

\mathcal{W} -extended Kac representations and integrable boundary conditions in the logarithmic minimal models $\mathcal{WLM}(1, p)$

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Abstract

We construct new Yang-Baxter integrable boundary conditions in the lattice approach to the logarithmic minimal model $\mathcal{WLM}(1, p)$ giving rise to reducible yet indecomposable representations of rank 1 in the continuum scaling limit. We interpret these \mathcal{W} -extended Kac representations as finitely-generated \mathcal{W} -extended Feigin-Fuchs modules over the triplet W -algebra $\mathcal{W}(p)$. The \mathcal{W} -extended fusion rules of these representations are inferred from the recently conjectured Virasoro fusion rules of the Kac representations in the underlying logarithmic minimal model $\mathcal{LM}(1, p)$. We also introduce the modules contragredient to the \mathcal{W} -extended Kac modules and work out the correspondingly-extended fusion algebra. Our results are in accordance with the Kazhdan-Lusztig dual of tensor products of modules over the restricted quantum universal enveloping algebra $\bar{U}_q(\mathfrak{sl}_2)$ at $q = e^{\pi i/p}$. Finally, polynomial fusion rings isomorphic with the various fusion algebras are determined, and the corresponding Grothendieck ring of characters is identified.

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1 Introduction

Physical systems described by logarithmic conformal field theories (CFTs) [1–3] include polymers [4–9], percolation [10–16], symplectic fermions [17, 18] and the abelian sandpile model [19–23]. In fact, an infinite series of logarithmic CFTs arises in the continuum scaling limit of certain two-dimensional lattice models of non-local statistical mechanical systems at criticality [24]. Polymers [7, 9], percolation [25, 26] and symplectic fermions [27] are all described by these logarithmic minimal models. Quantum spin chains with a non-diagonalizable Hamiltonian [28] likewise give rise to logarithmic CFTs.

Mathematically, vertex operator algebras (VOAs) [29–32] provide an algebraic pendant to CFTs. The abelian category of modules over a VOA associated with a rational CFT is semi-simple and contains only finitely many simple objects (irreducible representations). In order for a CFT to make sense on a higher-genus Riemann surface, the corresponding VOA must satisfy Zhu’s C_2 -cofiniteness condition [33]. The triplet W -algebra $\mathcal{W}(p)$ [34], where $p = 2, 3, \dots$, is an example of such a VOA, as demonstrated in [35, 36]. The corresponding abelian category of modules is *non*-semi-simple, however, and the associated CFT is *logarithmic* [17, 37–39]. VOAs of logarithmic CFTs are discussed more generally in [40–44].

In a series of papers [45–48], Feigin et al conjectured and examined a Kazhdan-Lusztig duality between the logarithmic CFTs based on the triplet W -algebra $\mathcal{W}(p)$ and the representation theory of the restricted quantum universal enveloping algebra $\bar{U}_q(\mathfrak{sl}_2)$ at $q = e^{\pi i/p}$ [49–52]. It was subsequently

proven [53] that the category of $\mathcal{W}(p)$ -modules and the category of finite-dimensional $\bar{U}_q(sl_2)$ -modules indeed are equivalent as abelian categories for all $p = 2, 3, \dots$. For $p > 2$, they are not, however, equivalent as braided quasi-tensor categories since their natural tensor structures are not fully compatible [54].

The present work concerns the logarithmic minimal models $\mathcal{LM}(p, p')$ [24] in the \mathcal{W} -extended picture [26, 27, 55] where they are denoted by $\mathcal{WLM}(p, p')$. The parameters p and p' constitute a pair of coprime positive integers $1 \leq p < p'$, and focus here is on the case $\mathcal{WLM}(1, p')$. For simplicity, this is denoted by $\mathcal{WLM}(1, p)$ where $p = 2, 3, \dots$, and the extension is believed to be with respect to the triplet W -algebra $\mathcal{W}(p)$. The logarithmic minimal model $\mathcal{WLM}(1, p)$ is thus conjectured to be associated with the VOA based on $\mathcal{W}(p)$, and we will assert this in the following.

Associated with a Yang-Baxter integrable boundary condition in the lattice approach to $\mathcal{LM}(1, p)$, there is a so-called Kac representation (r, s) for each pair of positive Kac labels $r, s \in \mathbb{N}$. The corresponding Virasoro modules were identified in [56] and conjectured to be finitely-generated Feigin-Fuchs modules [57]. The fusion algebras generated by the Kac representations and their contragredient counterparts $(r, s)^*$ were also determined in [56] and confirmed in [58] based on the Kazhdan-Lusztig duality conjectured in [59]. Here we lift the findings of [56] to the \mathcal{W} -extended picture using methods developed in [27].

In Section 2, we review the logarithmic minimal models $\mathcal{LM}(1, p)$ in the Virasoro picture. Following [56], we discuss the fusion properties of the Kac representations and their contragredient counterparts. In Section 3, we generalize the construction of \mathcal{W} -extended representations in [27] from \mathcal{W} -irreducible to general \mathcal{W} -extended Kac representations $(r, s)_{\mathcal{W}}$ where $r, s \in \mathbb{N}$. We thus introduce new Yang-Baxter integrable boundary conditions whose continuum scaling limits give rise to these \mathcal{W} -extended Kac representations. We interpret these representations as finitely-generated \mathcal{W} -extended Feigin-Fuchs modules over the triplet W -algebra $\mathcal{W}(p)$. The contragredient modules $(r, s)_{\mathcal{W}}^*$ of the \mathcal{W} -extended Kac representations $(r, s)_{\mathcal{W}}$ are also introduced, and the various fusion rules are inferred from the recently conjectured Kac fusion algebra in the Virasoro picture [56]. This Kac fusion algebra is summarized in Appendix A. In Section 4, we determine polynomial fusion rings isomorphic with the \mathcal{W} -extended Kac fusion algebra and its contragredient extension. We also identify the corresponding Grothendieck ring associated with the Virasoro characters of the \mathcal{W} -extended representations. Section 5 contains some concluding remarks and a comparison of our results on fusion with the tensor structure of the restricted quantum universal enveloping algebra $\bar{U}_q(sl_2)$ at $q = e^{\pi i/p}$ [54]. To facilitate this comparison, a dictionary relating the different notations is presented in Appendix B.

Notation

For $n, m \in \mathbb{Z}$ and modules A_n ,

$$\mathbb{Z}_{n,m} = \mathbb{Z} \cap [n, m], \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

$$\epsilon(n) = \frac{1 - (-1)^n}{2}, \quad n \cdot m = 1 + \epsilon(n + m), \quad \bigoplus_n^N A_n = \bigoplus_{n=\epsilon(N), \text{ by } 2}^N A_n \quad (1.1)$$

It is noted that $n \cdot m \in \mathbb{Z}_{1,2}$ and that this dot product is commutative and associative.

2 Logarithmic minimal model $\mathcal{LM}(1, p)$

The logarithmic minimal model $\mathcal{LM}(1, p)$ is a logarithmic CFT with central charge

$$c = 1 - 6 \frac{(p-1)^2}{p}, \quad p = 2, 3, \dots \quad (2.1)$$

In this section, we review the Virasoro representations associated with the boundary conditions appearing in the lattice approach to $\mathcal{LM}(1, p)$ as described in [24, 56, 60]. We also recall the associated contragredient modules introduced in [56] and review the corresponding fusion algebras.

2.1 Kac representations

There is a so-called Kac representation (r, s) for each pair of positive Kac labels $r, s \in \mathbb{N}$. It is associated with a Yang-Baxter integrable boundary condition in the lattice approach to $\mathcal{LM}(1, p)$ [24, 60] and arises in the continuum scaling limit. A classification of these Kac representations as modules over the Virasoro algebra was recently proposed in [56]. It was thus conjectured that they can be viewed as finitely-generated submodules of Feigin-Fuchs modules [57].

To describe these finitely-generated Feigin-Fuchs modules, we first consider the quotient module

$$Q_{r,s} = V_{r,s}/V_{r,-s}, \quad r, s \in \mathbb{N} \quad (2.2)$$

where $V_{r,s}$ is the Verma module of conformal weight

$$\Delta_{r,s} = \frac{(rp - s)^2 - (p - 1)^2}{4p}, \quad r, s \in \mathbb{Z} \quad (2.3)$$

The corresponding *irreducible* highest-weight module is denoted by $M_{r,s}$, where we set $M_{r,0} = M_{0,s} = 0$. Parameterizing the second Kac label as

$$s = s_0 + kp, \quad s_0 \in \mathbb{Z}_{1,p-1}, \quad k \in \mathbb{N}_0 \quad (2.4)$$

the structure diagram of the quotient module $Q_{r,s}$ is given by

$$Q_{r,s} : \quad M_{k-r+1,p-s_0} \rightarrow M_{k-r+2,s_0} \rightarrow M_{k-r+3,p-s_0} \rightarrow \dots \rightarrow M_{k+r-1,p-s_0} \rightarrow M_{k+r,s_0} \quad (2.5)$$

We can associate a pair of finitely-generated Feigin-Fuchs modules to every quotient module $Q_{r,s}$. For $2r - 1 < 2k$, the Feigin-Fuchs modules corresponding to $Q_{r,s}$ are characterized by the structure diagrams

$$\begin{aligned} Q_{r,s}^{\rightarrow} : \quad & M_{k-r+1,p-s_0} \rightarrow M_{k-r+2,s_0} \leftarrow M_{k-r+3,p-s_0} \rightarrow \dots \leftarrow M_{k+r-1,p-s_0} \rightarrow M_{k+r,s_0} \\ Q_{r,s}^{\leftarrow} : \quad & M_{k-r+1,p-s_0} \leftarrow M_{k-r+2,s_0} \rightarrow M_{k-r+3,p-s_0} \leftarrow \dots \rightarrow M_{k+r-1,p-s_0} \leftarrow M_{k+r,s_0} \end{aligned} \quad (2.6)$$

For $2r - 1 > 2k$, the Feigin-Fuchs modules corresponding to $Q_{r,s}$ are characterized by the structure diagrams

$$\begin{aligned} Q_{r,s}^{\rightarrow} : \quad & M_{r-k,s_0} \rightarrow M_{r-k+1,p-s_0} \leftarrow M_{r-k+2,s_0} \rightarrow \dots \rightarrow M_{r+k-1,p-s_0} \leftarrow M_{r+k,s_0} \\ Q_{r,s}^{\leftarrow} : \quad & M_{r-k,s_0} \leftarrow M_{r-k+1,p-s_0} \rightarrow M_{r-k+2,s_0} \leftarrow \dots \leftarrow M_{r+k-1,p-s_0} \rightarrow M_{r+k,s_0} \end{aligned} \quad (2.7)$$

By construction, the associated Virasoro characters satisfy

$$\chi[Q_{r,s}^{\rightarrow}](q) = \chi[Q_{r,s}^{\leftarrow}](q) = \chi[Q_{r,s}](q) \quad (2.8)$$

Letting

$$\text{ch}_{r,s}(q) = \chi[M_{r,s}](q) \quad (2.9)$$

denote the character of the irreducible module $M_{r,s}$, we thus have

$$\begin{aligned}\chi[Q_{r,s}](q) &= \sum_{j=0}^{\min(2r-1,2k)} \text{ch}_{r+k-j,(-1)^j s_0+(1-(-1)^j)p/2}(q) \\ &= \sum_{j=|r-k|+1, \text{ by 2}}^{r+k-1} \text{ch}_{j,p-s_0}(q) + \sum_{j=|r-k-1|+1, \text{ by 2}}^{r+k} \text{ch}_{j,s_0}(q)\end{aligned}\quad (2.10)$$

The range for s_0 in (2.4) can be extended from $\mathbb{Z}_{1,p-1}$ to $\mathbb{Z}_{0,p-1}$ such that s can be any positive integer $s \in \mathbb{N}$ (where we exclude $s_0 = k = 0$ for which $s = 0$). For $s_0 = 0$, the structure diagrams associated with $Q_{r,s}^{\rightarrow}$ and $Q_{r,s}^{\leftarrow}$ are separable (degenerate) and the modules are fully reducible

$$Q_{r,kp}^{\rightarrow} = Q_{r,kp}^{\leftarrow} = Q_{r,kp} = \bigoplus_{j=|r-k|+1, \text{ by 2}}^{r+k-1} M_{j,p}\quad (2.11)$$

It is noted that this decomposition is symmetric in r and k . It is also noted that, for $k = 0$, the finitely-generated Feigin-Fuchs modules associated with $Q_{r,s}$ are irreducible as we have

$$Q_{r,s_0}^{\rightarrow} = Q_{r,s_0}^{\leftarrow} = Q_{r,s_0} = M_{r,s_0}\quad (2.12)$$

From [60], we know that the Kac representation (r, s) is irreducible for $s \leq p$ and fully reducible for $s = kp$, with the set of irreducible modules denoted by

$$\mathcal{J}^{\text{Irr}} = \{(r, s); r \in \mathbb{N}, s \in \mathbb{Z}_{1,p}\}\quad (2.13)$$

A conjecture for the structure of the remaining Kac representations was presented in [56]. For general (r, s) with s given in (2.4), it was thus proposed that

$$(r, s) = \begin{cases} Q_{r,s}^{\rightarrow}, & 2r - 1 < 2k \\ Q_{r,s}^{\leftarrow}, & 2r - 1 > 2k \end{cases}\quad (2.14)$$

Here we adopt this assumption, but will return to it in Section 2.4. The associated Virasoro characters are denoted by $\chi_{r,s}(q)$ and by construction given by

$$\chi_{r,s}(q) = \chi[Q_{r,s}](q)\quad (2.15)$$

We note that the irreducibility of (r, s) for $s \leq p$ corresponds to (2.12), while the fully reducibility of (r, kp) corresponds to (2.11). The symmetry in r and k in (2.11) corresponds to the identification $(k, rp) \equiv (r, kp)$ which reduces to the identification $(1, rp) \equiv (r, p)$ of irreducible modules. This justifies the choice of notation in (2.13).

2.2 Contragredient Kac representations

In all cases (2.6) and (2.7), the finitely-generated Feigin-Fuchs modules $Q_{r,s}^{\rightarrow}$ and $Q_{r,s}^{\leftarrow}$ are *contragredient* to each other where the contragredient module A^* to a module A is obtained by reversing all structure arrows between its irreducible subfactors (subquotients). It follows, in particular, that $\chi[A^*](q) = \chi[A](q)$ and that $A^{**} = A$. Following [56], the contragredient Kac representations are introduced as

$$(r, s)^* = \begin{cases} Q_{r,s}^{\leftarrow}, & 2r - 1 < 2k \\ Q_{r,s}^{\rightarrow}, & 2r - 1 > 2k \end{cases}\quad (2.16)$$

whose Virasoro characters $\chi_{r,s}^*(q) = \chi[(r,s)^*](q)$ are given by $\chi_{r,s}^*(q) = \chi_{r,s}(q)$. We note that $(r,s)^* = (r,s)$ if and only if (r,s) is fully reducible, that is,

$$(r,s)^* = (r,s) \iff s \in \mathbb{Z}_{1,p-1} \cup p\mathbb{N} \quad (2.17)$$

2.3 Rank-2 and projective modules

The infinite family

$$\{\mathcal{R}_r^b; r \in \mathbb{N}, b \in \mathbb{Z}_{1,p-1}\} \quad (2.18)$$

of reducible yet indecomposable modules of rank 2 arises from repeated fusion of *irreducible* Kac representations [60]. The rank-2 module \mathcal{R}_r^b is characterized by the structure diagram

$$\mathcal{R}_1^b : \begin{array}{c} M_{2,b} \\ \swarrow \quad \nwarrow \\ M_{1,p-b} \longleftarrow M_{1,p-b} \end{array}, \quad \mathcal{R}_r^b : \begin{array}{c} M_{r+1,b} \\ \swarrow \quad \nwarrow \\ M_{r,p-b} \longleftarrow M_{r,p-b} \\ \swarrow \quad \nwarrow \\ M_{r-1,b} \end{array}, \quad r \in \mathbb{Z}_{\geq 2} \quad (2.19)$$

It is noted that the rank-2 modules are all self-contragredient

$$(\mathcal{R}_r^b)^* = \mathcal{R}_r^b \quad (2.20)$$

The Virasoro character of the rank-2 module \mathcal{R}_r^b follows from the structure diagram (2.19) and is given by

$$\chi[\mathcal{R}_r^b](q) = (1 - \delta_{r,1})\text{ch}_{r-1,b}(q) + 2\text{ch}_{r,p-b}(q) + \text{ch}_{r+1,b}(q) \quad (2.21)$$

According to the fusion algebra conjectured in [56] and reviewed in Appendix A, no additional rank-2 modules nor higher-rank modules are generated from repeated fusion of the *full* set of Kac representations (r,s) and contragredient Kac representations $(r,s)^*$.

These rank-2 modules are all projective modules, but not the only projective modules in the model. The Kac representations $(1,rp) \equiv (r,p)$ are both irreducible and projective as modules over the Virasoro algebra. It is thus convenient to introduce the alternative notation

$$\mathcal{R}_r^0 \equiv (1,rp) \equiv (r,p) \quad (2.22)$$

allowing us to write the set of projective modules as

$$\mathcal{J}^{\text{Proj}} = \{\mathcal{R}_r^b; r \in \mathbb{N}, b \in \mathbb{Z}_{0,p-1}\} \quad (2.23)$$

2.4 Fusion algebras

There are infinitely many fusion (sub)algebras associated with $\mathcal{LM}(1,p)$. Results on the corresponding fusion rules can be found in [61,62]. The *fundamental fusion algebra* [60]

$$\langle (1,1), (2,1), (1,2) \rangle \quad (2.24)$$

in particular, is generated from the two fundamental Kac representations $(2,1)$ and $(1,2)$ in addition to the identity $(1,1)$. This fusion algebra involves all the irreducible Kac representations and all the rank-2 representations (2.18). On the other hand, no reducible yet indecomposable Kac representations arise

as the result of repeated fusion of the fundamental Kac representations. The set of indecomposable modules partaking in the fundamental fusion algebra is simply given by

$$\mathcal{J}^{\text{Fund}} = \mathcal{J}^{\text{Irr}} \cup \mathcal{J}^{\text{Proj}} \quad (2.25)$$

This is not written as a disjoint union of sets since the modules (2.22) are both irreducible and projective.

The *Kac fusion algebra*

$$\langle (r, s); r, s \in \mathbb{N} \rangle \quad (2.26)$$

is generated by repeated fusion of the full set of Kac representations, where

$$(r, s) = (r, 1) \otimes (1, s), \quad \mathcal{R}_r^b = (r, 1) \otimes \mathcal{R}_1^b \quad (2.27)$$

A concrete conjecture for this fusion algebra was presented in [56]. It was subsequently demonstrated in [56] that this conjectured fusion algebra is generated by repeated fusion of four Kac representations

$$\langle (r, s); r, s \in \mathbb{N} \rangle = \langle (1, 1), (2, 1), (1, 2), (1, p+1) \rangle \quad (2.28)$$

The set of distinct, indecomposable modules partaking in this *Kac fusion algebra* is

$$\mathcal{J}^{\text{Kac}} = \mathcal{J}^{\text{Fund}} \cup \{(r, s); r \in \mathbb{N}, s \in \mathbb{N} \setminus (\mathbb{Z}_{1,p-1} \cup p\mathbb{N})\} \quad (2.29)$$

here written as a disjoint union of sets. It was also found that the fusion algebra generated by the contragredient Kac representations is isomorphic to the Kac fusion algebra, that is,

$$\langle (r, s)^*; r, s \in \mathbb{N} \rangle = \langle (1, 1)^*, (2, 1)^*, (1, 2)^*, (1, p+1)^* \rangle \simeq \langle \mathcal{J}^{\text{Kac}} \rangle \quad (2.30)$$

where

$$A^* = A \quad \text{if} \quad A \in \mathcal{J}^{\text{Fund}}, \quad A^* \neq A \quad \text{if} \quad A \in \mathcal{J}^{\text{Kac}} \setminus \mathcal{J}^{\text{Fund}} \quad (2.31)$$

The *contragrediently-extended Kac fusion algebra*

$$\langle (r, s), (r, s)^*; r, s \in \mathbb{N} \rangle = \langle (1, 1), (2, 1), (1, 2), (1, p+1), (1, p+1)^* \rangle \quad (2.32)$$

is generated by repeated fusion of Kac representations *and* contragredient Kac representations. As indicated, it is actually generated by repeated fusion of five modules only. Three of these five modules are self-contragredient: $(1, 1)^* = (1, 1)$, $(2, 1)^* = (2, 1)$ and $(1, 2)^* = (1, 2)$. The set of distinct, indecomposable modules partaking in this fusion algebra is

$$\mathcal{J}^{\text{Cont}} = \mathcal{J}^{\text{Kac}} \cup \{(r, s)^*; r \in \mathbb{N}, s \in \mathbb{N} \setminus (\mathbb{Z}_{1,p-1} \cup p\mathbb{N})\} \quad (2.33)$$

here written as a disjoint union of sets. The corresponding fusion rules are reviewed in Appendix A.

The set of projective modules $\mathcal{J}^{\text{Proj}}$ generates an *ideal* of the contragrediently-extended Kac fusion algebra and hence of the Kac fusion algebra itself as well as of the fundamental fusion algebra. We furthermore observe that, as a factor in a fusion product, a projective module is *insensitive* to the decomposability properties of the other fusion factor. That is,

$$\mathcal{R}_r^b \otimes A = \mathcal{R}_r^b \otimes \left(\bigoplus_n M_n \right), \quad \mathcal{R}_r^b \in \mathcal{J}^{\text{Proj}} \quad (2.34)$$

where $\bigoplus_n M_n$ is the direct sum of the irreducible subfactors (subquotients) of the module A . By construction, we thus have

$$\chi[A](q) = \sum_n \chi[M_n](q) \quad (2.35)$$

It follows from (2.34) that fusion by the projective module \mathcal{R}_r^b is an *exact functor* for all $\mathcal{R}_r^b \in \mathcal{J}^{\text{Proj}}$.

As discussed in [56], the lattice approach to the logarithmic minimal models seems incapable of distinguishing between the family of Kac representations and the family of contragredient Kac representations. The two families generate isomorphic fusion algebras (2.30) and the corresponding characters are identical $\chi_{r,s}(q) = \chi_{r,s}^*(q)$. It was simply asserted in [56] that the Yang-Baxter integrable boundary conditions likewise denoted by (r, s) in the lattice approach [24, 60] are associated with the Kac representations, even though they, a priori, could be associated with the contragredient Kac representations. We will return to this issue in Section 3.5 when discussing fusion in the \mathcal{W} -extended picture.

3 \mathcal{W} -extended logarithmic minimal model $\mathcal{WLM}(1, p)$

3.1 Integrable boundary conditions and \mathcal{W} -extended modules

It was found in [27] that the \mathcal{W} -extended vacuum boundary condition can be constructed by fusing three r -type integrable seams to the boundary

$$(1, 1)_{\mathcal{W}} := \lim_{n \rightarrow \infty} (2n - 1, 1) \otimes (2n - 1, 1) \otimes (2n - 1, 1) = \bigoplus_{n=1}^{\infty} (2n - 1) (2n - 1, 1) \quad (3.1)$$

thereby ensuring that the \mathcal{W} -extended vacuum boundary condition is a solution to the boundary Yang-Baxter equation. The corresponding \mathcal{W} -extended module $(1, 1)_{\mathcal{W}}$ is indecomposable (in fact, irreducible) with respect to the W -algebra $\mathcal{W}(p)$, but decomposable with respect to the Virasoro algebra. Its decomposition in terms of indecomposable Virasoro modules appears as the last expression in (3.1). These Virasoro modules are all irreducible.

Using the stability properties [27]

$$\begin{aligned} (2m - 1, s) \otimes (1, 1)_{\mathcal{W}} &= (2m - 1) \left(\bigoplus_{n=1}^{\infty} (2n - 1) (2n - 1, s) \right) \\ (2m, s) \otimes (1, 1)_{\mathcal{W}} &= 2m \left(\bigoplus_{n=1}^{\infty} 2n (2n, s) \right) \\ \mathcal{R}_{2m-1}^b \otimes (1, 1)_{\mathcal{W}} &= (2m - 1) \left(\bigoplus_{n=1}^{\infty} (2n - 1) \mathcal{R}_{2n-1}^b \right) \\ \mathcal{R}_{2m}^b \otimes (1, 1)_{\mathcal{W}} &= 2m \left(\bigoplus_{n=1}^{\infty} 2n \mathcal{R}_{2n}^b \right) \end{aligned} \quad (3.2)$$

for $s \in \mathbb{Z}_{1,p}$, $b \in \mathbb{Z}_{1,p-1}$ and $m \in \mathbb{N}$, one can identify integrable boundary conditions corresponding to the $2p$ \mathcal{W} -irreducible modules

$$\begin{aligned} (1, s)_{\mathcal{W}} &:= (1, s) \otimes (1, 1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n - 1) (2n - 1, s) \\ (2, s)_{\mathcal{W}} &:= \frac{1}{2} (2, s) \otimes (1, 1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n (2n, s) \end{aligned} \quad (3.3)$$

and the $2p - 2$ \mathcal{W} -reducible yet \mathcal{W} -indecomposable rank-2 modules

$$\begin{aligned}\hat{\mathcal{R}}_1^b &:= \mathcal{R}_1^b \otimes (1, 1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n - 1) \mathcal{R}_{2n-1}^b \\ \hat{\mathcal{R}}_2^b &:= \frac{1}{2} \mathcal{R}_2^b \otimes (1, 1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n \mathcal{R}_{2n}^b\end{aligned}\quad (3.4)$$

The set $\mathcal{J}_{\mathcal{W}}^{\text{Irr}}$ of \mathcal{W} -irreducible modules is given by

$$\mathcal{J}_{\mathcal{W}}^{\text{Irr}} = \{(r, s)_{\mathcal{W}}; r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}\} = \{\hat{M}_{r,s}; r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}\} \quad (3.5)$$

where we have introduced the notation $\hat{M}_{r,s}$ to denote a \mathcal{W} -irreducible module:

$$\hat{M}_{r,s} = (r, s)_{\mathcal{W}} \quad \text{if } r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p} \quad (3.6)$$

The structure diagrams of the rank-2 modules are of the form

$$\hat{\mathcal{R}}_r^b : \begin{array}{c} \hat{M}_{2-r,b} \\ \swarrow \quad \nwarrow \\ \hat{M}_{r,p-b} \longleftarrow \hat{M}_{r,p-b} \\ \swarrow \quad \nwarrow \\ \hat{M}_{2-r,b} \end{array} \quad (3.7)$$

Introducing

$$\hat{\mathcal{R}}_r^0 \equiv (r, p)_{\mathcal{W}} = \hat{M}_{r,p}, \quad r \in \mathbb{Z}_{1,2} \quad (3.8)$$

the set of \mathcal{W} -projective modules is

$$\mathcal{J}_{\mathcal{W}}^{\text{Proj}} = \{\hat{\mathcal{R}}_r^b; r \in \mathbb{Z}_{1,2}, b \in \mathbb{Z}_{0,p-1}\} \quad (3.9)$$

It is noted that the two modules in (3.8) are both \mathcal{W} -irreducible and \mathcal{W} -projective, and that they are the only such modules. The Virasoro characters of the \mathcal{W} -indecomposable modules (3.3) and (3.4) follow readily from the indicated decompositions in terms of Virasoro modules.

The work [27], in which the lattice construction of the \mathcal{W} -extended modules (3.3) and (3.4) first appeared, was focused on the construction of \mathcal{W} -irreducible and \mathcal{W} -projective modules and on their fusion properties. The ensuing fusion algebra thus corresponds to a lift of the fundamental fusion algebra $\langle \mathcal{J}_{\mathcal{W}}^{\text{Fund}} \rangle$ to the \mathcal{W} -extended fundamental fusion algebra

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Fund}} \rangle = \langle (1, 1)_{\mathcal{W}}, (2, 1)_{\mathcal{W}}, (1, 2)_{\mathcal{W}} \rangle = \langle \hat{M}_{1,1}, \hat{M}_{2,1}, \hat{M}_{1,2} \rangle \quad (3.10)$$

The set of \mathcal{W} -modules partaking in this fusion algebra is

$$\mathcal{J}_{\mathcal{W}}^{\text{Fund}} = \mathcal{J}_{\mathcal{W}}^{\text{Irr}} \cup \mathcal{J}_{\mathcal{W}}^{\text{Proj}} \quad (3.11)$$

This is not a disjoint union of sets since the modules in (3.8) are both \mathcal{W} -irreducible and \mathcal{W} -projective.

For later convenience, we introduce the redundant notation

$$(0, s)_{\mathcal{W}} \equiv (r, 0)_{\mathcal{W}} \equiv \hat{\mathcal{R}}_0^b \equiv 0 \quad (3.12)$$

3.2 \mathcal{W} -extended Kac representations

With the recent advances [56] in the understanding of the general structure and fusion of Kac representations reviewed in Section 2, we now turn to the construction of the corresponding lift to the \mathcal{W} -extended picture.

Since

$$(r, s) = (r, 1) \otimes (1, s) \quad (3.13)$$

it is readily seen that the validity of the stability properties (3.2) extends from $s \in \mathbb{Z}_{1,p}$ to $s \in \mathbb{N}$. For every pair of positive Kac labels $r, s \in \mathbb{N}$, we can therefore define the \mathcal{W} -extended Kac representation $(r, s)_{\mathcal{W}}$ by

$$(r, s)_{\mathcal{W}} := \frac{1}{r}(r, s) \otimes (1, 1)_{\mathcal{W}} \quad (3.14)$$

Occasionally, we will refer to these \mathcal{W} -extended Kac representations simply as \mathcal{W} -Kac representations. It follows from (3.2) that

$$(r, s)_{\mathcal{W}} = (1 \cdot r, s)_{\mathcal{W}} \quad (3.15)$$

and since $1 \cdot r \in \mathbb{Z}_{1,2}$, it thus suffices to define $(r, s)_{\mathcal{W}}$ for $r \in \mathbb{Z}_{1,2}$. Likewise, it is also sufficient to define the \mathcal{W} -extended rank-2 modules $\hat{\mathcal{R}}_r^b$ for $r \in \mathbb{Z}_{1,2}$ only, since

$$\hat{\mathcal{R}}_r^b := \frac{1}{r}\mathcal{R}_r^b \otimes (1, 1)_{\mathcal{W}} = \hat{\mathcal{R}}_{1 \cdot r}^b, \quad r \in \mathbb{N}, \quad b \in \mathbb{Z}_{1,p-1} \quad (3.16)$$

In the following, we therefore let $r \in \mathbb{Z}_{1,2}$.

Recalling $(1, kp) \equiv (k, p)$, we immediately obtain the identifications

$$(r, kp)_{\mathcal{W}} \equiv k(r \cdot k, p)_{\mathcal{W}} = k\hat{M}_{r \cdot k, p} \quad (3.17)$$

of \mathcal{W} -extended modules. It follows that the modules $(r, kp)_{\mathcal{W}}$ are fully reducible.

Based on the conjecture (2.14) for the structure of the Kac representations, we find that the \mathcal{W} -Kac representation $(r, s)_{\mathcal{W}}$ is the finitely-generated \mathcal{W} -extended Feigin-Fuchs module

$$(r, s_0 + kp)_{\mathcal{W}} = \hat{Q}_{r, s_0 + kp}^{\leftarrow}, \quad s_0 \in \mathbb{Z}_{0,p-1}, \quad k \in \mathbb{N}_0 \quad (3.18)$$

whose structure diagram is given by

$$\hat{Q}_{r, s_0 + kp}^{\leftarrow} : \underbrace{\hat{M}_{2 \cdot r \cdot k, s_0} \leftarrow \hat{M}_{r \cdot k, p - s_0} \rightarrow \hat{M}_{2 \cdot r \cdot k, s_0} \leftarrow \dots \leftarrow \hat{M}_{r \cdot k, p - s_0} \rightarrow \hat{M}_{2 \cdot r \cdot k, s_0}}_{\# = 2k+1} \quad (3.19)$$

Before justifying this claim, we note that for $k = 0$, it correctly reduces to

$$(r, s_0)_{\mathcal{W}} = \hat{Q}_{r, s_0}^{\leftarrow} = \hat{M}_{r, s_0} \quad (3.20)$$

while for $s_0 = 0$, it correctly reduces to $(r, kp)_{\mathcal{W}} = k\hat{M}_{r \cdot k, p}$. The \mathcal{W} -extended Kac characters following from (3.18) and (3.19) are given by

$$\begin{aligned} \hat{\chi}_{r, s_0 + kp}(q) &= \chi[(r, s_0 + kp)_{\mathcal{W}}](q) = k\hat{\chi}_{r \cdot k, p - s_0}(q) + (k+1)\hat{\chi}_{2 \cdot r \cdot k, s_0}(q) \\ &= \begin{cases} k \sum_{n \in \mathbb{N}} (2n-1) \text{ch}_{2n-1, p-s_0}(q) + (k+1) \sum_{n \in \mathbb{N}} 2n \text{ch}_{2n, s_0}(q), & r \cdot k = 1 \\ k \sum_{n \in \mathbb{N}} 2n \text{ch}_{2n, p-s_0}(q) + (k+1) \sum_{n \in \mathbb{N}} (2n-1) \text{ch}_{2n-1, s_0}(q), & r \cdot k = 2 \end{cases} \end{aligned} \quad (3.21)$$

Our argument for the structure (3.18) of the \mathcal{W} -Kac representation $(r, s)_{\mathcal{W}}$ is based on

- (i) the conjectured structure diagrams (2.14) of the Kac representations appearing in the decomposition of $(r, s)_{\mathcal{W}}$ in terms of indecomposable Virasoro modules;
- (ii) the assertion that the \mathcal{W} -indecomposable module $(r, s)_{\mathcal{W}}$ can be described by a structure diagram linking \mathcal{W} -irreducible modules only.

To determine the structure diagram of $(r, s)_{\mathcal{W}}$, we thus have to ‘add’ or ‘glue together’ the infinite sequence of structure diagrams associated with the participating Kac representations to form a single structure diagram involving \mathcal{W} -irreducible modules only.

First, for

$$s = s_0 + kp, \quad s_0 \in \mathbb{Z}_{0,p-1}, \quad k \in \mathbb{N}_0 \quad (3.22)$$

we see that the \mathcal{W} -Kac representation $(r, s)_{\mathcal{W}}$ as defined in (3.14) decomposes as

$$\begin{aligned} (1, s)_{\mathcal{W}} &= \left(\bigoplus_{n=1}^{\lceil \frac{k}{2} \rceil} (2n-1) Q_{2n-1,s}^{\rightarrow} \right) \oplus \left(\bigoplus_{n=\lfloor \frac{k+3}{2} \rfloor}^{\infty} (2n-1) Q_{2n-1,s}^{\leftarrow} \right) \\ (2, s)_{\mathcal{W}} &= \left(\bigoplus_{n=1}^{\lceil \frac{k-1}{2} \rceil} 2n Q_{2n,s}^{\rightarrow} \right) \oplus \left(\bigoplus_{n=\lfloor \frac{k+2}{2} \rfloor}^{\infty} 2n Q_{2n,s}^{\leftarrow} \right) \end{aligned} \quad (3.23)$$

in terms of indecomposable Virasoro modules. Here we have used the conjectured structure of the Kac representations discussed in Section 2.1. Using (2.10) and the sum formula

$$\sum_{j=|n-n'|+1, \text{ by } 2}^{n+n'-1} j = nn' \quad (3.24)$$

it is verified that the Virasoro characters of (3.23) agree with the Virasoro characters (3.21) of the proposed Feigin-Fuchs structures (3.18). The appearance of the indecomposable Virasoro modules $Q_{2n-1,s}^{\leftarrow}$ or $Q_{2n,s}^{\leftarrow}$ in (3.23) requires that similar indecomposable structures are present in the ambient \mathcal{W} -Kac representation as well. Following assertion (ii) above, we are thus led to the conjecture (3.18).

For $\mathcal{WLM}(1, p)$, $p \in \mathbb{Z}_{2,5}$, Kac tables of the conformal weights $\Delta_{r,s}$ of the \mathcal{W} -irreducible modules $\hat{M}_{r,s}$ (3.6) over the triplet \mathcal{W} -algebra $\mathcal{W}(p)$ appear in Figure 1.

3.3 \mathcal{W} -extended Kac representations in $\mathcal{WLM}(1, 2)$

Here we illustrate the structure diagrams of the \mathcal{W} -indecomposable modules given in (3.19) for the logarithmic minimal model $\mathcal{WLM}(1, 2)$. In the following, we let k denote a non-negative integer. The structure diagrams are

$$\begin{aligned} (1, 4k-1)_{\mathcal{W}} : & \quad \begin{array}{c} 1 & & 1 & & \cdots & & 1 \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ & 0 & & 0 & & \cdots & & 0 \end{array} & \quad \begin{array}{l} \# = 2k \\ \# = 2k - 1 \end{array} \\ (2, 4k-1)_{\mathcal{W}} : & \quad \begin{array}{c} & 1 & & 1 & & \cdots & & 1 \\ & \swarrow & & \searrow & & \swarrow & & \searrow \\ 0 & & 0 & & \cdots & & 0 \end{array} & \quad \begin{array}{l} \# = 2k - 1 \\ \# = 2k \end{array} \end{aligned} \quad (3.25)$$

| | | | | | |
|--|----------------------|----------------------|---|----------------------|----------------------|
| | | | | | $\mathcal{WLM}(1,5)$ |
| | | | | $\mathcal{WLM}(1,4)$ | |
| | $\mathcal{WLM}(1,2)$ | $\mathcal{WLM}(1,3)$ | | | |
| | 2 | 3 | 4 | 5 | |
| | 1 | 2 | 3 | 4 | 5 |
| | 1 | 1 | 1 | 1 | 1 |
| | 2 | 2 | 2 | 2 | 2 |
| | 1 | 1 | 1 | 1 | 1 |
| | 2 | 2 | 2 | 2 | 2 |

Figure 1: Kac tables of conformal weights $\Delta_{r,s}$ of the \mathcal{W} -irreducible \mathcal{W} -Kac representations in $\mathcal{WLM}(1,p)$ where $p \in \mathbb{Z}_{2,5}$. The corresponding central charges are $c = -2$, $c = -7$, $c = -25/2$ and $c = -91/5$, respectively. In a given table, the column index is $r \in \mathbb{Z}_{1,2}$, while the row index is $s \in \mathbb{Z}_{1,p}$. The pair of \mathcal{W} -Kac representations which are both \mathcal{W} -irreducible and \mathcal{W} -projective are indicated by darker shadings.

and

$$\begin{aligned}
(1, 4k+1)_{\mathcal{W}} : & \quad \begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & \swarrow & & \searrow & \swarrow & \searrow & \swarrow \\ 0 & & 0 & & \dots & & 0 \end{array} & \quad \begin{array}{l} \# = 2k \\ \# = 2k+1 \end{array} \\
(2, 4k+1)_{\mathcal{W}} : & \quad \begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & \swarrow & & \searrow & \swarrow & \searrow & \swarrow \\ & 0 & & 0 & & 0 & & 0 \end{array} & \quad \begin{array}{l} \# = 2k+1 \\ \# = 2k \end{array}
\end{aligned} \tag{3.26}$$

where the \mathcal{W} -irreducible module $\hat{M}_{r,s}$ (3.6) is represented by its conformal weight $\Delta_{r,s}$. For $k = 1$, we thus have

$$(1, 3)_{\mathcal{W}} : \begin{array}{ccc} & 1 & \\ & \swarrow & \searrow \\ & 0 & \end{array} \quad (2, 3)_{\mathcal{W}} : \begin{array}{ccc} & 1 & \\ & \swarrow & \searrow \\ 0 & & 0 \end{array} \tag{3.27}$$

and

$$(1, 5)_{\mathcal{W}} : \begin{array}{ccccc} & & 1 & & 1 \\ & \swarrow & & \searrow & \swarrow \\ 0 & & 0 & & 0 \end{array} \quad (2, 5)_{\mathcal{W}} : \begin{array}{ccccc} & & 1 & & 1 \\ & \swarrow & & \searrow & \swarrow \\ & 0 & & 0 & & 0 \end{array} \tag{3.28}$$

3.4 \mathcal{W} -extended Kac fusion algebra

The fusion rules in the \mathcal{W} -extended picture are inferred from the fusion rules in the Virasoro picture. Letting $\hat{\otimes}$ denote the fusion multiplication in the \mathcal{W} -extended picture, it is interpreted [27] as a limit of a rescaled fusion

$$(1, 1)_{\mathcal{W}} \hat{\otimes} (1, 1)_{\mathcal{W}} := \lim_{n \rightarrow \infty} \left(\frac{1}{(2n-1)^3} (2n-1, 1) \otimes (2n-1, 1) \otimes (2n-1, 1) \otimes (1, 1)_{\mathcal{W}} \right) = (1, 1)_{\mathcal{W}} \tag{3.29}$$

in the Virasoro picture of the logarithmic minimal model $\mathcal{LM}(1,p)$. This ensures that fusion in the extended picture has a natural implementation on the lattice.

Now, a representation \hat{A} in the \mathcal{W} -extended picture is constructed as the integrable boundary condition $A \otimes (1, 1)_{\mathcal{W}}$ where A is some Virasoro representation in the logarithmic minimal model.

Fusion in the extended picture is then computed as

$$\begin{aligned}\hat{A} \hat{\otimes} \hat{B} &= (A \otimes (1, 1)_{\mathcal{W}}) \hat{\otimes} (B \otimes (1, 1)_{\mathcal{W}}) = (A \otimes B) \otimes ((1, 1)_{\mathcal{W}} \hat{\otimes} (1, 1)_{\mathcal{W}}) \\ &= \left(\bigoplus_j C_j \right) \otimes (1, 1)_{\mathcal{W}} = \bigoplus_j \hat{C}_j\end{aligned}\quad (3.30)$$

where $A \otimes B = \bigoplus_j C_j$ is the fusion of the representations A and B in the Virasoro picture. This \mathcal{W} -extended fusion prescription is readily seen to be both associative and commutative. It is also immediately verified that $(1, 1)_{\mathcal{W}}$ is the identity of the ensuing fusion algebra

$$(1, 1)_{\mathcal{W}} \hat{\otimes} \hat{A} = ((1, 1) \otimes (1, 1)_{\mathcal{W}}) \hat{\otimes} (A \otimes (1, 1)_{\mathcal{W}}) = ((1, 1) \otimes A) \otimes (1, 1)_{\mathcal{W}} = \hat{A} \quad (3.31)$$

With this \mathcal{W} -extended fusion prescription, it follows that the \mathcal{W} -Kac representation $(r, s)_{\mathcal{W}}$ ‘separates’ in much the same way (3.13) as the original Kac representations, that is,

$$\begin{aligned}(r, s)_{\mathcal{W}} &= \frac{1}{r}(r, s) \otimes (1, 1)_{\mathcal{W}} = \left[\frac{1}{r}(r, 1) \otimes (1, s) \right] \otimes [(1, 1)_{\mathcal{W}} \hat{\otimes} (1, 1)_{\mathcal{W}}] \\ &= \left[\frac{1}{r}(r, 1) \otimes (1, 1)_{\mathcal{W}} \right] \hat{\otimes} [(1, s) \otimes (1, 1)_{\mathcal{W}}] \\ &= (r, 1)_{\mathcal{W}} \hat{\otimes} (1, s)_{\mathcal{W}}\end{aligned}\quad (3.32)$$

Using (3.24), we also find that

$$(r, 1)_{\mathcal{W}} \hat{\otimes} (r', 1)_{\mathcal{W}} = (r \cdot r', 1)_{\mathcal{W}}, \quad (r, 1)_{\mathcal{W}} \hat{\otimes} \hat{\mathcal{R}}_{r'}^b = \hat{\mathcal{R}}_{r \cdot r'}^b \quad (3.33)$$

and hence

$$(r, s)_{\mathcal{W}} \hat{\otimes} (r', s')_{\mathcal{W}} = (r \cdot r', 1)_{\mathcal{W}} \hat{\otimes} [(1, s)_{\mathcal{W}} \hat{\otimes} (1, s')_{\mathcal{W}}] \quad (3.34)$$

and

$$\hat{\mathcal{R}}_r^b \hat{\otimes} (r', s')_{\mathcal{W}} = (r \cdot r', 1)_{\mathcal{W}} \hat{\otimes} [\hat{\mathcal{R}}_1^b \hat{\otimes} (1, s')_{\mathcal{W}}], \quad \hat{\mathcal{R}}_r^b \hat{\otimes} \hat{\mathcal{R}}_{r'}^{b'} = (r \cdot r', 1)_{\mathcal{W}} \hat{\otimes} [\hat{\mathcal{R}}_1^b \hat{\otimes} \hat{\mathcal{R}}_1^{b'}] \quad (3.35)$$

Based on the fusion rules [56] in the Virasoro picture, summarized in Appendix A, we work out the \mathcal{W} -extended Kac fusion algebra

$$\langle (r, s)_{\mathcal{W}}; r \in \mathbb{Z}_{1,2}, s \in \mathbb{N} \rangle \quad (3.36)$$

generated by repeated fusion of the \mathcal{W} -Kac representations. Written as a disjoint union of sets, the set of distinct, \mathcal{W} -indecomposable modules partaking in this fusion algebra is

$$\mathcal{J}_{\mathcal{W}}^{\text{Kac}} = \{(r, s)_{\mathcal{W}}; r \in \mathbb{Z}_{1,2}, s \in \mathbb{N} \setminus p\mathbb{N}\} \cup \mathcal{J}_{\mathcal{W}}^{\text{Proj}} \quad (3.37)$$

For $r, r' \in \mathbb{Z}_{1,2}$, $b, b' \in \mathbb{Z}_{0,p-1}$ and $k, k' \in \mathbb{N}_0$, we find the underlying fusion rules to be given by

$$\begin{aligned}(r, b + kp)_{\mathcal{W}} \hat{\otimes} (r', b' + k'p)_{\mathcal{W}} &= kk' \left(\bigoplus_{\beta}^{p-|b-b'|-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k \cdot k'}^{\beta} \right) \oplus (k + k' + 1) \left(\bigoplus_{\beta}^{b+b'-p-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k \cdot k'}^{\beta} \right) \\ &\oplus (k + 1)k' \left(\bigoplus_{\beta}^{b-b'-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k \cdot k'}^{\beta} \right) \oplus k(k' + 1) \left(\bigoplus_{\beta}^{b'-b-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k \cdot k'}^{\beta} \right) \\ &\oplus \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p-|p-b-b'|-1} (r \cdot r', \beta + (k + k')p)_{\mathcal{W}}\end{aligned}\quad (3.38)$$

and

$$\begin{aligned}
\hat{\mathcal{R}}_r^b \hat{\otimes} (r', b' + k'p)_{\mathcal{W}} &= \left\{ k' \left(\bigoplus_{\beta}^{p-|b-b'|-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k'}^{\beta} \right) \oplus k' \left(\bigoplus_{\beta}^{|p-b-b'|-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k'}^{\beta} \right) \oplus 2 \left(\bigoplus_{\beta}^{b+b'-p-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k'}^{\beta} \right) \right. \\
&\quad \oplus (k' + 1) \left(\bigoplus_{\beta=|b-b'+1, \text{ by } 2}^{p-|p-b-b'|-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k'}^{\beta} \right) \oplus 2k' \left(\bigoplus_{\beta}^{|b-b'|-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k'}^{\beta} \right) \\
&\quad \left. \oplus 2 \left(\bigoplus_{\beta}^{b'-b-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k'}^{\beta} \right) \right\} / (1 + \delta_{b,0}) \tag{3.39}
\end{aligned}$$

as well as the known fusion rules

$$\begin{aligned}
\hat{\mathcal{R}}_r^b \hat{\otimes} \hat{\mathcal{R}}_{r'}^{b'} &= 2 \left(\bigoplus_{\beta}^{p-|b-b'|-1} \hat{\mathcal{R}}_{r \cdot r'}^{\beta} \oplus \bigoplus_{\beta}^{|p-b-b'|-1} \hat{\mathcal{R}}_{r \cdot r'}^{\beta} \right. \\
&\quad \left. \oplus \bigoplus_{\beta}^{p-|p-b-b'|-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r'}^{\beta} \oplus \bigoplus_{\beta}^{|b-b'|-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r'}^{\beta} \right) / \{(1 + \delta_{b,0})(1 + \delta_{b',0})\} \tag{3.40}
\end{aligned}$$

for the subalgebra generated by the projective modules $\hat{\mathcal{R}}_r^b$. This subalgebra is actually an *ideal*, in accordance with the modules $\hat{\mathcal{R}}_r^b$ being projective. The divisions in (3.39) and (3.40) by $(1 + \delta_{b,0})$ and $(1 + \delta_{b',0})$ ensure that the fusion rules for $\hat{\mathcal{R}}_r^0$ match those for $(r, p)_{\mathcal{W}}$.

We observe that, as a factor in a fusion product, the \mathcal{W} -projective module $\hat{\mathcal{R}}_r^b$ is *insensitive* to the indecomposable structure of the other \mathcal{W} -extended fusion factor, that is,

$$\begin{aligned}
\hat{\mathcal{R}}_r^b \hat{\otimes} \hat{\mathcal{R}}_{r'}^{b'} &= \hat{\mathcal{R}}_r^b \hat{\otimes} [2(r', p - b')_{\mathcal{W}} \oplus 2(2 \cdot r', b')_{\mathcal{W}}] \\
\hat{\mathcal{R}}_r^b \hat{\otimes} (r', b' + k'p)_{\mathcal{W}} &= \hat{\mathcal{R}}_r^b \hat{\otimes} [k'(r' \cdot k', p - b')_{\mathcal{W}} \oplus (k' + 1)(2 \cdot r' \cdot k', b')_{\mathcal{W}}] \tag{3.41}
\end{aligned}$$

Here we have $b \in \mathbb{Z}_{0,p-1}$, while we set $b' \in \mathbb{Z}_{1,p-1}$ for $\hat{\mathcal{R}}_{r'}^{b'}$ and $(r', b' + k'p)_{\mathcal{W}}$ to be reducible yet indecomposable. Similar to the situation in the Virasoro case, the insensitivity properties (3.41) imply that fusion by $\hat{\mathcal{R}}_r^b$ in $\langle \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \rangle$ is an *exact functor* for all $\hat{\mathcal{R}}_r^b \in \mathcal{J}_{\mathcal{W}}^{\text{Proj}}$.

3.5 Contragredient modules and their fusion properties

As in the Virasoro picture, we introduce the contragredient module to each \mathcal{W} -Kac representation $(r, s)_{\mathcal{W}}$ by reversing the arrows in the corresponding structure diagram (3.18) and (3.19). For $r \in \mathbb{N}$, $s_0 \in \mathbb{Z}_{0,p-1}$ and $k \in \mathbb{N}_0$, we thus have

$$(r, s_0 + kp)_{\mathcal{W}}^* = \hat{Q}_{r, s_0 + kp}^{\rightarrow} \tag{3.42}$$

whose structure diagram is given by

$$\hat{Q}_{r, s_0 + kp}^{\rightarrow} : \underbrace{\hat{M}_{2 \cdot r \cdot k, s_0} \rightarrow \hat{M}_{r \cdot k, p - s_0} \leftarrow \hat{M}_{2 \cdot r \cdot k, s_0} \rightarrow \dots \rightarrow \hat{M}_{r \cdot k, p - s_0} \leftarrow \hat{M}_{2 \cdot r \cdot k, s_0}}_{\# = 2k+1} \tag{3.43}$$

The corresponding character is denoted by $\hat{\chi}_{r,s}^*(q) = \chi[(r, s)_{\mathcal{W}}^*](q)$, and it follows that $(r, s)_{\mathcal{W}}^* = (r, s)_{\mathcal{W}}$ if and only if $(r, s)_{\mathcal{W}}$ is fully reducible, that is,

$$(r, s)_{\mathcal{W}}^* = (r, s)_{\mathcal{W}} \iff s \in \mathbb{Z}_{1,p-1} \cup p\mathbb{N} \tag{3.44}$$

As in the case of the \mathcal{W} -Kac representations themselves, we may restrict our considerations of $(r, s)_{\mathcal{W}}^*$ to $r \in \mathbb{Z}_{1,2}$ since $(r, s)_{\mathcal{W}}^* = (r \cdot 1, s)_{\mathcal{W}}^*$. For $\mathcal{WLM}(1, 2)$, the contragredient modules to the ones described explicitly in (3.27) and (3.28) are

$$(1, 3)_{\mathcal{W}}^* : \begin{array}{c} 1 \quad 1 \\ \searrow \quad \swarrow \\ 0 \end{array} \quad (2, 3)_{\mathcal{W}}^* : \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 0 \quad 0 \end{array} \quad (3.45)$$

and

$$(1, 5)_{\mathcal{W}}^* : \begin{array}{c} 1 \quad 1 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ 0 \quad 0 \quad 0 \end{array} \quad (2, 5)_{\mathcal{W}}^* : \begin{array}{c} 1 \quad 1 \quad 1 \\ \searrow \quad \swarrow \quad \searrow \quad \swarrow \\ 0 \quad 0 \quad 0 \end{array} \quad (3.46)$$

By construction, the \mathcal{W} -extended rank-2 modules are all self-contragredient:

$$(\hat{\mathcal{R}}_r^b)^* = \hat{\mathcal{R}}_r^b \quad (3.47)$$

To describe the fusion algebra

$$\langle (r, s)_{\mathcal{W}}, (r, s)_{\mathcal{W}}^*; r \in \mathbb{Z}_{1,2}, s \in \mathbb{N} \rangle \quad (3.48)$$

generated by repeated fusion of the \mathcal{W} -Kac representations and their contragredient counterparts, we mimic (A.3) and introduce

$$\hat{\mathcal{C}}_n[(r, s)_{\mathcal{W}}] = \begin{cases} (r, s)_{\mathcal{W}}, & n > 0 \\ (r, s)_{\mathcal{W}}^*, & n < 0 \end{cases} \quad (3.49)$$

In our applications, $\hat{\mathcal{C}}_0[(r, s)_{\mathcal{W}}]$ only appears if $(r, s)_{\mathcal{W}}$ is fully reducible in which case

$$\hat{\mathcal{C}}_0[(r, s)_{\mathcal{W}}] = (r, s)_{\mathcal{W}} = (r, s)_{\mathcal{W}}^*, \quad s \in \mathbb{Z}_{1,p-1} \cup p\mathbb{N} \quad (3.50)$$

The fusion rules involving the contragredient modules $(r, s)_{\mathcal{W}}^*$ are inferred from the corresponding fusion rules in the Virasoro picture [56]. We thus find that the decomposition of the fusion products

$$(r, s)_{\mathcal{W}}^* \hat{\otimes} (r', s')_{\mathcal{W}}^* = ((r, s)_{\mathcal{W}} \hat{\otimes} (r', s')_{\mathcal{W}})^*, \quad \hat{\mathcal{R}}_r^b \hat{\otimes} (r', s')_{\mathcal{W}}^* = \hat{\mathcal{R}}_r^b \hat{\otimes} (r', s')_{\mathcal{W}} \quad (3.51)$$

follow readily from the fusion rules underlying the \mathcal{W} -Kac fusion algebra discussed above, while

$$\begin{aligned} (r, b+kp)_{\mathcal{W}} \hat{\otimes} (r', b'+k'p)_{\mathcal{W}}^* &= kk' \left(\bigoplus_{\beta}^{p-b-b'-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k \cdot k'}^{\beta} \right) \oplus (k+1)(k'+1) \left(\bigoplus_{\beta}^{b+b'-p-1} \hat{\mathcal{R}}_{r \cdot r' \cdot k \cdot k'}^{\beta} \right) \\ &\oplus (kk' + \min(k, k')) \left(\bigoplus_{\beta}^{p-|p-b-b'|-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k \cdot k'}^{\beta} \right) \\ &\oplus |k-k'| \left(\bigoplus_{\beta}^{(b-b')\text{sg}(k'-k)-1} \hat{\mathcal{R}}_{2 \cdot r \cdot r' \cdot k \cdot k'}^{\beta} \right) \\ &\oplus \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p-|p-b-b'|-1} \hat{\mathcal{C}}_{k-k'}[(r \cdot r', \beta + |k-k'|p)_{\mathcal{W}}] \end{aligned} \quad (3.52)$$

The set of distinct, \mathcal{W} -indecomposable representations partaking in this contragrediently-extended \mathcal{W} -Kac fusion algebra is

$$\mathcal{J}_{\mathcal{W}}^{\text{Cont}} = \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \cup \{(r, s)_{\mathcal{W}}^*; r \in \mathbb{Z}_{1,2}, s \in \mathbb{N} \setminus (\mathbb{Z}_{1,p-1} \cup p\mathbb{N})\} \quad (3.53)$$

here written as a disjoint union of sets. It is noted that the fusion subalgebras generated by the \mathcal{W} -Kac representations and their contragredient counterparts, respectively, are isomorphic

$$\langle (r, s)_{\mathcal{W}}; r \in \mathbb{Z}_{1,2}, s \in \mathbb{N} \rangle \simeq \langle (r, s)_{\mathcal{W}}^*; r \in \mathbb{Z}_{1,2}, s \in \mathbb{N} \rangle \quad (3.54)$$

This resembles the situation (2.30) in the Virasoro picture.

Combining the second fusion property in (3.51) with the fact that the \mathcal{W} -projective modules form an ideal of the \mathcal{W} -Kac fusion algebra, we see that the \mathcal{W} -projective modules also form an ideal of the larger contragrediently-extended \mathcal{W} -Kac fusion algebra $\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle$. It also follows that the insensitivity properties (3.41) of the \mathcal{W} -projective modules are supplemented by

$$\hat{\mathcal{R}}_r^b \hat{\otimes} (r', b' + k'p)_{\mathcal{W}}^* = \hat{\mathcal{R}}_r^b \hat{\otimes} [k'(r' \cdot k', p - b')_{\mathcal{W}} \oplus (k' + 1)(2 \cdot r' \cdot k', b')_{\mathcal{W}}] \quad (3.55)$$

implying that fusion by $\hat{\mathcal{R}}_r^b$ is an *exact functor* in $\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle$ for all $\hat{\mathcal{R}}_r^b \in \mathcal{J}_{\mathcal{W}}^{\text{Proj}}$.

Following up on the discussion at the end of Section 2.4, the lattice issue with Kac representations versus contragredient Kac representations carries over to the \mathcal{W} -extended picture. Here we have merely reiterated the assertion [56] that it is the Kac representations (r, s) and not (in general) the contragredient Kac representations $(r, s)^*$ which are associated with Yang-Baxter integrable boundary conditions in the lattice approach. As a consequence of the way we have introduced the corresponding Yang-Baxter integrable boundary conditions in the \mathcal{W} -extended picture, it is the \mathcal{W} -Kac representations $(r, s)_{\mathcal{W}}$ and not (in general) their contragredient counterparts $(r, s)_{\mathcal{W}}^*$ which are associated with \mathcal{W} -extended boundary conditions. The two families generate isomorphic fusion algebras (3.54) and the corresponding characters are identical $\hat{\chi}_{r,s}(q) = \hat{\chi}_{r,s}^*(q)$.

4 Polynomial fusion rings

Our last objectives are to determine polynomial fusion rings isomorphic with the \mathcal{W} -Kac fusion algebra $\langle \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \rangle$ and its contragredient extension $\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle$, and to identify the corresponding Grothendieck ring of characters. This is a continuation of our recent work [63–65] on polynomial fusion rings in $\mathcal{WLM}(p, p')$. The corresponding constructions in the Virasoro picture were obtained in [66].

4.1 \mathcal{W} -extended Kac fusion algebra

Together with the fact that the \mathcal{W} -extended fundamental fusion algebra $\langle \mathcal{J}_{\mathcal{W}}^{\text{Fund}} \rangle$ is a subalgebra of the \mathcal{W} -Kac fusion algebra, the fusion rules

$$\begin{aligned} (1, 2)_{\mathcal{W}} \hat{\otimes} (1, b + kp)_{\mathcal{W}} &= (1, b - 1 + kp)_{\mathcal{W}} \oplus (1, b + 1 + kp)_{\mathcal{W}} \\ (1, p + 1)_{\mathcal{W}} \hat{\otimes} (1, b + kp)_{\mathcal{W}} &= k \left(\bigoplus_{\beta}^{p-b} \hat{\mathcal{R}}_{1,k}^{\beta} \right) \oplus (k + 1) \left(\bigoplus_{\beta}^{b-2} \hat{\mathcal{R}}_{2,k}^{\beta} \right) \oplus (1, b + (k + 1)p)_{\mathcal{W}} \end{aligned} \quad (4.1)$$

where $b \in \mathbb{Z}_{1,p-1}$, demonstrate that the \mathcal{W} -Kac fusion algebra is generated from repeated fusion of the four modules $(1, 1)_{\mathcal{W}}$, $(2, 1)_{\mathcal{W}}$, $(1, 2)_{\mathcal{W}}$ and $(1, p + 1)_{\mathcal{W}}$, that is,

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \rangle = \langle (1, 1)_{\mathcal{W}}, (2, 1)_{\mathcal{W}}, (1, 2)_{\mathcal{W}}, (1, p + 1)_{\mathcal{W}} \rangle \quad (4.2)$$

Since $(1, 1)_{\mathcal{W}} = \hat{M}_{1,1}$ is the algebra identity, it is therefore natural to expect that this fusion algebra is isomorphic to a polynomial ring in the three entities $X \leftrightarrow (2, 1)_{\mathcal{W}} = \hat{M}_{2,1}$, $Y \leftrightarrow (1, 2)_{\mathcal{W}} = \hat{M}_{1,2}$ and $Z \leftrightarrow (1, p + 1)_{\mathcal{W}}$. This is indeed what we find and it is the content of Proposition 1 below. In the

following, $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively, where we set $U_{-1}(x) = 0$.

Proposition 1. The \mathcal{W} -Kac fusion algebra is isomorphic to the polynomial ring generated by X , Y and Z modulo the ideal

$$\mathcal{I}_{\mathcal{W}}^{\text{Kac}} = (X^2 - 1, P_p(X, Y), Q_p(Y, Z)) \quad (4.3)$$

that is,

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \rangle \simeq \mathbb{C}[X, Y, Z] / \mathcal{I}_{\mathcal{W}}^{\text{Kac}} \quad (4.4)$$

where

$$P_p(X, Y) = [X - T_p(\frac{Y}{2})]U_{p-1}(\frac{Y}{2}), \quad Q_p(Y, Z) = [Z - U_p(\frac{Y}{2})]U_{p-1}(\frac{Y}{2}) \quad (4.5)$$

For $r \in \mathbb{Z}_{1,2}$, $b \in \mathbb{Z}_{0,p-1}$ and $k \in \mathbb{N}_0$, the isomorphism reads

$$\begin{aligned} (r, b + kp)_{\mathcal{W}} &\leftrightarrow X^{r-1} \left(U_{kp+b-1}(\frac{Y}{2}) + [Z^k - U_p^k(\frac{Y}{2})]U_{b-1}(\frac{Y}{2}) \right) \\ \hat{\mathcal{R}}_r^b &\leftrightarrow (2 - \delta_{b,0})X^{r-1}T_b(\frac{Y}{2})U_{p-1}(\frac{Y}{2}) \end{aligned} \quad (4.6)$$

Proof. The relation $P_p(X, Y) = 0$ corresponds to the identification $(1, 2p)_{\mathcal{W}} \equiv 2(2, p)_{\mathcal{W}}$, while the relation $Q_p(Y, Z) = 0$ corresponds to the fusion rule

$$(1, p)_{\mathcal{W}} \hat{\otimes} (1, p+1)_{\mathcal{W}} = 2(2, p)_{\mathcal{W}} \oplus \bigoplus_{\beta}^{p-2} \hat{\mathcal{R}}_1^{\beta} \quad (4.7)$$

when employing the identity

$$\sum_{\beta=\epsilon(p), \text{ by } 2}^p (2 - \delta_{\beta,0})T_{\beta}(x) = U_p(x) \quad (4.8)$$

The remaining fusion rules are then verified straightforwardly in the polynomial ring. Here we only demonstrate explicitly the two fusion rules in (4.1). The first of these follows immediately from the recursion relation for the Chebyshev polynomials. To show the second of the fusion rules, we follow [56] on the similar fusion rule in the Virasoro picture and note the basic decomposition rules

$$U_m(x)U_n(x) = \sum_{j=|m-n|, \text{ by } 2}^{m+n} U_j(x), \quad 2T_m(x)U_{n-1}(x) = U_{n+m-1}(x) + \text{sg}(n-m)U_{|n-m|-1}(x) \quad (4.9)$$

As a consequence, we have

$$U_{p-1}(x) \sum_{j=0}^{k-1} U_p^{k-j-1}(x)U_{jp+b-2}(x) = U_{b-1}(x)U_p^k(x) - U_{kp+b-1}(x) \quad (4.10)$$

which is established by induction in k and shows that the expression on the right side is divisible by $U_{p-1}(x)$. This is of importance when multiplied by Z due to the form of $Q_p(Y, Z)$. With the additional observation that

$$U_{r-1}(\frac{X}{2})U_{p-1}(\frac{Y}{2}) \equiv U_{rp-1}(\frac{Y}{2}) \pmod{P_p(X, Y)} \quad (4.11)$$

which follows by induction in r , the second fusion rule readily follows. \square

In [56], we demonstrated that the conjectured Kac fusion algebra $\langle \mathcal{J}^{\text{Kac}} \rangle$ in the Virasoro picture is isomorphic to the polynomial ring

$$\langle \mathcal{J}^{\text{Kac}} \rangle \simeq \mathbb{C}[X, Y, Z] / (P_p(X, Y), Q_p(Y, Z)) \quad (4.12)$$

In somewhat sloppy notation, we thus have the relation

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \rangle \simeq \langle \mathcal{J}^{\text{Kac}} \rangle / (X^2 - 1) \quad (4.13)$$

between the \mathcal{W} -Kac fusion algebra and the Kac fusion algebra itself.

4.2 Contragredient extension

Extending the arguments presented above for the \mathcal{W} -Kac fusion algebra, one finds that its contragredient extension $\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle$ is also generated from repeated fusion of a small number of modules, namely

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle = \langle \hat{M}_{1,1}, \hat{M}_{2,1}, \hat{M}_{1,2}, (1, p+1)_{\mathcal{W}}, (1, p+1)_{\mathcal{W}}^* \rangle \quad (4.14)$$

where it is recalled that

$$\hat{M}_{1,1} = (1, 1)_{\mathcal{W}} = (1, 1)_{\mathcal{W}}^*, \quad \hat{M}_{2,1} = (2, 1)_{\mathcal{W}} = (2, 1)_{\mathcal{W}}^*, \quad \hat{M}_{1,2} = (1, 2)_{\mathcal{W}} = (1, 2)_{\mathcal{W}}^* \quad (4.15)$$

Using

$$(1, p+1)_{\mathcal{W}} \hat{\otimes} (1, p+1)_{\mathcal{W}}^* = \hat{M}_{1,1} \oplus 2\hat{\mathcal{R}}_2^1 \oplus \bigoplus_{\beta}^{p-3} \hat{\mathcal{R}}_1^{\beta} \quad (4.16)$$

in particular, one also finds that $\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle$ is isomorphic to a polynomial ring in the four entities $X \leftrightarrow \hat{M}_{2,1}$, $Y \leftrightarrow \hat{M}_{1,2}$, $Z \leftrightarrow (1, p+1)_{\mathcal{W}}$ and $Z^* \leftrightarrow (1, p+1)_{\mathcal{W}}^*$ as demonstrated in Proposition 2 below.

Proposition 2. The contragrediently-extended \mathcal{W} -Kac fusion algebra is isomorphic to the polynomial ring generated by X , Y , Z and Z^* modulo the ideal

$$\mathcal{I}_{\mathcal{W}}^{\text{Cont}} = (X^2 - 1, P_p(X, Y), Q_p(Y, Z), Q_p(Y, Z^*), R_p(Y, Z, Z^*)) \quad (4.17)$$

that is,

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle \simeq \mathbb{C}[X, Y, Z, Z^*] / \mathcal{I}_{\mathcal{W}}^{\text{Cont}} \quad (4.18)$$

where the polynomials P_p and Q_p are defined in (4.5) while

$$R_p(Y, Z, Z^*) = ZZ^* - U_p^2\left(\frac{Y}{2}\right) \quad (4.19)$$

For $r \in \mathbb{Z}_{1,2}$, $b \in \mathbb{Z}_{0,p-1}$ and $k \in \mathbb{N}_0$, the isomorphism reads

$$\begin{aligned} (r, b+kp)_{\mathcal{W}} &\leftrightarrow X^{r-1} \left(U_{kp+b-1}\left(\frac{Y}{2}\right) + [Z^k - U_p^k\left(\frac{Y}{2}\right)] U_{b-1}\left(\frac{Y}{2}\right) \right) \\ (r, b+kp)_{\mathcal{W}}^* &\leftrightarrow X^{r-1} \left(U_{kp+b-1}\left(\frac{Y}{2}\right) + [(Z^*)^k - U_p^k\left(\frac{Y}{2}\right)] U_{b-1}\left(\frac{Y}{2}\right) \right) \\ \hat{\mathcal{R}}_r^b &\leftrightarrow (2 - \delta_{b,0}) X^{r-1} T_b\left(\frac{Y}{2}\right) U_{p-1}\left(\frac{Y}{2}\right) \end{aligned} \quad (4.20)$$

Proof. This proof is almost identical to the proof in [56] of the similar proposition in the Virasoro picture, but is included for completeness. Compared to the proof of Proposition 1, the essential new feature is the appearance of Z^* . The relation $Q_p(Y, Z^*) = 0$ plays the same role for the contragredient \mathcal{W} -Kac representations and Z^* as $Q_p(Y, Z) = 0$ does for the \mathcal{W} -Kac representations and Z . This yields the part of the polynomial ring corresponding to (3.54). The relation $R_p(Y, Z, Z^*) = 0$ corresponds to the fusion rule (4.16). To establish the general fusion rule (3.52) in the ring picture, we first use induction in n to establish

$$U_p^{2n}\left(\frac{Y}{2}\right) Z^m \equiv Z^m + \sum_{j=0}^{n-1} U_p^{m+2j}\left(\frac{Y}{2}\right) U_{p-1}\left(\frac{Y}{2}\right) U_{p+1}\left(\frac{Y}{2}\right) \pmod{Q_p(Y, Z)}, \quad n \in \mathbb{N} \quad (4.21)$$

and similarly for Z replaced by Z^* . This is needed when reducing

$$Z^k (Z^*)^{k'} \equiv U_p^{2\min(k,k')} \left(\frac{Y}{2}\right) \begin{cases} Z^{k-k'}, & k \geq k' \\ (Z^*)^{k'-k}, & k < k' \end{cases} \pmod{R_p(Y, Z, Z^*)} \quad (4.22)$$

For simplicity, we let $k \geq k'$ in which case we find

$$(1, b + kp)_{\mathcal{W}} \hat{\otimes} (1, b' + k'p)_{\mathcal{W}}^* \leftrightarrow [Z^{k-k'} - U_p^{k-k'} \left(\frac{Y}{2}\right)] U_{b-1} \left(\frac{Y}{2}\right) U_{b'-1} \left(\frac{Y}{2}\right) + U_{kp+b-1} \left(\frac{Y}{2}\right) U_{k'p+b'-1} \left(\frac{Y}{2}\right) \quad (4.23)$$

This polynomial expression is recognized as corresponding to the right side of (3.52). \square

In [56], we demonstrated that the conjectured contragrediently-extended Kac fusion algebra $\langle \mathcal{J}^{\text{Cont}} \rangle$ in the Virasoro picture is isomorphic to the polynomial ring

$$\langle \mathcal{J}^{\text{Cont}} \rangle \simeq C[X, Y, Z, Z^*] / (P_p(X, Y), Q_p(Y, Z), Q_p(Y, Z^*), R_p(Y, Z, Z^*)) \quad (4.24)$$

In somewhat sloppy notation, we thus have the relation

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle \simeq \langle \mathcal{J}^{\text{Cont}} \rangle / (X^2 - 1) \quad (4.25)$$

between the contragrediently-extended \mathcal{W} -Kac fusion algebra and the contragrediently-extended Kac fusion algebra itself. With this and (4.13) in mind, the proofs of Proposition 1 and 2 could have been reduced to an analysis of the consequences of $X^2 = 1$ since (4.12) and (4.24) were established in [56]. However, we found it more instructive to include direct and independent proofs of the two propositions above.

4.3 Grothendieck ring

The set of Virasoro characters in a CFT naturally forms a Grothendieck group whose generators are equivalence classes $[R]$ formed by the characters: $[R] = \chi[R](q)$. Its group operation is addition and is defined via direct summation of the representations of the equivalence classes

$$[R_1] + [R_2] = [R_1 \oplus R_2] \quad (4.26)$$

that is, by addition of characters. For rational CFTs, this Grothendieck group admits a ring structure whose multiplication follows from the fusion product of representations

$$[R_1] * [R_2] = [R_1 \otimes R_2] \quad (4.27)$$

For logarithmic models, on the other hand, the fusion of representations does not, in general, induce a product on the Grothendieck group in this way, see [67] for example. However, on the Grothendieck group associated with the fundamental fusion algebra of $\mathcal{WLM}(1, p)$, the fusion rules *do* induce a well-defined multiplication (4.27) thereby turning the group into a ring, as described in [68].

The Grothendieck group associated with the fundamental fusion algebra of $\mathcal{WLM}(1, p)$ is generated by the $2p$ generators

$$G_{r,s} = [\hat{M}_{r,s}], \quad r \in \mathbb{Z}_{1,2}, \quad s \in \mathbb{Z}_{1,p} \quad (4.28)$$

corresponding to the set of irreducible modules. Following from (3.7), the equivalence class of a rank-2 module thus decomposes as

$$[\hat{\mathcal{R}}_r^b] = 2G_{2,r,b} + 2G_{1,r,p-b}, \quad b \in \mathbb{Z}_{1,p-1} \quad (4.29)$$

The corresponding multiplication rules follow from the fusion rules and are given by

$$G_{r,s} * G_{r',s'} = \sum_{j=|s-s'|+1, \text{ by } 2}^{p-|p-s-s'|-1} G_{r \cdot r', j} + \sum_{\beta=\epsilon(s+s'-p-1), \text{ by } 2}^{s+s'-p-1} (2 - \delta_{\beta,0}) (G_{r \cdot r', p-\beta} + G_{2 \cdot r \cdot r', \beta}) \quad (4.30)$$

where $G_{r,0} \equiv 0$.

Let

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Grot}} \rangle, \quad \mathcal{J}_{\mathcal{W}}^{\text{Grot}} = \{G_{r,s}; r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p}\} \quad (4.31)$$

denote the Grothendieck ring associated with $\mathcal{WLM}(1,p)$, and let \sim denote the equivalence relation between modules in $\mathcal{J}_{\mathcal{W}}^{\text{Cont}}$ with identical characters such that

$$\hat{\mathcal{R}}_r^b \sim 2\hat{M}_{2 \cdot r, b} \oplus 2\hat{M}_{1 \cdot r, p-b}, \quad (r, s_0 + kp)_{\mathcal{W}} \sim (r, s_0 + kp)_{\mathcal{W}}^* \sim k\hat{M}_{r \cdot k, p-s_0} \oplus (k+1)\hat{M}_{2 \cdot r \cdot k, s_0} \quad (4.32)$$

Proposition 3. The Grothendieck ring associated with $\mathcal{WLM}(1,p)$ is isomorphic with the contragredient, the \mathcal{W} -Kac and the fundamental fusion algebra modulo the equivalence relation \sim , that is,

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Grot}} \rangle \simeq \left(\langle \mathcal{J}_{\mathcal{W}}^{\text{Cont}} \rangle / \sim \right) \simeq \left(\langle \mathcal{J}_{\mathcal{W}}^{\text{Kac}} \rangle / \sim \right) \simeq \left(\langle \mathcal{J}_{\mathcal{W}}^{\text{Fund}} \rangle / \sim \right) \quad (4.33)$$

It is also isomorphic with the following polynomial rings

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Grot}} \rangle \simeq \mathbb{C}[X, Y] / (X^2 - 1, X - T_p(\frac{Y}{2})), \quad \langle \mathcal{J}_{\mathcal{W}}^{\text{Grot}} \rangle \simeq \mathbb{C}[Y] / ((Y^2 - 4)U_{p-1}^2(\frac{Y}{2})) \quad (4.34)$$

The isomorphisms in (4.34) are given by

$$G_{r,s} \leftrightarrow X^{r-1}U_{s-1}(\frac{Y}{2}), \quad G_{r,s} \leftrightarrow T_p^{r-1}(\frac{Y}{2})U_{s-1}(\frac{Y}{2}), \quad r \in \mathbb{Z}_{1,2}, s \in \mathbb{Z}_{1,p} \quad (4.35)$$

respectively.

Proof. That the Grothendieck ring is isomorphic with $\langle \mathcal{J}_{\mathcal{W}}^{\text{Fund}} \rangle / \sim$ was established in [68]. The other two isomorphisms in (4.33) correspond to an elevation of this result to the \mathcal{W} -Kac fusion algebra and further to the contragredient extension thereof. These elevations are established by applying

$$[(r, s_0 + kp)_{\mathcal{W}}] = [(r, s_0 + kp)_{\mathcal{W}}^*] = kG_{r \cdot k, p-s_0} + (k+1)G_{2 \cdot r \cdot k, s_0} \quad (4.36)$$

and the multiplication rule (4.30) to the fusion rules (3.38), (3.39) and (3.52) involving $(r, s_0 + kp)_{\mathcal{W}}$ or $(r, s_0 + kp)_{\mathcal{W}}^*$, where (4.36) itself is a consequence of (4.32).

To establish the first isomorphism in (4.34), we note that $\hat{\mathcal{R}}_1^1 \sim 2\hat{M}_{2,1} \oplus 2\hat{M}_{1,p-1}$ implies the equivalence relation

$$X \sim T_1(\frac{Y}{2})U_{p-1}(\frac{Y}{2}) - U_{p-2}(\frac{Y}{2}) = T_p(\frac{Y}{2}) \quad (4.37)$$

As a consequence, $(1, p+1)_{\mathcal{W}} \sim \hat{M}_{1,p-1} \oplus \hat{M}_{2,1}$ implies the equivalence relation

$$Z \sim U_{p-2}(\frac{Y}{2}) + 2X \sim U_{p-2}(\frac{Y}{2}) + 2T_p(\frac{Y}{2}) = U_p(\frac{Y}{2}) \quad (4.38)$$

We likewise have $Z^* \sim U_p(\frac{Y}{2})$. The polynomials $X - T_p(\frac{Y}{2})$, $Z - U_p(\frac{Y}{2})$ and $Z^* - U_p(\frac{Y}{2})$ are divisors of the polynomials $P_p(X, Y)$, $Q_p(Y, Z)$ and $Q_p(Y, Z^*)$, respectively, as defined in (4.5), and since $R_p(Y, Z, Z^*)$ is trivial modulo $Z - U_p(\frac{Y}{2})$, $Z^* - U_p(\frac{Y}{2})$, they eliminate the dependence on Z and Z^* in the polynomial ring in Proposition 2. They also simplify the polynomials associated with the (contragredient) \mathcal{W} -Kac representations as we have

$$\begin{aligned} (r, b + kp)_{\mathcal{W}} &\sim (r, b + kp)_{\mathcal{W}}^* \sim X^{r-1}U_{kp+b-1}(\frac{Y}{2}) \\ &\equiv kX^{r \cdot k-1}U_{p-b-1}(\frac{Y}{2}) + (k+1)X^{2 \cdot r \cdot k-1}U_{b-1}(\frac{Y}{2}) \quad (\text{mod } X^2 - 1, X - T_p(\frac{Y}{2})) \\ &\sim k\hat{M}_{r \cdot k, p-b} \oplus (k+1)\hat{M}_{2 \cdot r \cdot k, b} \end{aligned} \quad (4.39)$$

| | | | | |
|-----------|-----------|-----------|-----------------------|-----------------------|
| * | $G_{1,1}$ | $G_{2,1}$ | $G_{1,2}$ | $G_{2,2}$ |
| $G_{1,1}$ | $G_{1,1}$ | $G_{2,1}$ | $G_{1,2}$ | $G_{2,2}$ |
| $G_{2,1}$ | $G_{2,1}$ | $G_{1,1}$ | $G_{2,2}$ | $G_{1,2}$ |
| $G_{1,2}$ | $G_{1,2}$ | $G_{2,2}$ | $2G_{1,1} + 2G_{2,1}$ | $2G_{1,1} + 2G_{2,1}$ |
| $G_{2,2}$ | $G_{2,2}$ | $G_{1,2}$ | $2G_{1,1} + 2G_{2,1}$ | $2G_{1,1} + 2G_{2,1}$ |

| | | | | |
|----------------|----------------|----------------|----------------|----------------|
| * | 0 | 1 | $-\frac{1}{8}$ | $\frac{3}{8}$ |
| 0 | 0 | 1 | $-\frac{1}{8}$ | $\frac{3}{8}$ |
| 1 | 1 | 0 | $\frac{3}{8}$ | $-\frac{1}{8}$ |
| $-\frac{1}{8}$ | $-\frac{1}{8}$ | $\frac{3}{8}$ | $2(0) + 2(1)$ | $2(0) + 2(1)$ |
| $\frac{3}{8}$ | $\frac{3}{8}$ | $-\frac{1}{8}$ | $2(0) + 2(1)$ | $2(0) + 2(1)$ |

Figure 2: Cayley tables of the multiplication rules for $\langle \mathcal{J}_{\mathcal{W}}^{\text{Grot}} \rangle$ in $\mathcal{WLM}(1, 2)$. In the second table, the generators $G_{r,s}$ are represented by the conformal weights of the corresponding \mathcal{W} -irreducible modules.

where the polynomial equivalence follows by induction in k . Likewise, the polynomial realizations of the rank-2 modules simplify as

$$\begin{aligned}
\hat{\mathcal{R}}_r^b &\sim 2X^{r-1}T_b\left(\frac{Y}{2}\right)U_{p-1}\left(\frac{Y}{2}\right) \equiv 2X^{2\cdot r-1}U_{b-1}\left(\frac{Y}{2}\right) + 2X^{r-1}U_{p-b-1}\left(\frac{Y}{2}\right) \pmod{X - T_p\left(\frac{Y}{2}\right)} \\
&\sim 2\hat{M}_{2\cdot r,b} \oplus 2\hat{M}_{1\cdot r,p-b}
\end{aligned} \tag{4.40}$$

This completes the proof of the first isomorphism in (4.34). The reduction of the polynomial ring in the two variables X and Y in (4.34) to the polynomial ring in the single variable Y follows from

$$X^2 - 1 \equiv T_p^2\left(\frac{Y}{2}\right) - 1 = \frac{1}{4}(Y^2 - 4)U_{p-1}^2\left(\frac{Y}{2}\right) \pmod{X - T_p\left(\frac{Y}{2}\right)} \tag{4.41}$$

□

As illustration of the structure of the Grothendieck rings, we follow [68] and consider $\mathcal{WLM}(1, 2)$ whose four-dimensional Grothendieck ring

$$\langle \mathcal{J}_{\mathcal{W}}^{\text{Grot}} \rangle \simeq \mathbb{C}[Y]/(Y^4 - 4Y^2) \tag{4.42}$$

is generated by

$$G_{1,1} \leftrightarrow 1, \quad G_{1,2} \leftrightarrow Y, \quad G_{2,1} \leftrightarrow \frac{1}{2}Y^2 - 1, \quad G_{2,2} \leftrightarrow \frac{1}{2}Y^3 - Y \tag{4.43}$$

The multiplication rules are given in the Cayley tables in Figure 2.

5 Discussion

We have constructed new Yang-Baxter integrable boundary conditions giving rise to reducible yet indecomposable rank-1 representations in the \mathcal{W} -extended logarithmic minimal model $\mathcal{WLM}(1, p)$ where $p = 2, 3, \dots$. These \mathcal{W} -Kac representations $(r, s)_{\mathcal{W}}$ correspond to finitely-generated \mathcal{W} -extended Feigin-Fuchs modules over the W -algebra $\mathcal{W}(p)$, and their fusion properties were inferred from the fusion rules in the Virasoro picture $\mathcal{LM}(1, p)$ of the logarithmic minimal model. The contragredient modules $(r, s)_{\mathcal{W}}^*$ to the \mathcal{W} -Kac representations were also introduced, and the correspondingly-extended fusion algebra was derived. Polynomial fusion rings isomorphic with the various fusion algebras were subsequently determined, and the corresponding Grothendieck ring of characters was identified.

The results presented here pertain to the \mathcal{W} -extended logarithmic minimal models $\mathcal{WLM}(1, p)$ and are based on the work [56] on the same models in the Virasoro picture $\mathcal{LM}(1, p)$. The methods used to obtain the various results, on the other hand, are expected to be applicable also in the general cases $\mathcal{LM}(p, p')$ and $\mathcal{WLM}(p, p')$, at least after implementation of the disentangling procedure employed in [55] when extending the work [27] on $\mathcal{WLM}(1, p)$ to $\mathcal{WLM}(p, p')$. We hope to discuss these generalizations elsewhere, in particular for critical percolation as described by $\mathcal{LM}(2, 3)$ and $\mathcal{WLM}(2, 3)$.

As already mentioned, the category of $\mathcal{W}(p)$ -modules and the category of finite-dimensional $\bar{U}_q(sl_2)$ -modules at $q = e^{\pi i/p}$ are equivalent as abelian categories for all $p \geq 2$ [45–48, 53]. For $p \geq 3$, however, it was found [54] that they are not equivalent as braided tensor categories due to complications arising from the presence of the modules $\mathcal{E}_s^\pm(n; \lambda)$. These ‘circular’ modules were denoted by $\mathcal{O}_s^\pm(n, z)$ in [46] where they first appeared. We note that the subcategory of $\bar{U}_q(sl_2)$ -modules obtained by excluding these circular modules closes under tensor products. Likewise, we can define the ‘contragrediently-extended boundary $\mathcal{W}(p)$ -category’ as the subcategory of $\mathcal{W}(p)$ -modules associated with the $\mathcal{WLM}(1, p)$ boundary conditions constructed in [27] and in Section 3, supplemented by the \mathcal{W} -reducible yet \mathcal{W} -indecomposable contragredient \mathcal{W} -Kac representations (thus counting the \mathcal{W} -irreducible representations only once, cf. (3.44)). Our results then suggest that this contragrediently-extended boundary $\mathcal{W}(p)$ -category and the above subcategory of $\bar{U}_q(sl_2)$ -modules are equivalent as tensor categories. That is, these proposed subcategories are not only equivalent as abelian categories, but we have verified that their tensor structures are compatible. To facilitate this verification, we refer to the dictionary in Appendix B. Without extending the category of boundary $\mathcal{W}(p)$ -modules by the contragredient \mathcal{W} -Kac representations, the boundary category *itself* is equivalent to the corresponding subcategory of $\bar{U}_q(sl_2)$ -modules as tensor categories. These affirmative observations provide substantial evidence for the conjectured Kac fusion algebra of [56] and its elevation to the \mathcal{W} -extended picture discussed in the present work. They also support the Kazhdan-Lusztig dualities of [45–48] and [58, 59].

The recent works [69, 70] on the structure of bulk logarithmic CFTs and their relation with boundary logarithmic CFTs have greatly advanced our understanding of logarithmic CFT. However, these results are based on the ‘rational’ part of the \mathcal{W} -extended picture formed by the *finitely* many \mathcal{W} -irreducible and \mathcal{W} -projective modules on the boundary side. The present work suggests a much richer boundary model, even in the logarithmic minimal model $\mathcal{WLM}(1, p)$, and it would be interesting to readdress the relation with the bulk model in light of these new findings.

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A Fusion rules in $\mathcal{LM}(1, p)$

A.1 Kac fusion rules

From [56, 60], for $b, b' \in \mathbb{Z}_{0, p-1}$ and $k, k' \in \mathbb{N}_0$, we have the fusion rules

$$\begin{aligned}
(1, b + kp) \otimes (1, b' + k'p) &= \bigoplus_{j=|k-k'|+1, \text{ by 2}}^{k+k'-1} \bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_j^\beta \oplus \bigoplus_{j=|k-k'+\text{sg}(b-b')|+1, \text{ by 2}}^{k+k'} \bigoplus_{\beta}^{|b-b'|-1} \mathcal{R}_j^\beta \\
&\oplus \bigoplus_{\beta}^{b+b'-p-1} \mathcal{R}_{k+k'+1}^\beta \oplus \bigoplus_{\beta=|b-b'|+1, \text{ by 2}}^{p-|p-b-b'|-1} (1, \beta + (k+k')p) \\
\mathcal{R}_1^b \otimes (1, b' + k'p) &= \left(\bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_{k'}^\beta \oplus \bigoplus_{\beta}^{p-b-b'-1} \mathcal{R}_{k'}^\beta \oplus (1 - \delta_{k',0}) \bigoplus_{\beta}^{b-b'-1} (\mathcal{R}_{k'-1}^\beta \oplus \mathcal{R}_{k'+1}^\beta) \right. \\
&\oplus \left. \bigoplus_{\beta=|b-b'|+1, \text{ by 2}}^{p-|p-b-b'|-1} \mathcal{R}_{k'+1}^\beta \oplus 2 \left(\bigoplus_{\beta}^{b'-b-1} \mathcal{R}_{k'+1}^\beta \right) \oplus \bigoplus_{\beta}^{b+b'-p-1} \mathcal{R}_{k'+2}^\beta \right) / (1 + \delta_{b,0}) \\
\mathcal{R}_1^b \otimes \mathcal{R}_1^{b'} &= \left(\bigoplus_{\beta=|p-b-b'|+1, \text{ by 2}}^{p-|b-b'|-1} \mathcal{R}_1^\beta \oplus \bigoplus_{\beta}^{p-|b-b'|-1} \mathcal{R}_1^\beta \oplus 3 \left(\bigoplus_{\beta}^{p-b-b'-1} \mathcal{R}_1^\beta \right) \oplus \bigoplus_{\beta}^{|b-b'|-1} \mathcal{R}_2^\beta \right. \\
&\oplus \left. \bigoplus_{\beta}^{p-|p-b-b'|-1} \mathcal{R}_2^\beta \oplus \bigoplus_{\beta}^{b+b'-p-1} \mathcal{R}_3^\beta \right) / \{(1 + \delta_{b,0})(1 + \delta_{b',0})\} \tag{A.1}
\end{aligned}$$

where $\mathcal{R}_0^b \equiv 0$. The divisions by $(1 + \delta_{b,0})$, for example, ensure that the fusion rules for \mathcal{R}_1^0 match those for $(1, p)$. Due to (2.27), and using

$$(r, 1) \otimes (r', 1) = \bigoplus_{j=|r-r'|+1, \text{ by 2}}^{r+r'-1} (j, 1) \tag{A.2}$$

the complete set of fusion rules underlying the Kac fusion algebra is obtained straightforwardly. It can be found in [56].

A.2 Contragredient Kac fusion rules

Following [56], we introduce

$$\mathcal{C}_n[(r, s)] = \begin{cases} (r, s), & n > 0 \\ (r, s)^*, & n < 0 \end{cases} \tag{A.3}$$

In our applications, $\mathcal{C}_0[(r, s)]$ only appears if (r, s) is fully reducible in which case

$$\mathcal{C}_0[(r, s)] = (r, s) = (r, s)^*, \quad s \in \mathbb{Z}_{1, p-1} \cup p\mathbb{N} \tag{A.4}$$

The fusion rules involving contragredient Kac representations are given by or follow readily from

$$(r, s)^* \otimes (r', s')^* = ((r, s) \otimes (r', s'))^*, \quad \mathcal{R}_r^b \otimes (r', s')^* = \mathcal{R}_r^b \otimes (r', s') \tag{A.5}$$

and

$$\begin{aligned}
(1, b + kp) \otimes (1, b' + k'p)^* &= \bigoplus_{j=|k-k'|+2, \text{ by } 2}^{k+k'} \bigoplus_{\beta}^{p-|p-b-b'|-1} \mathcal{R}_j^\beta \oplus \bigoplus_{j=|k-k'|+1, \text{ by } 2}^{k+k'-\text{sg}(p-b-b')} \bigoplus_{\beta}^{|p-b-b'|-1} \mathcal{R}_j^\beta \\
&\oplus \bigoplus_{\beta}^{(b-b')\text{sg}(k'-k)-1} \mathcal{R}_{|k-k'|}^\beta \oplus \bigoplus_{\beta=|b-b'|+1, \text{ by } 2}^{p-|p-b-b'|-1} \mathcal{C}_{k-k'}[(1, \beta + |k - k'|p)] \quad (\text{A.6})
\end{aligned}$$

where $b, b' \in \mathbb{Z}_{0,p-1}$ and $k, k' \in \mathbb{N}_0$. Since $(r, 1)$ is irreducible, we thus have

$$(r, s)^* = (r, 1)^* \otimes (1, s)^* = (r, 1) \otimes (1, s)^* \quad (\text{A.7})$$

from which it follows that the general fusion product $(r, s) \otimes (r', s')^*$ can be computed as

$$(r, s) \otimes (r', s')^* = ((r, 1) \otimes (r', 1)) \otimes ((1, s) \otimes (1, s')^*) \quad (\text{A.8})$$

The complete set of fusion rules underlying the contragrediently-extended Kac fusion algebra can be found in [56].

B Dictionary

Here we present a dictionary for translating the notation used in [54] (and similarly in [45–49, 53]) for indecomposable quantum-group modules to the one employed here for \mathcal{W} -extended modules. For $s, s' \in \mathbb{Z}_{1,p}$, $a \in \mathbb{Z}_{1,p-1}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{X}_s^+ &\leftrightarrow (1, s)_{\mathcal{W}}, & \mathcal{X}_s^- &\leftrightarrow (2, s)_{\mathcal{W}} \\
\mathcal{P}_a^+ &\leftrightarrow \hat{\mathcal{R}}_1^{p-a}, & \mathcal{P}_a^- &\leftrightarrow \hat{\mathcal{R}}_2^{p-a} \\
\mathcal{M}_a^+(n) &\leftrightarrow (2 \cdot n, p - a + (n - 1)p)_{\mathcal{W}}, & \mathcal{M}_a^-(n) &\leftrightarrow (1 \cdot n, p - a + (n - 1)p)_{\mathcal{W}} \\
\mathcal{W}_a^+(n) &\leftrightarrow (1 \cdot n, a + (n - 1)p)_{\mathcal{W}}^*, & \mathcal{W}_a^-(n) &\leftrightarrow (2 \cdot n, a + (n - 1)p)_{\mathcal{W}}^*
\end{aligned} \quad (\text{B.1})$$

where

$$\mathcal{M}_{p-s}^\mp(1) = \mathcal{W}_s^\pm(1) = \mathcal{X}_s^\pm \quad (\text{B.2})$$

Direct sums over of the index sets $I_{s,s'}$ and $J_{s,s'}$ in [54] correspond to

$$\bigoplus_{t \in I_{s,s'}} A_t = \bigoplus_{t=|s-s'|+1, \text{ by } 2}^{p-|p-s-s'|-1} A_t, \quad \bigoplus_{t \in J_{s,s'}} A_t = \bigoplus_t^{s+s'-p-1} A_{p-t} \quad (\text{B.3})$$

that is,

$$\begin{aligned}
\bigoplus_{t \in I_{s,s'}} \mathcal{X}_t^\alpha &\leftrightarrow \bigoplus_{j=|s-s'|+1, \text{ by } 2}^{p-|p-s-s'|-1} (\alpha, j)_{\mathcal{W}} \\
\bigoplus_{t \in I_{s,s'}} \mathcal{P}_t^\alpha &\leftrightarrow \bigoplus_{\beta=|p-s-s'|+1, \text{ by } 2}^{p-|s-s'|-1} \hat{\mathcal{R}}_\alpha^\beta \\
\bigoplus_{t \in J_{s,s'}} \mathcal{P}_t^\alpha &\leftrightarrow \bigoplus_{\beta}^{s+s'-p-1} \hat{\mathcal{R}}_\alpha^\beta \quad (\text{B.4})
\end{aligned}$$

Here $\alpha = +$ ($\alpha = -$) on the quantum-group side corresponds to $\alpha = 1$ ($\alpha = 2$) on the logarithmic CFT side.

References

- [1] V. Gurarie, *Logarithmic operators in conformal field theory*, Nucl. Phys. **B410** (1993) 535–549, arXiv:hep-th/9303160.
- [2] M. Flohr, *Bits and pieces in logarithmic conformal field theory*, Int. J. Mod. Phys. **A18** (2003) 4497–4591, arXiv:hep-th/0111228.
- [3] M.R. Gaberdiel, *An algebraic approach to logarithmic conformal field theory*, Int. J. Mod. Phys. **A18** (2003) 4593–4638, arXiv:hep-th/0111260.
- [4] H. Saleur, *Conformal invariance for polymers and percolation*, J. Phys. A: Math. Gen. **20** (1987) 455–470.
- [5] B. Duplantier, *Exact critical exponents for two-dimensional dense polymers*, J. Phys. A: Math. Gen. **19** (1986) L1009–L1014.
- [6] H. Saleur, *Polymers and percolation in two dimensions and twisted $N = 2$ supersymmetry*, Nucl. Phys. **B382** (1992) 486–531, arXiv:hep-th/9111007.
- [7] P.A. Pearce, J. Rasmussen, *Solvable critical dense polymers*, J. Stat. Mech. (2007) P02015, arXiv:hep-th/0610273.
- [8] A. Nigro, *Integrals of motion for critical dense polymers and symplectic fermions*, J. Stat. Mech. (2009) P10007, arXiv:0903.5051 [hep-th].
- [9] P.A. Pearce, J. Rasmussen, S.P. Villani, *Solvable critical dense polymers on the cylinder*, J. Stat. Mech. (2010) P02010, arXiv:0910.4444 [hep-th].
- [10] J.L. Cardy, *Critical percolation in finite geometries*, J. Phys. A: Math. Gen. **25** (1992) L201–L206, arXiv:hep-th/9111026.
- [11] G.M.T. Watts, *A crossing probability for critical percolation in two dimensions*, J. Phys. A: Math. Gen. **29** (1996) L363–L368, arXiv:cond-mat/9603167.
- [12] J.L. Cardy, *Conformal invariance and percolation*, arXiv:math-ph/0103018.
- [13] S. Smirnov, *Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits*, Comptes Rendus de l’Académie des Sciences, Series I - Mathematics, **333** (2001) 239–244.
- [14] P. Mathieu, D. Ridout, *From percolation to logarithmic conformal field theory*, Phys. Lett. **B657** (2007) 120–129, arXiv:0708.0802 [hep-th].
- [15] D. Ridout, *On the percolation BCFT and the crossing probability of Watts*, Nucl. Phys. **B810** (2009) 503–526, arXiv:0808.3530 [hep-th].
- [16] J. Dubail, J.L. Jacobsen, H. Saleur, *Conformal field theory at central charge $c=0$: a measure of the indecomposability (b) parameters*, Nucl. Phys. **B834** (2010) 399–422, arXiv:1001.1151 [math-ph].
- [17] H.G. Kausch, *Curiosities at $c = -2$* , arXiv:hep-th/9510149.
- [18] H.G. Kausch, *Symplectic fermions*, Nucl. Phys. **B583** (2000) 513–541, arXiv:hep-th/0003029.
- [19] D. Dhar, *The Abelian sandpile and related models*, Physica **A263** (1999) 4–25.

- [20] S. Mahieu, P. Ruelle, *Scaling fields in the two-dimensional Abelian sandpile model*, Phys. Rev. **E64** (2001) 066130, arXiv:hep-th/0107150.
- [21] P. Ruelle, *A $c = -2$ boundary changing operator for the Abelian sandpile model*, Phys. Lett, **B539** (2002) 172–177, arXiv:hep-th/0203105.
- [22] S. Moghimi-Araghi, M.A. Rajabpour, S. Rouhani, *Abelian sandpile model: a conformal field theory point of view*, Nucl. Phys. **B718** (2005) 362–370, arXiv:cond-mat/0410434.
- [23] V.S. Poghosyan, S.Y. Grigorev, V.B. Priezzhev, P. Ruelle, *Pair correlations in sandpile model: a check of logarithmic conformal field theory*, Phys. Lett. **B659** (2008) 768–772, arXiv:0710.3051 [cond-mat.stat-mech].
- [24] P.A. Pearce, J. Rasmussen, J.-B. Zuber, *Logarithmic minimal models*, J. Stat. Mech. (2006) P11017, arXiv:hep-th/0607232.
- [25] J. Rasmussen, P.A. Pearce, *Fusion algebra of critical percolation*, J. Stat. Mech. (2007) P09002, arXiv:0706.2716 [hep-th].
- [26] J. Rasmussen, P.A. Pearce, *\mathcal{W} -extended fusion algebra of critical percolation*, J. Phys. A: Math. Theor. **41** (2008) 295208, arXiv:0804.4335 [hep-th].
- [27] P.A. Pearce, J. Rasmussen, P. Ruelle, *Integrable boundary conditions and \mathcal{W} -extended fusion in the logarithmic minimal models $\mathcal{LM}(1, p)$* , J. Phys. A: Math. Theor. **41** (2008) 295201, arXiv:0803.0785 [hep-th].
- [28] N. Read, H. Saleur, *Associative-algebraic approach to logarithmic conformal field theories*, Nucl. Phys. **B777** (2007) 316–351, arXiv:hep-th/0701117.
- [29] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the monster*, Proc. Natl. Acad. Sci. USA **83** (1986) 3068–3071.
- [30] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the monster*, Pure Appl. Math. **134**, Academic Press (1988).
- [31] V. Kac, *Vertex algebras for beginners*, Univ. Lecture Series **10**, 2nd edition, Amer. Math. Society (1998).
- [32] E. Frenkel, D. Ben-Zvi, *Vertex algebras and algebraic curves*, Math. Surveys and Monographs **88**, 2nd edition, Amer. Math. Society (2004).
- [33] Y.C. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996) 237–302.
- [34] H.G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, Phys. Lett. **B259** (1991) 448–455.
- [35] N. Carqueville, M. Flohr, *Nonmeromorphic operator product expansion and C_2 -cofiniteness for a family of \mathcal{W} -algebras*, J. Phys. A: Math. Gen. **39** (2006) 951–966, arXiv:math-ph/0508015.
- [36] D. Adamović, A. Milas, *On the triplet vertex algebra $\mathcal{W}(p)$* , Adv. Math. **217** (2008) 2664–2699, arXiv:0707.1857 [math.QA].

- [37] M.R. Gaberdiel, H.G. Kausch, *A rational logarithmic conformal field theory*, Phys. Lett. **B386** (1996) 131–137, arXiv:hep-th/9606050.
- [38] J. Fuchs, S. Hwang, A.M. Semikhatov, I.Yu. Tipunin, *Nonsemisimple fusion algebras and the Verlinde formula*, Commun. Math. Phys. **247** (2004) 713–742, arXiv:hep-th/0306274.
- [39] M.R. Gaberdiel, I. Runkel, *From boundary to bulk in logarithmic CFT*, J. Phys. A: Math. Theor. **41** (2008) 075402, arXiv:0707.0388 [hep-th].
- [40] A. Milas, *Weak modules and logarithmic intertwining operators for vertex operator algebras*, Contemp. Math. **297** (2002) 201–225, arXiv:math/0101167 [math.QA].
- [41] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C_2 -cofiniteness*, Duke Math. J. **122** (2004) 51–91, arXiv:math/0209101 [math.QA].
- [42] Y.-Z. Huang, J. Lepowsky, L. Zhang, *Logarithmic tensor product theory for generalized modules for a conformal vertex algebra*, arXiv:0710.2687 [math.QA].
- [43] Y.-Z. Huang, *Cofiniteness conditions, projective covers and the logarithmic tensor product theory*, J. Pure Appl. Alg. **213** (2009) 458–475, arXiv:0712.4109 [math.QA].
- [44] Y.-Z. Huang, *Representations of vertex operator algebras and braided finite tensor categories*, arXiv:0903.4233 [math.QA].
- [45] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Commun. Math. Phys. **265** (2006) 47–93, arXiv:hep-th/0504093.
- [46] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Kazhdan–Lusztig correspondence for the representation category of the triplet W -algebra in logarithmic CFT*, Theor. Math. Phys. **148** (2006) 1210–1235, arXiv:math/0512621.
- [47] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Logarithmic extensions of minimal models: characters and modular transformations*, Nucl. Phys. **B757** (2006) 303–343, arXiv:hep-th/0606196.
- [48] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, I.Yu. Tipunin, *Kazhdan–Lusztig-dual quantum group for logarithmic extensions of Virasoro minimal models*, J. Math. Phys. **48** (2007) 032303, arXiv:math/0606506.
- [49] R. Suter, *Modules over $U_q(sl_2)$* , Commun. Math. Phys. **163** (1994) 359–393.
- [50] V. Chari, A. Premet, *Indecomposable restricted representations of quantum sl_2* , Publ. RIMS **30** (1994) 335–352.
- [51] J. Xiao, *Finite dimensional representations of $U_t(sl_2)$ at root of unity*, Can. J. Math. **49** (1997) 772–787.
- [52] Y. Arike, *Symmetric linear functions of the restricted quantum group $\bar{U}_q sl_2(\mathbb{C})$* , arXiv:0706.1113 [math.QA].
- [53] K. Nagatomo, A. Tsuchiya, *The triplet vertex operator algebra $W(p)$ and the restricted quantum group $\bar{U}_q(sl_2)$ at $q = e^{\pi i/p}$* , arXiv:0902.4607 [math.QA].

- [54] H. Kondo, Y. Saito, *Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to sl_2* , Journal of Algebra **330** (2011) 103–129, arXiv:0901.4221 [math.QA].
- [55] J. Rasmussen, *\mathcal{W} -extended logarithmic minimal models*, Nucl. Phys. **B807** (2009) 495–533, arXiv:0805.2991 [hep-th].
- [56] J. Rasmussen, *Classification of Kac representations in the logarithmic minimal models $\mathcal{LM}(1, p)$* , arXiv:1012.5190 [hep-th].
- [57] B.L. Feigin, D.B. Fuchs, *Representations of the Virasoro algebra*, in *Representations of infinite-dimensional Lie groups and Lie algebras*, Gordon and Breach, New York (1989).
- [58] P.V. Bushlanov, A.M. Gainutdinov, I. Yu. Tipunin, *Kazhdan-Lusztig equivalence and fusion of Kac modules in Virasoro logarithmic models*, arXiv:1102.0271 [hep-th].
- [59] P.V. Bushlanov, B.L. Feigin, A.M. Gainutdinov, I. Yu. Tipunin, *Lusztig limit of quantum $sl(2)$ at root of unity and fusion of $(1, p)$ Virasoro logarithmic minimal models*, Nucl. Phys. **B818** (2009) 179–195, arXiv:0901.1602 [hep-th].
- [60] J. Rasmussen, P.A. Pearce, *Fusion algebras of logarithmic minimal models*, J. Phys. A: Math. Theor. **40** (2007) 13711–13733, arXiv:0707.3189 [hep-th].
- [61] M.R. Gaberdiel, H.G. Kausch, *Indecomposable fusion products*, Nucl. Phys. **B477** (1996) 293–318, arXiv:hep-th/9604026.
- [62] M.A.I. Flohr, *On fusion rules in logarithmic conformal field theories*, Int. J. Mod. Phys. **A12** (1997) 1943–1958, arXiv:hep-th/9605151.
- [63] J. Rasmussen, *Polynomial fusion rings of \mathcal{W} -extended logarithmic minimal models*, J. Math. Phys. **50** (2009) 043512, arXiv:0812.1070 [hep-th].
- [64] J. Rasmussen, *Fusion of irreducible modules in $\mathcal{WLM}(p, p')$* , J. Phys. A: Math. Theor. **43** (2010) 045210, arXiv:0906.5414 [hep-th].
- [65] J. Rasmussen, *Fusion matrices, generalized Verlinde formulas and partition functions in $\mathcal{WLM}(1, p)$* , J. Phys. A: Math. Theor. **43** (2010) 105201, arXiv:0908.2014 [hep-th].
- [66] J. Rasmussen, P.A. Pearce, *Polynomial fusion rings of logarithmic minimal models*, J. Phys. A: Math. Theor. **41** (2008) 175210, arXiv:0709.3337 [hep-th].
- [67] M.R. Gaberdiel, I. Runkel, S. Wood, *Fusion rules and boundary conditions in the $c = 0$ triplet model*, J. Phys. A: Math. Theor. **42** (2009) 325403, arXiv:0905.0916 [hep-th].
- [68] P.A. Pearce, J. Rasmussen, P. Ruelle, *Grothendieck ring and Verlinde-like formula for the \mathcal{W} -extended logarithmic minimal model $\mathcal{WLM}(1, p)$* , J. Phys. A: Math. Theor. **43** (2010) 045211, arXiv:0907.0134 [hep-th].
- [69] M.R. Gaberdiel, I. Runkel, S. Wood, *A modular invariant bulk theory for the $c = 0$ triplet model*, J. Phys. A: Math. Theor. **44** (2011) 015204, arXiv:1008.0082 [hep-th].
- [70] P.A. Pearce, J. Rasmussen, *Coset graphs in bulk and boundary logarithmic minimal models*, Nucl. Phys. **B846** (2011) 616–649, arXiv:1010.5328 [hep-th].