbf{J}$ following formula $\tilde{\theta}(\mathbf{X},\mathbf{Y}) := \tilde{\mathbf{g}}(\tilde{\mathbf{J}}\mathbf{X},\mathbf{Y})$ for any d–vectors $\mathbf{X}$ and $\mathbf{Y}$. By straightforward computations, we can prove that $d\tilde{\omega} = \tilde{\theta}$. This states on TM an almost Hermitian (symplectic) structure nonholonomically induced by F. Considering ${}^{\theta}\mathbf{D} \equiv \tilde{\mathbf{D}}$ as an almost symplectic d–connection, we can prove that

$${}^{\theta}\mathbf{D}_{\mathbf{X}}\tilde{\theta} = \mathbf{0} \text{ and } {}^{\theta}\mathbf{D}_{\mathbf{X}}\tilde{\mathbf{J}} = \mathbf{0}.$$

The data $(F;\tilde{\theta},\tilde{\mathbf{J}},{}^{\theta}\mathbf{D})$ define a nonholonomic almost Kähler space.

It should be noted that canonical almost symplectic/Kähler variables $\tilde{\theta},\tilde{\mathbf{J}}$, and ${}^{\theta}\mathbf{D}$ can be introduced for any TM endowed with d–metric, $\mathbf{g}$, and N–connection, $\mathbf{N}$, structures. For this, we have to prescribe an effective generating function F and compute $\tilde{\mathbf{e}}_\alpha$ and $\tilde{\mathbf{g}}$. Solving a quadratic algebraic equation to construct $\tilde{\mathbf{e}}_\gamma \to \mathbf{e}_{\gamma'} = e^{\gamma}{}_{\gamma'}\tilde{\mathbf{e}}_\gamma$, we encode equivalently and data $(TM,\mathbf{g})$ as a Cartan–Finsler model, $(F;\tilde{\mathbf{g}},\tilde{\mathbf{N}},\tilde{\mathbf{D}})$, and/or an almost Kähler–Finsler model, $(F;\tilde{\theta},\tilde{\mathbf{J}},{}^{\theta}\mathbf{D})$. Such results were used for deformation quantization of Lagrange–Finsler spaces [38]. Finally, we cite an alternative approach with Kähler structures associated to Berwald or Randers metrics etc [39, 40, 41].

3 Metric Noncompatible Finsler Spaces

There were developed alternative approaches to constructing geometric models determined by a fundamental Finsler function $F(x,y)$. In some sense, mathematicians attempted to formulate a more "simple" version of Finsler geometry than the Cartan model, not mimicking on tangent bundles

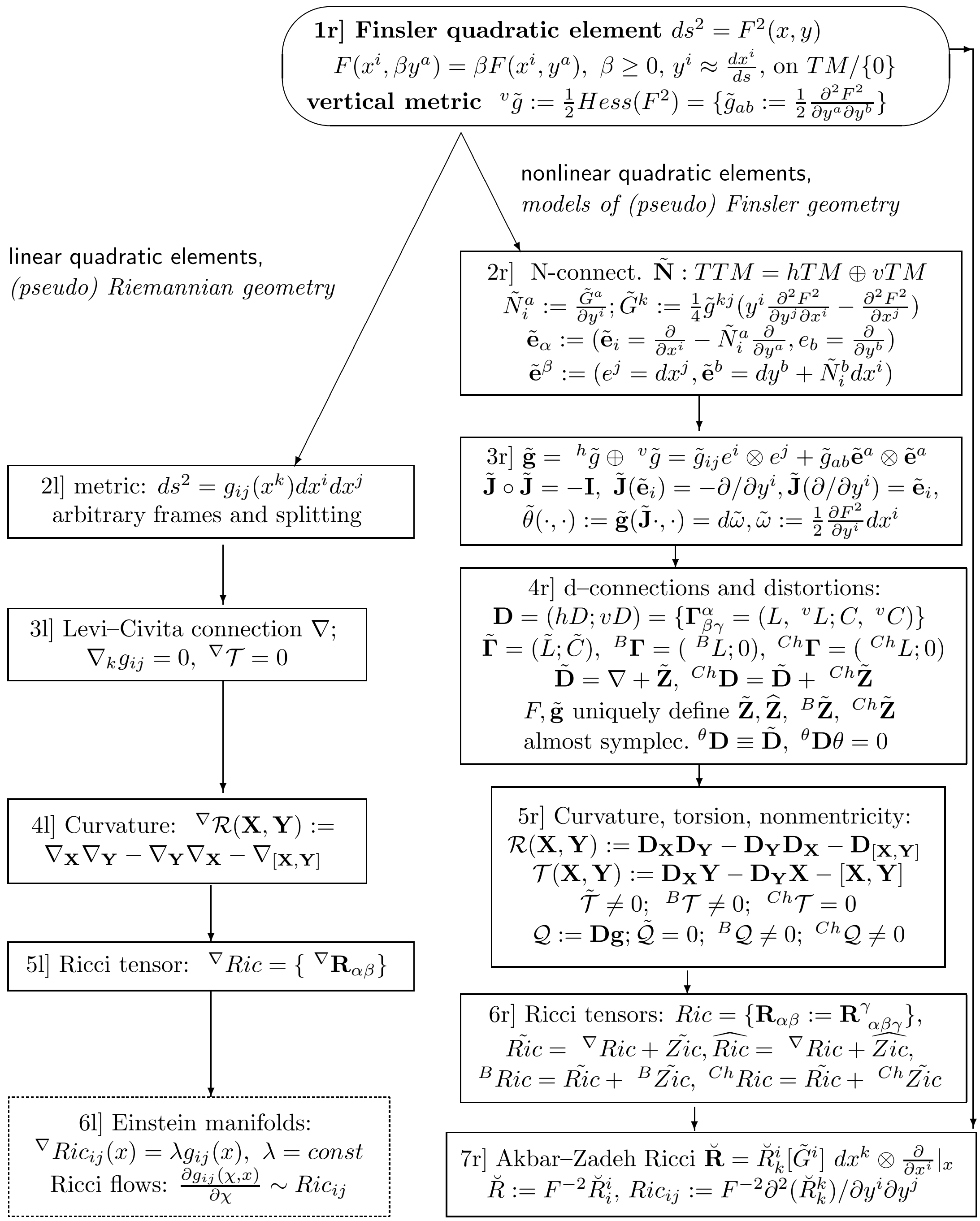

Figure 1: Riemann and Finsler spaces generated by $g_{ij}(x^k)$ and $F(x^i, y^a)$

a variant of nonholonomic Riemann space. Chronologically, the first metric noncompatible models were proposed by L. Berwald [35] and S. Chern [42] (see details in [18]). More recently, a different "nonstandard" construction for the Ricci curvature was proposed by H. Akbar–Zadeh [30]. In this section, we outline three important models of "non–Cartan" Finsler spaces following Fugure 2.

- The **Berwald d–connection** is ${}^{B}\mathbf{D} := ({}^{B}L^{i}{}_{jk} = \partial \tilde{N}^{i}_{j}/\partial y^{k},\ {}^{B}C^{i}_{jc} = 0)$, when the h–covariant derivative is defined by the first v–derivatives of Cartan's N–connection structure $\tilde{N}$ (8) and an additional constraint that the v–covariant derivative is zero is imposed.

- The **Chern d–connection** is ${}^{Ch}\mathbf{D} := ({}^{Ch}L^{i}{}_{jk} = \tilde{L}^{i}{}_{jk},\ {}^{Ch}C^{i}_{jc} = 0)$, when the h–covariant derivative is the Cartan's one computed as in the first formula for the normal d–connection (20), with an additional constraint that the v–covariant derivative is zero is imposed.

Both geometries with ${}^{B}\mathbf{D}$ and/or ${}^{Ch}\mathbf{D}$ can be modelled on hTM. The Chern's d–connection keeps all properties of the Levi–Civita connection for geometric constructions on the h–subspace. Nevertheless, both d–connections are not metric compatible on total space of TM, i.e. there are nontrivial nonmetricity fields, $\mathcal{Q} := \mathbf{Dg}$, ${}^{B}\mathcal{Q} \neq 0$ and ${}^{Ch}\mathcal{Q} \neq 0$. Such nonmetricities, in general, present substantial difficulties in constructing well–defined minimal Finsler extensions of the standard models of Finsler gravity, see critical remarks in Refs. [11, 10, 13] (for instance, there are problems with physical interpretation of nonmetricity fields, definition of spinors and constructing Dirac operators, formulating conservation laws etc).

Applying formulas (13) and (14), we compute respectively the torsions ${}^{B}\mathcal{T} \neq 0$, ${}^{Ch}\mathcal{T} = 0$ and curvatures ${}^{B}\mathcal{R} \neq 0$, ${}^{Ch}\mathcal{R} \neq 0$ as 2–forms.

In Refs. [43, 30], it is used as curvature in Finsler geometry the value

$$\breve{\mathbf{R}} = \breve{R}^{i}_{k}\ dx^{k} \otimes \frac{\partial}{\partial x^{i}}|_{x} : T_{x}M \to T_{x}M,$$

(this type of "curvature" is considered in a manner different that definitions of curvatures with associated 2–forms) where

$$\breve{R}^{i}_{k} = 2\frac{\partial \tilde{G}^{i}}{\partial x^{k}} - y^{j}\frac{\partial^{2}\tilde{G}^{i}}{\partial x^{j}\partial y^{k}} + 2\tilde{G}^{j}\frac{\partial^{2}\tilde{G}^{i}}{\partial y^{j}\partial y^{k}} - \frac{\partial \tilde{G}^{i}}{\partial y^{j}}\frac{\partial \tilde{G}^{j}}{\partial y^{k}} \qquad (25)$$

is determined by semi–spray $\tilde{G}^{k}$ (7). Such values are convenient for study geometric objects in $T_{x}M$ for a point $x \in M$.

3.1 Nonholonomic deformations and distortions

We note that above presented formulas for metric compatible and noncommpatible d–connections in Finsler geometry are uniquely related via certain distortion tensors of type (23) and (19). In order to derive deformations of fundamental tensor objects (for instance, torsions and curvature) in N–adapted form it is convenient to perform all constructions for the Cartan d–connection and then to compute distortions for necessary tensors and differential forms. We can write

$$ {}^{B}\mathbf{D}= \tilde{\mathbf{D}}+ {}^{B}\tilde{\mathbf{Z}} \text{ and } \quad {}^{Ch}\mathbf{D}= \tilde{\mathbf{D}}+ {}^{Ch}\tilde{\mathbf{Z}}, \qquad (26)$$

where all d–connections and distorting tensors are uniquely computed using components of $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{N}}$ for a chosen fundamental Finsler function F. Both d–connections are with nontrivial nonmetricity ${}^{B}\mathcal{Q}\neq 0$ and ${}^{Ch}\mathcal{Q}\neq 0$. Nevertheless, such tensor objects are not arbitrary ones but completely induced by respective ${}^{B}\tilde{\mathbf{Z}}$ and ${}^{Ch}\tilde{\mathbf{Z}}$.

3.1.1 Distortions of Ricci tensors for Finsler d–connections

Hereafter, we shall denote by ${}^{F}\mathbf{D}$ any d–connection (metric compatible or not, for instance, of type (26)) uniquely determined by F. Computing the curvature 2–form (16) for ${}^{F}\mathbf{D}$,

$$ {}^{F}\mathcal{R}^{\alpha}_{\ \beta} \doteqdot {}^{F}\mathbf{D}\ {}^{F}\mathbf{\Gamma}^{\alpha}_{\ \beta} = d\ {}^{F}\mathbf{\Gamma}^{\alpha}_{\ \beta} - {}^{F}\mathbf{\Gamma}^{\gamma}_{\ \beta}\wedge {}^{F}\mathbf{\Gamma}^{\alpha}_{\ \gamma} = {}^{F}\mathbf{R}^{\alpha}_{\ \beta\gamma\delta}\mathbf{e}^{\gamma}\wedge\mathbf{e}^{\delta}, $$

we express

$$ {}^{F}\mathbf{R}^{\alpha}_{\ \beta\gamma\delta} = \tilde{\mathbf{R}}^{\alpha}_{\ \beta\gamma\delta} + \tilde{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\delta}. \qquad (27)$$

Contracting indices, ${}^{F}\mathbf{R}_{\ \beta\gamma} := {}^{F}\mathbf{R}^{\alpha}_{\ \beta\gamma\alpha}$, we obtain the N–adapted coefficients for the Ricci tensor ${}^{F}\mathcal{R}ic \doteqdot \{ {}^{F}\mathbf{R}_{\beta\gamma} = (R_{ij}, R_{ia}, R_{ai}, R_{ab})\}$. This tensor, in general, is not symmetric, ${}^{F}\mathbf{R}_{\beta\gamma}\neq {}^{F}\mathbf{R}_{\gamma\beta}$, and corresponding Bianchi identities result in constraints

$$ {}^{F}\mathbf{D}^{\beta}\left({}^{F}\mathbf{R}_{\beta\gamma} - \frac{1}{2}\mathbf{g}_{\beta\gamma}\ {}^{F}_{s}R\right) := {}^{F}\mathbf{J}_{\gamma}\neq 0 \qquad (28)$$

even for metric compatible ${}^{F}\mathbf{D}$. In above formulas, the scalar curvature is by definition

$$ {}^{F}_{s}R := \mathbf{g}^{\beta\gamma}\ {}^{F}\mathbf{R}_{\beta\gamma} = g^{ij}R_{ij} + h^{ab}R_{ab}. \qquad (29)$$

The Einstein tensor ${}^{F}\mathbf{E}_{\beta\gamma}$ can be postulated in standard form for any ${}^{F}\mathbf{D}$,

$$ {}^{F}\mathbf{E}_{\beta\gamma} := {}^{F}\mathbf{R}_{\beta\gamma} - \frac{1}{2}\mathbf{g}_{\beta\gamma}\ {}^{F}_{s}R. \qquad (30)$$

The relations (28) can be considered as a nonholonomic "unique" deformation of standard relations $\nabla_\alpha E^{\alpha\beta} = 0$, with Einstein tensor $E^{\alpha\beta}$ for the Levi–Civita connection ∇_α, in general relativity. Using distorting relations (26), we can always compute the source ${}^F\mathbf{J}_\gamma$ and define an associated set of constraints as in nonholonomic mechanics.

The distortions of connections will be used in the nonholonomic geometric flow theory as follows. Contracting indices in (27), we compute

$$ {}^F\mathbf{R}_{\ \beta\gamma} = \tilde{\mathbf{R}}_{\ \beta\gamma} + \tilde{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\delta}, \qquad (31) $$

$$\begin{aligned}
\tilde{\mathbf{R}}_{\ \beta\gamma} &:= \tilde{\mathbf{R}}^{\alpha}_{\ \beta\gamma\alpha} = \mathbf{e}_\alpha \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma} - \mathbf{e}_\gamma\ \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \beta\alpha} \\
&\quad + \tilde{\mathbf{\Gamma}}^{\varphi}_{\ \beta\gamma}\ \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \varphi\alpha} - \tilde{\mathbf{\Gamma}}^{\varphi}_{\ \beta\alpha}\ \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \varphi\gamma} + \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \beta\varphi} W^{\varphi}_{\gamma\alpha} \\
\tilde{\mathbf{Z}}_{\ \beta\gamma} &:= \tilde{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\alpha} = \mathbf{e}_\alpha\ \tilde{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} - \mathbf{e}_\gamma\ \tilde{\mathbf{Z}}^{\alpha}_{\ \beta\alpha} + \tilde{\mathbf{Z}}^{\varphi}_{\ \beta\gamma}\ \tilde{\mathbf{Z}}^{\alpha}_{\ \varphi\alpha} - \tilde{\mathbf{Z}}^{\varphi}_{\ \beta\alpha}\ \tilde{\mathbf{Z}}^{\alpha}_{\ \varphi\gamma} + \\
&\quad \tilde{\mathbf{\Gamma}}^{\varphi}_{\ \beta\gamma}\ \tilde{\mathbf{Z}}^{\alpha}_{\ \varphi\alpha} - \tilde{\mathbf{\Gamma}}^{\varphi}_{\ \beta\alpha}\ \tilde{\mathbf{Z}}^{\alpha}_{\ \varphi\gamma} + \tilde{\mathbf{Z}}^{\varphi}_{\ \beta\gamma}\ \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \varphi\alpha} - \tilde{\mathbf{Z}}^{\varphi}_{\ \beta\alpha}\ \tilde{\mathbf{\Gamma}}^{\alpha}_{\ \varphi\gamma} + \tilde{\mathbf{Z}}^{\alpha}_{\ \beta\varphi} W^{\varphi}_{\gamma\alpha}.
\end{aligned}$$

Introducing in above formulas $\tilde{\mathbf{Z}} = {}^B\tilde{\mathbf{Z}}$, or $\tilde{\mathbf{Z}} = {}^{Ch}\tilde{\mathbf{Z}}$, we get explicit formulas for distortions of the Ricci tensor (31) for the Berwald, or Chern, d–connection (for simplicity, we omit such technical results in this work). Equivalent distortions can be computed if we fix, for instance, as a "background" connection just the Levi–Civita connection ∇ but such constructions are not adapted to the N–connection splitting. Other fundamental geometric objects derived for ${}^B\mathbf{D}$ and/or ${}^{Ch}\mathbf{D}$ can be generated by noholonomic deformations from analogous ones for the metric compatible, and almost Kähler, d–connection ${}^\theta\mathbf{D} \equiv \tilde{\mathbf{D}}$. This property is very important because it allows us to construct, for instance, Dirac operators and define generalized Perelman's functionals (see next section) even such geometric models are metric noncompatible.

We conclude that metric compatible Finsler geometry models with d–connections uniquely defined by respective metric structures (and induced by fundamental Finsler functions and Hessians) play a preferred role both for elaborating geometric and physical theories on TM. Using one of the connections $\tilde{\mathbf{D}}$, or $\hat{\mathbf{D}}$, we work as in usual Riemann–Cartan geometry and/or the Ricci flow theory of Riemannian metrics. It is also possible to reformulate the theories as almost Kähler geometries. Then, such constructions can be nonholonomically deformed into metric noncompatible structures by considering respective distortion tensors.

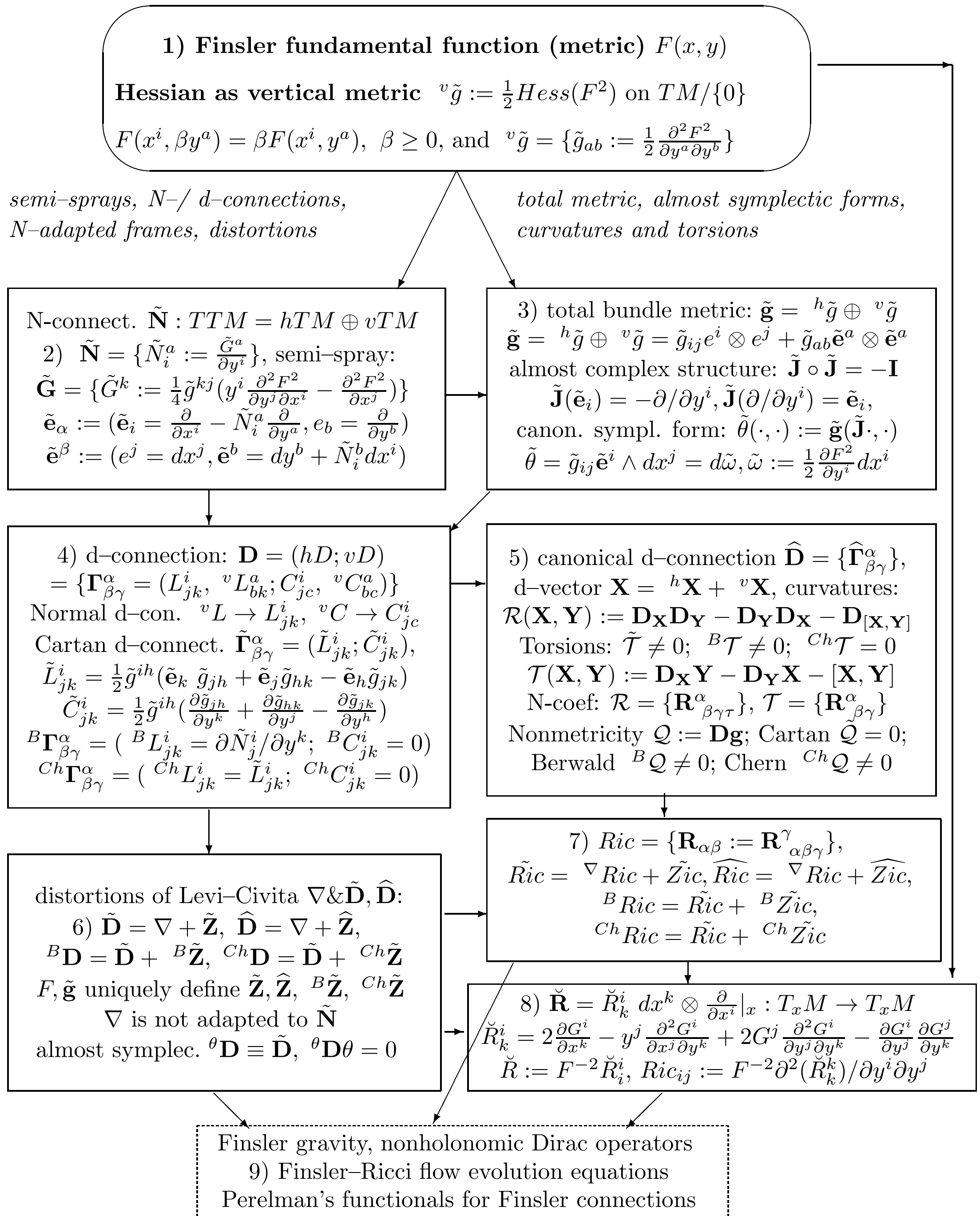

Figure 2: Fundamental Objects for Finsler Geometries on TM

3.1.2 A Ricci tensor constructed by Akbar–Zadeh

For a class of geometric and flow models on T_xM, see Refs. [30, 9, 43, 29], a different than (31) Ricci tensor is used. As noted above, there is an alternative curvature tensor $\breve{R}^i_k$ (25) completely determined by semi–spray $\tilde{G}^k$ (7). Contracting indices, we introduce a scalar function $\breve{R}(x,y) := F^{-2}\breve{R}^i_i$ and define a variant of the Ricci tensor,

$$\breve{R}ic_{jk} := F^{-2}\frac{\partial^2 \breve{R}}{\partial y^j \partial y^k}. \tag{32}$$

The geometric object $\breve{R}ic_{jk}$ (32) is induced by the Finsler metric F via inverse Hessian $\tilde{g}^{ij}$, see $\tilde{g}_{ij}$ (1), and $\tilde{G}^k$ not involving in such a model the N–connection structure, lifts on metrics on total space of TM, and d–connections. By definition, the scalar $\breve{R}$ is positive homogeneous of degree 0 in v–variables y^a.

Following such an approach to Finsler geometry, the Einstein metrics $\tilde{g}_{ij}$ are those for which $\breve{R}ic_{jk} = \lambda(x)\tilde{g}_{jk}$, i.e. when the scalar function $\breve{R}(x,y) = \lambda(x)$ is a function only on h–variables x^k. This class of Finsler spaces is by definition different from that derived for a Ricci d–tensor ${}^F\mathcal{R}ic$ (31) on TM. The priority of $\breve{R}ic_{jk}$ (32) is that it is always symmetric (by definition) and "simplified" to consider a Ricci field and/or evolution dynamics in any point T_xM. Nevertheless, such a nonholonomically constrained model does not allow us to study, for instance, mutual transforms of Riemann and Finsler metrics with general nonsymmetric Ricci d–tensor $\tilde{\mathbf{R}}_{\beta\gamma}$, and respective Einstein d–tensor (30) on total space of TM and nonholonomic (pseudo) Riemannian manifolds [21, 22, 24] (a series of works from 2006–2008).

A variant of geometric evolution equations for Finsler metrics $F(\chi, x, y)$ using $\breve{R}ic_{jk}$ was published in 2007 in Ref. [9],

$$\frac{\partial \tilde{g}_{ij}}{\partial \chi} = -2\breve{R}ic_{jk}, \tag{33}$$

when the Hessian $\tilde{g}_{ij}(\chi, x, y)$ and the volume element

$$\upsilon := (\partial F/\partial y^i)dx^i \tag{34}$$

depend on $\chi \in [-\epsilon, \epsilon] \subset \mathbb{R}$ and $\epsilon > 0$ is sufficiently small. These equations consist an example of heuristic Finsler evolution equations (2) when ${}^F Ric_{ij} \sim \breve{R}ic_{jk}$. In order to elaborate a self–consistent Ricci flow theory, at first steps, we have to prove the conditions when $\breve{R}ic_{jk} \sim {}^F\Delta$, for a Finsler Laplacian, and find certain analogs of Perelman's functionals from which (33). One of the aims of the present paper is to show that this type of evolution models belong to a class of nonholonomicaly constrained system (in general, with metric noncompatible Finsler connections) which can be uniquely defined via corresponding nonholonomic deformations and constraints from theories flows of the canonical and/or Cartan d–connection [19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

For any $F(\chi) = F(\chi, x, y)$, and respective $\tilde{g}_{ij}$ and $\tilde{N}_i^a$, we can compute a family of Ricci d–tensors,

$$\tilde{\mathbf{R}}_{\ \beta\gamma}(\chi) = \{\tilde{R}_{\ hj} := \tilde{R}^i{}_{hji}, \tilde{P}_{\ ja} := \tilde{P}^i{}_{jia},\ \tilde{S}_{\ bc} :=\ \tilde{S}^a{}_{bca}\},$$

for the d–connection $\tilde{\mathbf{D}}$, by constructing respective tensors in (22), or

$$\widehat{\mathbf{R}}_{\ \beta\gamma}(\chi) = \widehat{R}_{ij} \doteqdot \widehat{R}^k{}_{ijk},\ \ \widehat{R}_{ia} \doteqdot -\widehat{R}^k{}_{ika},\ \widehat{R}_{ai} \doteqdot \widehat{R}^b{}_{aib},\ \widehat{R}_{ab} \doteqdot \widehat{R}^c{}_{abc}, \quad (35)$$

for the canonical d–connection $\widehat{\mathbf{D}}$ (18) (see explicit formulas for h–v–components in Refs. [10, 13, 16]). Because all values $\breve{R}ic_{jk}, \tilde{R}_{\ hj}$ and $\widehat{R}_{ij}$ are generated by the same Finsler metric, we can compute in unique forms (up to frame transforms) the distortions

$$\breve{R}ic_{jk} = \tilde{R}_{\ jk} + \tilde{Z}ic_{jk},\ \breve{R}ic_{jk} = \widehat{R}_{\ jk} + \widehat{Z}ic_{jk}, \quad (36)$$

if values $\tilde{\mathbf{R}}_{\ \beta\gamma}$ and $\widehat{\mathbf{R}}_{\ \beta\gamma}$ are defined on TM. Similar splitting can be computed in unique forms for Ricci d–tensors corresponding to $\ ^{B}\mathbf{D}= \tilde{\mathbf{D}}+\ ^{B}\tilde{\mathbf{Z}}$ and $\ ^{Ch}\mathbf{D}= \tilde{\mathbf{D}}+\ ^{Ch}\tilde{\mathbf{Z}}$ from (26),

$$\begin{aligned} \ ^{B}\mathbf{R}_{\ \beta\gamma} &= \tilde{\mathbf{R}}_{\ \beta\gamma} +\ ^{B}\tilde{\mathbf{Z}}ic_{\beta\gamma},\ \ ^{Ch}\mathbf{R}_{\ \beta\gamma} = \tilde{\mathbf{R}}_{\ \beta\gamma} +\ ^{Ch}\tilde{\mathbf{Z}}ic_{\beta\gamma}; \quad (37)\\ \ ^{B}\mathbf{R}_{\ \beta\gamma} &= \widehat{\mathbf{R}}_{\ \beta\gamma} +\ ^{B}\widehat{\mathbf{Z}}ic_{\beta\gamma},\ \ ^{Ch}\mathbf{R}_{\ \beta\gamma} = \widehat{\mathbf{R}}_{\ \beta\gamma} +\ ^{Ch}\widehat{\mathbf{Z}}ic_{\beta\gamma}. \quad (38)\end{aligned}$$

If we construct a geometric evolution model for $F(\chi)$ with $\tilde{g}_{ij}(\chi)$ derived for $\tilde{\mathbf{R}}_{\ \beta\gamma}$, such constructions are preferred for almost Kähler models and deformation quantization [38], or for $\widehat{\mathbf{R}}_{\ \beta\gamma}$ (this is important to study evolution of exact solutions in gravity theories [36, 37]), we can always "extract" and follow evolution of geometric objects and metric noncompatible Finsler geometries and/or with "nonstandard" curvature (25).

3.2 Einstein–Finsler spaces

Following the classification presented in Figure 2, we can work equivalently with any d–connection $\tilde{\mathbf{D}}$, $\hat{\mathbf{D}}$, $^{B}\mathbf{D}$, $\ ^{Ch}\mathbf{D}$ (all these geometric objects are uniquely determined by F and/or $\widetilde{\mathbf{g}}$. To study possible physical applications with generalized gravitational field/ evolution equations is important to decide which type of connection and nonholonomic constraints are used for elaborating physical theories.

It should be noted that all constructions provided in previous sections can be performed not only on tangent bundle TM with N–connection splitting (5) but on any manifold $\mathbf{V}$ with "conventional" h–v–splitting (called also as a nonholonomic manifold) defined by a Whitney sum

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}. \tag{39}$$

Such a nonintegrable distribution, for instance, can be introduced always on a Lorenz manifold $\mathbf{V}$ in general relativity (GR) defining a so–called $2+2$ splitting.[9] More than that, GR and various modifications can be described equivalently in Finsler and/or almost Kähler variables, see details in Refs. [10, 12, 13, 36, 37]. There is an unified formalism for geometrical/physical models which can be elaborated any nonholonomic manifold, $\mathbf{V}$, or tangent bundle space, $\mathbf{V} = TM$. Physically, the y–variables are treated differently: On a general $\mathbf{V}$, such values/coordinates are certain nonholonomically constrained ones; on TM, the values y^a as some "velocities" (for dual configurations on T^*M, there are considered "momenta").

The Einstein equations in GR were postulated in standard form using the Levi–Civita connection ∇,

$$R_{\ \beta\delta} - \frac{1}{2} g_{\beta\delta} R = \varkappa T_{\beta\delta}, \tag{40}$$

written for the Levi–Civita connection $\nabla = \{\Gamma^{\gamma}_{\ \alpha\beta}\}$. In formulas (40), $R_{\ \beta\delta}$ and R are respectively the Ricci tensor and scalar curvature of ∇; it is also considered the energy–momentum tensor for matter, $T_{\alpha\beta}$, where $\varkappa = const$. Various tetradic, spinor, connection etc variables were used with various purposes to construct exact solutions and quantize gravity, see standard monographs [44, 45]. Using conventional Finsler variables, the gravitational field equations (40) can be re–written equivalently using the canonical d–connection $\widehat{\mathbf{D}}$ (18),

$$\widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ \ _{s}\widehat{R} = \widehat{\mathbf{\Upsilon}}_{\beta\delta}, \tag{41}$$

$$\widehat{L}^{c}_{aj} = e_a(N^c_j),\ \widehat{C}^{i}_{jb} = 0,\ \Omega^{a}_{\ ji} = 0, \tag{42}$$

for $\widehat{\mathbf{\Upsilon}}_{\beta\delta} \rightarrow T_{\beta\delta}$ if $\widehat{\mathbf{D}} \rightarrow \nabla$. The constraints (42) are equivalent to the condition of vanishing of torsion (15), the distortion d–tensors $\widehat{\mathbf{Z}} = 0$, which

[9] we use "boldface" letters for manifolds, bundles endowed with N–connection structure and for geometric objects adapted to corresponding h-v–splitting

results in $\widehat{\mathbf{D}}=\nabla$, see formulas (19). The system of equations (41) and (42) have a very important property of decoupling with respect to N–adapted frames (9) and (10) which allows to integrate the Einstein and geometric evolution equations in very general forms [10, 12, 13, 36, 37, 19, 20, 26, 27, 28].[10]

On TM, for metric compatible Finsler geometry models, constraints of type (42), or (44), are not necessary. Using distortion relations (19), (23) and (26), we can compute other types of distortions,

$$\tilde{\nabla} = {}^{Ch}\mathbf{D} - {}^{Ch}_{\nabla}\mathbf{Z} = {}^{B}\mathbf{D} - {}^{B}_{\nabla}\mathbf{Z} = \tilde{\mathbf{D}} - \tilde{\mathbf{Z}} = \widehat{\mathbf{D}} - \widehat{\mathbf{Z}}, \qquad (45)$$

where all geometric objects are determined by $F(u)$ via $\tilde{g}_{ij}(u)$. Such nonholonomic constraints show that in any model of Finsler geometry we can consider equivalently "not–adapted" (to N–connection) geometric constructions with $\tilde{\nabla}$ defined by a (pseudo) Riemannian metric

$$\tilde{g}_{\underline{\alpha}\underline{\beta}} = \left[\begin{array}{cc} \tilde{g}_{ij} + \tilde{N}_i^a \ \tilde{N}_j^b \ \tilde{g}_{ab} & \tilde{N}_j^e \ \tilde{g}_{ae} \\ \tilde{N}_i^e \ \tilde{g}_{be} & \tilde{g}_{ab} \end{array} \right], \qquad (46)$$

where the coefficients $\tilde{g}_{\underline{\alpha}\underline{\beta}}$ are those for the Finsler d–metric (12) re–defined with respect to a coordinate co-basis, $du^{\underline{\alpha}} = (dx^{\underline{i}}, dy^{\underline{a}})$. The nonholonomic structure is encoded into vielbeins $\tilde{\mathbf{e}}_\alpha = \tilde{\mathbf{e}}_\alpha^{\ \underline{\alpha}}(u)\partial_{\underline{\alpha}}$ with coefficients

$$\tilde{\mathbf{e}}_\alpha^{\ \underline{\alpha}}(u) = \left[\begin{array}{cc} \tilde{e}_i^{\ \underline{i}}(u) & \tilde{N}_i^b(u) \ \tilde{e}_b^{\ \underline{a}}(u) \\ 0 & \tilde{e}_a^{\ \underline{a}}(u) \end{array} \right], \qquad (47)$$

when $\tilde{g}_{ij}(u) = \tilde{e}_i^{\ \underline{i}}(u) \ \tilde{e}_j^{\ \underline{j}}(u)\eta_{\underline{i}\underline{j}}$, for $\eta_{\underline{i}\underline{j}} = diag[\pm 1, ... \pm 1]$ fixing a corresponding local metric signature on TM.

[10] Up to frame/coordinate transforms the equations (40), and/or (41) and (42), are equivalent to

$$\tilde{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\tilde{\mathbf{g}}_{\beta\delta} \ {}_s\tilde{R} = \tilde{\mathbf{\Upsilon}}_{\beta\delta}, \qquad (43)$$

$$\tilde{L}^a_{\ bi} = e_b(\tilde{N}_i^a), \ \tilde{C}^i_{\ jc} = 0, \ \tilde{\Omega}^a_{ij} = 0, \qquad (44)$$

when the d–connection is chosen to be the Cartan one ${}^{\theta}\mathbf{D} \equiv \tilde{\mathbf{D}}$. The conditions (44) are for zero torsion (21) when $\tilde{\mathbf{Z}} = 0$ and $\tilde{\mathbf{D}} = \tilde{\nabla}$, in (23). Here, we note that, in general, $\tilde{\mathbf{\Upsilon}}_{\beta\delta}$ is different from $\widehat{\mathbf{\Upsilon}}_{\beta\delta}$. The priority of system (43) written in Cartan d–metric and d–connection Finsler variables is a the possibility to re–define the geometric objects in almost Kähler variables with a further deformation quantization [38]. The "decoupling effect" for gravitational field equations also exists but the zero torsion conditions seem to be "more rigid" for such configurations.

We conclude this section with the remark that models of Finsler geometry on TM [11] with Cartan/ canonical d–connection, Berwald and/or Chern d–connections can be reconsidered equivalently as certain nonholonomic (pseudo) Riemannian ones endowed with nonholonomic h–v–splitting and corresponding unique distortions of $\tilde{\nabla}$. The distortion relations (45) play a crucial role in constructing models of Finsler–Ricci flow evolution uniquely related to standard theory of Ricci flows for Riemannian geometries. The main theorems can be proven using $\tilde{\nabla}$ and then the results for Finsler flows are stated by "uniquely" defined nonholonomic distortions and constraints.

4 Finsler–Ricci Flows and Distortions

In this section we show how a self–consistent approach to geometric flows with metric noncompatible connections can be elaborated if there are used special classes of nonholonomic deformations/distortions of metric compatible flows.

4.1 The Perelman's Functionals on Finsler Spaces

G. Perelman's idea [3] was to derive the Ricci flow equations of (pseudo) Riemannian geometries as gradient flows for some functionals defined by the Levi–Civita connection ∇ and respective scalar curvature $\ _{\nabla}R$. Considering a compact region $\mathcal{V} \subset TM$ (in general, we can take any nonholonomic manifold $\mathbf{V}$ instead of TM), with $\tilde{\nabla}$ computed for $\tilde{g}_{\underline{\alpha\beta}}$ (46). This family of geometric objects is induced by a family of Finsler generating function $F(\tau, x, y)$ parametrized by a flow parameter $\tau \in [-\epsilon, \epsilon] \subset \mathbb{R}$ with a sufficiently small $\epsilon > 0$. It is possible to introduce such functionals in Finsler geometry (we use our system of denotations),

$$
\begin{aligned}
\ _{\shortmid}\mathcal{F}(\tilde{\mathbf{g}}, \tilde{\nabla}, f) &= \int_{\mathcal{V}} \left(\ _{\nabla}\tilde{R} + \left| \tilde{\nabla} f \right|^2 \right) e^{-f} \ dV, \qquad (48) \\
\ _{\shortmid}\mathcal{W}(\tilde{\mathbf{g}}, \tilde{\nabla}, f, \tau) &= \int_{\mathcal{V}} \left[\tau \left(\ _{\nabla}\tilde{R} + \left| \tilde{\nabla} f \right| \right)^2 + f - 2n) \right] \mu \ dV,
\end{aligned}
$$

where dV is the volume form of $\mathbf{g} \sim \tilde{\mathbf{g}}$ (up to frame transforms), integration is taken over $\mathcal{V}, \dim \mathcal{V} = 2n$. Via frame transforms and for a parameter $\tau > 0$, we can fix $\int_{\mathcal{V}} dV = 1$ when $\mu = (4\pi\tau)^{-n} e^{-f}$. Working with $\tilde{\nabla}$, we can model in "not N–adapted" form different types of Ricci flow evolutions of Finsler geometries by imposing nonholonomic constraints with a distortion relation

[11] and/or any nonholonomic manifold $\mathbf{V}$ with N–connection splitting

(45). In this approach, the Finsler–Ricci flows can be considered as evolving nonholonomic dynamical systems on the space of Riemannian metrics on TM and the functionals ${}_{\shortmid}\mathcal{F}$ and ${}_{\shortmid}\mathcal{W}$ are of Lyapunov type. Levi–Civita Ricci flat configurations are defined as "fixed" on τ points of the corresponding dynamical systems.

The goal of this section is to re–define the functionals (48) in N–adapted form when the evolution of Finsler geometries with Sasaki type metrics (12) on $\widetilde{TM}$ will be extracted by a corresponding fixing ${}^{F}\mathbf{D} = {}^{Ch}\mathbf{D}$, or ${}^{B}\mathbf{D}$ (the variants with ${}^{F}\mathbf{D} = \tilde{\mathbf{D}}$, and/or $=\widehat{\mathbf{D}}$ where studied in Refs. [22, 24]).

Lemma 4.1 *For a Finsler geometry model with d–connection ${}^{F}\mathbf{D}$ completely determined by F and $\tilde{\mathbf{g}}$, the Perelman's functionals (48) can be re–written equivalently in N–adapted form by considering distortion relations for scalar curvature and Ricci tensor (31),*

$$
\begin{aligned}
{}^{F}\mathcal{F}(\tilde{\mathbf{g}},\ {}^{F}\mathbf{D},\breve{f}) &= \int_{\mathcal{V}} (\ {}_{s}^{F}R + |\ {}^{F}\mathbf{D}\breve{f}|^{2})e^{-\breve{f}}\ dV, && (49)\\
{}^{F}\mathcal{W}(\tilde{\mathbf{g}},\ {}^{F}\mathbf{D},\breve{f},\breve{\tau}) &= \int_{\mathcal{V}} [\breve{\tau}(\ {}_{s}^{F}R + |\ {}^{h}D\breve{f}| + |\ {}^{v}D\breve{f}|)^{2} + \breve{f} - 2n]\breve{\mu} dV, && (50)
\end{aligned}
$$

where the scalar curvature ${}_{s}^{F}R$ (29) is computed for ${}^{F}\mathbf{D} = (\ {}_{h}^{F}D,\ \ {}_{v}^{F}D)$, $\left|\ {}^{F}\mathbf{D}\breve{f}\right|^{2} = \left|\ {}_{h}^{F}D\breve{f}\right|^{2} + \left|\ {}_{v}^{F}D\breve{f}\right|^{2}$, and the new scaling function $\breve{f}$ satisfies $\int_{\mathcal{V}} \breve{\mu} dV = 1$ for $\breve{\mu} = (4\pi\breve{\tau})^{-n}\, e^{-\breve{f}}$ and $\breve{\tau} > 0$.

Proof. The proof of this Lemma is similar to that for Claim 3.1 in Ref. [22] for nonholonomic manifolds (for a prescribed canonical d–connection). On $\widetilde{TM}$, such a statement transforms into a Lemma similar to that in original Perelman's work [3] if we consider models of Finsler geometry with ${}^{F}\mathbf{D}$ is related to $\tilde{\nabla}$ via a unique distortion relation (45). For simplicity, we can use $\breve{\tau} = \ {}^{h}\tau = \ {}^{v}\tau$ for a couple of possible h– and v–flows parameters, $\breve{\tau} = (\ {}^{h}\tau,\ {}^{v}\tau)$, and introduce a new function $\breve{f}$, instead of f. This scalar function is re–defined in such a form that in formulas (48) the distortion of Ricci tensor (31) and d–connection under $\tilde{\nabla} \rightarrow\ {}^{F}\mathbf{D}$ results in

$$
(\ {}_{\shortmid}\tilde{R} + |\tilde{\nabla}f|^{2})e^{-f} = (\ {}_{s}^{F}R + |\ {}^{F}\mathbf{D}\breve{f}|^{2})e^{-\breve{f}}\ + \Phi \qquad (51)
$$

for (49). Similarly, we re–scale the parameter $\tau \rightarrow \breve{\tau}$ to have

$$
[\tau(\ {}_{\shortmid}\tilde{R} + |\tilde{\nabla}f|)^{2} + f - 2n)]\mu = [\breve{\tau}(\ {}_{s}^{F}R + |\ {}^{h}D\breve{f}| + |\ {}^{v}D\breve{f}|)^{2} + \breve{f} - 2n]\breve{\mu} + \Phi_{1} \qquad (52)
$$

for some Φ and Φ_{1} for which $\int_{\mathcal{V}} \Phi dV = 0$ and $\int_{\mathcal{V}} \Phi_{1} dV = 0$. This results in formula (50). Finally, in this proof, we conclude that both in metric compatible and noncompatible Finsler models uniquely determined by F and $\tilde{\mathbf{g}}$, the Perelman functionals are certain nonholonomic deformations of those for $\tilde{\nabla}$.

$\square$

A similar proof with redefinition to a corresponding function $\breve{f} \rightarrow \underline{f}$ and parameter $\breve{\tau} \rightarrow \underline{\tau}$, can be used for proof of

Corollary 4.1 *Fixing a point $x \in TM$ and a compact region $\mathcal{V}_x$ and via distortions (36), respectively, we can transform (49) and (50) into*

$$^F\mathcal{F}(\tilde{g}_{ij},\ \tilde{D}, \breve{f}) = \int_{\mathcal{V}} (\ \tilde{g}^{jk} \breve{R}ic_{jk} + |\tilde{D}\breve{f}|^2) e^{-\underline{f}}\ dV, \quad (53)$$

$$^F\mathcal{W}(\tilde{g}_{ij},\ \tilde{D}, \breve{f}, \breve{\tau}) = \int_{\mathcal{V}} [\underline{\tau}(\ \tilde{g}^{jk} \breve{R}ic_{jk} + |\tilde{D}\breve{f}|)^2 + \underline{f} - n] \breve{\mu} dV, \quad (54)$$

defining a nonholonomic dynamics related to Akbar–Zadeh definition of the Ricci tensor $\breve{R}ic_{jk}$ (32).

In above formulas, integrals of type $\int_{\mathcal{V}} \{\ldots\} dV$ can be transformed into computations on "spherical" bundle SM, see details [30, 43, 29],

$$\int_{SM} \{\ldots\} \frac{(-1)^{n(n-1)/2}}{(n-1)!} \upsilon \wedge (d\upsilon)^{n-1} = \int_{SM} \{\ldots\} dV_{SM},$$

where the volume element υ is determined by F following formula (34).

4.2 On N–adapted geometric structures

Following the classification of fundamental geometric objects for Finsler geometry models presented in Figures 1 and 2 , we conclude that any geometric configuration and Ricci flow evolution formula for Riemannian metrics containing the Levi–Civita connection ∇ can be transformed into its analogous on TM for Finsler spaces following such rules:

Conclusion 4.1 ***(Rules)***

1. *Consider a h-v–splitting determined by $F(\chi) := F(\chi, u)$ via flows of canonical N–connection $\tilde{\mathbf{N}}\ (\chi)$ and adapted frames*

   $$\partial_{\underline{\alpha}} \rightarrow \mathbf{e}_\alpha(\chi) = (\mathbf{e}_i(\chi) = \partial_i - N_i^b(\chi)\partial_b, e_a = \partial_a),$$
   $$du^{\underline{\alpha}} \rightarrow \mathbf{e}^\alpha(\chi) = \left(e^i = dx^i, e^a = dy^a + N_k^a(\chi) dx^k\right),$$

   related to $\tilde{\mathbf{e}}_\alpha(\chi)$ (9) and $\tilde{\mathbf{e}}^\alpha(\chi)$ (10) by any convenient frame/coordinate transforms.

2. *Metrics* $\tilde{g}_{\underline{\alpha\beta}}(\chi)$ *(46) are transformed equivalently into d–metrics* $\tilde{\mathbf{g}}(\chi)$ *(12) and/or any related via frame transforms* $\mathbf{g}$ *(17).*

3. *Via distortion relations (45), we construct necessary chains of distortions of connections,* $\tilde{\nabla}(\chi) \to \nabla(\chi) \to \widehat{\mathbf{D}}(\chi) = \left({}^{h}D(\chi),\ {}^{v}D(\chi)\right) \to {}^{F}\mathbf{D}(\chi)$, *where* ${}^{F}\mathbf{D} = \tilde{\mathbf{D}}, = {}^{Ch}\mathbf{D}$, *or* $= {}^{B}\mathbf{D}$.

4. *Using such distortions of connections, we can compute distortions of curvature tensors and related Ricci tensors (see (31), (37), (38) and (36)) and scalar curvatures.*

5. *Changing data* $(f,\tau) \to (\breve{f},\breve{\tau})$ *given by formulas of type (51) and (52), we compute distortions of the Perelman's functionals (48), i.e* ${}_{\shortmid}\mathcal{F}$ *and* ${}_{\shortmid}\mathcal{W}$ *, into* ${}^{F}\mathcal{F}$ *and* ${}^{F}\mathcal{W}$, *respectively, (49) and (50).*

In this work, we shall omit detailed proofs if they can be obtained using metric compatible constructions in (pseudo) Riemannian and Lagrange–Finsler geometry as in Refs. [3, 6, 21, 22, 24] following rules stated in Conclusion 4.1.

4.3 Hamilton equations for metric noncompatible Finsler spaces

For the canonical d–connection $\widehat{\mathbf{D}}$ (similarly, for $\tilde{\mathbf{D}}$), we can construct the canonical Laplacian operator, $\widehat{\Delta} := \widehat{\mathbf{D}}\ \widehat{\mathbf{D}}$, h- and v–components of the Ricci tensor, $\widehat{R}_{ij}$ and $\widehat{R}_{ab}$, and consider parameter $\tau(\chi)$, $\partial\tau/\partial\chi = -1$ (for simplicity, we do not include the normalized term).

Theorem 4.1 *The Finsler–Ricci flows for* ${}^{F}\mathbf{D}$ *preserving a symmetric metric structure* $\mathbf{g} = \tilde{\mathbf{g}}$ *and nonholonomic constraints*

$$\tilde{\nabla} = {}^{F}\mathbf{D} - {}_{\nabla}^{F}\mathbf{Z},\ \widehat{\mathbf{D}} = {}^{F}\mathbf{D} - {}^{F}\widehat{\mathbf{Z}}, \tag{55}$$

resulting in distortions

$$\begin{aligned}
\widehat{\Delta} &= \widehat{\mathbf{D}}_{\alpha}\ \widehat{\mathbf{D}}^{\alpha} = {}^{F}\Delta + {}^{Z}\widehat{\Delta}, \qquad (56)\\
{}^{F}\Delta &= {}^{F}\mathbf{D}_{\alpha}\ {}^{F}\mathbf{D}^{\alpha},\ \ {}^{Z}\widehat{\Delta} = {}^{F}\widehat{\mathbf{Z}}_{\alpha}\ {}^{F}\widehat{\mathbf{Z}}^{\alpha} - [\ {}^{F}\mathbf{D}_{\alpha}(\ {}^{F}\widehat{\mathbf{Z}}^{\alpha}) + {}^{F}\widehat{\mathbf{Z}}_{\alpha}(\ {}^{F}\mathbf{D}^{\alpha})];\\
\widehat{\mathbf{R}}_{\ \beta\gamma} &= {}^{F}\mathbf{R}_{\ \beta\gamma} - {}^{F}\widehat{\mathbf{Z}}ic_{\beta\gamma},\ \ {}_{s}\widehat{R} = {}_{s}^{F}R - \mathbf{g}^{\beta\gamma}\ {}^{F}\widehat{\mathbf{Z}}ic_{\beta\gamma} = {}_{s}^{F}R - {}_{s}^{F}\widehat{\mathbf{Z}},\\
{}_{s}^{F}\widehat{\mathbf{Z}} &= \mathbf{g}^{\beta\gamma}\ {}^{F}\widehat{\mathbf{Z}}ic_{\beta\gamma} = {}_{h}^{F}\widehat{Z} + {}_{v}^{F}\widehat{Z},\ {}_{h}^{F}\widehat{Z} = g^{ij}\ {}^{F}\widehat{\mathbf{Z}}ic_{ij},\ {}_{v}^{F}\widehat{Z} = h^{ab}\ {}^{F}\widehat{\mathbf{Z}}ic_{ab};\\
{}_{s}^{F}R &= {}_{h}^{F}R + {}_{v}^{F}R,\ \ {}_{h}^{F}R := g^{ij}\ {}^{F}R_{ij},\ {}_{v}^{F}R = h^{ab}\ {}^{F}R_{ab},
\end{aligned}$$

can be characterized by two equivalent systems of geometric flow equations:

1. *Evolution with distortions of the canonical d–connections introduced in metric compatible nonholonomic Ricci flow equations,*

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \chi} &= -2\left({}^{F}\mathbf{R}_{\ ij} - {}^{F}\widehat{\mathbf{Z}}ic_{ij}\right),\ \frac{\partial g_{ab}}{\partial \chi} = -2\left({}^{F}\mathbf{R}_{\ ij} - {}^{F}\widehat{\mathbf{Z}}ic_{ij}\right), \\
{}^{F}\mathbf{R}_{\ ia} &= {}^{F}\widehat{\mathbf{Z}}ic_{ia},\ {}^{F}\mathbf{R}_{\ ai} = {}^{F}\widehat{\mathbf{Z}}ic_{ai}, \qquad (57)\\
\frac{\partial \widehat{f}}{\partial \chi} &= -\left({}^{F}\Delta + {}^{Z}\widehat{\Delta}\right)\widehat{f} + \left|\left({}^{F}\mathbf{D} - {}^{F}\widehat{\mathbf{Z}}\right)\widehat{f}\right|^{2} - {}_{s}^{F}R + {}_{s}^{F}\widehat{\mathbf{Z}},
\end{aligned}$$

and the property that

$$\begin{aligned}
&\frac{\partial}{\partial \chi}\widehat{\mathcal{F}}(\mathbf{g},\ \widehat{\mathbf{D}},\widehat{f}) = \\
&2\int_{\mathcal{V}}[| \ {}^{F}R_{\ ij} - {}^{F}\widehat{\mathbf{Z}}ic_{ij} + ({}^{F}\mathbf{D}_{i} - {}^{F}\widehat{\mathbf{Z}}_{i})({}^{F}\mathbf{D}_{j} - {}^{F}\widehat{\mathbf{Z}}_{j})\widehat{f}|^{2} + \\
&| \ {}^{F}R_{ab} - {}^{F}\widehat{\mathbf{Z}}ic_{ab} + ({}^{F}\mathbf{D}_{a} - {}^{F}\widehat{\mathbf{Z}}_{a})({}^{F}\mathbf{D}_{b} - {}^{F}\widehat{\mathbf{Z}}_{b})\widehat{f}|^{2}]e^{-\widehat{f}}dV,
\end{aligned}$$

when $\int_{\mathcal{V}} e^{-\widehat{f}}dV$ *is constant.*

2. *Evolution derived from distorted Perelman's functional* ${}^{F}\mathcal{F}(\tilde{\mathbf{g}},\ {}^{F}\mathbf{D},\check{f})$ *(49),*

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \chi} &= -2\left({}^{F}\mathbf{R}_{\ ij} - {}^{F}\widehat{\mathbf{Z}}ic_{ij}\right),\ \frac{\partial g_{ab}}{\partial \chi} = -2\left({}^{F}\mathbf{R}_{\ ij} - {}^{F}\widehat{\mathbf{Z}}ic_{ij}\right), \\
{}^{F}\mathbf{R}_{\ ia} &= {}^{F}\widehat{\mathbf{Z}}ic_{ia},\ {}^{F}\mathbf{R}_{\ ai} = {}^{F}\widehat{\mathbf{Z}}ic_{ai}, \qquad (58)\\
\frac{\partial \check{f}}{\partial \chi} &= - {}^{F}\Delta\check{f} + \left| {}^{F}\mathbf{D}\check{f}\right|^{2} - {}_{s}^{F}R\ ,
\end{aligned}$$

and the property that

$$\frac{\partial}{\partial \chi}\ {}^{F}\mathcal{F}(\tilde{\mathbf{g}},\ {}^{F}\mathbf{D},\check{f}) = 2\int_{\mathcal{V}}[| \ {}^{F}\mathbf{R}_{\ \beta\gamma} + {}^{F}\mathbf{D}_{\beta}\ {}^{F}\mathbf{D}_{\gamma}\check{f}|^{2}]e^{-\check{f}}dV,$$

when $\int_{\mathcal{V}} e^{-\check{f}}dV = const.$

Proof. The distortions (45) can be written in an equivalent form (55) which allows us to compute respective splitting for Laplacians and, following formula (29), the decomposition of necessary types Ricci and scalar curvature operators (56). This reduces the constructions to a corresponding system of Ricci flow evolution equations for $\widehat{\mathbf{D}}$, see proofs in Refs. [22, 24],

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \chi} &= -2\widehat{R}_{ij},\ \frac{\partial \underline{g}_{ab}}{\partial \chi} = -2\widehat{R}_{ab}, \qquad (59)\\
\frac{\partial \widehat{f}}{\partial \chi} &= -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathbf{D}}\widehat{f}\right|^{2} - {}^{h}\widehat{R} - {}^{v}\widehat{R},
\end{aligned}$$

derived from the functional $\widehat{\mathcal{F}}(\tilde{\mathbf{g}}, \widehat{\mathbf{D}}, \widehat{f}) = \int_{\mathcal{V}}(\ _s\widehat{R} + |\widehat{\mathbf{D}}\widehat{f}|^2)e^{-\widehat{f}}\ dV$.

Such metric compatible canonical Finsler–Ricci flow equations are equivalent (via nonholonomic transforms $\nabla \to \widehat{\mathbf{D}}$) to those proposed for Riemannian spaces by G. Perelman [3] (details of the proof with ∇ are given in Proposition 1.5.3 of [6]). We must impose the conditions $\widehat{R}_{ia} = 0$ and $\widehat{R}_{ai} = 0$ if we wont to keep the total metric to be symmetric under Ricci evolution. If such conditions are not satisfied, we generate nonsymmetric metrics under nonholonomic geometric evolution because the Ricci tensor may be nonsymmetric for Finsler spaces, see details in [23].

The system of equations (57) is just that for the canonical d–connection (59) but rewritten in terms of (in general, metric noncompatible) $\ ^F\mathbf{D}$. This means that we can follow a metric noncompatible evolution derived from a Perlman type functional $\widehat{\mathcal{F}}$ formulated in terms of the canonical d–connection and respective scalar function $\widehat{f}$.

In another turn, the system of evolution equations (58) is derived directly from $\ ^F\mathcal{F}(\tilde{\mathbf{g}},\ ^F\mathbf{D}, \breve{f})$ applying the same methods from [3, 6] and, for spaces with N–connection structure, in [21, 22, 23, 24]. The equivalence of both systems (57) and (58) can be proven for Finsler connections which are related mutually via distortions completely and uniquely determined by F and $\tilde{\mathbf{g}}$, for instance, the Berwald and Chern connections. In such cases, via frame transforms and nonholonomic deformations $\widehat{\mathcal{F}}$ can be transformed into $\ ^F\mathcal{F}$ and inversely. Such an equivalence and, in general, Perelman functionals can not be introduced in self–consistent form if $\ ^F\mathbf{D}$ is with arbitrary nonmetricity.

Finally, we note that the functional $\widehat{\mathcal{F}}(\mathbf{g}, \widehat{\mathbf{D}}, \widehat{f})$ is nondecreasing in time and the monotonicity is strict unless we are on a steady N–adapted gradient solution (see details in [22]). This property may not "survive" under nonholonomic deformations to certain $\ ^F\mathbf{D}$. This is not surprising for metric noncompatible geometric evolutions. Nevertheless, such distortions can be computed in unique forms due to relations (45) and kept under control via nonholonomic constraints which allows us to construct $\ ^F\mathcal{F}(\tilde{\mathbf{g}},\ ^F\mathbf{D}, \breve{f})$ and derive metric noncompatible evolution equations (58). $\square$

The above theorem can be reformulated in terms of distortions from the Cartan d–connection, when $\tilde{\mathbf{D}} =\ ^F\mathbf{D} -\ ^F\tilde{\mathbf{Z}}$ is used in (55) (instead of $\widehat{\mathbf{D}} =\ ^F\mathbf{D} -\ ^F\widehat{\mathbf{Z}}$). To consider an almost Kähler model of Cartan–Finsler space is important because following such an approach we work with almost symplectic variables, see an explicit construction in section 2.2.4. This way, it is possible to perform deformation quantization of the Finsler–Ricci flow theory [38] and develop noncommutative models [25] applying standard geometric quantization methods.

The Finsler–Ricci evolution equations derived in this work are with respect to N–adapted frames (9) and (10) which in their turn are subjected to geometric evolution. Using vielbein parametrizations (47) and similar formulas for Riemannian spaces [6, 7, 8] (see also models of geometric evolution with N–connections in [21, 22]),

Corollary 4.2 *The evolution, for all time $\tau \in [0, \tau_0)$, of N–adapted frames in a Finsler space,*

$$\tilde{\mathbf{e}}_\alpha(\tau) = \tilde{\mathbf{e}}_\alpha^{\ \underline{\alpha}}(\tau, u)\partial_{\underline{\alpha}},$$

up to frame/coordinate transforms, is defined by the coefficients

$$\tilde{\mathbf{e}}_\alpha^{\ \underline{\alpha}}(\tau, u) = \begin{bmatrix} e_i^{\ \underline{i}}(\tau, u) & \tilde{N}_i^b(\tau, u)\ e_b^{\ \underline{a}}(\tau, u) \\ 0 & e_a^{\ \underline{a}}(\tau, u) \end{bmatrix},$$

$$\tilde{\mathbf{e}}^\alpha_{\ \underline{\alpha}}(\tau, u) = \begin{bmatrix} e^i_{\ \underline{i}} = \delta^i_{\underline{i}} & e^b_{\ \underline{i}} = -\tilde{N}_k^b(\tau, u)\ \ \delta^k_{\underline{i}} \\ e^i_{\ \underline{a}} = 0 & e^a_{\ \underline{a}} = \delta^a_{\underline{a}} \end{bmatrix},$$

with $\tilde{g}_{ij}(\tau) = e_i^{\ \underline{i}}(\tau, u)\ e_j^{\ \underline{j}}(\tau, u)\eta_{\underline{ij}}$ and $\tilde{g}_{ab}(\tau) = e_a^{\ \underline{a}}(\tau, u)\ e_b^{\ \underline{b}}(\tau, u)\eta_{\underline{ab}}$, where $\eta_{\underline{ij}} = diag[\pm 1, ... \pm 1]$ and $\eta_{\underline{ab}} = diag[\pm 1, ... \pm 1]$ fix a signature of $\tilde{\mathbf{g}}^{[0]}_{\alpha\beta}(u)$, is given by equations

$$\frac{\partial}{\partial \tau}\tilde{\mathbf{e}}^\alpha_{\ \underline{\alpha}} = \tilde{\mathbf{g}}^{\alpha\beta}\ \widehat{\mathbf{R}}_{\beta\gamma}\ \ \tilde{\mathbf{e}}^\gamma_{\ \underline{\alpha}} \qquad (60)$$

if we prescribe that the geometric constructions are derived by the canonical d–connection.

Finally, we emphasize that $\mathbf{g}^{\alpha\beta}\ \widehat{\mathbf{R}}_{\beta\gamma} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}$ in (60) selects for evolution only the symmetric components of the Ricci d–tensor for the canonical d–connection. The formulas for a distortion $\widehat{\mathbf{R}}_{\ \beta\gamma} = {}^F\mathbf{R}_{\ \beta\gamma} - {}^F\widehat{\mathbf{Z}}ic_{\beta\gamma}$ allow us to compute flow contributions defined by metric noncompatibe flows with ${}^F\mathbf{R}_{\ \beta\gamma}$.

4.4 Statistical analogy and thermodynamics of Finsler–Ricci flows

The functional ${}_{\shortmid}\mathcal{W}$ is in a sense analogous to minus entropy [3] and this property was proven for metric compatible Finsler–Ricci flows [22, 24] with functionals $\widehat{\mathcal{W}}$ and/or $\tilde{\mathcal{W}}$, respectively written for $\widehat{\mathbf{D}}$ and $\tilde{\mathbf{D}}$. This allows us to associate some thermodynamical values characterizing (non) holonomic geometric evolution. The aim of this section is to show how a statistical/thermodynamic analogy can be provided for metric noncompatible Ricci flows.

For the functionals $\widehat{\mathcal{W}}$ and ${}^{F}\mathcal{W}$ (50), we can prove two systems of equations as in Theorem 4.1 (we omit such considerations in this work). For simplicity, we provide an equivalent result for stated for $\tilde{\mathcal{W}}$.

Theorem 4.2 *For any d–metric* $\mathbf{g}(\chi)$ *(17),* $\tilde{\mathbf{D}} = {}^{F}\mathbf{D} - {}^{F}\tilde{\mathbf{Z}}$*, and functions* $\widehat{f}(\chi)$ *and* $\widehat{\tau}(\chi)$ *being solutions of the system of equations*

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \chi} &= -2\left({}^{F}R_{\ ij} - {}^{F}\tilde{\mathbf{Z}}ic_{ij}\right),\ \frac{\partial g_{ab}}{\partial \chi} = -2\left({}^{F}R_{\ ab} - {}^{F}\tilde{\mathbf{Z}}ic_{ab}\right), \\
\frac{\partial \tilde{f}}{\partial \chi} &= -({}^{F}\Delta + {}^{Z}\tilde{\Delta})\tilde{f} + \left|({}^{F}\mathbf{D} - {}^{F}\tilde{\mathbf{Z}})\tilde{f}\right|^{2} - {}_{s}\tilde{R} + \frac{2n}{\hat{\tau}},\ \frac{\partial \tilde{\tau}}{\partial \chi} = -1,
\end{aligned}$$

it is satisfied the condition

$$\begin{aligned}
&\frac{\partial}{\partial \chi}\tilde{\mathcal{W}}(\mathbf{g}(\chi),\tilde{f}(\chi),\tilde{\tau}(\chi)) = 2\int_{\mathcal{V}} \tilde{\tau}[| \ {}^{F}\mathbf{R}_{\alpha\beta} - {}^{F}\tilde{\mathbf{Z}}ic_{\alpha\beta} + \\
&({}^{F}\mathbf{D}_{\alpha} - {}^{F}\tilde{\mathbf{Z}}_{\alpha})({}^{F}\mathbf{D}_{\alpha} - {}^{F}\tilde{\mathbf{Z}}_{\alpha})\tilde{f} - \frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\alpha\beta}|^{2}](4\pi\tilde{\tau})^{-n}e^{-\tilde{f}}dV,
\end{aligned}$$

for $\int_{\mathcal{V}} e^{-\tilde{f}}dV = const.$ *This functional is N–adapted nondecreasing if it is both h– and v–nondecreasing.*

Proof. We apply the rules from Conclusion 4.1 using, for instance, a proof with N–adapted modification of Proposition 1.5.8 in [6] containing the details of the original result from [3]). For metric compatible Lagrange and/or Finsler flows, there are proofs [22, 24] that for $\widehat{\mathbf{D}}$, the equations

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \chi} &= -2\widehat{R}_{ij},\ \frac{\partial g_{ab}}{\partial \chi} = -2\widehat{R}_{ab}, \\
\frac{\partial \widehat{f}}{\partial \chi} &= -\widehat{\Delta}\widehat{f} + \left|\widehat{\mathbf{D}}\widehat{f}\right|^{2} - {}_{s}\widehat{R} + \frac{2n}{\hat{\tau}},\ \frac{\partial \hat{\tau}}{\partial \chi} = -1
\end{aligned}$$

result in the condition

$$\begin{aligned}
&\frac{\partial}{\partial \chi}\widehat{\mathcal{W}}(\mathbf{g}(\chi),\widehat{f}(\chi),\hat{\tau}(\chi)) = 2\int_{\mathcal{V}} \hat{\tau}[|\widehat{R}_{ij} + \widehat{D}_{i}\widehat{D}_{j}\widehat{f} - \frac{1}{2\hat{\tau}}g_{ij}|^{2} + \\
&|\widehat{R}_{ab} + \widehat{D}_{a}\widehat{D}_{b}\widehat{f} - \frac{1}{2\hat{\tau}}g_{ab}|^{2}](4\pi\hat{\tau})^{-n}e^{-\widehat{f}}dV.
\end{aligned}$$

The rules from Conclusion 4.1 allows us to write for $\tilde{\mathbf{D}}$, rescaling correspondingly the functions $\widehat{f}(\chi) \rightarrow \tilde{f}(\chi), \hat{\tau}(\chi) \rightarrow \tilde{\tau}(\chi)$, that

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial \chi} &= -2\tilde{R}_{ij}, \ \frac{\partial g_{ab}}{\partial \chi} = -2\tilde{R}_{ab}, \\
\frac{\partial \tilde{f}}{\partial \chi} &= -\tilde{\triangle} \tilde{f} + \left| \tilde{\mathbf{D}} \tilde{f} \right|^2 - \ {}_s\tilde{R} + \frac{2n}{\tilde{\tau}}, \frac{\partial \tilde{\tau}}{\partial \chi} = -1
\end{aligned}$$

and (for another functional, $\tilde{\mathcal{W}}$)

$$\frac{\partial}{\partial \chi} \tilde{\mathcal{W}}(\mathbf{g}(\chi), \tilde{f}(\chi), \tilde{\tau}(\chi)) = 2 \int_{\mathcal{V}} \tilde{\tau} [|\tilde{\mathbf{R}}_{\alpha\beta} + \tilde{\mathbf{D}}_\alpha \tilde{\mathbf{D}}_\beta \tilde{f} - \frac{1}{2\tilde{\tau}} \tilde{\mathbf{g}}_{\alpha\beta}|^2] (4\pi\tilde{\tau})^{-n} e^{-\tilde{f}} dV.$$

In the above formulas, we introduce the distorting relations

$$\tilde{\nabla} = \ {}^F\mathbf{D} - \ {}_\nabla^F\mathbf{Z}, \ \widehat{\mathbf{D}} = \ {}^F\mathbf{D} - \ {}^F\widehat{\mathbf{Z}}, \ \tilde{\mathbf{D}} = \ {}^F\mathbf{D} - \ {}^F\tilde{\mathbf{Z}} \tag{61}$$

and

$$\begin{aligned}
\tilde{\Delta} &= \tilde{\mathbf{D}}_\alpha \ \tilde{\mathbf{D}}^\alpha = \ {}^F\Delta + \ {}^Z\tilde{\Delta}, \qquad (62) \\
{}^F\Delta &= \ {}^F\mathbf{D}_\alpha \ {}^F\mathbf{D}^\alpha, \ \ {}^Z\tilde{\Delta} = \ {}^F\tilde{\mathbf{Z}}_\alpha \ {}^F\tilde{\mathbf{Z}}^\alpha - [\ {}^F\mathbf{D}_\alpha(\ {}^F\tilde{\mathbf{Z}}^\alpha) + \ {}^F\tilde{\mathbf{Z}}_\alpha(\ {}^F\mathbf{D}^\alpha)]; \\
\tilde{\mathbf{R}}_{\ \beta\gamma} &= \ {}^F\mathbf{R}_{\ \beta\gamma} - \ {}^F\tilde{\mathbf{Z}}ic_{\beta\gamma}, \ \ {}_s\tilde{R} = \ {}_s^F R - \mathbf{g}^{\beta\gamma} \ {}^F\tilde{\mathbf{Z}}ic_{\beta\gamma} = \ {}_s^F R - \ {}_s^F\tilde{\mathbf{Z}}, \\
{}_s^F\tilde{\mathbf{Z}} &= \mathbf{g}^{\beta\gamma} \ {}^F\tilde{\mathbf{Z}}ic_{\beta\gamma} = \ {}_h^F\tilde{Z} + \ {}_v^F\tilde{Z}, \ {}_h^F\tilde{Z} = g^{ij} \ {}^F\tilde{\mathbf{Z}}ic_{ij}, \ {}_v^F\tilde{Z} = h^{ab} \ {}^F\tilde{\mathbf{Z}}ic_{ab}; \\
{}_s^F R &= \ {}_h^F R + \ {}_v^F R, \ \ {}_h^F R := g^{ij} \ {}^F R_{ij}, \ {}_v^F R = h^{ab} \ {}^F R_{ab},
\end{aligned}$$

resulting in the equations from the conditions of theorem. □

Ricci flows with ∇, $\widehat{\mathbf{D}}$ and $\tilde{\mathbf{D}}$ are characterized by respective thermodynamic values, see section 5 in [3] and, for metric compatible Finsler spaces, Refs. [22, 24]. Such constructions can be noholonomically deformed into metric noncompatible configurations.

In order to provide a statistical analogy, we consider a partition function $Z = \int \exp(-\beta E) d\omega(E)$ for the canonical ensemble at temperature β^{-1} being defined by the measure taken to be the density of states $\omega(E)$. The thermodynamical values are computed in standard form for the average energy, $\langle E \rangle := -\partial \log Z / \partial \beta$, the entropy $S := \beta \langle E \rangle + \log Z$ and the fluctuation $\sigma := \left\langle (E - \langle E \rangle)^2 \right\rangle = \partial^2 \log Z / \partial \beta^2$.

Theorem 4.3 *Any family of Finsler geometries for which the conditions of Theorem 4.2 are satisfied is characterized by thermodynamic values*

$$
\begin{aligned}
\left\langle\ ^{F}E\right\rangle &= -\tilde{\tau}^{2}\int_{\mathcal{V}}\left(\ _{s}^{F}R+|\ ^{F}\mathbf{D}\tilde{f}|^{2}-\frac{n}{\tilde{\tau}}\right)\tilde{\mu}\ dV, \\
^{F}S &= -\int_{\mathcal{V}}\left[\tilde{\tau}\left(\ _{s}^{F}R+|\ ^{F}\mathbf{D}\tilde{f}|^{2}\right)+\tilde{f}-2n\right]\tilde{\mu}\ dV, \\
^{F}\sigma &= 2\ \tilde{\tau}^{4}\int_{\mathcal{V}}[|\ ^{F}\mathbf{R}_{\alpha\beta}-\ ^{F}\tilde{\mathbf{Z}}ic_{\alpha\beta}+ \\
&\quad (\ ^{F}\mathbf{D}_{\alpha}-\ ^{F}\tilde{\mathbf{Z}}_{\alpha})(\ ^{F}\mathbf{D}_{\beta}-\ ^{F}\tilde{\mathbf{Z}}_{\beta})\tilde{f}-\frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\alpha\beta}|^{2}]\tilde{\mu}\ dV.
\end{aligned} \tag{63}
$$

Proof. There are two possibilities to prove this theorem. The first one is to use the partition function $\breve{Z}=\exp\left\{\int_{\mathcal{V}}[-\breve{f}+n]\ \breve{\mu}dV\right\}$ and compute values (63) using methods from [3, 6], changing $\nabla\rightarrow\ ^{F}\mathbf{D}$ following rules from Conclusion 4.1 and rescaling $\breve{f}\rightarrow\tilde{f}$ and $\breve{\tau}\rightarrow\tilde{\tau}$ (such a rescaling is useful if we wont to compare thermodynamical values for different Finsler connections). A similar proof is possible if metric compatible Finsler connections are used. For instance, considering $\tilde{\mathbf{D}}\rightarrow\ ^{F}\mathbf{D}$ and $\tilde{Z}=\exp\left\{\int_{\mathcal{V}}[-\tilde{f}+n]\ \tilde{\mu}dV\right\}$, we compute [22, 24]

$$
\begin{aligned}
\left\langle\tilde{E}\right\rangle &= -\tilde{\tau}^{2}\int_{\mathcal{V}}\left(\ _{s}\tilde{R}+|\tilde{\mathbf{D}}\tilde{f}|^{2}-\frac{n}{\tilde{\tau}}\right)\tilde{\mu}\ dV, \\
\tilde{S} &= -\int_{\mathcal{V}}\left[\tilde{\tau}\left(\ _{s}\tilde{R}+|\tilde{\mathbf{D}}\tilde{f}|^{2}\right)+\tilde{f}-2n\right]\tilde{\mu}\ dV, \\
\tilde{\sigma} &= 2\ \tilde{\tau}^{4}\int_{\mathcal{V}}[|\tilde{\mathbf{R}}_{\alpha\beta}+\tilde{\mathbf{D}}_{\alpha}\tilde{\mathbf{D}}_{\beta}\tilde{f}-\frac{1}{2\tilde{\tau}}\tilde{\mathbf{g}}_{\alpha\beta}|^{2}]\tilde{\mu}\ dV.
\end{aligned}
$$

Introducing distortions (61) and (62) into the thermodynamical values for $\tilde{\mathbf{D}}$, we generate analogous thermodynamical values (63) for $^{F}\mathbf{D}$. □

In general, for a given fundamental Finsler function $F(u)$, we can construct an infinite number of metric compatible and noncompatible d–connections $^{F}\mathbf{D}$. The theory of Finsler–Ricci flows allows us to solve the problem which Finsler configuration is more "optimal" thermodynamically. Fixing two d–connections $_{1}^{F}\mathbf{D}$ and $_{2}^{F}\mathbf{D}$ generated by the same F, we can compute two triples of data $\left[\left\langle\ _{1}^{F}E\right\rangle,\ _{1}^{F}S,\ _{1}^{F}\sigma\right]$ and $\left[\left\langle\ _{2}^{F}E\right\rangle,\ _{2}^{F}S,\ _{2}^{F}\sigma\right]$.

Conclusion 4.2 *Two models of Finsler geometry,* $\left(F,\tilde{\mathbf{g}},_{1}^{F}\mathbf{D}\right)$ *and* $\left(F,\tilde{\mathbf{g}},_{2}^{F}\mathbf{D}\right)$ *generated by the same fundamental Finsler function* F *are thermodynamically more (less, equivalent) convenient if* $_{1}^{F}S<\ _{2}^{F}S$ $(\ _{1}^{F}S>\ _{2}^{F}S,\ _{1}^{F}S=$

${}^F_2S$). *Similar statements on energetic convenience can be formulated comparing* $\langle\ {}^F_1E \rangle$ *and* $\langle\ {}^F_2E \rangle$.

The theorems and conclusions provided in this section can be formulated and proved separately on h- and v–subspaces of a nonholonomic manifold $\mathbf{V}$ and/or a tangent bundle TM. Some geometric and physical models with the Akbar-Zadeh curvature or other "preferred" Finsler connection (Berwald, Chern types etc) can be more/less/equivalent to alternative ones, but generated by the same F. An exact answer is possible if a value F is fixed following certain geometric/physical arguments.

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