# Graded Hopf Maps and Fuzzy Superspheres

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**Abstract**

We argue supersymmetric generalizations of fuzzy two- and four-spheres based on the unitary-orthosymplectic algebras, $UOSp(N|2)$ and $UOSp(N|4)$, respectively. Supersymmetric version of Schwinger construction is applied to derive graded fully symmetric representation for fuzzy superspheres. As a classical counterpart of fuzzy superspheres, graded versions of 1st and 2nd Hopf maps are also introduced, and their basic geometrical structures are studied. It is shown that fuzzy superspheres are represented as a "superposition" of fuzzy superspheres with lower supersymmetries. We investigate algebraic structures of fuzzy two- and four-superspheres to identify $SU(2|N)$ and $SU(4|N)$ as their enhanced algebraic structures, respectively. Evaluation of correlation functions manifests such enhanced structure as quantum fluctuations of fuzzy supersphere.

# Contents

# 1 Introduction

About two decades ago, fuzzy two-sphere and field theory on it were formulated by Madore [1]. The fuzzy two-sphere is one of the simplest curved fuzzy manifolds whose coordinates satisfy the $SU(2)$ algebra. Few years after, Grosse et al. introduced four-dimensional fuzzy spheres [2] and supersymmetric (SUSY) generalizations of fuzzy spheres in sequel works [3, 4]. Field theory defined on fuzzy manifolds naturally contain a "cut-off", and such fuzzy field theory was expected to have weaker infinity than that of the conventional field theory. Furthermore in the developments of string theory in late 90's, researchers recognized that the geometry of D-branes is described by fuzzy geometry [5, 6, 7] (as reviews) and fuzzy manifolds arise as classical solutions of Matrix theory, e.g. [8, 9]. It is also known that fuzzy superspheres provide a set-up for field theory on SUSY lattice regularization [3, 10, 11], and realize as a classical solution of supermatrix model [12, 13]. For such important properties, fuzzy spheres and their variants have attracted a great deal of attentions [14, 15, ] (as reviews). Fuzzy physics also found its applications to gravity [18] and even to condensed matter physics [19, 20]. Recently, the mathematics of fuzzy geometry is applied to construction of topologically non-trivial many-body states on bosonic manifolds [21, 22, 23] and on supermanifolds [24, 25] as well.

In this paper, we apply close relations between fuzzy spheres and Hopf maps [26] to generalize fuzzy superspheres in higher dimensions. A useful mathematical tool for the construction is the Schwinger operator formalism [27, 16]. Specifically, the two-dimensional fuzzy sphere coordinates are simply obtained by sandwiching the Pauli matrices with two-component Schwinger operators:

$$X_i = \Phi^\dagger \sigma_i \Phi. \tag{1}$$

With the Schwinger operator, it is quite straightforward to derive fully symmetric representation, which corresponds to a finite number of states on fuzzy sphere. In general, a finite number of states on $2k$-dimensional fuzzy spheres are given by fully symmetric representation of $SO(2k+1)$ [28]. The Schwinger operator is regarded as the "square root" of the fuzzy sphere coordinates, and play fundamental roles rather than the fuzzy sphere coordinates themselves. Meanwhile, with $\phi$ denoting a normalized two-component complex spinor, the (1st) Hopf map is represented as

$$x_i = \phi^\dagger \sigma_i \phi. \tag{2}$$

Comparison between (1) and (2) finds that the (1st) Hopf map can be regarded as the "classical" counterpart of the (Schwinger) operator construction of fuzzy two-sphere.

In the construction of fuzzy superspheres [29, 30], nice algebraic structures and relations between the Hopf map and fuzzy sphere are inherited. The fuzzy two-superspheres[1] constructed by Grosse et al. [3, 4] are based on the $UOSp(1|2)$ algebra that includes $SU(2) \simeq USp(2)$:

$$SU(2) \subset UOSp(1|2). \tag{3}$$

(The classical counter part of the fuzzy two-supersphere, the graded 1st Hopf map, was first given in Refs.[31, 32]. See also Refs.[33, 30].) The coordinates of the fuzzy two-supersphere are introduced by replacing the $SU(2)$ Pauli matrices with the $UOSp(1|2)$ matrices of fundamental representation. As the $UOSp(1|2)$ contains the $SU(2)$ as its maximal bosonic subalgebra, the fuzzy two-supersphere "contains" the fuzzy two-sphere as its fuzzy body. The construction is based on the graded Lie algebra, and fuzzy super-geometry is transparent. We want to maintain such nice features. To this end, we utilize a graded Lie algebra whose maximal bosonic subalgebra is $SO(5)$. The minimal graded Lie algebra that suffices for this requirement is $UOSp(1|4)$, since $SO(5) \simeq USp(4)$:

$$SO(5) \subset UOSp(1|4). \tag{4}$$

We adopt $UOSp(1|4)$ version of Schwinger operator in the construction of fuzzy four-supersphere and also introduce the graded 2nd Hopf map as its classical counterpart. We further extend such formulation to include more supersymmetries with use of $UOSp(N|2)$ and $UOSp(N|4)$. Representation theory of the graded Lie algebra is rather complicated, however if restricted to graded fully symmetric representation[2], investigations are greatly simplified. By dealing with the Schwinger operator as fundamental quantity, we observe "enhancement" of symmetry of fuzzy superspheres. This mechanism is similar to the symmetry enhancement reported in higher dimensional fuzzy spheres [35, 36]. We also reconsider such enhancement in view of quantum fluctuations of fuzzy superspheres.

Some comments are added to clarify difference to related works. In Ref.[37], supersymmetric Hopf maps were introduced in the context of SUSY non-linear sigma models. In the construction, the fermionic parts are introduced to incorporate $N = 4$ supersymmetry. Though the bosonic parts are related to Hopf maps, the fermionic parts themselves are not directly related. In the present construction, together with bosonic components, the fermionic components themselves constitute graded Hopf maps. Supersymmetric quantum mechanics in monopole background related to the Hopf map is well investigated recently [38, 39, 40, 41, 42, 43, 44]. Works about higher dimensional fuzzy super-manifolds of which the author is aware are Ref.[45, 46, 47]. The fuzzy complex projective space was constructed in Ref.[45] based on the super unitary algebra.

[1] In this paper, two-supersphere is referred to as the supersphere whose body is two-dimensional sphere. The two-supersphere with $N$ supersymmetry is denoted as $S^{2|2N}$ whose bosonic dimension is two and the fermionic dimension is $2N$, and hence the total dimension is $2+2N$. Similarly, fuzzy four-supersphere consists of four-sphere body and extra fermionic coordinates.

[2] We adopt the terminology, "graded fully symmetric representation" to indicate a representation constructed by a supersymmetric version of Schwinger operator. The graded fully symmetric representation is totally symmetric for the bosonic part and totally *antisymmetric* for the fermionic part. It is also referred to as harmonic oscillator representation in several literatures. For general representation theory of graded Lie groups, one may for instance consult Ref.[34] and references therein.

Such construction is similar to the spirit of the present work, and is indeed closed related as we shall discuss. In [47], fuzzy superspheres are formulated in any dimensions. However, the fuzzy two-supersphere provided by the formulation is not same as of Grosse et al. In the present, though the construction is restricted to two and four-dimensions, the underlying algebraic geometry is transparent and the fuzzy two-supersphere of Grosse et al. is naturally reproduced.

The paper is organized as follows. In Sec.2, we briefly introduce the unitary-orthosymplectic algebra $UOSp(N|M)$. In Sec.3, we review the construction of fuzzy two-supersphere as well as 1st graded Hopf map. $N=2$ fuzzy two-supersphere and the corresponding 1st graded Hopf map are provided, too. In Sec.4, we argue construction of $N=1$ and $N=2$ fuzzy superspheres and the graded 2nd Hopf maps. More supersymmetric extensions are explored in Sec.5. In Sec.6, we give supercoherent states on fuzzy two- and four-superspheres and investigate quantum fluctuations of fuzzy superspheres. Sec.7 is devoted to summary and discussions.

## 2 $UOSp(N|M)$

Generators of the orthosymplectic algebra, $OSp(N|M)$, are defined so as to satisfy

$$\Sigma_{AB}^{st}\begin{pmatrix} J & 0 \\ 0 & 1_N \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & 1_N \end{pmatrix}\Sigma_{AB} = 0, \tag{5}$$

where $1_N$ denotes $N \times N$ unit matrix and $J$ represents the invariant matrix of the symplectic group

$$J = \begin{pmatrix} 0 & 1_{M/2} \\ -1_{M/2} & 0 \end{pmatrix}, \tag{6}$$

and the supertranspose, $st$, is defined as

$$\begin{pmatrix} B & F \\ F' & B' \end{pmatrix}^{st} \equiv \begin{pmatrix} B^t & F'^t \\ -F^t & B'^t \end{pmatrix}. \tag{7}$$

Here, $t$ stands for the ordinary transpose, and $B$ and $B'$ signify bosonic components while $F$ and $F'$ fermionic components. $\Sigma_{AB}$ can be expressed by a linear combination of

$$\Sigma_{\alpha\beta} = \begin{pmatrix} \sigma_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_{lm} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{lm} \end{pmatrix}, \quad \Sigma_{l\alpha} = \begin{pmatrix} 0 & \sigma_{l\alpha} \\ -(J\sigma_{l\alpha})^t & 0 \end{pmatrix}, \tag{8}$$

where $\alpha, \beta$ are the indices of $Sp(M)$ $(\alpha, \beta = 1, 2, \cdots, M)$ and $l, m$ those of $O(N)$ $(l, m = 1, 2, \cdots, N)$. $\sigma_{l\alpha}$ denote arbitrary $M \times N$ matrices, while $\sigma_{\alpha\beta}$ and $\sigma_{lm}$ signify $M \times M$ and $N \times N$ matrices that respectively satisfy

$$\sigma_{lm}{}^t + \sigma_{lm} = 0, \tag{9a}$$

$$\sigma_{\alpha\beta}{}^t J + J\sigma_{\alpha\beta} = 0. \tag{9b}$$

The $OSp(M|N)$ algebra contains the maximal bosonic subalgebra, $Sp(M) \oplus O(N)$, whose generators are $\Sigma_{\alpha\beta}$ and $\Sigma_{lm}$. The off-diagonal block matrices $\Sigma_{l\alpha}$ are called fermionic generators that transform as fundamental representation under each of $Sp(M)$ and $O(N)$. Then, the $SO(N)$ matrix $\sigma_{lm}$ is an antisymmetric real matrix (9a) with real degrees of freedom $N(N-1)/2$. The indices of $\sigma_{lm}$ can be taken to be antisymmetric, $\sigma_{lm} = -\sigma_{ml}$. Meanwhile, from the relation (9b) $\sigma_{\alpha\beta}$ takes the form of

$$\sigma_{\alpha\beta} = \begin{pmatrix} k & s \\ s' & -k^t \end{pmatrix}, \tag{10}$$

where $k$ stands for a $M/2 \times M/2$ complex matrix, and $s$ and $s'$ are $M/2 \times M/2$ symmetric complex matrices. If the hermiticity condition instead of the reality condition is imposed, $\sigma_{\alpha\beta}$ are reduced to generators of $USp(M)$ and take the form of

$$\sigma_{\alpha\beta} = \begin{pmatrix} h & s \\ s^\dagger & -h^* \end{pmatrix}, \tag{11}$$

where $h$ represents hermitian matrix and $s$ symmetric complex matrix. The real independent degrees of freedom of $\sigma_{\alpha\beta}$ is $M(M+1)/2$. Then, for $USp(M)$, the indices can be taken to be symmetric, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$. Meanwhile, the real degrees of freedom of the fermionic generators $\Sigma_{l\alpha}$ is $MN$. As a result, th real degrees of freedom of $UOSp(N|M)$ are given by

$$\dim[UOSp(N|M)] = \frac{1}{2}(M^2 + N^2 + M - N)|MN = \frac{1}{2}((M+N)^2 + M - N). \tag{12}$$

There are isometries between the unitary-symplectic and orthogonal algebras only for

$$USp(2) \simeq SO(3), \qquad USp(4) \simeq SO(5),$$

Taking advantage of the isomorphism, we construct fuzzy two- and four-superspheres based on $UOSp(N|2)$ and $UOSp(N|4)$.

# 3 Graded 1st Hopf maps and fuzzy two-superspheres

Here, we review relations between fuzzy two-sphere and 1st Hopf map, and their supersymmetric version. We also explore a construction of $N = 2$ fuzzy supersphere with use of typical representation of $UOSp(2|2)$ algebra.

## 3.1 The 1st Hopf map and fuzzy two-sphere

To begin with, we introduce relations between fuzzy two-sphere and 1st Hopf map

$$S^3 \xrightarrow{S^1} S^2. \tag{13}$$

With a normalized complex two-component spinor $\phi = (\phi_1, \phi_2)^t$ subject to $\phi^\dagger\phi = 1$, the 1st Hopf map is realized as

$$\phi \to x_i = \phi^\dagger \sigma_i \phi, \tag{14}$$

where $\sigma_i$ $(i = 1, 2, 3)$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

$\phi$ is regarded as coordinates on $S^3$ from the normalization condition, and $x_i$ denote coordinates of $S^2$:

$$x_i x_i = (\phi^\dagger \phi)^2 = 1. \tag{16}$$

Coordinates of fuzzy two-sphere $S_F^2$ are constructed as

$$X_i = \Phi^\dagger \sigma_i \Phi, \tag{17}$$

where $\Phi = (\Phi_1, \Phi_2)^t$ stands for two-component Schwinger operator that satisfies $[\Phi_\alpha, \Phi_\beta^\dagger] = \delta_{\alpha\beta}$ and $[\Phi_\alpha, \Phi_\beta] = 0$ $(\alpha, \beta = 1, 2)$. Usually, in front of the right-hand side of (17), the non-commutative parameter of dimension of length is added, however for notational brevity, we omit it throughout the paper. $X_i$ satisfy

$$[X_i, X_j] = 2i\epsilon_{ijk} X_k, \tag{18}$$

and square of the radius of fuzzy two-sphere is given by

$$X_i X_i = (\Phi^\dagger \Phi)(\Phi^\dagger \Phi + 2) = \hat{n}(\hat{n} + 2). \tag{19}$$

Here, $\hat{n}$ is the number operator $\hat{n} = \Phi^\dagger \Phi$ and its eigenvalues are non-negative integers that specify fully symmetric representation. The fully symmetric representation is simply obtained by acting the Schwinger operators to the vacuum:

$$|l_1, l_2\rangle = \frac{1}{\sqrt{l_1!\, l_2!}} {\Phi_1^\dagger}^{l_1} {\Phi_2^\dagger}^{l_2} |0\rangle, \tag{20}$$

where $l_1$ and $l_2$ are non-negative integers satisfying $l_1 + l_2 = n$. Physically, $|l_1, l_2\rangle$ represent a finite number of states on fuzzy two-sphere, and their 3rd-components are

$$X_3 = l_1 - l_2 = n - 2k, \tag{21}$$

where $k = l_2 = 0, 1, 2, \cdots, n$. The dimension of (20) is

$$d(n) = n + 1. \tag{22}$$

The Hopf map (14) is regarded as a classical counterpart of the Schwinger construction of fuzzy sphere (17) with the replacement

$$\Phi \to \phi, \qquad \Phi^\dagger \to \phi^*, \tag{23}$$

and (19) is reduced to (16) except for the "zero-point energy", stemming from the non-commutativity of two bosonic components of the Schwinger operator.

### 3.2 $N = 1$ fuzzy two-supersphere

Here, we extend the above discussions to the graded 1st Hopf map [31, 32] and $N = 1$ fuzzy two-supersphere [3, 4] along Refs.[30, 33].

#### 3.2.1 $UOSp(1|2)$ algebra

The $UOSp(1|2)$ algebra contains the $SU(2)$ algebra as its maximal bosonic subalgebra, and consists of five generators three of which are bosonic $L_i$ ($i = 1, 2, 3$) and two of which are fermionic $L_\alpha$ ($\alpha = \theta_1, \theta_2$). They satisfy

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, L_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_\beta, \quad \{L_\alpha, L_\beta\} = \frac{1}{2}(\epsilon\sigma_i)_{\alpha\beta}L_i, \tag{24}$$

where $\epsilon = i\sigma_2$ is the $SU(2)$ charge conjugation matrix. One may find that $L_i$ transform as an $SU(2)$ vector, while $L_\alpha$ an $SU(2)$ spinor. The $UOSp(1|2)$ Casimir is constructed as

$$\mathcal{C} = L_iL_i + \epsilon_{\alpha\beta}L_\alpha L_\beta, \tag{25}$$

and its eigenvalues are given by $j(j+1/2)$ with $j$ referred to as superspin that takes non-negative integers and half-integers, $j = 0, 1, 2, 1, 3/2, \cdots$. The $UOSp(1|2)$ irreducible representation specified by the superspin index $j$ consists of $SU(2)$ $j$ and $j-1/2$ spin representations and hence the dimension of the $UOSp(1|2)$ representation with superspin $j = n/2$ is

$$d(n) + d(n-1) = 2n + 1, \tag{26}$$

where $d(n)$ is the dimension of the $SU(2)$ spin $n/2$ (22). For $UOSp(1|M)$, there exists a "square root" of the Casimir, the Scasimir [48, 34]. In the present, Scasimir is given by

$$\mathcal{S} = 2\epsilon_{\alpha\beta}L_\alpha L_\beta - \frac{1}{4}, \tag{27}$$

which satisfies

$$\mathcal{S}^2 = \mathcal{C} + \frac{1}{16}. \tag{28}$$

Then, the eigenvalues of Scasimir are $\pm j(j+1/4)$. The Scasimir is commutative with the bosonic generators and anticommutative with the fermionic ones,

$$[L_i, \mathcal{S}] = \{L_\alpha, \mathcal{S}\} = 0. \tag{29}$$

#### 3.2.2 $N = 1$ graded 1st Hopf map

The graded 1st Hopf map is given by

$$S^{3|2} \xrightarrow{S^1} S^{2|2}, \tag{30}$$

where left index to the slash indicates the number of bosonic coordinates, while the right index fermionic coordinates. The bosonic part of (30) is exactly equivalent to the 1st Hopf map. The coordinates on the total manifold $S^{3|2}$ is represented by a normalized three-component superspinor

$\psi = (\psi_1, \psi_2, \eta)^t$ whose first two components are Grassmann even and the third component is Grassmann odd. A normalization condition is imposed as

$$\psi^\ddagger \psi = \psi_1^* \psi_1 + \psi_2^* \psi_2 - \eta^* \eta = 1, \tag{31}$$

where $\psi^\ddagger = (\psi_1^*, \psi_2^*, -\eta^*)$ and $*$ represents the pseudo-conjugation[3]. The graded 1st Hopf map is realized as [31, 32]

$$\psi \quad \rightarrow \quad x_i = 2\psi^\ddagger L_i \psi, \quad \theta_\alpha = 2\psi^\ddagger L_\alpha \psi, \tag{32}$$

where $L_i$ and $L_\alpha$ are the fundamental representation matrices of $UOSp(1|2)$

$$L_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad L_\alpha = \frac{1}{2}\begin{pmatrix} 0_2 & \tau_\alpha \\ -(\epsilon\tau_\alpha)^t & 0 \end{pmatrix}, \tag{33}$$

with $\epsilon = i\sigma_2$, $\tau_1 = (1,0)^t$ and $\tau_2 = (0,1)^t$. They are "hermitian" in the sense

$$L_i^\ddagger = L_i, \quad L_\alpha^\ddagger = \epsilon_{\alpha\beta} L_\beta, \tag{34}$$

where $\ddagger$ is the super-adjoint defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\ddagger = \begin{pmatrix} A^\dagger & C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}. \tag{35}$$

From (32), we see that $x_i$ and $\theta_\alpha$ are coordinates on $S^{2|2}$:

$$x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = (\psi^\ddagger\psi)^2 = 1, \tag{36}$$

and from (34),

$$x_i^* = x_i, \quad \theta_\alpha^* = \epsilon_{\alpha\beta}\theta_\beta. \tag{37}$$

Notice that $x_i$ are Grassmann even but not usual c-number, since the square of $x_i$ is not c-number as observed in (36). Instead, we can introduce c-number $y_i$ as

$$y_i = \frac{1}{\sqrt{1 - \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta}}\, x_i, \tag{38}$$

which satisfy $y_i y_i = 1$ and denote coordinates on $S^2$, the body of $S^{2|2}$. The original normalized $SU(2)$ spinor is "embedded" in $\psi$ as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{2}{2 + \eta^*\eta}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{39}$$

With $y_i$, $\phi$ can be written as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2(1+y_3)}}\begin{pmatrix} 1 + y_3 \\ y_1 + iy_2 \end{pmatrix} e^{i\chi}, \tag{40}$$

[3]The pseudo-conjugation is imposed as $(\eta^*)^* = -\eta$ and $(\eta_1\eta_2)^* = \eta_1^*\eta_2^*$ for Grassmann odd quantities. See Ref.[34] for instance.

where $e^{i\chi}$ denotes arbitrary $U(1)$ phase. Represent the Grassmann odd component $\eta$ as

$$\eta = \phi_1 \mu + \phi_2 \nu, \tag{41}$$

with $\mu$ and $\nu$ being real and imaginary components of $\eta$. They satisfy

$$\mu^* = \nu, \qquad \nu^* = -\mu. \tag{42}$$

The map (32) immediately determines the relations between $\theta_1$, $\theta_2$ and $\mu$, $\nu$:

$$\mu = \theta_1, \qquad \nu = \theta_2. \tag{43}$$

Consequently, $\psi$ can be expressed as

$$\begin{aligned}\psi &= \frac{1}{\sqrt{1-\eta^*\eta}}\begin{pmatrix}\phi_1\\ \phi_2\\ \eta\end{pmatrix} = \frac{1}{\sqrt{1+\theta_1\theta_2}}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_1\theta_1+\phi_2\theta_2\end{pmatrix}\\ &= \frac{1}{\sqrt{2(1+y_3)(1+\theta_1\theta_2)}}\begin{pmatrix}1+y_3\\ y_1+iy_2\\ (1+y_3)\theta_1+(y_1+iy_2)\theta_2\end{pmatrix}e^{i\chi}.\end{aligned} \tag{44}$$

The last expression on the right-hand side manifests the $N=1$ graded Hopf fibration, $S^{3|2} \sim S^{2|2} \otimes S^1$: the $S^1(\simeq U(1))$-fibre, $e^{i\chi}$, is canceled in the graded Hopf map (32), and the remaining quantities, $y_i$ and $\theta_\alpha$, correspond to the coordinates on $S^{2|2}$.

#### 3.2.3 $N=1$ fuzzy two-supersphere

Coordinates on fuzzy supersphere are constructed by the graded version of the Schwinger construction[4] [29]:

$$X_i = 2\Psi^\dagger L_i \Psi, \quad \Theta_\alpha = 2\Psi^\dagger L_\alpha \Psi, \tag{45}$$

where $\Psi$ stands for a graded Schwinger operator

$$\Psi = (\Psi_1, \Psi_2, \tilde{\Psi})^t, \tag{46}$$

with bosonic operators $\Psi_1$ and $\Psi_2$ and fermionic one $\tilde{\Psi}$ satisfying

$$\begin{aligned}&[\Psi_\alpha, \Psi_\beta^\dagger] = \delta_{\alpha\beta}, \quad \{\tilde{\Psi}, \tilde{\Psi}^\dagger\} = 1, \quad [\Psi_\alpha, \tilde{\Psi}^\dagger] = 0,\\ &[\Psi_\alpha, \Psi_\beta] = \{\tilde{\Psi}, \tilde{\Psi}\} = [\Psi_\alpha, \tilde{\Psi}] = 0.\end{aligned} \tag{47}$$

It is straightforward to see that (45) satisfy the algebra

$$[X_i, X_j] = 2i\epsilon_{ijk}X_k, \qquad [X_i, \Theta_\alpha] = (\sigma_i)_{\beta\alpha}\Theta_\beta, \qquad \{\Theta_\alpha, \Theta_\beta\} = (\epsilon\sigma_i)_{\alpha\beta}X_i. \tag{48}$$

Radius of the square of fuzzy supersphere is given by the $UOSp(1|2)$ Casimir

$$X_iX_i + \epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta = (\Psi^\dagger\Psi)(\Psi^\dagger\Psi + 1), \tag{49}$$

[4] In (45) we adopted the ordinary definition of the Hermitian conjugate $\dagger$, so $\Theta_\alpha^\dagger \neq \epsilon_{\alpha\beta}\Theta_\beta$ unlike $\theta_\alpha^* = \epsilon_{\alpha\beta}\theta_\beta$.

where we used

$$X_i X_i = \hat{n}_B(\hat{n}_B + 2),$$
$$\epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta = -\hat{n}_B + 2\hat{n}_B\hat{n}_F + 2\hat{n}_F, \tag{50}$$

with $\hat{n}_B = \Psi_1^\dagger\Psi_1 + \Psi_2^\dagger\Psi_2$, $\hat{n}_F = \tilde{\Psi}^\dagger\tilde{\Psi}$, and $\hat{n}_F^2 = \hat{n}_F$. $\Psi^\dagger\Psi$ denotes the total number-operator $\hat{n} = \Psi^\dagger\Psi = \hat{n}_B + \hat{n}_F$. Notice the zero-point energy in (49) reflects the difference between the bosonic and fermionic degrees of freedom of the Schwinger operator. The Scasimir is also expressed as

$$\mathcal{S} = (\hat{n}_F - \frac{1}{2})(\hat{n} + \frac{1}{2}). \tag{51}$$

From (49) and (51), one may readily show (28).

Graded fully symmetric representation specified by the superspin $j = n/2$ is given by

$$|l_1, l_2\rangle = \frac{1}{\sqrt{l_1!\ l_2!}}\Psi_1^{\dagger\, l_1}\Psi_2^{\dagger\, l_2}|0\rangle, \tag{52a}$$
$$|m_1, m_2) = \frac{1}{\sqrt{m_1!\ m_2!}}\Psi_1^{\dagger\, m_1}\Psi_2^{\dagger\, m_2}\tilde{\Psi}^\dagger|0\rangle, \tag{52b}$$

where $l_1 + l_2 = m_1 + m_2 + 1 = n$ with non-negative integers, $l_1, l_2, m_1$ and $m_2$. $|m_1, m_2)$ are the fermionic counterpart of $|l_1, l_2\rangle$, and thus they exhibit $N = 1$ SUSY. The bosonic and fermionic states[5] are classified by the sign of Scasimir (51). Scasimir takes the values

$$\mathcal{S} = \pm\frac{1}{4}(2n + 1), \tag{53}$$

with $+$ and $-$ for the bosonic (52a) and fermionic (52b) states, respectively. The degrees of freedom of bosonic and fermionic states are respectively

$$d_B = d(n) = n + 1, \quad d_F = d(n-1) = n, \tag{54}$$

and then the total degrees of freedom is

$$d_T = d_B + d_F = 2n + 1. \tag{55}$$

$X_3$-coordinates of these states are

$$X_3 = n - k, \tag{56}$$

where $k = 0, 1, 2, \cdots, 2n$. For even $k$, the eigenvalues of $X_3$ correspond to the bosonic states (52a), while for odd $k$, the fermionic states (52b). Compare the $X_3$ eigenvalues of fuzzy supersphere (56) and those of the fuzzy (bosonic) sphere (21): the degrees of freedom of fuzzy supersphere for even $k$ are accounted for those of fuzzy sphere with radius $n$, while those for odd $k$ are for those of fuzzy sphere with radius $n - 1$. Thus, the bosonic and fermionic degrees of freedom are same as of the fuzzy spheres with radius $n$ and radius $n - 1$, respectively. Consequently, the fuzzy

[5] In the paper, the bosonic and fermionic states refer to states with even and odd number of fermion operators, respectively. They are eigenstates of the fermion parity $(-1)^{\hat{n}_F}$ with the eigenvalues $+1$ and $-1$.

two-supersphere of radius $n$ is intuitively understood as a "superposition" of two fuzzy spheres whose radii are $n$ and $n-1$. Schematically,

$$S_F^{2|2}(n) \simeq S_F^2(n) \oplus S_F^2(n-1). \tag{57}$$

It is noted that though we only utilized the $UOSp(1|2)$ algebra, fuzzy two-supersphere itself is invariant under the larger $SU(2|1)$ symmetry: indeed, the right-hand side of (49) is invariant under the $SU(2|1)$ rotation of the Schwinger operator $\Psi$. In this sense, the symmetry of fuzzy two-supersphere is $SU(2|1)$ rather than $UOSp(1|2)$. Also notice that the graded fully symmetric representation (52) is regarded as a (atypical) representation of $SU(2|1)$.

### 3.3 $N = 2$ fuzzy two-supersphere

We utilized the $UOSp(1|2)$ algebra to construct $N = 1$ fuzzy supersphere $S_F^{2|2}$. Here, we apply $UOSp(2|2)$ algebra to construct $N = 2$ fuzzy supersphere $S_F^{2|4}$.

#### 3.3.1 $UOSp(2|2)$ algebra

$UOSp(2|2)$ algebra contains the $USp(2) \simeq SU(2)$ and $O(2) \simeq U(1)$ as bosonic algebras, and the fermionic generators transform as a $SU(2)$ spinor and carry $U(1)$ charge as well. $UOSp(2|2)$ is isomorphic to $SU(2|1)$, and its dimension is

$$\dim[UOSp(2|2)] = \dim[SU(2|1)] = 4|4 = 8. \tag{58}$$

We denote the four bosonic generators as $L_i$ $(i = 1, 2, 3)$ and $\Gamma$, and the four fermionic generators as $L_\alpha$ and $L'_\alpha$ $(\alpha = \theta_1, \theta_2)$. The $UOSp(2|2)$ algebra is given by

$$\begin{aligned}
&[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, L_{\alpha\sigma}] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_{\beta\sigma}, \quad \{L_{\alpha\sigma}, L_{\beta\tau}\} = \frac{1}{2}\delta_{\sigma\tau}(\epsilon\sigma_i)_{\alpha\beta}L_i + \frac{1}{2}\epsilon_{\sigma\tau}\epsilon_{\alpha\beta}\Gamma, \\
&[\Gamma, L_i] = 0, \qquad [\Gamma, L_{\alpha\sigma}] = \frac{1}{2}\epsilon_{\tau\sigma}L_{\alpha\tau},
\end{aligned} \tag{59}$$

where $L_{\alpha\sigma} = (L_\alpha, L'_\alpha)$ [6]. $L_i$ and $L_\alpha$ form the $UOSp(1|2)$ subalgebra. There are two sets of fermionic generators, $L_\alpha$ and $L'_\alpha$, which bring $N = 2$ SUSY. The fundamental representation is 3 dimensional representation, as expected from $UOSp(2|2) \simeq SU(2|1)$. The $UOSp(2|2)$ algebra has two Casimirs, quadratic and cubic [49]. The quadratic Casimir is given by

$$\mathcal{C} = L_iL_i + \epsilon_{\alpha\beta}L_\alpha L_\beta + \epsilon_{\alpha\beta}L'_\alpha L'_\beta + \Gamma^2. \tag{61}$$

The irreducible representation is classified into two categories; typical representation and atypical representation (see Appendix A.1.1 for details). Since the Casimir eigenvalues (61) are identically

[6] The algebra (59) coincides with the $UOSp(2|2)$ algebra usually found in literature by the following redefinitions,

$$L_i \to L_i, \quad L_\alpha \to L_\alpha, \quad L'_\alpha \to iD_\alpha, \quad \Gamma \to -i\Gamma. \tag{60}$$

zero for atypical representation, we utilize typical representation to construct $N=2$ fuzzy two-superspheres. The minimal dimensional matrices of typical representation is given by the following $4\times 4$ matrices:

$$L_i=\frac{1}{2}\begin{pmatrix}\sigma_i & 0_2\\ 0_2 & 0_2\end{pmatrix},\quad L_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & \tau_\alpha & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad L'_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & 0 & \tau_\alpha\\ 0 & 0 & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\end{pmatrix},\quad \Gamma=\frac{1}{2}\begin{pmatrix}0_2 & 0_2\\ 0_2 & \epsilon\end{pmatrix}. \tag{62}$$

(These are equivalent to those given in Ref.[49].)

#### 3.3.2 $N=2$ fuzzy two-supersphere

Applying the Schwinger construction to (62), we introduce $N=2$ fuzzy supersphere coordinates as

$$X_i=2\Psi^\dagger L_i\Psi,\qquad \Theta_\alpha=2\Psi^\dagger L_\alpha\Psi,\qquad \Theta'_\alpha=2\Psi^\dagger L'_\alpha\Psi,\qquad G=2\Psi^\dagger\Gamma\Psi, \tag{63}$$

where $\Psi$ denotes the four-component Schwinger operator

$$\Psi=(\Psi_1,\Psi_2,\tilde{\Psi}_1,\tilde{\Psi}_2)^t. \tag{64}$$

$\Psi_\alpha$ ($\alpha=1,2$) are bosonic operators while $\tilde{\Psi}_\sigma$ ($\sigma=1,2$) are fermionic ones satisfying

$$\begin{aligned}&[\Psi_\alpha,\Psi_\beta^\dagger]=\delta_{\alpha\beta},\quad \{\tilde{\Psi}_\sigma,\tilde{\Psi}_\tau^\dagger\}=\delta_{\sigma\tau},\\ &[\Psi_\alpha,\Psi_\beta]=\{\tilde{\Psi}_\sigma,\tilde{\Psi}_\tau\}=[\Psi_\alpha,\tilde{\Psi}_\sigma]=0.\end{aligned} \tag{65}$$

Square of the radius of $N=2$ fuzzy two-supersphere is evaluated as

$$X_iX_i+\epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta+\epsilon_{\alpha\beta}\Theta'_\alpha\Theta'_\beta+G^2=(\Psi^\dagger\Psi)^2. \tag{66}$$

Here, we used

$$\begin{aligned}&X_iX_i=\hat{n}_B(\hat{n}_B+2),\\ &\epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta+\epsilon_{\alpha\beta}\Theta'_\alpha\Theta'_\beta=-\hat{n}_B+2\hat{n}_B\hat{n}_F+2\hat{n}_F,\\ &G^2=4\hat{n}_F(\hat{n}_F-2),\end{aligned} \tag{67}$$

where $\hat{n}_B=\sum_{\alpha=1}^2\Psi_\alpha^\dagger\Psi_\alpha$, $\hat{n}_F=\sum_{\sigma=1}^2\tilde{\Psi}_\sigma^\dagger\tilde{\Psi}_\sigma$. For $\Psi^\dagger\Psi=n$, the graded fully symmetric representation is derived as

$$|l_1,l_2\rangle=\frac{1}{\sqrt{l_1!\;l_2!}}{\Psi_1^\dagger}^{l_1}{\Psi_2^\dagger}^{l_2}|0\rangle, \tag{68a}$$

$$|m_1,m_2)=\frac{1}{\sqrt{m_1!\;m_2!}}{\Psi_1^\dagger}^{m_1}{\Psi_2^\dagger}^{m_2}\tilde{\Psi}_1^\dagger|0\rangle, \tag{68b}$$

$$|m'_1,m'_2)=\frac{1}{\sqrt{m'_1!\;m'_2!}}{\Psi_1^\dagger}^{m'_1}{\Psi_2^\dagger}^{m'_2}\tilde{\Psi}_2^\dagger|0\rangle, \tag{68c}$$

$$|n_1,n_2\rangle=\frac{1}{\sqrt{n_1!\;n_2!}}{\Psi_1^\dagger}^{n_1}{\Psi_2^\dagger}^{n_2}\tilde{\Psi}_1^\dagger\tilde{\Psi}_2^\dagger|0\rangle, \tag{68d}$$

where $l_1 + l_2 = m_1 + m_2 + 1 = m'_1 + m'_2 + 1 = n_1 + n_2 + 2 = n$ with non-negative integers, $l_1$, $l_2$, $m_1$, $m_2$, $m'_1$, $m'_2$, $n_1$, $n_2$. We have two sets of bosonic states, $|l_1, l_2\rangle$ and $|n_1, n_2\rangle$, and two sets of fermionic states, $|m_1, m_2)$ and $|m'_1, m'_2)$ as well. The degrees of freedom of bosonic and fermionic states are equally given by

$$\begin{aligned} d_B &= d(n) + d(n-2) = 2n, \\ d_F &= 2 \times d(n-1) = 2n, \end{aligned} \tag{69}$$

with $d(n) = n + 1$, and the total is

$$d_T = d_B + d_F = 4n. \tag{70}$$

Square of the radius of $N = 2$ fuzzy two-supersphere (66) does not have the zero-pint energy since the bosonic and fermionic degrees of freedom are equal. The first two sets, (68a) and (68b), are $UOSp(1|2)$ $j = n/2$ irreducible representations, and the other two, (68c) and (68d), are $UOSp(1|2)$ $j = n/2 - 1/2$. In this sense, the $N = 2$ fuzzy two-supersphere with radius $n$ is regarded as a "superposition" of two $N = 1$ fuzzy superspheres whose radii are $n$ and $n - 1$. Remember that $N = 1$ fuzzy two-supersphere can be regarded as a superposition of two bosonic fuzzy spheres. Consequently, $N = 2$ fuzzy sphere is realized as a superposition of four fuzzy spheres whose radii are $n$, $n - 1$, $n - 1$ and $n - 2$. Schematically,

$$\begin{aligned} S_F^{2|4}(n) &\simeq S_F^{2|2}(n) \oplus S_F^{2|2}(n-1) \\ &\simeq S_F^2(n) \oplus S_F^2(n-1) \oplus S_F^2(n-1) \oplus S_F^2(n-2). \end{aligned} \tag{71}$$

Notice that such particular feature is a consequence of the adoption of graded fully symmetric representation. The corresponding latitudes of the states (68) are given by

$$X_3 = n - k \tag{72}$$

with $k = 0, 1, 2, \cdots, 2n$. The even $k$ correspond to the bosonic states, (68a) and (68d), while odd $k$ the fermionic states, (68b) and (68c). Except for non-degenerate states at the north and south poles $X_3 = \pm n$, the eigenvalues of $X_3$ (72) are doubly-degenerate.

Since the right-hand side of (66) is invariant under the $SU(2|2)$ rotation of $\Psi$, the symmetry of $N = 2$ fuzzy two-supersphere is considered as $SU(2|2)$ rather than $UOSp(2|2)$.

#### 3.3.3 $N = 2$ graded 1st Hopf map

Based on the Schwinger construction of $N = 2$ fuzzy two-supersphere, we introduce $N = 2$ version of the graded 1st Hopf map. With (62), we define

$$x_i = 2\psi^\ddagger L_i \psi, \quad \theta_\alpha = 2\psi^\ddagger L_\alpha \psi, \quad \theta'_\alpha = 2\psi^\ddagger L'_\alpha \psi, \quad g = \psi^\ddagger \Gamma \psi. \tag{73}$$

Here, $\psi$ denotes a four-component spinor $\psi = (\psi_1, \psi_2, \eta_1, \eta_2)^t$ normalized as

$$\psi^\ddagger \psi = \psi_1^* \psi_1 + \psi_2^* \psi_2 - \eta_1^* \eta_1 - \eta_2^* \eta_2 = 1, \tag{74}$$

and then is regarded as coordinates on $S^{3|4}$. The coordinates (73) satisfy the relation

$$x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta + \epsilon_{\alpha\beta}\theta'_\alpha\theta'_\beta + g^2 = (\psi^\ddagger\psi) = 1. \tag{75}$$

Notice all of the quantities (73) are not independent[7]. This can typically be seen from $\theta_1\theta_2\theta'_1\theta'_2 g = 0$. (If $\theta_\alpha$, $\theta'_\alpha$ and $g$ were independent, their product would not be zero.) Rewrite $\psi$ as

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \eta_1\\ \eta_2\end{pmatrix} = \sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2}\begin{pmatrix}\phi_1\\ \phi_2\\ \sqrt{1-\eta_2^*\eta_2}\ \eta_1\\ \sqrt{1-\eta_1^*\eta_1}\ \eta_2\end{pmatrix}, \tag{76}$$

where $(\phi_1,\phi_2)$ denotes the normalized $SU(2)$ spinor (40). Also, we express $\eta_1$ and $\eta_2$ as

$$\begin{aligned}\eta_1 &= \phi_1\mu_1 + \phi_2\nu_1,\\ \eta_2 &= \phi_1\mu_2 + \phi_2\nu_2,\end{aligned} \tag{77}$$

where $\mu_1$ $\mu_2$ represent the real parts of the Grassmann odd quantities and $\nu_1$ $\nu_2$ represent the imaginary parts. The map (73) determines the relations between $\mu_{1,2}$, $\nu_{1,2}$ and $\theta_{1,2}$, $\theta'_{1,2}$ as

$$\begin{aligned}\theta_1 &= \sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2}\ \mu_1, &\qquad \theta_2 &= \sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2}\ \nu_1,\\ \theta'_1 &= \sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2}\ \mu_2, &\qquad \theta'_2 &= \sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2}\ \nu_2.\end{aligned} \tag{78}$$

Then,

$$\theta_1\theta_2 + \theta'_1\theta'_2 = -(\eta_1^*\eta_1+\eta_2^*\eta_2)(1+\eta_1^*\eta_1+\eta_2^*\eta_2) = -\frac{\eta_1^*\eta_1+\eta_2^*\eta_2}{1-\eta_1^*\eta_1-\eta_2^*\eta_2}, \tag{79}$$

or inversely,

$$\eta_1^*\eta_1+\eta_2^*\eta_2 = -\frac{\theta_1\theta_2+\theta'_1\theta'_2}{1-\theta_1\theta_2-\theta'_1\theta'_2} = -\theta_1\theta_2 - \theta'_1\theta'_2 - 2\theta_1\theta_2\theta'_1\theta'_2. \tag{80}$$

Therefore, from (78) and (80), $\mu_1$, $\nu_1$, $\mu_2$ and $\nu_2$ are represented as

$$\begin{aligned}\mu_1 &= \frac{1}{\sqrt{1-\theta'_1\theta'_2}}\theta_1, &\qquad \nu_1 &= \frac{1}{\sqrt{1-\theta'_1\theta'_2}}\theta_2,\\ \mu_2 &= \frac{1}{\sqrt{1-\theta_1\theta_2}}\theta'_1, &\qquad \nu_2 &= \frac{1}{\sqrt{1-\theta_1\theta_2}}\theta'_2.\end{aligned} \tag{81}$$

Consequently, $\psi$ is given by

$$\psi = \frac{1}{\sqrt{2(1+y_3)(1+\theta_1\theta_2+\theta'_1\theta'_2+4\theta_1\theta_2\theta'_1\theta'_2)}}\begin{pmatrix}1+y_3\\ y_1+iy_2\\ (1+\theta'_1\theta'_2)(\theta_1(1+y_3)+\theta_2(y_1+iy_2))\\ (1+\theta_1\theta_2)(\theta'_1(1+y_3)+\theta'_2(y_1+iy_2))\end{pmatrix}e^{i\chi}, \tag{82}$$

[7] This situation is similar to Schwinger construction of fuzzy complex projective space. The coordinates on fuzzy $CP^{N-1}$ are represented by the $SU(N)$ generators sandwiched by Schwinger operators. Though the real dimension of $CP^{N-1}$ is $2N-2$, the dimension of $SU(N)$ generator is $N^2-1$. This "discrepancy" is resolved by noticing all of the $SU(N)$ generators in the Schwinger construction are not independent and satisfy a set of constraints. See [50] for more details.

where $e^{i\chi}$ denotes arbitrary $U(1)$ phase factor. $x_i$ and $y_i$ are related as

$$y_i = \frac{1}{1+\eta_1^*\eta_1+\eta_2^*\eta_2}x_i = \frac{1}{1-\theta_1\theta_2-\theta'_1\theta'_2-2\theta_1\theta_2\theta'_1\theta'_2}x_i. \tag{83}$$

Thus, $\psi$ can be expressed by $x_i$, $\theta_\alpha$, $\theta'_\alpha$, the coordinates on $S^{2|4}$, and arbitrary $U(1)$ phase factor. Obviously, the $U(1)(\simeq S^1)$ phase is canceled in (73). Then, the bilinear map (73) represents

$$S^{3|4} \ \overset{S^1}{\longrightarrow} \ S^{2|4}, \tag{84}$$

which we call the $N=2$ graded 1st Hopf map. We have four bosonic and four fermionic coordinates in (73), but $g=-\eta_1^*\eta_2+\eta_2^*\eta_1$ is a redundant coordinate. Indeed, with (82), $g$ is expressed by $y_i$, $\theta_\alpha$ and $\theta'_\alpha$ as

$$g = y_1(\theta_1\theta'_1-\theta_2\theta'_2) - iy_2(\theta_1\theta'_1+\theta_2\theta'_2) - y_3(\theta_1\theta'_2+\theta_2\theta'_1). \tag{85}$$

It can also be shown that the following "renormalization",

$$\begin{aligned} & x_i \rightarrow \sqrt{1-g^2}\ x_i = (1-\frac{1}{2}g^2)x_i, \qquad \theta_\alpha \rightarrow \sqrt{1-g^2}\ \theta_\alpha = \theta_\alpha, \\ & \theta'_\alpha \rightarrow \sqrt{1-g^2}\ \theta'_\alpha = \theta'_\alpha, \end{aligned} \tag{86}$$

eliminates $g$: the renormalized coordinates satisfy the ordinary condition of $S^{2|4}$,

$$x_ix_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta + \epsilon_{\alpha\beta}\theta'_\alpha\theta'_\beta = 1. \tag{87}$$

One might attempt to introduce more supersymmetry. In principle, it is probable to do so by utilizing $UOSp(N|2)$ algebras for $N \geq 3$. However, the radius of the $N=2$ fuzzy two-supersphere (66) already saturates the "classical bound" (75). In general, square of the radius of fuzzy supersphere with $N$-SUSY is proportional to $n(n+2-N)$ and becomes negative for "sufficiently small" $n$ that satisfies $n < N-2$. Hence we stop at $N=2$.

# 4 Graded 2nd Hopf maps and fuzzy four-superspheres

In this section, we extend the previous formulation to fuzzy four-supersphere.

## 4.1 The 2nd Hopf map and fuzzy four-sphere

The 2nd Hopf map

$$S^7 \ \overset{S^3}{\longrightarrow} \ S^4 \tag{88}$$

is represented as

$$\phi \ \ \rightarrow \ \ x_a = \phi^\dagger\gamma_a\phi, \tag{89}$$

where $\phi = (\phi_1,\phi_2,\phi_3,\phi_4)^t$ is a normalized four-component complex spinor $\phi^\dagger\phi = 1$, representing coordinates on $S^7$. $\gamma_a$ $(a=1,2,3,4,5)$ are $SO(5)$ gamma matrices that satisfy $\{\gamma_a,\gamma_b\} = 2\delta_{ab}$

with Kronecker delta $\delta_{ab}$. $\gamma_a$ can be taken as

$$\gamma_1 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix},$$
$$\gamma_4 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \tag{90}$$

where $1_2$ denotes $2 \times 2$ unit matrix. From (89), we have

$$x_a x_a = (\phi^\dagger \phi)^2 = 1. \tag{91}$$

Thus, $x_a$ (89) are coordinates on four-sphere.

Coordinates on fuzzy four-sphere $S_F^4$ are constructed as [2]

$$X_a = \Phi^\dagger \gamma_a \Phi, \tag{92}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^t$ represents a four-component Schwinger operator satisfying $[\Phi_\alpha, \Phi_\beta^\dagger] = \delta_{\alpha\beta}$ and $[\Phi_\alpha, \Phi_\beta] = 0$ $(\alpha, \beta = 1, 2, 3, 4)$. Square of the radius of fuzzy four-sphere is derived as

$$X_a X_a = (\Phi^\dagger \Phi)(\Phi^\dagger \Phi + 4). \tag{93}$$

The zero-point energy corresponds to the number of the four-components of the Schwinger operator. Let $n$ be the eigenvalues of the number operator $\hat{n} = \Phi^\dagger \Phi$. The corresponding eigenstates are fully symmetric representation:

$$|l_1, l_2, l_3, l_4\rangle = \frac{1}{\sqrt{l_1!\, l_2!\, l_3!\, l_4!}} \Phi_1^{\dagger\, l_1} \Phi_2^{\dagger\, l_2} \Phi_3^{\dagger\, l_3} \Phi_4^{\dagger\, l_4} |0\rangle, \tag{94}$$

with $l_1 + l_2 + l_3 + l_4 = n$ for non-negative integers $l_1$, $l_2$, $l_3$, $l_4$. The degeneracy is

$$D(n) = \frac{1}{3!}(n+1)(n+2)(n+3). \tag{95}$$

Notice, for the fully symmetric representation, square of the radius (93) is equal to the $SO(5)$ Casimir:

$$X_a X_a = (\Phi^\dagger \Phi)(\Phi^\dagger \Phi + 4) = 2 X_{ab} X_{ab}, \tag{96}$$

where $X_{ab}$ are the $SO(5)$ generators given by

$$[X_a, X_b] = 4i X_{ab}, \tag{97}$$

with

$$\gamma_{ab} = -i\frac{1}{4}[\gamma_a, \gamma_b]. \tag{98}$$

$\gamma_{ab}$ are explicitly

$$\gamma_{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma_{13} = \frac{1}{2}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \gamma_{14} = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix},$$
$$\gamma_{15} = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_{23} = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma_{24} = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},$$
$$\gamma_{25} = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma_{34} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma_{35} = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$
$$\gamma_{45} = \frac{1}{2}\begin{pmatrix} 0 & i1_2 \\ -i1_2 & 0 \end{pmatrix}. \tag{99}$$

Inversely, the sum of $SO(5)$ generators can be "converted" to that of gamma matrices as long as the fully symmetric representation is adopted. Such conversion is crucial in constructing fuzzy four-superspheres as we shall see.

In total, the fifteen operators, $X_a$ and $X_{ab}$, satisfy a closed algebra:

$$[X_a, X_b] = 4iX_{ab}, \quad [X_a, X_{bc}] = -i(\delta_{ab}X_c - \delta_{ac}X_b),$$
$$[X_{ab}, X_{cd}] = i(\delta_{ac}X_{bd} - \delta_{ad}X_{bc} + \delta_{bc}X_{ad} - \delta_{bd}X_{ac}). \tag{100}$$

By identifying $X_{a6} = \frac{1}{2}X_a$ and $X_{ab} = X_{ab}$, one may find that (100) is equivalent to $SO(6) \simeq SU(4)$ algebra,

$$[X_{AB}, X_{CD}] = i(\delta_{AC}X_{BD} - \delta_{AD}X_{BC} + \delta_{BC}X_{AD} - \delta_{BD}X_{AC}), \tag{101}$$

where $A, B = 1, 2, \ldots, 6$. Thus, the underlying algebra of fuzzy foursphere is considered as $SU(4)$. The $SU(4)$ structure of the fuzzy four-sphere can also be deduced from the $SU(4)$ invariance of the right-hand side of (93). The states $|l_1, l_2, l_3, l_4\rangle$ (94) ring the four-sphere at latitudes

$$X_5 = n - 2k, \tag{102}$$

where $k = 0, 1, 2, \cdots, n$, and is related to $l_1$, $l_2$, $l_3$, $l_4$ as

$$k = l_3 + l_4 = n - l_1 - l_2 \tag{103}$$

or

$$l_1 + l_2 = n - k, \qquad l_3 + l_4 = k. \tag{104}$$

From (104), one may find, unlike the fuzzy two-sphere case, at $X_5 = n - 2k$, there is degeneracy

$$D_k(n) = d(n-k) \cdot d(k) = (n-k+1)(k+1), \tag{105}$$

where $d(k)$ is the number of the states on fuzzy two-sphere (22). (95) is reproduced as

$$D(n) = \sum_{k=0}^{n} D_k(n) = \sum_{k=0}^{n} d(n-k) \cdot d(k). \tag{106}$$

With increase of $k$, $D_k(n)$ monotonically increases from the north-pole to the equator $k = n/2$, and monotonically decreases from the equator to the south-pole. $D_k(n)$ is symmetric under $k \leftrightarrow n-k$, which corresponds to the inversion symmetry of sphere with respect to the equator. Since $d(n-k)$ and $d(k)$ represent the degrees of freedom of fuzzy two-spheres with radii $n-k$ and $k$, respectively, (105) and (106) imply the "internal" degrees of freedom of fuzzy four-sphere: fuzzy four-sphere is constituted of four-sphere and fibre consisting of two fuzzy two-spheres (whose radii are $(n+X_5)/2$ and $(n-X_5)/2$ at the latitude $X_5$). Schematically,

$$S_F^4(n)|_{X_5=n-2k} \simeq S_F^2(n-k) \otimes S_F^2(k). \tag{107}$$

In particular, at the north-pole, *i.e.* $X_5 = n$, we have only one fuzzy two-sphere fibre with radius $n$: $S_F^4(n)|_{X_5=n} \simeq S_F^2(n)$. Coordinates of the two "internal" fuzzy two-spheres are respectively given by

$$R_i = \frac{1}{2}\epsilon_{ijk}X_{jk} + X_{i4}, \qquad R'_i = \frac{1}{2}\epsilon_{ijk}X_{jk} - X_{i4}. \tag{108}$$

They satisfy

$$[R_i, R_j] = -2i\epsilon_{ijk}R_k, \qquad [R'_i, R'_j] = 2i\epsilon_{ijk}R'_k, \qquad [R_i, R'_j] = 0. \tag{109}$$

Then, naturally, $|l_1, l_2, l_3, l_4\rangle$ are regarded as the states on the fuzzy manifold spanned by $X_a$ and $X_{ab}$. The three independent quantities of $l_1$, $l_2$, $l_3$, $l_4$, specify three latitudes of the four-sphere and two "internal" fuzzy two-spheres:

$$\begin{aligned} X_5 &= l_1 + l_2 - l_3 - l_4, \\ R_3 &= l_1 - l_2, \\ R'_3 &= l_3 - l_4. \end{aligned} \tag{110}$$

Inversely, $|l_1, l_2, l_3, l_4\rangle$ is uniquely specified by the eigenvalues of $X_5$, $X_{12}$ and $X_{34}$:

$$\begin{aligned} l_1 &= \frac{1}{4}n + \frac{1}{4}X_5 + \frac{1}{2}R_3, \qquad l_2 = \frac{1}{4}n + \frac{1}{4}X_5 - \frac{1}{2}R_3, \\ l_3 &= \frac{1}{4}n - \frac{1}{4}X_5 + \frac{1}{2}R'_3, \qquad l_4 = \frac{1}{4}n - \frac{1}{4}X_5 - \frac{1}{2}R'_3. \end{aligned} \tag{111}$$

Thus, as emphasized in Refs.[35, 36], the fuzzy four-sphere has such "extra-fuzzy space" that does not have counterpart in the original four-sphere[8]. The existence of th fuzzy fibre $S_F^2$ can naturally be understood in the context of the 2nd Hopf map. The $SO(5)$ spinor $\phi$ denotes coordinates on $S^7 \sim S^4 \otimes S^3$, and the $U(1)$ phase of $\phi$ is factored out to obtain $\mathbb{C}P^3 \simeq S^7/S^1 \sim S^4 \otimes S^2$ [52]: we have $S^2$-fibred $S^4$ as the classical counterpart of $S_F^4$, not just $S^4$. Such enhancement mechanism is inherited to the supersymmetric cases.

### 4.2 $N = 1$ fuzzy four-supersphere

Here, we utilize $UOSp(1|4)$ algebra to construct fuzzy four-superspheres with $N = 1$ SUSY.

[8] One could truncate the extra fuzzy spaces, however in such a case, non-associative product has to be implemented [51].

#### 4.2.1 $UOSp(1|4)$ algebra

The $UOSp(1|4)$ algebra is constituted of fourteen generators, ten of which are bosonic $\Gamma_{ab} = -\Gamma_{ba}$ $(a, b = 1, 2, \cdots, 5)$, and the remaining four are fermionic $\Gamma_\alpha$ $(\alpha = 1, 2, 3, 4)$,

$$\dim[UOSp(1|4)] = 10|4 = 14. \tag{112}$$

The $UOSp(1|4)$ algebra is given by

$$\begin{aligned}
&[\Gamma_{ab}, \Gamma_{cd}] = i(\delta_{ac}\Gamma_{bd} - \delta_{ad}\Gamma_{bc} - \delta_{bc}\Gamma_{ad} + \delta_{bd}\Gamma_{ac}), \\
&[\Gamma_{ab}, \Gamma_\alpha] = (\gamma_{ab})_{\beta\alpha}\Gamma_\beta, \\
&\{\Gamma_\alpha, \Gamma_\beta\} = \sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab},
\end{aligned} \tag{113}$$

where $C$ is the $SO(5)$ charge conjugation matrix

$$C = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \tag{114}$$

with $\epsilon = i\sigma_2$ (see Appendix B for detail properties of $C$). $\Gamma_{ab}$ act as $SO(5)$ generators and $\Gamma_\alpha$ as a $SO(5)$ spinor. The $UOSp(1|4)$ quadratic Casimir is given by

$$\mathcal{C} = \sum_{a<b}\Gamma_{ab}\Gamma_{ab} + C_{\alpha\beta}\Gamma_\alpha\Gamma_\beta, \tag{115}$$

and Scasimir is

$$\mathcal{S} = \frac{1}{\sqrt{2}}(C_{\alpha\beta}\Gamma_\alpha\Gamma_\beta - \frac{3}{4}). \tag{116}$$

Similar to the $UOSp(1|2)$ case, the Scasimir satisfies

$$[\Gamma_{ab}, \mathcal{S}] = \{\Gamma_\alpha, \mathcal{S}\} = 0, \tag{117}$$

and

$$\mathcal{S}^2 = \mathcal{C} + \frac{9}{8}. \tag{118}$$

The fundamental representation matrices are constructed as follows. First, we introduce

$$\Gamma_a = \begin{pmatrix} \gamma_a & 0 \\ 0 & 0 \end{pmatrix} \tag{119}$$

with $\gamma_a$ (90), to yield $SO(5)$ generators

$$\Gamma_{ab} = -i\frac{1}{4}[\Gamma_a, \Gamma_b], \tag{120}$$

or

$$\Gamma_{ab} = \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{121}$$

with $\gamma_{ab}$ (99). The fermionic generators are

$$\Gamma_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 0_4 & \tau_\alpha \\ -(C\tau_\alpha)^t & 0 \end{pmatrix}, \tag{122}$$

where

$$\tau_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{123}$$

More explicitly,

$$\Gamma_{\theta_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_{\theta_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_{\theta_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma_{\theta_4} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{124}$$

They satisfy the "hermiticity" condition

$$\Gamma_a^\ddagger = \Gamma_a, \qquad \Gamma_{ab}^\ddagger = \Gamma_{ab}, \qquad \Gamma_\alpha^\ddagger = C_{\alpha\beta}\Gamma_\beta. \tag{125}$$

#### 4.2.2 $N = 1$ graded 2nd Hopf map

Generalizing the procedure in Sec.3.2.2, we construct $N = 1$ graded version of the 2nd Hopf map. We first introduce an $UOSp(1|4)$ spinor

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \eta)^t, \tag{126}$$

where $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$, are Grassmann even while $\eta$ is Grassmann odd. $\psi$ is normalized as

$$\psi^\ddagger \psi = 1, \tag{127}$$

where

$$\psi^\ddagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*, -\eta^*) \tag{128}$$

with pseudo-complex conjugation $*$. From (127), we find that $\psi$ denotes coordinates on $S^{7|2}$. We give $N = 1$ graded 2nd Hopf map as

$$\psi \quad \longrightarrow \quad x_a = \psi^\ddagger \Gamma_a \psi, \quad \theta_\alpha = \psi^\ddagger \Gamma_\alpha \psi, \tag{129}$$

where $\Gamma_a$ and $\Gamma_\alpha$ are (119) and (124), respectively. In detail,

$$\begin{aligned}
x_1 &= i\psi_1^*\psi_4 + i\psi_2^*\psi_3 - i\psi_3^*\psi_2 - i\psi_4^*\psi_1,\\
x_2 &= \psi_1^*\psi_4 - \psi_2^*\psi_3 - \psi_3^*\psi_2 + \psi_4^*\psi_1,\\
x_3 &= i\psi_1^*\psi_3 - i\psi_2^*\psi_4 - i\psi_3^*\psi_1 + i\psi_4^*\psi_2,\\
x_4 &= \psi_1^*\psi_3 + \psi_2^*\psi_4 + \psi_3^*\psi_1 + \psi_4^*\psi_2,\\
x_5 &= \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4,\\
\theta_1 &= \frac{1}{\sqrt{2}}(\psi_1^*\eta - \eta^*\psi_2),\\
\theta_2 &= \frac{1}{\sqrt{2}}(\psi_2^*\eta + \eta^*\psi_1),\\
\theta_3 &= \frac{1}{\sqrt{2}}(\psi_3^*\eta - \eta^*\psi_4),\\
\theta_4 &= \frac{1}{\sqrt{2}}(\psi_4^*\eta + \eta^*\psi_3).
\end{aligned} \tag{130}$$

From $(\eta^*)^* = -\eta$, we have $x_a^* = x_a$ and $\theta_\alpha^* = C_{\alpha\beta}\theta_\beta$. It is straightforward to see

$$x_a x_a + 2C_{\alpha\beta}\theta_\alpha\theta_\beta = (\psi^\ddagger\psi)^2 = 1. \tag{131}$$

If $x_a$ and $\theta_\alpha$ were independent, (131) was the definition of four-supersphere with four (pseudo-real) fermionic coordinates, $S^{4|4}$. However, $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ are *not* independent to each other, since they are constructed from only one Grassmann odd quantity $\eta$ that carries two real (Grassmann odd) degrees of freedom. Indeed,

$$\begin{aligned}
\theta_1\theta_2 &= -\frac{1}{2}\eta^*\eta(\psi_1^*\psi_1 + \psi_2^*\psi_2),\\
\theta_3\theta_4 &= -\frac{1}{2}\eta^*\eta(\psi_3^*\psi_3 + \psi_4^*\psi_4),
\end{aligned} \tag{132}$$

and then, for instance, $\theta_1\theta_2\theta_3 = 0$. Also we find

$$C_{\alpha\beta}\theta_\alpha\theta_\beta = -\eta^*\eta(\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4) = -\eta^*\eta, \tag{133}$$

and the relation (131) can be rewritten as

$$x_a x_a - 2\eta^*\eta = 1, \tag{134}$$

which corresponds to $S^{4|2}$. Thus, $x_a$ and $\theta_\alpha$ are regarded as coordinates on $S^{4|2}$ rather than $S^{4|4}$. As a consequence, (129) represents

$$S^{7|2} \quad \xrightarrow{S^3} \quad S^{4|2} \subset S^{4|4}. \tag{135}$$

The cancellation of $S^3$ can be understood by the following arguments. The original normalized $SO(5)$ spinor is embedded in the $UOSp(1|4)$ spinor as

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{\sqrt{1+\eta^*\eta}} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{136}$$

From (127), the normalization of $\phi$ follows

$$\phi^\dagger \phi = 1. \tag{137}$$

Then, the map

$$\phi \quad \rightarrow \quad y_a = \phi^\dagger \gamma_a \phi, \tag{138}$$

signifies the 2nd Hopf map (88). $y_a$ are coordinates of $S^4$; the body of $S^{4|2}$. With $y_a$, $\phi$ is expressed as

$$\phi = \frac{1}{\sqrt{2(1+y_5)}} \begin{pmatrix} (1+y_5) \begin{pmatrix} u \\ v \end{pmatrix} \\ (y_4 - i y_i \sigma_i) \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix}, \tag{139}$$

where $(u,v)^t$ is an arbitrary two-component spinor subject to the normalization $u^* u + v^* v = 1$ representing $S^3$-fibre. Such $S^3$-fibre is canceled in (138) to yield the coordinates on $S^4$. In the graded 2nd Hopf map (130), the cancellation of $S^3$ can also be shown. Write the Grassmann odd component $\eta$ as

$$\eta = u\mu + v\nu, \tag{140}$$

with $\mu$ and $\nu$ being real and imaginary Grassmann odd quantities that satisfy

$$\mu^* = \nu, \qquad \nu^* = -\mu. \tag{141}$$

By inserting (139) and (140) to (130), one may show

$$\begin{aligned} x_a &= (1-\mu\nu)\ y_a, \\ \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} &= \frac{\sqrt{1+y_5}}{2} \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \\ \begin{pmatrix} \theta_3 \\ \theta_4 \end{pmatrix} &= \frac{1}{2\sqrt{1+y_5}} (y_4 + i y_i {\sigma_i}^t) \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \end{aligned} \tag{142}$$

where $\eta^* \eta = -\mu\nu$ was utilized. Notice that $S^3$-fibre denoted by $(u,v)$ vanish in the expression of $x_a$ and $\theta_\alpha$ (142). Furthermore, $\theta_{\alpha=3,4}$ are not independent with $\theta_{\alpha=1,2}$, but related as

$$\begin{pmatrix} \theta_3 \\ \theta_4 \end{pmatrix} = \frac{1}{1+y_5} (y_4 + i y_i {\sigma_i}^t) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{143}$$

The $N=1$ graded Hopf fibration, $S^{7|2} \sim S^{4|2} \otimes S^3$, is obvious from the expression

$$\psi = \frac{1}{\sqrt{1-\eta^*\eta}} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \eta \end{pmatrix} = \frac{1}{\sqrt{2(1+y_5)}} \begin{pmatrix} \sqrt{1-\mu\nu}\ (1+y_5) \begin{pmatrix} u \\ v \end{pmatrix} \\ \sqrt{1-\mu\nu}\ (y_4 - i y_i \sigma_i) \begin{pmatrix} u \\ v \end{pmatrix} \\ \sqrt{2(1+y_5)}\ (u\mu + v\nu) \end{pmatrix}, \tag{144}$$

where $S^3$-fibre, $(u,v)^t$, is canceled in (130), and $y_a$ and $\mu, \nu$, respectively account for bosonic and fermionic coordinates on $S^{4|2}$. With $\theta_1$ and $\theta_2$, $\psi$ is rewritten as

$$\psi = \frac{1}{\sqrt{2(1+y_5)}} \begin{pmatrix} \sqrt{1-\frac{4}{1+y_5}\theta_1\theta_2}\ (1+y_5) \begin{pmatrix} u \\ v \end{pmatrix} \\ \sqrt{1-\frac{4}{1+y_5}\theta_1\theta_2}\ (y_4 - iy_i\sigma_i) \begin{pmatrix} u \\ v \end{pmatrix} \\ 2\sqrt{2}\ (u\theta_1 + v\theta_2) \end{pmatrix}, \tag{145}$$

where $y_a$ are related to $x_a$ as

$$y_a = \left(1 - \frac{4}{1+x_5}\theta_1\theta_2\right) x_a. \tag{146}$$

The relations between $x_a$ and $y_a$ are "singular" with $\theta_{1,2}$, but not with $\mu$, $\nu$ (see the first equation in (142)).

#### 4.2.3 $N = 1$ fuzzy four-supersphere

The target manifold of the graded Hopf map is $S^{4|2}$, and we denote the corresponding fuzzy four-supersphere as $S_F^{4|2}$. Coordinates of $S_F^{4|2}$, $X_a$ and $\Theta_\alpha$, are constructed as

$$X_a = \Psi^\dagger \Gamma_a \Psi, \qquad \Theta_\alpha = \Psi^\dagger \Gamma_\alpha \Psi, \tag{147}$$

where $\Psi$ is a five-component graded Schwinger operator

$$\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \tilde{\Psi})^t, \tag{148}$$

with $\Psi_\alpha$ $(\alpha = 1,2,3,4)$ being bosonic operators and $\tilde{\Psi}$ a fermionic operator:

$$[\Psi_\alpha, \Psi_\beta^\dagger] = \delta_{\alpha\beta}, \qquad \{\tilde{\Psi}, \tilde{\Psi}^\dagger\} = 1, \qquad [\Psi_\alpha^\dagger, \tilde{\Psi}] = [\Psi_\alpha, \tilde{\Psi}] = \{\tilde{\Psi}, \tilde{\Psi}\} = 0. \tag{149}$$

Square of the radius of fuzzy four-supersphere is derived as

$$X_a X_a + 2C_{\alpha\beta}\Theta_\alpha\Theta_\beta = (\Psi^\dagger\Psi)(\Psi^\dagger\Psi + 3). \tag{150}$$

With the Schwinger construction, the Casimir (115) is represented as

$$\mathcal{C} = \sum_{a<b} X_{ab}X_{ab} + C_{\alpha\beta}\Theta_\alpha\Theta_\beta = \frac{1}{2}(\Psi^\dagger\Psi)(\Psi^\dagger\Psi + 3), \tag{151}$$

where

$$X_{ab} = \Psi^\dagger \Gamma_{ab} \Psi. \tag{152}$$

We used

$$\begin{aligned} &X_a X_a = 2\sum_{a<b} X_{ab}X_{ab} = \hat{n}_B(\hat{n}_B + 4), \\ &C_{\alpha\beta}\Theta_\alpha\Theta_\beta = -\frac{1}{2}\hat{n}_B + \hat{n}_B\hat{n}_F + 2\hat{n}_F, \end{aligned} \tag{153}$$

with $\hat{n}_B = \sum_{\alpha=1,2,3,4} \Psi_\alpha^\dagger \Psi_\alpha$ and $\hat{n}_F = \tilde{\Psi}^\dagger \tilde{\Psi}$. The Casimir (151) is identical to (150) up to the proportional factor. The graded fully symmetric representation is given by

$$|l_1, l_2, l_3, l_4\rangle = \frac{1}{\sqrt{l_1!\ l_2!\ l_3!\ l_4!}} \Psi_1^{\dagger\, l_1} \Psi_2^{\dagger\, l_2} \Psi_3^{\dagger\, l_3} \Psi_4^{\dagger\, l_4} |0\rangle, \tag{154a}$$

$$|m_1, m_2, m_3, m_4) = \frac{1}{\sqrt{m_1!\ m_2!\ m_3!\ m_4!}} \Psi_1^{\dagger\, m_1} \Psi_2^{\dagger\, m_2} \Psi_3^{\dagger\, m_3} \Psi_4^{\dagger\, m_4} \tilde{\Psi}^\dagger |0\rangle, \tag{154b}$$

where $l_1 + l_2 + l_3 + l_4 = m_1 + m_2 + m_3 + m_4 + 1 = n$. The Scasimir (116) is given by

$$\mathcal{S} = \frac{1}{2\sqrt{2}} (2\hat{n} + 3)(2\hat{n}_F - 1), \tag{155}$$

and the bosonic (154a) and fermionic (154b) states are classified by the sign of Scasimir eigenvalues,

$$\mathcal{S} = \pm \frac{1}{2\sqrt{2}} (2n + 3). \tag{156}$$

The dimensions of bosonic and fermionic states are respectively given by

$$D_B = D(n) \equiv \frac{1}{3!} (n + 1)(n + 2)(n + 3), \tag{157a}$$

$$D_F = D(n - 1) = \frac{1}{3!} n(n + 1)(n + 2), \tag{157b}$$

and the total dimension is

$$D_T = D_B + D_F = \frac{1}{6} (n + 1)(n + 2)(2n + 3). \tag{158}$$

Similar to the case of $N = 1$ fuzzy two-supersphere, the bosonic degrees of freedom (157a) are accounted for the fuzzy four-sphere with radius $n$ and the fermionic degrees of freedom (157b) are for the one with radius $n - 1$. Thus, the $N = 1$ fuzzy four-supersphere is a "superposition" of two fuzzy four-spheres with radii $n$ and $n - 1$. Schematically,

$$S_F^{4|2}(n) \ \simeq \ S_F^4(n) \oplus S_F^4(n - 1). \tag{159}$$

$X_5$ eigenvalues for the states (154) are

$$X_5 = n - k, \tag{160}$$

with $k = 0, 1, 2, \cdots, 2n$. The degeneracies for even $k = 2l$ and for odd $k = 2l + 1$ are respectively given by

$$\begin{aligned} D_{k=2l}(n) &= d(n - l) \cdot d(l) = (n - l + 1)(l + 1), \\ D_{k=2l+1}(n) &= d(n - l - 1) \cdot d(l) = (n - l)(l + 1), \end{aligned} \tag{161}$$

which give rise to

$$\sum_{l=0}^{l} D_{k=2l}(n) = D_B, \qquad \sum_{l=0}^{n-1} D_{k=2l+1}(n) = D_F. \tag{162}$$

Therefore, at latitude $X_5 = n - 2l$, we have fuzzy fibre consisting of two fuzzy two-spheres with radii $n - l$ and $l$, while at latitude $X_5 = n - 2l - 1$ two fuzzy two-spheres with radii $n - l - 1$ and $l$. In other words, as fuzzy fibre at $X_5 = n - 2l$, we have two fuzzy two-spheres with radii $(n + X_5)/2$ and $(n - X_5)/2$, while at $X_5 = n - 2l - 1$, two fuzzy two-spheres with radii $(n + X_5)/2 - 1/2$ and $(n - X_5)/2 - 1/2$.

#### 4.2.4 Algebraic structure

The $N=1$ fuzzy four-supersphere (150) is invariant under the $SU(4|1)$ rotation of the Schwinger operator $\Psi$. This implies hidden $SU(4|1)$ structure of $N=1$ fuzzy four-supersphere. Here, we demonstrate the $SU(4|1)$ structure of fuzzy four-supersphere based on algebraic approach. Notice that the fuzzy four-supersphere coordinates $X_a$, $\Theta_\alpha$ do not satisfy a closed algebra by themselves,

$$[X_a, X_b] = 4iX_{ab}, \quad [X_a, \Theta_\alpha] = (\gamma_a)_{\beta\alpha}\varTheta_\beta, \quad \{\Theta_\alpha, \Theta_\beta\} = \sum_{a<b}(C\gamma_{ab})_{\alpha\beta}X_{ab}. \tag{163}$$

The "new" operators that appear on the right-hand sides of (163) are

$$X_{ab} = \Psi^\dagger \Gamma_{ab}\Psi, \quad \varTheta_\alpha = \Psi^\dagger D_\alpha \Psi, \tag{164}$$

with $\Gamma_{ab}$ (121) and $D_\alpha$[9]

$$D_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_4 & \tau_\alpha \\ (C\tau_\alpha)^t & 0 \end{pmatrix}. \tag{167}$$

$X_{ab}$ and $\varTheta_\alpha$ respectively act as $SO(5)$ generators and spinor. Commutation relations including them are

$$\begin{aligned} &[X_a, \Theta_\alpha] = (\gamma_a)_{\beta\alpha}\varTheta_\beta, && [X_a, \varTheta_\alpha] = (\gamma_a)_{\beta\alpha}\Theta_\beta, \\ &[X_{ab}, \Theta_\alpha] = (\gamma_{ab})_{\beta\alpha}\Theta_\beta, && [X_{ab}, \varTheta_\alpha] = (\gamma_{ab})_{\beta\alpha}\varTheta_\beta, \\ &\{\Theta_\alpha, \varTheta_\beta\} = \sum_{a<b}(C\gamma_{ab})_{\alpha\beta}X_{ab}, && \{\varTheta_\alpha, \varTheta_\beta\} = -\sum_{a<b}(C\gamma_{ab})_{\alpha\beta}X_{ab}, \\ &\{\Theta_\alpha, \varTheta_\beta\} = \frac{1}{4}(C\gamma_a)_{\alpha\beta}X_a + \frac{1}{4}C_{\alpha\beta}Z. \end{aligned} \tag{168}$$

The last equation further yield a new operator

$$Z = \Psi^\dagger H\Psi, \tag{169}$$

with[10]

$$H = \begin{pmatrix} 1_4 & 0 \\ 0 & 4 \end{pmatrix}. \tag{171}$$

The commutation relations concerned with $Z$ are given by

$$[Z, X_a] = [Z, X_{ab}] = 0, \quad [Z, \Theta_\alpha] = -3\varTheta_\alpha, \quad [Z, \varTheta_\alpha] = -3\Theta_\alpha. \tag{172}$$

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[9] $D_\alpha$ have the properties

$$D_\alpha^\ddagger = -C_{\alpha\beta}D_\beta, \quad D_\alpha = -C_{\alpha\beta}\Gamma_\beta^\dagger. \tag{165}$$

$D_\alpha$ can be constructed by

$$D_\alpha = \frac{2}{5}\sum_{a<b}(\gamma_{ab})_{\beta\alpha}\{\Gamma_{ab}, \Gamma_\beta\}, \tag{166}$$

similarly to the $UOSp(1|2)$ case (see Appendix A.1.1).

[10] $H$ is constructed as

$$H = \frac{6}{5}\Big(C_{\alpha\beta}\Gamma_\alpha\Gamma_\beta + \frac{4}{3}\Big). \tag{170}$$

(172) does not yield further new operators. After all, for the closure of the algebra of the fuzzy coordinates $X_a$ and $\Theta_\alpha$, we have to introduce new fuzzy coordinates $X_{ab}$, $\Theta_\alpha$ and $Z$,[11] and such twenty four operators amount to the $SU(4|1)$ algebra (see Appendix A.2.1). The basic concept of the non-commutative geometry is "algebraic construction of geometry". Thus, the algebraic geometry of fuzzy four-supersphere is considered as $SU(4|1)$ rather than $UOSp(1|4)$. We revisit the $SU(4|1)$ structure in Sec.6.

### 4.3 $N = 2$ fuzzy four-supersphere

We proceed to the construction of $N = 2$ version of fuzzy four-sphere, $S_F^{4|8}$ based on the $UOSp(2|4)$ algebra.

#### 4.3.1 $UOSp(2|4)$ algebra

The dimension of $UOSp(2|4)$ is

$$\dim[UOSp(2|4)] = 11|8 = 19. \tag{174}$$

We denote the eleven bosonic generators as $\Gamma_{ab} = -\Gamma_{ba}$ $(a, b = 1, 2, 3, 4, 5)$ and $\Gamma$, and the eight fermionic generators as $\Gamma_\alpha$ and $\Gamma'_\alpha$ $(\alpha = \theta_1, \theta_2, \theta_3, \theta_4)$. The $UOSp(2|4)$ algebra is given by

$$\begin{aligned}
&[\Gamma_{ab}, \Gamma_{cd}] = i(\delta_{ac}\Gamma_{bd} - \delta_{ad}\Gamma_{bc} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}),\\
&[\Gamma_{ab}, \Gamma_\alpha] = (\gamma_{ab})_{\beta\alpha}\Gamma_\beta, \qquad [\Gamma_{ab}, \Gamma'_\alpha] = (\gamma_{ab})_{\beta\alpha}\Gamma'_\beta,\\
&\{\Gamma_\alpha, \Gamma_\beta\} = \{\Gamma'_\alpha, \Gamma'_\beta\} = \sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab},\\
&\{\Gamma_\alpha, \Gamma'_\beta\} = \frac{1}{2}C_{\alpha\beta}\Gamma, \qquad [\Gamma_{ab}, \Gamma] = 0,\\
&[\Gamma_\alpha, \Gamma] = -\Gamma'_\alpha, \qquad [\Gamma'_\alpha, \Gamma] = \Gamma_\alpha.
\end{aligned} \tag{175}$$

The $UOSp(2|4)$ quadratic Casimir is

$$\mathcal{C} = \sum_{a<b}\Gamma_{ab}\Gamma_{ab} + C_{\alpha\beta}\Gamma_\alpha\Gamma_\beta + C_{\alpha\beta}\Gamma'_\alpha\Gamma'_\beta + \frac{1}{2}\Gamma^2. \tag{176}$$

The fundamental representation of the $UOSp(2|4)$ generators is expressed by the following $6 \times 6$ matrices

$$\Gamma_{ab} = \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & 0_2 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\Gamma_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_4 & \tau_\alpha & 0 \\ -(C\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma'_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_4 & 0 & \tau_\alpha \\ 0 & 0 & 0 \\ -(C\tau_\alpha)^t & 0 & 0 \end{pmatrix}, \tag{177}$$

[11] With use of $\Theta_\alpha$ and $Z$, the $UOSp(1|4)$ invariant quantity is given by

$$C_{\alpha\beta}\Theta_\alpha\Theta_\beta + \frac{1}{6}Z^2 = \frac{1}{6}(\Psi^\dagger\Psi)(\Psi^\dagger\Psi + 3). \tag{173}$$

with $\gamma_{ab}$ (99) and $\tau_\alpha$ (123). The corresponding gamma matrices are also

$$\Gamma_a = \begin{pmatrix} \gamma_a & 0 \\ 0 & 0_2 \end{pmatrix}, \tag{178}$$

with $\gamma_a$ (90).

#### 4.3.2 $N = 2$ fuzzy four-supersphere

With a Schwinger operator $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \tilde{\Psi}_1, \tilde{\Psi}_2)^t$, we introduce $N = 2$ fuzzy four-supersphere coordinates

$$X_a = \Psi^\dagger \Gamma_a \Psi, \quad \Theta_\alpha = \Psi^\dagger \Gamma_\alpha \Psi, \quad \Theta'_\alpha = \Psi^\dagger \Gamma'_\alpha \Psi, \quad G = \Psi^\dagger \Gamma \Psi. \tag{179}$$

As emphasized in Sec.4.1, in Schwinger construction, the $SO(5)$ Casimir can be replaced with the inner product of $SO(5)$ gamma matrices,

$$\sum_{a<b} X_{ab} X_{ab} = \frac{1}{2} X_a X_a, \tag{180}$$

and from (176), square of the radius of $N = 2$ fuzzy four-supersphere is obtained as

$$X_a X_a + 2C_{\alpha\beta}\Theta_\alpha\Theta_\beta + 2C_{\alpha\beta}\Theta'_\alpha\Theta'_\beta + G^2 = (\Psi^\dagger\Psi)(\Psi^\dagger\Psi + 2). \tag{181}$$

For $\Psi^\dagger\Psi = n$, the graded fully symmetric representation is constructed as

$$|l_1, l_2, l_3, l_4\rangle = \frac{1}{\sqrt{l_1!\ l_2!\ l_3!\ l_4!}} \Psi_1^{\dagger l_1} \Psi_2^{\dagger l_2} \Psi_3^{\dagger l_3} \Psi_4^{\dagger l_4} |0\rangle, \tag{182a}$$

$$|m_1, m_2, m_3, m_4) = \frac{1}{\sqrt{m_1!\ m_2!\ m_3!\ m_4!}} \Psi_1^{\dagger m_1} \Psi_2^{\dagger m_2} \Psi_3^{\dagger m_3} \Psi_4^{\dagger m_4} \tilde{\Psi}_1^\dagger |0\rangle, \tag{182b}$$

$$|m'_1, m'_2, m'_3, m'_4) = \frac{1}{\sqrt{m'_1!\ m'_2!\ m'_3!\ m'_4!}} \Psi_1^{\dagger m'_1} \Psi_2^{\dagger m'_2} \Psi_3^{\dagger m'_3} \Psi_4^{\dagger m'_4} \tilde{\Psi}_2^\dagger |0\rangle, \tag{182c}$$

$$|n_1, n_2, n_3, n_4\rangle = \frac{1}{\sqrt{n_1!\ n_2!\ n_3!\ n_4!}} \Psi_1^{\dagger n_1} \Psi_2^{\dagger n_2} \Psi_3^{\dagger n_3} \Psi_4^{\dagger n_4} \tilde{\Psi}_1^\dagger \tilde{\Psi}_2^\dagger |0\rangle, \tag{182d}$$

where $l_1+l_2+l_3+l_4 = m_1+m_2+m_3+m_4+1 = m'_1+m'_2+m'_3+m'_4+1 = n_1+n_2+n_3+n_4+2 = n$. The first two are $UOSp(1|4)$ representation of the index $n$ (154) while the other two are that of $n-1$. In passing from $|l_1, l_2, l_3, l_4\rangle$ to $|n_1, n_2, n_3, n_4\rangle$ via either $|m_1, m_2, m_3, m_4)$ or $|m'_1, m'_2, m'_3, m'_4)$, we perform supersymmetric transformations twice, and hence we have $N = 2$ SUSY. Dimensions of bosonic and fermionic states are respectively

$$D_B = d(n) + d(n-2) = \frac{1}{3}(n+1)(n^2+2n+3),$$
$$D_F = 2d(n-1) = \frac{1}{3}(n+2)(n+1)n. \tag{183}$$

The total dimension is

$$D_T = D_B + D_F = \frac{1}{3}(2n^2+4n+3)(n+1). \tag{184}$$

As in the case of fuzzy two-supersphere, $N=2$ fuzzy four-supersphere is a "superposition" of two $N=1$ fuzzy four-superspheres. Schematically,

$$S_F^{4|4}(n) \simeq S_F^{4|2}(n) \oplus S_F^{4|2}(n-1) \\ \simeq S_F^4(n) \oplus S_F^4(n-1) \oplus S_F^4(n-1) \oplus S_F^4(n-2). \tag{185}$$

The last expression corresponds to the degrees of freedom of (182). The states (182) are eigenstates of $X_5$ with eigenvalues

$$X_5 = n-k, \tag{186}$$

where $k=0,1,2,\cdots,2n$. The degeneracy at $X_5 = n-2l$ ($l=0,1,2,\cdots,n$) is accounted for the bosonic states (182a) and (182d):

$$D_B^{k=2l} = D_l(n) + D_{l-1}(n-2) = 2l(n-l)+n+1 \tag{187}$$

with $D_l(n)$ (105), while that at $X_5 = n-2l-1$ ($l=0,1,2,\cdots,n-1$) is accounted for the fermionic states (182b) and (182c):

$$D_F^{k=2l+1} = 2D_l(n-1) = 2(l+1)(n-l). \tag{188}$$

#### 4.3.3 $N=2$ graded 2nd Hopf map

The derivation of the corresponding Hopf map is straightforward. With a normalized $UOSp(2|4)$ spinor $\psi$

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \eta_1, \eta_2)^t, \tag{189}$$

subject to $\psi^\ddagger \psi = 1$, $N=2$ graded 2nd Hopf map is given by

$$x_a = \psi^\ddagger \Gamma_a \psi, \quad \theta_\alpha = \psi^\ddagger \Gamma_\alpha \psi, \quad \theta'_\alpha = \psi^\ddagger \Gamma'_\alpha \psi, \quad g = \psi^\ddagger \Gamma \psi. \tag{190}$$

The normalization indicates that $\psi$ is coordinates on $S^{7|4}$. (190) satisfy

$$x_a x_a + 2C_{\alpha\beta}\theta_\alpha\theta_\beta + 2C_{\alpha\beta}\theta'_\alpha\theta'_\beta + g^2 = (\psi^\ddagger\psi)^2 = 1. \tag{191}$$

Then, we have eight (pseudo-)Majorana fermionic coordinates, $\theta_\alpha$ and $\theta'_\alpha$. However they are not independent, since they contain only four real Grassmann odd degrees of freedom coming from $\eta_1$ and $\eta_2$. With the renormalization,

$$x_a \to \sqrt{1-g^2}\ x_a = (1-\frac{1}{2}g^2)x_a, \quad \theta \to \sqrt{1-g^2}\ \theta_\alpha = \theta_\alpha, \\ \theta'_\alpha \to \sqrt{1-g^2}\ \theta'_\alpha = \theta'_\alpha, \quad g \to \sqrt{1-g^2}\ g = g, \tag{192}$$

(191) is restated as

$$x_a x_a + 2C_{\alpha\beta}\theta_\alpha\theta_\beta + 2C_{\alpha\beta}\theta'_\alpha\theta'_\beta = 1. \tag{193}$$

This would represent $S^{4|8}$ provided $\theta_\alpha$ and $\theta'_\alpha$ were independent. The original $SO(5)$ normalized spinor $\phi$ (139) is embedded as

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{\sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2}} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{194}$$

One may demonstrate the cancellation of $S^3$-fibre in the map (190) by following similar arguments in Sec.4.2.2. Write the two Grassmann odd components as

$$\begin{aligned} \eta_1 &= u\mu_1 + v\nu_1, \\ \eta_2 &= u\mu_2 + v\nu_2, \end{aligned} \tag{195}$$

where $u$ and $v$ denote the coordinates on $S^3$ ($u^*u + v^*v = 1$), and $\mu_{1,2}$ and $\nu_{1,2}$ are respectively real and imaginary components of $\eta_{1,2}$. From (190), we have

$$\begin{aligned} &x_a = (1-\mu_1\nu_1-\mu_2\nu_2)\ y_a, \\ &\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \frac{1}{2}\sqrt{(1+y_5)(1-\mu_1\nu_1-\mu_2\nu_2)} \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix}, \qquad \begin{pmatrix} \theta_3 \\ \theta_4 \end{pmatrix} = \frac{1}{2}\sqrt{\frac{1-\mu_1\nu_1-\mu_2\nu_2}{1+y_5}}(y_4+iy_i{\sigma_i}^t) \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix}, \\ &\begin{pmatrix} \theta'_1 \\ \theta'_2 \end{pmatrix} = \frac{1}{2}\sqrt{(1+y_5)(1-\mu_1\nu_1-\mu_2\nu_2)} \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}, \qquad \begin{pmatrix} \theta'_3 \\ \theta'_4 \end{pmatrix} = \frac{1}{2}\sqrt{\frac{1-\mu_1\nu_1-\mu_2\nu_2}{1+y_5}}(y_4+iy_i{\sigma_i}^t) \begin{pmatrix} \mu_2 \\ \nu_2 \end{pmatrix}, \end{aligned} \tag{196}$$

where $\eta_1^*\eta_1 = -\mu_1\nu_1$ and $\eta_2^*\eta_2 = -\mu_2\nu_2$ were utilized. Notice that $(u, v)$ does not appear in (196). Besides, $\theta_{3,4}$ and $\theta'_{3,4}$ are respectively related to $\theta_{1,2}$ and $\theta'_{1,2}$ as

$$\begin{pmatrix} \theta_3 \\ \theta_4 \end{pmatrix} = \frac{1}{1+y_5}(y_4+iy_i{\sigma_i}^t) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \qquad \begin{pmatrix} \theta'_3 \\ \theta'_4 \end{pmatrix} = \frac{1}{1+y_5}(y_4+iy_i{\sigma_i}^t) \begin{pmatrix} \theta'_1 \\ \theta'_2 \end{pmatrix}. \tag{197}$$

Thus, with the representation

$$\psi = \sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \sqrt{1-\eta_2^*\eta_2}\ \eta_1 \\ \sqrt{1-\eta_1^*\eta_1}\ \eta_2 \end{pmatrix} = \sqrt{\frac{1-\mu_1\nu_1-\mu_2\nu_2}{2(1+y_5)}} \begin{pmatrix} (1+y_5)\begin{pmatrix} u \\ v \end{pmatrix} \\ (y_4-iy_i\sigma_i)\begin{pmatrix} u \\ v \end{pmatrix} \\ \sqrt{2(1+y_5)(1+\mu_2\nu_2)}\ (u\mu_1+v\nu_1) \\ \sqrt{2(1+y_5)(1+\mu_1\nu_1)}\ (u\mu_2+v\nu_2) \end{pmatrix}, \tag{198}$$

the $S^3$-fibre denoted by $(u, v)$ is canceled in $x_a$, $\theta_\alpha$ and $\theta'_\alpha$ (190). From (196), we have

$$\theta_1\theta_2 = -\frac{1+y_5}{4(1+\mu_2\nu_2)}\mu_1\nu_1, \qquad \theta'_1\theta'_2 = -\frac{1+y_5}{4(1+\mu_1\nu_1)}\mu_2\nu_2, \tag{199}$$

and hence

$$\theta_1\theta_2+\theta'_1\theta'_2=-\frac{1+y_5}{4(1+\mu_1\nu_1+\mu_2\nu_2)}(\mu_1\nu_1+\mu_2\nu_2), \tag{200}$$

or inversely,

$$\mu_1\nu_1+\mu_2\nu_2=\frac{4}{1+y_5-4(\theta_1\theta_2+\theta'_1\theta'_2)}(\theta_1\theta_2+\theta'_1\theta'_2). \tag{201}$$

Then,

$$\begin{aligned}\begin{pmatrix}\mu_1\\ \nu_1\end{pmatrix}&=\frac{2}{\sqrt{1+y_5-4(\theta_1\theta_2+\theta'_1\theta'_2)}}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix},\\ \begin{pmatrix}\mu_2\\ \nu_2\end{pmatrix}&=\frac{2}{\sqrt{1+y_5-4(\theta_1\theta_2+\theta'_1\theta'_2)}}\begin{pmatrix}\theta'_1\\ \theta'_2\end{pmatrix},\end{aligned} \tag{202}$$

Therefore, with the coordinates on $S^{4|4}$, $y_a$. $\theta_{1,2}$ and $\theta'_{1,2}$, (198) is rewritten as

$$\psi=\frac{1}{\sqrt{2(1+y_5-4(\theta_1\theta_2+\theta'_1\theta'_2))}}\begin{pmatrix}\sqrt{1-\frac{8}{1+y_5}(\theta_1\theta_2+\theta'_1\theta'_2)}\;(1+y_5)\begin{pmatrix}u\\ v\end{pmatrix}\\ \sqrt{1-\frac{8}{1+y_5}(\theta_1\theta_2+\theta'_1\theta'_2)}\;(y_4-iy_i\sigma_i)\begin{pmatrix}u\\ v\end{pmatrix}\\ 2\sqrt{2}\;(u\theta_1+v\theta_2)\\ 2\sqrt{2}\;(u\theta'_1+v\theta'_2)\end{pmatrix}, \tag{203}$$

where $y_a$ are related to $x_a$ as

$$\begin{aligned}y_a&=\left(1-\frac{4}{1+y_5-4(\theta_1\theta_2+\theta'_1\theta'_2)}(\theta_1\theta_2+\theta'_1\theta'_2)\right)\,x_a\\ &=\left(1-\frac{4}{1+x_5}(\theta_1\theta_2+\theta'_1\theta'_2)-\frac{16}{(1+x_5)^3}(1+2x_5)(\theta_1\theta_2+\theta'_1\theta'_2)^2\right)\,x_a.\end{aligned} \tag{204}$$

Meanwhile, $g(=-\eta_1^*\eta_2+\eta_1^*\eta_2)$ is given by

$$g=2\mu_1\mu_2uv^*-(\mu_1\nu_2+\nu_1\mu_2)(u^*u-v^*v)-2\nu_1\nu_2u^*v, \tag{205}$$

which depends on the $S^3$-fibre, $(u,v)$. $S^3$-fibre is canceled in $g^2$:

$$g^2=2\mu_1\nu_1\mu_2\nu_2=\frac{32}{(1+y_5)^2}\theta_1\theta_2\theta'_1\theta'_2. \tag{206}$$

Thus, though the $S^3$ cancellation is not "complete" in (190) (because of $g$), with the renormalization (192) in which only $g^2$ is concerned, $S^3$ is completely projected out to yield coordinates on $S^{4|4}$. Consequently, the map (190) with (192) represents

$$S^{7|4}\;\overset{S^3}{\longrightarrow}\;S^{4|4}\subset S^{4|8}. \tag{207}$$

The basemanifold is $S^{4|4}$, and then the corresponding fuzzy manifold is $S_F^{4|4}$.

# 5 More supersymmetries

One may incorporate more supersymmetries based on $UOSp(N|4)$ algebra with $N \geq 3$. The dimension of $UOSp(N|4)$ is

$$\dim[UOSp(N|4)] = 10 + \frac{1}{2}N(N-1)|4N = 10 + \frac{1}{2}N(N+7). \tag{208}$$

We denote bosonic generators as $\Gamma_{ab} = -\Gamma_{ba}$ $(a, b = 1, 2, 3, 4, 5)$, $\tilde{\Gamma}_{lm} = -\tilde{\Gamma}_{ml}$ $(l, m = 1, 2, \cdots, N)$ and fermionic generators as $\Gamma_{l\alpha}$ $(\alpha = 1, 2, 3, 4)$. They satisfy

$$\begin{aligned}
&[\Gamma_{ab}, \Gamma_{cd}] = i(\delta_{ac}\Gamma_{bd} - \delta_{ad}\Gamma_{bc} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}),\\
&[\Gamma_{ab}, \Gamma_{l\alpha}] = (\gamma_{ab})_{\beta\alpha}\Gamma_{l\beta},\\
&[\Gamma_{ab}, \tilde{\Gamma}_{lm}] = 0, ,\\
&\{\Gamma_{l\alpha}, \Gamma_{m\beta}\} = \sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab}\delta_{lm} + \frac{1}{4}C_{\alpha\beta}\tilde{\Gamma}_{lm},\\
&[\Gamma_{l\alpha}, \tilde{\Gamma}_{mn}] = (\gamma_{mn})_{pl}\Gamma_{p\alpha},\\
&[\tilde{\Gamma}_{lm}, \tilde{\Gamma}_{np}] = -\delta_{ln}\tilde{\Gamma}_{mp} + \delta_{lp}\tilde{\Gamma}_{mn} - \delta_{mp}\tilde{\Gamma}_{ln} + \delta_{mn}\tilde{\Gamma}_{lp},
\end{aligned} \tag{209}$$

where $C$ is the $SO(5)$ charge conjugation matrix (114) and $\gamma_{lm} = -\gamma_{ml}$ $(l < m)$ are $SO(N)$ generators given by

$$(\gamma_{lm})_{np} = \delta_{ln}\delta_{mp} - \delta_{lp}\delta_{mn}. \tag{210}$$

The $UOSp(N|4)$ quadratic Casimir is

$$\mathcal{C} = \sum_{a<b}\Gamma_{ab}\Gamma_{ab} + C_{\alpha\beta}\sum_{l=1}^{N}\Gamma_{l\alpha}\Gamma_{l\beta} + \frac{1}{2}\sum_{l<m=1}^{N}\tilde{\Gamma}_{lm}\tilde{\Gamma}_{lm}. \tag{211}$$

The fundamental representation generators are given by

$$\Gamma_{ab} = \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & 0_N \end{pmatrix}, \quad \Gamma_{l\alpha} = \begin{pmatrix} 0_{3+l} & \tau_\alpha & 0 \\ -(C\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0_{N-l} \end{pmatrix}, \quad \tilde{\Gamma}_{lm} = \begin{pmatrix} 0_4 & 0 \\ 0 & \gamma_{lm} \end{pmatrix}, \tag{212}$$

where $0_k$ signify $k \times k$ zero-matrices, and $\tau_\alpha = (0, \cdots, 0, \overset{\alpha}{\check{1}}, 0, \cdots, 0)^t$. Notice that $\tilde{\Gamma}_{lm}$ are taken as anti-hermitian, $\tilde{\Gamma}_{lm}^\dagger = -\tilde{\Gamma}_{lm}$. We apply the Schwinger construction to (212) and define

$$X_a = \Psi^\dagger\Gamma_a\Psi, \quad X_{ab} = \Psi^\dagger\Gamma_{ab}\Psi, \quad \Theta_\alpha^{(l)} = \Psi^\dagger\Gamma_{l\alpha}\Psi, \quad Y_{lm} = \Psi^\dagger\tilde{\Gamma}_{lm}\Psi, \tag{213}$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \tilde{\Psi}_1, \tilde{\Psi}_2, \cdots, \tilde{\Psi}_N)^t$ in which $\Psi_\alpha$ $(\alpha = 1, 2, 3, 4)$ are bosonic Schwinger operators while $\tilde{\Psi}_l$ $(l = 1, 2, \cdots, N)$ are fermionic ones. Square of the radius of $N$-SUSY fuzzy four-supersphere is derived as

$$X_aX_a + 2\sum_{l=1}^{N}C_{\alpha\beta}\Theta_\alpha^{(l)}\Theta_\beta^{(l)} + \sum_{l<m=1}^{N}Y_{lm}Y_{lm} = \hat{n}(\hat{n} + 4 - N), \tag{214}$$

with $\hat{n} = \Psi^\dagger\Psi$. Here, we utilized

$$\sum_{a=1}^{5} X_a X_a = 2\sum_{a<b=1}^{5} X_{ab}X_{ab} = \hat{n}_B(\hat{n}_B+4),$$
$$\sum_{l=1}^{N} C_{\alpha\beta}\Theta_\alpha^{(l)}\Theta_\beta^{(l)} = -\frac{N}{2}\hat{n}_B + \hat{n}_B\hat{n}_F + 2\hat{n}_F,$$
$$\sum_{l<m=1}^{N} Y_{lm}Y_{lm} = \hat{n}_F(\hat{n}_F - N), \tag{215}$$

with $\hat{n}_B = \sum_{\alpha=1}^{4}\Psi_\alpha^\dagger\Psi_\alpha$ and $\hat{n}_F = \sum_{\sigma=1}^{N}\tilde{\Psi}_\sigma^\dagger\tilde{\Psi}_\sigma$. $X_a$, $\Theta_\alpha^{(l)}$, $Y_{lm}$ do not satisfy a closed algebra by themselves. As a similar manner to Sec.4.2.4, one may readily show that the minimally extended algebra that includes $X_a$, $\Theta_\alpha^{(l)}$ and $Y_{lm}$ is $SU(4|N)$.

For $UOSp(N|4)$ with $\Psi^\dagger\Psi = n$, the graded fully symmetric representation is given by

$$|l_1,l_2,l_3,l_4\rangle = \frac{1}{\sqrt{l_1!\ l_2!\ l_3!\ l_4!}}{\Psi_1^\dagger}^{l_1}{\Psi_2^\dagger}^{l_2}{\Psi_3^\dagger}^{l_3}{\Psi_4^\dagger}^{l_4}|0\rangle,$$
$$|m_1,m_2,m_3,m_4)_{i_1} = \frac{1}{\sqrt{m_1!\ m_2!\ m_3!\ m_4!}}{\Psi_1^\dagger}^{m_1}{\Psi_2^\dagger}^{m_2}{\Psi_3^\dagger}^{m_3}{\Psi_4^\dagger}^{m_4}\tilde{\Psi}_{i_1}^\dagger|0\rangle$$
$$|n_1,n_2,n_3,n_4\rangle_{i_1<i_2} = \frac{1}{\sqrt{n_1!\ n_2!\ n_3!\ n_4!}}{\Psi_1^\dagger}^{n_1}{\Psi_2^\dagger}^{n_2}{\Psi_3^\dagger}^{n_3}{\Psi_4^\dagger}^{n_4}\tilde{\Psi}_{i_1}^\dagger\tilde{\Psi}_{i_2}^\dagger|0\rangle$$
$$\vdots$$
$$|q_1,q_2,q_3,q_4\rangle_{i_1<i_2<\cdots<i_{N-1}} = \frac{1}{\sqrt{q_1!q_2!q_3!q_4!}}{\Psi_1^\dagger}^{q_1}{\Psi_2^\dagger}^{q_2}{\Psi_3^\dagger}^{q_3}{\Psi_4^\dagger}^{q_4}\ \tilde{\Psi}_{i_1}^\dagger\tilde{\Psi}_{i_2}^\dagger\tilde{\Psi}_{i_3}^\dagger\cdots\tilde{\Psi}_{i_{N-1}}^\dagger|0\rangle,$$
$$|r_1,r_2,r_3,r_4) = \frac{1}{\sqrt{r_1!r_2!r_3!r_4!}}{\Psi_1^\dagger}^{r_1}{\Psi_2^\dagger}^{r_2}{\Psi_3^\dagger}^{r_3}{\Psi_4^\dagger}^{r_4}\ \tilde{\Psi}_1^\dagger\tilde{\Psi}_2^\dagger\tilde{\Psi}_3\cdots\tilde{\Psi}_{N-1}^\dagger\tilde{\Psi}_N^\dagger|0\rangle, \tag{216}$$

where $m_1+m_2+m_3+m_4 = n_1+n_2+n_3+n_4+1 = l_1+l_2+l_3+l_4+2 = \cdots = q_1+q_2+q_3+q_4+N-1 = r_1+r_2+r_3+r_4+N = n$. Therefore, with $D(n)$ (95), the dimension of (216) is derived as

$$D_T = \sum_{l=0}^{N} {}_N C_l \cdot D(n-l) = \frac{1}{3}(2n+4-N)\biggl((2n+4-N)^2 - 4 + 3N\biggr)2^{N-4}, \tag{217}$$

for $n \geq N-3$. (One may readily confirm that (217) reproduces the previous results (158), (184) for $N = 1, 2$.) For odd $N$, $N = 2l+1$, the degeneracies of bosonic and fermionic states are respectively given by $D_B = \sum_{k=0}^{l} {}_{2l+1}C_{2k}\cdot D(n-2k)$ and $D_F = \sum_{k=0}^{l} {}_{2l+1}C_{2k+1}\cdot D(n-2k-1)$. Meanwhile, for even $N$, $N = 2l$, the degeneracies are respectively $D_B = \sum_{k=0}^{l} {}_{2l}C_{2k}\cdot D(n-2k)$ and $D_F = \sum_{k=0}^{l-1} {}_{2l}C_{2k+1}\cdot D(n-2k-1)$. Schematically, $S_F^{4|2N}(n)$ is expressed as a superposition of fuzzy four-superspheres with lower supersymmetries, $S_F^{4|2N-2l}$, with different radii, $n$, $n-1$,

$n-2, \cdots, n-l$:

$$S_F^{4|2N}(n) \simeq \sum_{m=0}^{l} {}_lC_m \cdot S_F^{4|2N-2l}(n-m)$$
$$\simeq S_F^{4|2N-2l}(n) \oplus l \cdot S_F^{4|2N-2l}(n-1) \oplus \frac{l(l-1)}{2!} \cdot S_F^{4|2N-2l}(n-2) \oplus \cdots \oplus S_F^{4|2N-2l}(n-l). \tag{218}$$

Explicitly,

$$\begin{aligned} S_F^{4|2N}(n) &\simeq S_F^{4|2N-2}(n) \oplus S_F^{4|2N-2}(n-1) \\ &\simeq S_F^{4|2N-4}(n) \oplus 2S_F^{4|2N-4}(n-1) \oplus S_F^{4|2N-4}(n-2), \\ &\simeq S_F^{4|2N-6}(n) \oplus 3S_F^{4|2N-6}(n-1) \oplus 3S_F^{4|2N-6}(n-2) \oplus S_F^{4|2N-6}(n-3), \\ &\simeq \cdots . \end{aligned} \tag{219}$$

Replacing the Schwinger operator with a normalized $UOSp(4|N)$ spinor, *i.e.*, $\Psi \rightarrow \psi$ and $\Psi^\dagger \rightarrow \psi^\ddagger$ ($\psi^\ddagger \psi = 1$) in (213), we introduce $x_a$, $\theta_\alpha^{(l)}$ and $y_{lm}$ that satisfy

$$x_a x_a + 2\sum_{l=1}^{N} C_{\alpha\beta}\theta_\alpha^{(l)}\theta_\beta^{(l)} + \sum_{l<m=1}^{N} y_{lm}y_{lm} = (\psi^\ddagger\psi)^2 = 1. \tag{220}$$

The original $SO(5)$ normalized spinor is embedded as

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{\sqrt{1+\eta_1^*\eta_1+\eta_2^*\eta_2+\cdots+\eta_k^*\eta_k}} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{221}$$

The normalized $UOSp(4|N)$ spinor $\psi$ has the dimension $(7|2N)$. One may readily demonstrate the cancellation of the $S^3$-fibre in the graded Hopf map by following the similar arguments presented in the previous sections (especially Sec.4.3.3), and hence the present graded Hopf map signifies

$$S^{7|2N} \stackrel{S^3}{\longrightarrow} S^{4|2N}. \tag{222}$$

Compare (220) with (214). Due to the existence of fermionic degrees of freedom, the zero-point energy in (214) decreases with increase of the number of supersymmetry. For $N=4$, the square of the radius of fuzzy supersphere (214) "saturates" the classical bound (220). In this sense, $N=4$ is the "maximum", otherwise the square of the radius takes negative value for sufficiently small $n$ that satisfies $n < N-4$. We have already discussed $N=0,1,2$ cases. In the following subsections, we argue the remaining cases, $N=3$ and 4.

### 5.1 $N=3$ graded 2nd Hopf map and fuzzy four-supersphere

The dimension of the $UOSp(3|4)$ algebra is

$$\dim[UOSp(3|4)] = 13|12 = 25. \tag{223}$$

From (210), we derive the $SO(3)$ generators $\gamma_{ij}$ $(i,j=1,2,3)$ as

$$\gamma_{12}=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \gamma_{23}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},\ \gamma_{31}=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}. \tag{224}$$

With the identification $\tilde{\Gamma}_i=-\frac{1}{2}\epsilon_{ijk}\tilde{\Gamma}_{jk}$, $\Gamma_i$ satisfy the $SU(2)$ algebra

$$[\tilde{\Gamma}_i,\tilde{\Gamma}_j]=\epsilon_{ijk}\tilde{\Gamma}_k. \tag{225}$$

Then, with

$$Y_i=\Psi^{\dagger}\tilde{\Gamma}_i\Psi, \tag{226}$$

square of the radius of $N=3$ fuzzy four-supersphere is obtained as

$$X_aX_a+2\sum_{i=1}^{3}C_{\alpha\beta}\Theta_{\alpha}^{(i)}\Theta_{\beta}^{(i)}+\sum_{i=1}^{3}Y_iY_i=(\Psi^{\dagger}\Psi)(\Psi^{\dagger}\Psi+1). \tag{227}$$

The corresponding classical relation is

$$x_ax_a+2\sum_{i=1}^{3}C_{\alpha\beta}\theta_{\alpha}^{(i)}\theta_{\beta}^{(i)}+\sum_{i=1}^{3}y_iy_i=(\psi^{\ddagger}\psi)^2=1. \tag{228}$$

For $\Psi^{\dagger}\Psi=n$, the dimensions of the bosonic and fermionic states in (216) with $N=3$ are respectively given by

$$\begin{aligned}D_B&=D(n)+3D(n-2)=\frac{1}{3}(2n^2+n+3)(n+1),\\ D_F&=3D(n-1)+D(n-3)=\frac{1}{3}(2n^2+3n+4)n,\end{aligned} \tag{229}$$

and the total dimension is

$$D_T=D_B+D_F=\frac{1}{3}(2n+1)(2n^2+2n+3). \tag{230}$$

## 5.2 $N=4$ graded 2nd Hopf map and fuzzy four-supersphere

The dimension of the $UOSp(4|4)$ algebra is

$$\dim[UOSp(4|4)]=16|16=32. \tag{231}$$

From (210), the $SO(4)$ generators are obtained as

$$\begin{aligned}&\gamma_{12}=\begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ \gamma_{13}=\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ \gamma_{14}=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix},\\ &\gamma_{23}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ \gamma_{24}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix},\ \gamma_{34}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}.\end{aligned} \tag{232}$$

Since $SO(4) \simeq SU(2) \oplus SU(2)$, two independent sets of $SU(2)$ generators can be constructed with the $SO(4)$ generators $\tilde{\Gamma}_{ij}$ $(i,j,k = 1,2,3)$:

$$\tilde{\Gamma}_i = -\frac{1}{4}\epsilon_{ijk}\tilde{\Gamma}_{jk} + \frac{1}{2}\tilde{\Gamma}_{i4}, \qquad \tilde{\Gamma}'_i = -\frac{1}{4}\epsilon_{ijk}\tilde{\Gamma}_{jk} - \frac{1}{2}\tilde{\Gamma}_{i4}, \tag{233}$$

which satisfy

$$[\tilde{\Gamma}_i, \tilde{\Gamma}_j] = \epsilon_{ijk}\tilde{\Gamma}_k, \qquad [\tilde{\Gamma}'_i, \tilde{\Gamma}'_j] = \epsilon_{ijk}\tilde{\Gamma}'_k, \qquad [\tilde{\Gamma}_i, \tilde{\Gamma}'_j] = 0. \tag{234}$$

Then, square of the radius of $N = 4$ fuzzy four-supersphere is written as

$$X_a X_a + C_{\alpha\beta}\sum_{l=1}^{4}\Theta^{(l)}_\alpha\Theta^{(l)}_\beta + \sum_{i=1}^{3} Y_i Y_i + \sum_{i=1}^{3} Y'_i Y'_i = (\Psi^\dagger\Psi)^2, \tag{235}$$

and the corresponding classical relation is

$$x_a x_a + C_{\alpha\beta}\sum_{l=1}^{4}\theta^{(l)}_\alpha\theta^{(l)}_\beta + \sum_{i=1}^{3} y_i y_i + \sum_{i=1}^{3} y'_i y'_i = (\psi^\ddagger\psi)^2 = 1. \tag{236}$$

The cancellation of the zero-point energy in (235) suggests equal numbers of bosonic and the fermionic states. Indeed,

$$\begin{aligned} D_B &= D(n) + 6D(n-2) + D(n-4) = \frac{4}{3}n(n^2+2), \\ D_F &= 4D(n-1) + 4D(n-3) = \frac{4}{3}n(n^2+2). \end{aligned} \tag{237}$$

The total dimension is

$$D_T = D_B + D_F = \frac{8}{3}n(n^2+2). \tag{238}$$

# 6 Symmetry enhancement as quantum fluctuations

As discussed in Sec.4.2.4, the algebraic structure of $N = 1$ fuzzy four-supersphere is given by $SU(4|1)$. In this section, we provide a physical interpretation of the $SU(4|1)$ structure by evaluating quantum fluctuations of fuzzy two- and four-superspheres exemplified by correlation functions. The method is taken from Balachandran et al.[53, 27]. We only discuss $N = 1$ fuzzy two- and four-superspheres, but generalizations to more SUSY cases are straightforward.

## 6.1 $N = 1$ fuzzy two-supersphere

We first define the supercoherent state on $N = 1$ fuzzy two-supersphere. With the coordinates $X_i$ (45) and $x_i$ (32), the super-coherent state, $|\omega\rangle$, is defined so as to satisfy

$$(x_i X_i + \epsilon_{\alpha\beta}\theta_\alpha\Theta_\beta)|\omega\rangle = n|\omega\rangle. \tag{239}$$

$|\omega\rangle$ is derived as

$$|\omega\rangle = \frac{1}{\sqrt{n!}}(\Psi^\dagger\psi)^n|0\rangle = \frac{1}{\sqrt{n!}}(\psi_1\Psi_1^\dagger + \psi_2\Psi_2^\dagger - \eta\tilde{\Psi}^\dagger)^n|0\rangle, \tag{240}$$

where $\Psi$ is the graded Schwinger operator and $\psi$ the normalized spinor related to $X_i$, $\Theta_\alpha$ and $x_i$, $\theta_\alpha$ by (45) and (32) respectively[12]. $|\omega\rangle$ is $n$th order polynomials expanded by the graded fully symmetric representation (52). $|\omega\rangle$ is normalized as

$$\langle\!\langle\omega|\omega\rangle = 1, \tag{242}$$

with dual state $\langle\!\langle\omega|$ given by

$$\langle\!\langle\omega| = \frac{1}{\sqrt{n!}}\langle 0|(\psi^\ddagger\Psi)^n. \tag{243}$$

The expectation values of $X_i$ and $\Theta_\alpha$ are calculated as

$$\langle\!\langle\omega|X_i|\omega\rangle = nx_i, \quad \langle\!\langle\omega|\Theta_\alpha|\omega\rangle = n\theta_\alpha. \tag{244}$$

Meanwhile, the correlation functions are

$$\begin{aligned}
&\langle\!\langle\omega|X_iX_j|\omega\rangle = n^2x_ix_j + 4n\psi^\ddagger L_iP_-L_j\psi,\\
&\langle\!\langle\omega|X_i\Theta_\alpha|\omega\rangle = n^2x_i\theta_\alpha + 4n\psi^\ddagger L_iP_-L_\alpha\psi,\\
&\langle\!\langle\omega|\Theta_\alpha\Theta_\beta|\omega\rangle = n^2\theta_\alpha\theta_\beta + 4n\psi^\ddagger L_\alpha P_-L_\beta\psi,
\end{aligned} \tag{245}$$

where $P_-$ denotes a projection operator [13]

$$P_- = 1 - \psi\psi^\ddagger. \tag{247}$$

In the classical limit $n \to \infty$, the first terms of the order $n^2$ are dominant and the fuzzy two-supersphere is reduced to the ordinary commutative supersphere. The second terms of the order $n$ exhibit quantum fluctuations particular to fuzzy geometry. The second terms are evaluated as

$$\begin{aligned}
&4\psi^\ddagger L_iP_-L_j\psi = -x_ix_j + i\epsilon_{ijk}x_k + \delta_{ij},\\
&4\psi^\ddagger L_iP_-L_\alpha\psi = -x_i\theta_\alpha + \frac{1}{2}(\sigma_i)_{\beta\alpha}(\theta_\beta + \vartheta_\beta),\\
&4\psi^\ddagger L_\alpha P_-L_\beta\psi = -\theta_\alpha\theta_\beta + \frac{1}{2}(\epsilon\sigma_i)_{\alpha\beta}x_i + \frac{3}{2}\epsilon_{\alpha\beta}g - 2\epsilon_{\alpha\beta}.
\end{aligned} \tag{248}$$

Here, $\vartheta_\alpha$ and $g$ are defined by

$$\vartheta_\alpha = 2\psi^\ddagger D_\alpha\psi, \qquad z = \psi^\ddagger H\psi, \tag{249}$$

[12] $\Psi$ and $\psi$ respectively satisfy

$$\begin{aligned}
&L_i\Psi\cdot X_i + \epsilon_{\alpha\beta}L_\alpha\Psi\cdot\Theta_\beta = \frac{1}{2}\Psi(\Psi^\dagger\Psi + 1),\\
&L_i\psi\cdot x_i + \epsilon_{\alpha\beta}L_\alpha\psi\cdot\theta_\beta = \frac{1}{2}\psi.
\end{aligned} \tag{241}$$

[13] With $P_+ = \psi\psi^\ddagger$, $P_-$ (247) satisfies the following relations,

$$\begin{aligned}
&P_+\psi = \psi, \quad P_-\psi = 0,\\
&P_+ + P_- = 1, \quad {P_\pm}^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0.
\end{aligned} \tag{246}$$

where $D_\alpha$ and $H$ are

$$D_\alpha = \frac{1}{2}\begin{pmatrix} 0 & -\tau_\alpha \\ -(\epsilon\tau_\alpha)^t & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{250}$$

with $\tau_1 = (1,0)^t$, $\tau_2 = (0,1)^t$ and $\epsilon = i\sigma_2$. They are considered as "new emerging coordinates" by quantum fluctuation. The corresponding fuzzy coordinates of $x_i$, $\theta_\alpha$, $\vartheta_\alpha$ and $z$ are $X_i$, $\Theta_\alpha$, $\mathit{\Theta}_\alpha$ and $Z$ defined in (297) (see Appendix A.1.1), and they amount to the $SU(2,2)$ algebra. Thus, the hidden $SU(2|1)$ structure appears as quantum fluctuation of fuzzy two-supersphere.

### 6.2 $N = 1$ fuzzy four-supersphere

With similar manner to Sec.6.1, the supercoherent state on $N = 1$ fuzzy four-supersphere, $|\omega\rangle$, is introduced as

$$(x_a X_a + 2C_{\alpha\beta}\theta_\alpha\Theta_\beta)|\omega\rangle = n|\omega\rangle, \tag{251}$$

where $X_a, \Theta_\alpha$ (147) and $x_a, \theta_\alpha$ (129) are coordinates on $S_F^{4|2}$ and $S^{4|2}$, respectively. Explicitly, $|\omega\rangle$ is given by

$$|\omega\rangle = \frac{1}{\sqrt{n!}}(\Psi^\dagger\psi)^n|0\rangle = \frac{1}{\sqrt{n!}}(\psi_1\Psi_1^\dagger + \psi_2\Psi_2^\dagger + \psi_3\Psi_3^\dagger + \psi_4\Psi_4^\dagger - \eta\tilde{\Psi}^\dagger)^n|0\rangle, \tag{252}$$

where $\Psi$ and $\psi$ are respectively the graded Schwinger operator (148) and normalized spinor (126)[14]. $|\omega\rangle$ can be expanded by the graded fully symmetric representation (154). The dual state $\langle\langle\omega|$ satisfying

$$\langle\langle\omega|\omega\rangle = 1, \tag{254}$$

is given by

$$\langle\langle\omega| = \frac{1}{\sqrt{n!}}\langle 0|(\psi^\ddagger\Psi)^n. \tag{255}$$

The expectation values of $X_a$ and $\Theta_\alpha$ are

$$\langle\langle\omega|X_a|\omega\rangle = nx_a, \qquad \langle\langle\omega|\Theta_\alpha|\omega\rangle = n\theta_\alpha. \tag{256}$$

The correlation functions are

$$\begin{aligned} \langle\langle\omega|X_aX_b|\omega\rangle &= n^2x_ax_b + n\psi^\ddagger\Gamma_aP_-\Gamma_b\psi, \\ \langle\langle\omega|X_a\Theta_\alpha|\omega\rangle &= n^2x_a\theta_\alpha + n\psi^\ddagger\Gamma_aP_-L_\alpha\psi, \\ \langle\langle\omega|\Theta_\alpha\Theta_\beta|\omega\rangle &= n^2\theta_\alpha\theta_\beta + n\psi^\ddagger\Gamma_\alpha P_-\Gamma_\beta\psi, \end{aligned} \tag{257}$$

[14] $\Psi$ and $\psi$ satisfy

$$\begin{aligned} &\Gamma_a\Psi\cdot X_a + 2C_{\alpha\beta}\Gamma_\alpha\Psi\cdot\Theta_\beta = \Psi(\Psi^\dagger\Psi + 3), \\ &\Gamma_a\psi\cdot x_a + 2C_{\alpha\beta}\Gamma_\alpha\psi\cdot\theta_\beta = \psi. \end{aligned} \tag{253}$$

where

$$P_- = 1 - \psi\psi^\ddagger. \tag{258}$$

The second terms of the right-hand side of (257) are calculated as

$$\begin{aligned}
&\psi^\ddagger \Gamma_a P_- \Gamma_b \psi = -x_a x_b + 2i x_{ab} + \delta_{ab} g + \frac{4}{3}\delta_{ab}, \\
&\psi^\ddagger \Gamma_a P_- \Gamma_\alpha \psi = -x_a \theta_\alpha + \frac{1}{2}(\gamma_a)_{\beta\alpha}(\theta_\beta + \vartheta_\beta), \\
&\psi^\ddagger \Gamma_\alpha P_- \Gamma_\beta \psi = -\theta_\alpha \theta_\beta - \frac{1}{2}\sum_{a<b}(C\gamma_{ab})_{\alpha\beta} x_{ab} - \frac{1}{8}(C\gamma_a)_{\alpha\beta} x_a + \frac{5}{24} C_{\alpha\beta} g - \frac{1}{3} C_{\alpha\beta}.
\end{aligned} \tag{259}$$

Thus, for fuzzy four-supersphere, we have new coordinates, $x_{ab}$, $\vartheta_\alpha$, $z$, defined by

$$x_{ab} = \psi^\ddagger \Gamma_{ab} \psi, \quad \vartheta_\alpha = \psi^\ddagger D_\alpha \psi, \quad z = \psi^\ddagger H \psi. \tag{260}$$

Here, $\Gamma_{ab}$, $D_\alpha$ and $H$ are respectively (121), (167) and (171). The fuzzy coordinates corresponding to twenty four coordinates, $x_a$, $\theta_\alpha$, $x_{ab}$, $\vartheta_\alpha$, $z$, are, $X_a$, $\Theta_\alpha$, $X_{ab}$, $\mathit{\Theta}_\alpha$, $Z$, defined in Sec.4.2.4, which satisfy the $SU(4|1)$ algebra (see also Appendix A.2.1). Thus, we confirmed, similar to the fuzzy two-supersphere case, the enhanced $SU(4|1)$ structure is brought by quantum fluctuation of fuzzy four-supersphere.

# 7 Summary

We performed a systematic study of fuzzy superspheres and graded Hopf maps based on $UOSp(N|2)$ and $UOSp(N|4)$, respectively. For the positive definiteness of square of radius, the construction of fuzzy two-superspheres is restricted to $N = 1, 2$, and fuzzy four-superspheres to $N = 1, 2, 3, 4$ (see Table 1 and 2). The graded Hopf maps were introduced as the classical counterpart of the fuzzy superspheres. We derived the explicit realizations of the 1st and 2nd graded Hopf maps:

$$S^{3|2N} \xrightarrow{S^1} S^{2|2N}, \qquad S^{7|2N} \xrightarrow{S^3} S^{4|2N}. \tag{261}$$

The particular feature of the present construction is based on the super Lie algebraic structures. With use of the graded Schwinger operators, super Lie group symmetries are naturally incorporated and the graded fully symmetric representation is readily derived. Adoption of the graded fully symmetric representation brings a particular feature to fuzzy superspheres: fuzzy superspheres are represented as a "superposition" of fuzzy superspheres with lower supersymmetries. The algebras of the fuzzy two- and four-superspheres are enhanced from the original algebras, $UOSp(N|2)$ and $UOSp(N|4)$, to the larger algebras, $SU(2|N)$ and $SU(4|N)$, respectively. We also argued such enhancement in view of quantum fluctuation of fuzzy spheres by evaluating correlation functions.

Since the present work is a natural generalization of precedent low dimensional fuzzy superspheres, one could pursue similar applications performed in low dimensions, such as realization in string theory, construction of supersymmetric gauge theories on fuzzy superspheres. Applications to topologically nontrivial many-body models would be interesting, too. The Hopf maps have

| Fuzzy manifold | $S_F^2$ | $S_F^{2\|2}$ | $S_F^{2\|4}$ |
|---|---|---|---|
| Number of supersymmetry | $N=0$ | $N=1$ | $N=2$ |
| Original symmetry | $SO(3)$ | $UOSp(1\|2)$ | $UOSp(2\|2)$ |
| Enhanced symmetry | $SU(2)$ | $SU(2\|1)$ | $SU(2\|2)$ |
| Square of the radius | $n(n+2)$ | $n(n+1)$ | $n^2$ |

Table 1: Fuzzy two-superspheres and symmetries.

| Fuzzy manifold | $S_F^4$ | $S_F^{4\|2}$ | $S_F^{4\|4}$ | $S_F^{4\|6}$ | $S_F^{4\|8}$ |
|---|---|---|---|---|---|
| Number of supersymmetry | $N=0$ | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| Original symmetry | $SO(5)$ | $UOSp(1\|4)$ | $UOSp(2\|4)$ | $UOSp(3\|4)$ | $UOSp(4\|4)$ |
| Enhanced symmetry | $SU(4)$ | $SU(4\|1)$ | $SU(4\|2)$ | $SU(4\|3)$ | $SU(4\|4)$ |
| Square of the radius | $n(n+4)$ | $n(n+3)$ | $n(n+2)$ | $n(n+1)$ | $n^2$ |

Table 2: Fuzzy four-superspheres and symmetries.

applications in physics widely [54] and also in quantum computation [55]. It may be intriguing to see the roles of the graded Hopf maps in the context of superqubits [56].

In this work, we focused on the construction of fuzzy supersphere whose bosonic dimension is two or four. This is because of the restriction of isomorphism between unitary-symplectic and orthogonal groups, $USp(2) \simeq SO(3)$, $USp(4) \simeq SO(5)$. At the present, we do not know how to generalize the present construction to even higher dimensions. Another remaining mathematical issue we have not fully discussed is the bundle structure of the graded Hopf maps. At least, these may deserve further investigations.

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## Appendix

## A $SU(M|N)$ and fuzzy complex projective superspace

We summarize formulae about $SU(M|N)$ algebra. The dimension is

$$\dim[SU(M|N)] = M^2 + N^2 - 1|2MN = M^2 + 2MN + N^2 - 1. \tag{262}$$

The maximal bosonic subalgebra of $SU(M|N)$ is $SU(M) \oplus SU(N) \oplus U(1)$, and its fundamental representation matrices are given by

$$S_A = \begin{pmatrix} s_A & 0 \\ 0 & 0 \end{pmatrix}, \quad T_P = \begin{pmatrix} 0 & 0 \\ 0 & t_P \end{pmatrix}, \quad H = \frac{1}{N} \begin{pmatrix} N \cdot 1_M & 0 \\ 0 & M \cdot 1_N \end{pmatrix}, \tag{263}$$

with $A = 1, 2, \cdots, M^2 - 1$ and $P = 1, 2, \cdots, N^2 - 1$. $s_A$ and $t_P$ in (263) satisfy

$$[s_A, s_B] = i f_{ABC} s_C, \qquad [t_P, t_Q] = i f'_{PQR} t_R, \tag{264}$$

with $SU(M)$ and $SU(N)$ structure constants, $f_{ABC}$ and $f'_{PQR}$. The fermion generators are

$$Q_{\alpha\sigma} = \begin{pmatrix} 0_{M+\sigma-1} & \tau_\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_{N-\sigma} \end{pmatrix}, \qquad \tilde{Q}_{\sigma\alpha} = \begin{pmatrix} 0_{M+\sigma-1} & 0 & 0 \\ \tau_\alpha^t & 0 & 0 \\ 0 & 0 & 0_{N-\sigma} \end{pmatrix}, \tag{265}$$

where $\alpha$ stands for the $SU(M)$ spinor index ($\alpha = 1, 2, \cdots, M$), and $\sigma$ does the $SU(N)$ index ($\sigma = 1, 2, \cdots, N$), and

$$\tau_\alpha = (0, \cdots, 0, \overset{\alpha}{\check{1}}, 0, \cdots, 0)^t. \tag{266}$$

Therefore, the only non-zero components of $Q_{\alpha\sigma}$ and $\tilde{Q}_{\sigma\alpha}$ (265) are, $(\alpha, M + \sigma)$ and $(M + \sigma, \alpha)$, respectively:

$$(Q_{\alpha\sigma})_{\beta\tau} = \delta_{\alpha\beta}\delta_{M+\sigma,\tau}, \qquad (\tilde{Q}_{\sigma\alpha})_{\beta\tau} = \delta_{\beta,M+\sigma}\delta_{\tau\alpha}. \tag{267}$$

Then,

$$(Q_{\alpha\sigma})^t = \tilde{Q}_{\sigma\alpha}. \tag{268}$$

The $SU(M|N)$ algebra is given by

$$\begin{aligned}
&[S_A, S_B] = i f_{ABC} S_C, \qquad [S_A, Q_{\alpha\sigma}] = (s_A)_{\beta\alpha} Q_{\beta\sigma}, \qquad [S_A, \tilde{Q}_{\sigma\alpha}] = -(s_A)_{\alpha\beta} \tilde{Q}_{\sigma\beta}, \\
&[S_A, T_P] = 0, \qquad \{Q_{\alpha\sigma}, Q_{\beta\tau}\} = \{\tilde{Q}_{\sigma\alpha}, \tilde{Q}_{\tau\beta}\} = 0, \\
&\{Q_{\alpha\sigma}, \tilde{Q}_{\tau\beta}\} = 2\delta_{\sigma\tau}(s_A)_{\beta\alpha} S_A + 2\delta_{\alpha\beta}(t_P)_{\sigma\tau} T_P + \frac{1}{M}\delta_{\sigma\tau}\delta_{\alpha\beta} H, \\
&[Q_{\alpha\sigma}, T_P] = (t_P)_{\sigma\tau} Q_{\alpha\tau}, \qquad [\tilde{Q}_{\sigma\alpha}, T_P] = -(t_P)_{\tau\sigma} \tilde{Q}_{\tau\alpha}, \\
&[T_P, T_Q] = i f'_{PQR} T_R, \qquad [S_A, \Gamma] = [T_P, H] = 0, \\
&[Q_{\alpha\sigma}, H] = \frac{M - N}{N} Q_{\alpha\sigma}, \qquad [\tilde{Q}_{\sigma\alpha}, H] = -\frac{M - N}{N} \tilde{Q}_{\sigma\alpha},
\end{aligned} \tag{269}$$

and the Casimir is

$$\mathcal{C} = 2 \sum_{A=1}^{M^2-1} S_A S_A - \sum_{\alpha=1}^{M} \sum_{\sigma=1}^{N} (Q_{\alpha\sigma}\tilde{Q}_{\sigma\alpha} - \tilde{Q}_{\sigma\alpha} Q_{\alpha\sigma}) - 2 \sum_{P=1}^{N^2-1} T_P T_P - \frac{N}{M(M-N)} H^2. \tag{270}$$

We apply the Schwinger construction to $X = S_A, Q_{\alpha\sigma}, \tilde{Q}_{\sigma\alpha}, T_P$, $H$:

$$\hat{X} = \Psi^\dagger X \Psi, \tag{271}$$

where

$$\Psi = (\Psi_1, \Psi_2, \cdots, \Psi_M, \tilde{\Psi}_1, \tilde{\Psi}_2, \cdots, \tilde{\Psi}_N)^t, \tag{272}$$

satisfying

$$[\Psi_\alpha, \Psi_\beta^\dagger] = \delta_{\alpha\beta}, \quad \{\tilde{\Psi}_\sigma, \tilde{\Psi}_\tau^\dagger\} = \delta_{\sigma\tau}. \tag{273}$$

Inserting (271) to (270), the Casimir is expressed as

$$\mathcal{C} = \frac{M-N-1}{M-N}\hat{n}(\hat{n}+M-N), \tag{274}$$

with $\hat{n} = \Psi^\dagger\Psi$. Here, we used

$$\begin{aligned}
&\sum_{A=1}^{M^2-1} \hat{S}_A\hat{S}_A = \frac{M-1}{2M}\hat{n}_B(\hat{n}_B+M),\\
&\sum_{P=1}^{N^2-1} \hat{T}_P\hat{T}_P = -\frac{N+1}{2N}\hat{n}_F(\hat{n}_F-N),\\
&\sum_{\alpha=1}^{M}\sum_{\sigma=1}^{N}(\hat{Q}_{\alpha\sigma}\hat{\tilde{Q}}_{\sigma\alpha} - \hat{\tilde{Q}}_{\sigma\alpha}\hat{Q}_{\alpha\sigma}) = N\hat{n}_B - 2\hat{n}_B\hat{n}_F - M\hat{n}_F,\\
&\hat{H}^2 = \frac{1}{N^2}(N\hat{n}_B + M\hat{n}_F)^2,
\end{aligned} \tag{275}$$

with $\hat{n}_B = \Sigma_{\alpha=1}^{M}\Psi_\alpha^\dagger\Psi_\alpha$, $\hat{n}_F = \sum_{\sigma=1}^{N}\tilde{\Psi}_\sigma^\dagger\tilde{\Psi}_\sigma$. The Casimir eigenvalues are regarded as the square of the radius of fuzzy complex projective superspaces, $\mathbb{C}P_F^{M-1|N}$. Notice that the coefficient of the right-hand side of (274) vanishes for $M = N+1$, and is not well defined for $M = N$. However, the ratio of the two different fuzzy complex projective superspaces are well defined even in such cases, and then square of the radius of $\mathbb{C}P_F^{M-1|N}$ may be considered as

$$n(n+M-N), \tag{276}$$

up to a proportional factor.

The classical counterpart of (271) reads as

$$x = \psi^\dagger X\psi, \tag{277}$$

where $\psi$ is a normalized $SU(M|N)$ spinor $\psi = (\psi_1, \psi_2, \cdots, \psi_M, \tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_N)^t$ with $\psi^\dagger\psi = 1$ regarded as coordinates on $S^{2M-1|2N}$. $U(1)$ phase of $\psi$ is canceled in (277), and thus (277) signifies a generalized graded 1st Hopf map,

$$S^{2M-1|2N} \xrightarrow{S^1} \mathbb{C}P^{M-1|N}. \tag{278}$$

See Ref.[45] for more details about $\mathbb{C}P_F^{M-1|N}$.

### A.1 $SU(2|N)$ and $\mathbb{C}P_F^{1|N}$

The dimension of $SU(2|N)$ algebra is

$$dim[SU(2|N)] = N^2 + 3|4N = N^2 + 4N + 3. \tag{279}$$

The bosonic generators (263) are given by

$$L_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad T_P = \begin{pmatrix} 0 & 0 \\ 0 & t_P \end{pmatrix}, \quad H = \frac{1}{N}\begin{pmatrix} N \cdot 1_2 & 0 \\ 0 & 2 \cdot 1_N \end{pmatrix}, \tag{280}$$

which respectively correspond to $SU(2)$, $SU(N)$ and $U(1)$ generators. To clarify relations to the subalgebra $UOSp(N|2)$, we separate the $SU(N)$ generators into symmetric and antisymmetric matrices:

$${T_S}^t = T_S, \qquad {T_I}^t = -T_I, \tag{281}$$

with $S = 1, 2, \cdots, N(N+1)/2 - 1$ and $I = 1, 2, \cdots, N(N-1)/2$. Note $T_I$ are pure imaginary antisymmetric matrices that satisfy the $SO(N)$ algebra by themselves. Instead of $Q_{\alpha\sigma}$ and $\tilde{Q}_{\sigma\alpha}$ (265), we introduce the following fermionic generators

$$L_{\alpha\sigma} = \frac{1}{2}\begin{pmatrix} 0_{1+\sigma} & \tau_\alpha & 0 \\ -(\epsilon\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0_{N-\sigma} \end{pmatrix}, \quad D_{\alpha\sigma} = \frac{1}{2}\begin{pmatrix} 0_{1+\sigma} & -\tau_\alpha & 0 \\ -(\epsilon\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0_{N-\sigma} \end{pmatrix}, \tag{282}$$

related to $Q_{\alpha\sigma}$ and $\tilde{Q}_{\sigma\alpha}$ as

$$Q_{\alpha\sigma} = L_{\alpha\sigma} - D_{\alpha\sigma}, \qquad \tilde{Q}_{\sigma\alpha} = -\epsilon_{\alpha\beta}(L_{\beta\sigma} + D_{\beta\sigma}), \tag{283}$$

or

$$L_{\alpha\sigma} = \frac{1}{2}(Q_{\alpha\sigma} + \epsilon_{\alpha\beta}\tilde{Q}_{\sigma\beta}), \qquad D_{\alpha\sigma} = -\frac{1}{2}(Q_{\alpha\sigma} - \epsilon_{\alpha\beta}\tilde{Q}_{\sigma\beta}). \tag{284}$$

Therefore,

$$Q_{\alpha\sigma}\tilde{Q}_{\sigma\alpha} - \tilde{Q}_{\sigma\alpha}Q_{\alpha\sigma} = -2\epsilon_{\alpha\beta}(L_{\alpha\sigma}L_{\beta\sigma} - D_{\alpha\sigma}D_{\beta\sigma}). \tag{285}$$

(282) are naturally regarded as $UOSp(N|2)$ spinors as we shall see below. With such matrices, $SU(2|N)$ algebra is written as

$$\begin{aligned}
&[L_i, L_j] = i\epsilon_{ijk}L_k, \\
&[L_i, L_{\alpha\sigma}] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_{\beta\sigma}, \qquad [L_i, D_{\alpha\sigma}] = \frac{1}{2}(\sigma_i)_{\beta\alpha}D_{\beta\sigma}, \\
&\{L_{\alpha\sigma}, L_{\beta\tau}\} = -\{D_{\alpha\sigma}, D_{\beta\tau}\} = \frac{1}{2}\delta_{\sigma\tau}(\epsilon\sigma_i)_{\alpha\beta}L_i - \epsilon_{\alpha\beta}(t_I)_{\sigma\tau}T_I, \\
&\{L_{\alpha\sigma}, D_{\beta\tau}\} = -\epsilon_{\alpha\beta}(t_S)_{\sigma\tau}T_S - \frac{1}{4}\epsilon_{\alpha\beta}\delta_{\sigma\tau}H, \\
&[L_{\alpha\sigma}, T_S] = -(t_S)_{\sigma\tau}D_{\alpha\tau}, \qquad [L_{\alpha\sigma}, T_I] = (t_I)_{\sigma\tau}L_{\alpha\tau}, \\
&[D_{\alpha\sigma}, T_S] = -(t_S)_{\sigma\tau}L_{\alpha\tau}, \qquad [D_{\alpha\sigma}, T_I] = (t_I)_{\sigma\tau}D_{\alpha\tau}, \\
&[L_{\alpha\sigma}, H] = -\frac{2-N}{N}D_{\alpha\sigma}, \qquad [D_{\alpha\sigma}, H] = -\frac{2-N}{N}L_{\alpha\sigma}, \\
&[T_P, T_Q] = if'_{PQR}T_R.
\end{aligned} \tag{286}$$

From (286), one may see that $L_i$, $L_{\alpha\sigma}$, $T_I$ satisfy a closed subalgebra, the $UOSp(N|2)$.

We introduce the fuzzy coordinates of $\mathbb{C}P_F^{1|N}$ as

$$\begin{aligned}
&X_i = 2\Psi^\dagger L_i \Psi, \qquad \Theta_\alpha^{(\sigma)} = 2\Psi^\dagger L_{\alpha\sigma}\Psi, \qquad \mathit{\Theta}_\alpha^{(\sigma)} = 2\Psi^\dagger D_{\alpha\sigma}\Psi, \\
&Y_P = 2\Psi^\dagger T_P \Psi, \qquad Z = \Psi^\dagger H \Psi,
\end{aligned} \tag{287}$$

with the Schwinger operator $\Psi = (\Psi_1, \Psi_2, \tilde{\Psi}_1, \cdots, \tilde{\Psi}_N)^t$. Square of the radius of $\mathbb{C}P_F^{1|N}$ is derived as

$$\begin{aligned}
&\sum_{i=1}^{3} X_i X_i + \sum_{\alpha,\beta=1}^{2}\sum_{\sigma=1}^{N} \epsilon_{\alpha\beta}(\Theta_\alpha^{(\sigma)}\Theta_\beta^{(\sigma)} - \mathit{\Theta}_\alpha^{(\sigma)}\mathit{\Theta}_\beta^{(\sigma)}) - \sum_{P=1}^{N^2-1} Y_P Y_P - \frac{N}{2-N} Z^2 \\
&= \frac{2(1-N)}{2-N}\hat{n}(\hat{n}+2-N),
\end{aligned} \tag{288}$$

where $\hat{n} = \Psi^\dagger\Psi$. Here, we used

$$\begin{aligned}
&X_i X_i = \hat{n}_B(\hat{n}_B + 2), \\
&\epsilon_{\alpha\beta}(\Theta_\alpha^{(\sigma)}\Theta_\beta^{(\sigma)} - \mathit{\Theta}_\alpha^{(\sigma)}\mathit{\Theta}_\beta^{(\sigma)}) = -2N\hat{n}_B + 4\hat{n}_B\hat{n}_F + 4\hat{n}_F, \\
&Y_P Y_P = -\frac{2(N+1)}{N}\hat{n}_F(\hat{n}_F - N), \\
&Z^2 = (\hat{n}_B + \frac{2}{N}\hat{n}_F)^2,
\end{aligned} \tag{289}$$

with $\hat{n}_B = \Psi_1^\dagger\Psi_1 + \Psi_2^\dagger\Psi_2$ and $\hat{n}_F = \sum_{\sigma=1}^{N} \tilde{\Psi}_\sigma^\dagger \tilde{\Psi}_\sigma$. For $\mathbb{C}P_F^{1|N}$, square of the radius is proportional to

$$n(n+2-N). \tag{290}$$

Notice that, for $n < N-2$, (290) becomes negative. This situation is similar to $S_F^{2|2N}$ (see the discussions below (87)).

#### A.1.1 $SU(2|1)$

The dimension of $SU(2|1)$ is

$$\dim[SU(2|1)] = 4|4 = 8. \tag{291}$$

From (286), the $SU(2|1)$ algebra reads as

$$\begin{aligned}
&[L_i, L_j] = i\epsilon_{ijk}L_k, \qquad [L_i, L_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_\beta, \qquad [L_i, D_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}D_\beta, \\
&\{L_\alpha, L_\beta\} = -\{D_\alpha, D_\beta\} = \frac{1}{2}\delta_{\sigma\tau}(\epsilon\sigma_i)_{\alpha\beta}L_i, \qquad \{L_\alpha, D_\beta\} = -\frac{1}{4}H, \\
&[L_\alpha, H] = -D_\alpha, \qquad [D_\alpha, H] = -L_\alpha.
\end{aligned} \tag{292}$$

The $SU(2|1)$ algebra (292) is isomorphic to $UOSp(2|2)$ (59) with the identification $(L_\alpha, iD_\alpha) = L_{\alpha\sigma}$ and $H = 2i\Gamma$. The maximal bosonic subalgebra of $SU(2|1)$ is $SU(2)\oplus U(1)$. The $UOSp(1|2)$ is realized as the subalgebra by $L_i$ and $L_\alpha$ in (292). The $SU(2|1)$ irreducible representation is specified by "superspin" indices $j$ (integers or half-integers) and $g$ (complex value). For details, see Refs.[49, 57]. $SU(2|1)$ has two Casimirs, quadratic and cubic. The quadratic Casimir are

$$\mathcal{C} = L_i L_i + \epsilon_{\alpha\beta} L_\alpha L_\beta - \epsilon_{\alpha\beta} D_\alpha D_\beta - \frac{1}{4} H^2, \tag{293}$$

and the eigenvalues of quadratic Casimir are

$$\mathcal{C} = j^2 - g^2. \tag{294}$$

- Atypical Representation

When $g = \pm j$, the irreducible representation is called atypical representation. Since there is automorphism, $D_\alpha \to -D_\alpha$ and $H \to -H$ in (292), we discuss only the case $g = +j$. The dimension of the atypical representation is $4j+1$, which is already irreducible for the subalgebra $UOSp(1|2)$. In the present case, the quadratic Casimir eigenvalues (294) vanish identically. (Also, the cubic Casimir eigenvalues vanish since the eigenvalues are proportional to $g(j^2 - g^2)$ [49].) Thus, the two Casimirs do not specify atypical representation. The fundamental representation of $SU(2|1)$ is the simplest atypical representation given by the following $3 \times 3$ matrices[15]:

$$L_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad L_\alpha = \frac{1}{2}\begin{pmatrix} 0_2 & \tau_\alpha \\ -(\epsilon\tau_\alpha)^t & 0 \end{pmatrix}, \quad D_\alpha = \frac{1}{2}\begin{pmatrix} 0_2 & -\tau_\alpha \\ -(\epsilon\tau_\alpha)^t & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{296}$$

where $\epsilon = i\sigma_2$, $\tau_1 = (1,0)^t$ and $\tau_2 = (0,1)^t$. For the matrices (296), one may readily check that $\mathcal{C}$ (293) vanishes. The Schwinger construction

$$X_i = 2\Psi^\dagger L_i \Psi, \quad \Theta_\alpha = 2\Psi^\dagger L_\alpha \Psi, \quad \mathit{\Theta}_\alpha = 2\Psi^\dagger D_\alpha \Psi, \quad Z = \Psi^\dagger H \Psi, \tag{297}$$

corresponds to atypical representation. The $UOSp(1|2)$ invariant quantity is given by

$$X_i X_i + \epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta = \epsilon_{\alpha\beta}\mathit{\Theta}_\alpha\mathit{\Theta}_\beta + Z^2 = \hat{n}(\hat{n}+1), \tag{298}$$

where $\hat{n} = \Psi^\dagger\Psi$. The $SU(2|1)$ Casimir for the Schwinger construction identically vanishes:

$$X_i X_i + \epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta - \epsilon_{\alpha\beta}\mathit{\Theta}_\alpha\mathit{\Theta}_\beta - Z^2 = 0, \tag{299}$$

which implies that the Schwinger construction corresponds to atypical representation with $g = j = n/2$.

- Typical Representation

[15] $D_\alpha$ and $H$ in (296) are constructed as

$$D_\alpha = -\frac{1}{2(j+\frac{1}{4})}(\sigma_i)_{\beta\alpha}\{L_i, L_\beta\}, \qquad H = \frac{1}{j+\frac{1}{4}}\Big(\epsilon_{\alpha\beta}L_\alpha L_\beta + 2j\Big(j+\frac{1}{2}\Big)\Big), \tag{295}$$

with $j = 1/2$.

The typical representation refers to $g \neq \pm j$. The simplest matrices of the typical representation are the following $4\times 4$ matrices[16]

$$L_i=\frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & 0_2\end{pmatrix},\quad L_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & \tau_\alpha & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad L'_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & 0 & \tau_\alpha\\ 0 & 0 & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\end{pmatrix},\quad \Gamma=\frac{1}{2}\begin{pmatrix}0_2 & 0\\ 0 & \epsilon\end{pmatrix}. \tag{302}$$

In the Schwinger construction with $\Psi=(\Psi_1,\Psi_2,\tilde{\Psi}_1,\tilde{\Psi}_2)^t$,

$$\hat{L}_i=\Psi^\dagger L_i\Psi,\quad \hat{L}_\alpha=\Psi^\dagger L_\alpha\Psi,\quad \hat{L}'_\alpha=\Psi^\dagger L'_\alpha\Psi,\quad \hat{\Gamma}=\Psi^\dagger\Gamma\Psi, \tag{303}$$

the quadratic Casimir is derived as

$$\mathcal{C}=\hat{L}_i\hat{L}_i+\epsilon_{\alpha\beta}\hat{L}_\alpha\hat{L}_\beta+\epsilon_{\alpha\beta}\hat{L}'_\alpha\hat{L}'_\beta+\hat{\Gamma}^2=\frac{1}{4}(\Psi^\dagger\Psi)^2. \tag{304}$$

The eigenvalue is $n^2/4$ and the corresponding eigenstates are given by (68). The $SU(2|1)$ typical representation for $(j,g)$ consists of $|j,j_3,g\rangle$, $|j-1/2,j_3,g+1/2\rangle$, $|j-1/2,j_3,g-1/2\rangle$ and $|j-1,j_3,g\rangle$ [49] with the Casimir eigenvalue (294). It is straightforward to see that the Schwinger construction corresponds to $(j,g)=(n/2,0)$.

### A.1.2 $SU(2|2)$

The dimension of $SU(2|2)$ is

$$\dim[SU(2|2)]=7|8=15. \tag{305}$$

The fundamental representation matrices are

$$L_i=\frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & 0_2\end{pmatrix},\quad L_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & \tau_\alpha & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad D_\alpha=-\frac{1}{2}\begin{pmatrix}0_2 & \tau_\alpha & 0\\ (\epsilon\tau_\alpha)^t & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$
$$L'_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & 0 & \tau_\alpha\\ 0 & 0 & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\end{pmatrix},\quad D'_\alpha=-\frac{1}{2}\begin{pmatrix}0_2 & 0 & \tau_\alpha\\ 0 & 0 & 0\\ (\epsilon\tau_\alpha)^t & 0 & 0\end{pmatrix},\quad T_i=\frac{1}{2}\begin{pmatrix}0_2 & 0\\ 0 & \sigma_i\end{pmatrix},$$
$$H=1_4. \tag{306}$$

[16] The typical representation matrices (302) are superficially different from those in Ref.[49]:

$$L_i=\frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & 0_2\end{pmatrix},\quad L_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & \tau_\alpha & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad D'_\alpha=\frac{1}{2}\begin{pmatrix}0_2 & 0 & -\tau_\alpha\\ 0 & 0 & 0\\ -(\epsilon\tau_\alpha)^t & 0 & 0\end{pmatrix},\quad \Gamma=\frac{1}{2}\begin{pmatrix}0_2 & 0\\ 0 & \sigma_1\end{pmatrix}. \tag{300}$$

In the case, the corresponding quadratic Casimir is given by

$$\mathcal{C}=L_iL_i+\epsilon_{\alpha\beta}L_\alpha L_\beta-\epsilon_{\alpha\beta}D'_\alpha D'_\beta-\Gamma^2. \tag{301}$$

With these, the $SU(2|2)$ algebra is expressed as[17]

$$
\begin{aligned}
&[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, L_{\alpha\sigma}] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_{\beta\sigma}, \quad [L_i, D_{\alpha\sigma}] = \frac{1}{2}(\sigma_i)_{\beta\alpha}D_{\beta\sigma},\\
&\{L_{\alpha\sigma}, L_{\beta\tau}\} = -\{D_{\alpha\sigma}, D_{\beta\tau}\} = \frac{1}{2}\delta_{\sigma\tau}(\epsilon\sigma_i)_{\alpha\beta}L_i + i\frac{1}{2}\epsilon_{\sigma\tau}\epsilon_{\alpha\beta}T_2,\\
&\{L_{\alpha\sigma}, D_{\beta\tau}\} = -\frac{1}{2}(\sigma_1)_{\sigma\tau}\epsilon_{\alpha\beta}T_1 - \frac{1}{2}(\sigma_3)_{\sigma\tau}\epsilon_{\alpha\beta}T_3 - \frac{1}{4}\delta_{\sigma\tau}\epsilon_{\alpha\beta}H,\\
&[L_{\alpha\sigma}, T_1] = -\frac{1}{2}(\sigma_1)_{\tau\sigma}D_{\alpha\tau}, \quad [D_{\alpha\sigma}, T_1] = -\frac{1}{2}(\sigma_1)_{\tau\sigma}L_{\alpha\tau},\\
&[L_{\alpha\sigma}, T_2] = -\frac{1}{2}(\sigma_2)_{\tau\sigma}L_{\alpha\tau}, \quad [D_{\alpha\sigma}, T_2] = -\frac{1}{2}(\sigma_2)_{\tau\sigma}D_{\alpha\tau},\\
&[L_{\alpha\sigma}, T_3] = -\frac{1}{2}(\sigma_3)_{\tau\sigma}D_{\alpha\tau}, \quad [D_{\alpha\sigma}, T_3] = -\frac{1}{2}(\sigma_3)_{\tau\sigma}L_{\alpha\tau},\\
&[L_i, T_j] = [L_{\alpha\sigma}, H] = [D_{\alpha\sigma}, H] = 0, \qquad (308)
\end{aligned}
$$

where $L_{\alpha\sigma} = (L_\alpha, L'_\alpha)$ and $D_{\alpha\sigma} = (D_\alpha, D'_\alpha)$. The $UOSp(2|2)$ algebra (292) is a subalgebra of (308) realized by $L_i$, $L_{\alpha\sigma}$ and $\Gamma = iT_2$.

### A.2 $SU(4|N)$ and $\mathbb{C}P_F^{3|N}$

The dimension of the $SU(4|N)$ is given by

$$\dim[SU(4|N)] = 15 + N^2|8N = N^2 + 8N + 15. \qquad (309)$$

For instance,

$$
\begin{aligned}
&\dim[SU(4|1)] = 16|8 = 24, \qquad \dim[SU(4|2)] = 19|16 = 35,\\
&\dim[SU(4|3)] = 24|24 = 48, \qquad \dim[SU(4|4)] = 31|32 = 63. \qquad (310)
\end{aligned}
$$

To clarify relations to $UOSp(N|4)$, we adopt the following "decomposition". We separate the $SU(4)$ generators into $SO(5)$ vector and antisymmetric 2 rank tensor:

$$S_A = \frac{1}{2\sqrt{2}}\Gamma_a = \frac{1}{\sqrt{2}}\begin{pmatrix}\gamma_a & 0\\ 0 & 0_N\end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\Gamma_{ab} = \frac{1}{\sqrt{2}}\begin{pmatrix}\gamma_{ab} & 0\\ 0 & 0_N\end{pmatrix}, \qquad (311)$$

with $\gamma_a$ (90) and $\gamma_{ab}$ (99). Notice that $\gamma_a$ and $\gamma_{ab}$ have different properties under transpose

$$(C\gamma_a)^t = -C\gamma_a, \qquad (C\gamma_{ab})^t = C\gamma_{ab}, \qquad (312)$$

[17]Since $H = 1_4$ commutes with all of the other fourteen generators, $H$ is the center of the $SU(2|2)$ algebra. The $pSU(2|2)$ algebra is defined by quenching $H$, and then

$$\dim\ [pSU(2|2)] = 6|8. \qquad (307)$$

There do not exit $4 \times 4$ dimensional matrices that satisfy $pSU(2|2)$ algebra. The minimum dimension matrices of $pSU(2|2)$ are $14 \times 14$ matrices, *i.e.* the adjoint representation. With $4 \times 4$ fundamental representation matrices (306), one may nevertheless discuss $pSU(2|2)$ by identifying matrices modulo $H$.

with $C$ being an $SO(5)$ charge conjugation matrix (114). We also separate the $SU(N)$ generators $T_P$ $(P = 1, 2, \cdots, N^2-1)$ into symmetric and antisymmetric matrices:

$$T_S{}^t = T_S, \qquad T_I{}^t = -T_I, \tag{313}$$

with $S = 1, 2, \cdots, N(N+1)/2 - 1$ and $I = 1, 2, \cdots, N(N-1)/2$. $T_I$ satisfy the $SO(N)$ algebra by themselves. Also, the $U(1)$ generator is given by

$$H = \frac{1}{N}\begin{pmatrix} N \cdot 1_4 & 0 \\ 0 & 4 \cdot 1_N \end{pmatrix}. \tag{314}$$

The $SU(4|N)$ fermionic generators are

$$\Gamma_{\alpha\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_{3+\sigma} & \tau_\alpha & 0 \\ -(C\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0_{k-\sigma} \end{pmatrix}, \qquad D_{\alpha\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_{3+\sigma} & \tau_\alpha & 0 \\ (C\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0_{k-\sigma} \end{pmatrix}, \tag{315}$$

which are related to (265) as

$$Q_{\alpha\sigma} = \frac{1}{\sqrt{2}}(\Gamma_{\alpha\sigma} + D_{\alpha\sigma}), \qquad \tilde{Q}_{\sigma\alpha} = -\frac{1}{\sqrt{2}}C_{\alpha\beta}(\Gamma_{\beta\sigma} - D_{\beta\sigma}), \tag{316}$$

or

$$\Gamma_{\alpha\sigma} = \frac{1}{\sqrt{2}}(Q_{\alpha\sigma} + C_{\alpha\beta}\tilde{Q}_{\sigma\beta}), \qquad D_{\alpha\sigma} = \frac{1}{\sqrt{2}}(Q_{\alpha\sigma} - C_{\alpha\beta}\tilde{Q}_{\sigma\beta}). \tag{317}$$

Therefore,

$$Q_{\alpha\sigma}\tilde{Q}_{\sigma\alpha} - \tilde{Q}_{\sigma\alpha}Q_{\alpha\sigma} = -C_{\alpha\beta}(\Gamma_{\alpha\sigma}\Gamma_{\beta\sigma} - D_{\alpha\sigma}D_{\beta\sigma}). \tag{318}$$

The $SU(4|N)$ commutation relations (269) concerned with the fermionic generators read as

$$\begin{aligned}
&[\Gamma_a, \Gamma_{\alpha\sigma}] = (\gamma_a)_{\beta\alpha}D_{\beta\sigma}, \qquad [\Gamma_{ab}, \Gamma_{\alpha\sigma}] = (\gamma_{ab})_{\beta\alpha}\Gamma_{\beta\sigma}, \\
&[\Gamma_a, D_{\alpha\sigma}] = (\gamma_a)_{\beta\alpha}\Gamma_{\beta\sigma}, \qquad [\Gamma_{ab}, D_{\alpha\sigma}] = (\gamma_{ab})_{\beta\alpha}D_{\beta\sigma}, \\
&\{\Gamma_{\alpha\sigma}, \Gamma_{\beta\tau}\} = -\{D_{\alpha\sigma}, D_{\beta\tau}\} = \delta_{\sigma\tau}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab} - 2C_{\alpha\beta}(t_I)_{\sigma\tau}T_I, \\
&\{\Gamma_{\alpha\sigma}, D_{\beta\tau}\} = \frac{1}{4}\delta_{\sigma\tau}(C\gamma_a)_{\alpha\beta}\Gamma_a + 2C_{\alpha\beta}(t_S)_{\sigma\tau}T_S + \frac{1}{4}C_{\alpha\beta}\delta_{\sigma\tau}H, \\
&[\Gamma_{\alpha\sigma}, T_S] = (t_S)_{\sigma\tau}D_{\alpha\tau}, \qquad [\Gamma_{\alpha\sigma}, T_I] = (t_I)_{\sigma\tau}\Gamma_{\alpha\tau}, \\
&[D_{\alpha\sigma}, T_S] = (t_S)_{\sigma\tau}\Gamma_{\alpha\tau}, \qquad [D_{\alpha\sigma}, T_I] = (t_I)_{\sigma\tau}D_{\alpha\tau}, \\
&[\Gamma_{\alpha\sigma}, H] = \frac{4-N}{N}D_{\alpha\sigma}, \qquad [D_{\alpha\sigma}, H] = \frac{4-N}{N}\Gamma_{\alpha\sigma}.
\end{aligned} \tag{319}$$

Thus, one may find that $\Gamma_{ab}$, $\Gamma_{\alpha\sigma}$, $T_I$ satisfy a closed algebra, the $UOSp(N|4)$.

Square of the radius of $\mathbb{C}P_F^{3|N}$ is derived as

$$\begin{aligned}
&\sum_{A=1}^{15}\hat{S}_A\hat{S}_A - \frac{1}{2}\sum_{\alpha=1}^{4}\sum_{\sigma=1}^{k}(\hat{Q}_{\alpha\sigma}\hat{\tilde{Q}}_{\sigma\alpha} - \hat{\tilde{Q}}_{\sigma\alpha}\hat{Q}_{\alpha\sigma}) - \sum_{P=1}^{N^2-1}\hat{T}_P\hat{T}_P - \frac{N}{8(4-N)}\hat{Z}^2 \\
=&\frac{1}{8}\sum_{a=1}^{5}X_aX_a + \frac{1}{2}\sum_{a<b=1}^{5}X_{ab}X_{ab} + \frac{1}{2}\sum_{\alpha,\beta=1}^{4}\sum_{\sigma=1}^{N}C_{\alpha\beta}(\Theta_\alpha^{(\sigma)}\Theta_\beta^{(\sigma)} - \mathit{\Theta}_\alpha^{(\sigma)}\mathit{\Theta}_\beta^{(\sigma)}) - \frac{1}{4}\sum_{P=1}^{N^2-1}Y_PY_P - \frac{N}{8(4-N)}Z^2 \\
=&\frac{3-N}{2(4-N)}\hat{n}(\hat{n}+4-N),
\end{aligned} \tag{320}$$

where $\hat{n} = \Psi^\dagger\Psi$,

$$X_a = \Psi^\dagger\Gamma_a\Psi, \qquad X_{ab} = \Psi^\dagger\Gamma_{ab}\Psi, \qquad \Theta^\sigma_\alpha = \Psi^\dagger\Gamma_{\alpha\sigma}\Psi,$$
$$\mathit{\Theta}^{(\sigma)}_i = \Psi^\dagger D_{\alpha\sigma}\Psi, \quad Y_P = 2\Psi^\dagger T_P\Psi, \quad Z = \Psi^\dagger H\Psi. \tag{321}$$

We utilized

$$\sum_{A=1}^{15}\hat{S}_A\hat{S}_A = \frac{1}{8}\sum_a X_aX_a + \frac{1}{2}\sum_{a<b}X_{ab}X_{ab} = \frac{3}{8}\hat{n}_B(\hat{n}_B+4),$$
$$\sum_{\alpha=1}^{4}\sum_{\sigma=1}^{N}(\hat{Q}_{\alpha\sigma}\hat{\tilde{Q}}_{\sigma\alpha} - \hat{\tilde{Q}}_{\sigma\alpha}\hat{Q}_{\alpha\sigma}) = -C_{\alpha\beta}(\Theta^{(\sigma)}_\alpha\Theta^{(\sigma)}_\beta - \mathit{\Theta}^{(\sigma)}_\alpha\mathit{\Theta}^{(\sigma)}_\beta) = N\hat{n}_B - 2\hat{n}_B\hat{n}_F - 4\hat{n}_F,$$
$$\sum_{P=1}^{N^2-1}\hat{T}_P\hat{T}_P = \frac{1}{4}Y_PY_P = -\frac{N+1}{2N}\hat{n}_F(\hat{n}_F - N),$$
$$Z^2 = (\hat{n}_B + \frac{4}{N}\hat{n}_F)^2. \tag{322}$$

Thus, for $\mathbb{C}P^{3|N}_F$, square of the radius is proportional to

$$n(n+4-N). \tag{323}$$

#### A.2.1 $SU(4|1)$

The $24(=16|8)$ generators of $SU(4|1)$ are represented as

$$\Gamma_a = \begin{pmatrix}\gamma_a & 0\\ 0 & 0\end{pmatrix}, \quad \Gamma_{ab} = \begin{pmatrix}\gamma_{ab} & 0\\ 0 & 0\end{pmatrix}, \quad H = \begin{pmatrix}1_4 & 0\\ 0 & 4\end{pmatrix},$$
$$\Gamma_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix}0_4 & \tau_\alpha\\ -(C\tau_\alpha)^t & 0\end{pmatrix}, \quad D_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix}0_4 & \tau_\alpha\\ (C\tau_\alpha)^t & 0\end{pmatrix}, \tag{324}$$

which satisfy

$$[\Gamma_a,\Gamma_b] = 4i\Gamma_{ab}, \qquad [\Gamma_a,\Gamma_{bc}] = -i(\delta_{ab}\Gamma_c - \delta_{ac}\Gamma_b), \qquad [\Gamma_{ab},\Gamma_{cd}] = i(\delta_{ac}\Gamma_{bd} - \delta_{ad}\Gamma_{bc} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}),$$
$$[\Gamma_a,\Gamma_\alpha] = (\gamma_a)_{\beta\alpha}D_\beta, \quad [\Gamma_a, D_\alpha] = (\gamma_a)_{\beta\alpha}\Gamma_\beta,$$
$$[\Gamma_{ab},\Gamma_\alpha] = (\gamma_{ab})_{\beta\alpha}\Gamma_\beta, \quad [\Gamma_{ab}, D_\alpha] = (\gamma_{ab})_{\beta\alpha}D_\beta,$$
$$\{\Gamma_\alpha,\Gamma_\beta\} = \sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab}, \quad \{D_\alpha, D_\beta\} = -\sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab}, \quad \{\Gamma_\alpha, D_\beta\} = \frac{1}{4}(C\gamma_a)_{\alpha\beta}\Gamma_a + \frac{1}{4}C_{\alpha\beta}H,$$
$$[\Gamma_\alpha, H] = 3D_\alpha, \qquad [D_\alpha, H] = 3\Gamma_\alpha. \tag{325}$$

$\Gamma_{ab}$ and $\Gamma_\alpha$ satisfy a closed algebra, the $UOSp(1|4)$.

### A.2.2 $SU(4|2)$

The $SU(4|2)$ contain $35(=19|16)$ generators. The bosonic and fermionic generators are given by

$$\begin{aligned}&\Gamma_a, \quad \Gamma_{ab}, \quad T_i, \quad H,\\ &\Gamma_\alpha, \quad D_\alpha, \quad \Gamma'_\alpha, \quad D'_\alpha,\end{aligned} \tag{326}$$

where $\Gamma_a$, $\Gamma_{ab}$, $\Gamma_\alpha$ and $\Gamma'_\alpha$ are (178) and (177), $T_i$ and $H$ are $U(2)$ generators

$$T_i = \frac{1}{2}\begin{pmatrix} 0_4 & 0 \\ 0 & \sigma_i \end{pmatrix}, \qquad H = \begin{pmatrix} 1_4 & 0 \\ 0 & 2\cdot 1_2 \end{pmatrix}, \tag{327}$$

and $D_\alpha$ and $D'_\alpha$ are

$$D_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_4 & \tau_\alpha & 0 \\ (C\tau_\alpha)^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D'_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 0_4 & 0 & \tau_\alpha \\ 0 & 0 & 0 \\ (C\tau_\alpha)^t & 0 & 0 \end{pmatrix}. \tag{328}$$

The $SU(4|2)$ generators (326) satisfy

$$\begin{aligned}
&\{\Gamma_{\alpha\sigma}, \Gamma_{\beta\tau}\} = -\{D_{\alpha\sigma}, D_{\beta\tau}\} = \delta_{\sigma\tau}\sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab} + i\frac{1}{2}\epsilon_{\sigma\tau}C_{\alpha\beta}T_2,\\
&\{\Gamma_{\alpha\sigma}, D_{\beta\tau}\} = \frac{1}{4}\delta_{\sigma\tau}(C\gamma_a)_{\alpha\beta}\Gamma_a + (\sigma_1)_{\sigma\tau}C_{\alpha\beta}T_1 + (\sigma_3)_{\sigma\tau}C_{\alpha\beta}T_3 + \frac{1}{4}\delta_{\sigma\tau}C_{\alpha\beta}H,\\
&[\Gamma_{\alpha\sigma}, T_1] = \frac{1}{2}(\sigma_1)_{\tau\sigma}D_{\alpha\tau}, \qquad [D_{\alpha\sigma}, T_1] = \frac{1}{2}(\sigma_1)_{\tau\sigma}\Gamma_{\alpha\tau},\\
&[\Gamma_{\alpha\sigma}, T_2] = -\frac{1}{2}(\sigma_2)_{\tau\sigma}\Gamma_{\alpha\tau}, \qquad [D_{\alpha\sigma}, T_2] = -\frac{1}{2}(\sigma_2)_{\tau\sigma}D_{\alpha\tau},\\
&[\Gamma_{\alpha\sigma}, T_3] = \frac{1}{2}(\sigma_3)_{\tau\sigma}D_{\alpha\tau}, \qquad [D_{\alpha\sigma}, T_3] = \frac{1}{2}(\sigma_3)_{\tau\sigma}\Gamma_{\alpha\tau},\\
&[\Gamma_{\alpha\sigma}, H] = D_{\alpha\sigma}, \qquad [D_{\alpha\sigma}, H] = \Gamma_{\alpha\sigma},\\
&[T_i, T_j] = i\epsilon_{ijk}T_k, \qquad [\Gamma_a, H] = [\Gamma_{ab}, H] = [T_i, H] = 0,\\
&[\Gamma_a, T_i] = [\Gamma_{ab}, T_i] = 0,
\end{aligned} \tag{329}$$

where $\Gamma_{\alpha\sigma} = (\Gamma_\alpha, \Gamma'_\alpha)$ and $D_{\alpha\sigma} = (D_\alpha, D'_\alpha)$. The $UOSp(2|4)$ (175) is realized as a subalgebra of $SU(4|2)$ (329) with $\Gamma_{ab}, \Gamma_{\alpha\sigma}$ and $\Gamma = 2iT_2$.

# B Charge conjugation matrices of $SO(5)$ and $UOSp(1|4)$

The complex representation of $SO(5)$, $\gamma_a$ (90) and $\gamma_{ab}$ (99), is given by

$$\tilde{\gamma}_a = \gamma^* = \gamma_a^t, \qquad \tilde{\gamma}_{ab} = -i\frac{1}{4}[\tilde{\gamma}_a, \tilde{\gamma}_b] = -\gamma_{ab}^* = -\gamma_{ab}^t. \tag{330}$$

The $SO(5)$ charge conjugation matrix (114) acts as

$$C^t\gamma_a C = \tilde{\gamma}_a, \qquad C^t\gamma_{ab}C = \tilde{\gamma}_{ab}. \tag{331}$$

$C\gamma_{ab}$ and $\gamma_{ab}C$ are symmetric matrices, while $C\gamma_a$ and $\gamma_a C$ are anti-symmetric matrices. $C$ has the following properties

$$C^\dagger = C^t = C^{-1} = -C, \qquad C^2 = -1, \tag{332}$$

and is related to the $USp(4)$ invariant matrix (see Sec.2)

$$J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \tag{333}$$

by unitary transformation, $J = V^\dagger C V$, with

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{334}$$

The unitary matrix (334) relates the $SO(5)$ matrices (99) to the bases of $USp(4)$ matrix (11), also.

The complex representation of $UOSp(1|4)$, $\Gamma_a$, $\Gamma_{ab}$ and $\Gamma_\alpha$ defined in Sec.4.2.1, is given by

$$\tilde{\Gamma}_a = \Gamma_a^t = \Gamma_a^*, \qquad \tilde{\Gamma}_{ab} = -\Gamma_{ab}^* = -\Gamma_{ab}^t, \qquad \tilde{\Gamma}_\alpha = C_{\alpha\beta}\Gamma_\beta. \tag{335}$$

The complex representation is related to the original representation as

$$\mathcal{R}^t\Gamma_a\mathcal{R} = \tilde{\Gamma}_a, \qquad \mathcal{R}^t\Gamma_{ab}\mathcal{R} = \tilde{\Gamma}_{ab}, \qquad \mathcal{R}^t\Gamma_\alpha\mathcal{R} = \tilde{\Gamma}_\alpha, \tag{336}$$

with the charge conjugation matrix

$$\mathcal{R} = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}. \tag{337}$$

# C Relations for matrix products

## C.1 $SU(2|1)$

With $3\times 3$ unit matrix $1_3$, the $SU(2|1)$ fundamental representation generators (296) span arbitrary $3\times 3$ matrix, and hence their products can be given by their linear combination. For $UOSp(1|2)$ matrices, $L_i$ and $L_\alpha$, their products are represented as

$$\begin{aligned} L_iL_j &= \frac{1}{4}\delta_{ij} + i\frac{1}{2}\epsilon_{ijk}L_k, \\ L_iL_\alpha &= \frac{1}{4}(\sigma_i)_{\beta\alpha}(L_\beta - D_\beta), \\ L_\alpha L_\beta &= \frac{1}{4}(\epsilon\sigma_i)_{\alpha\beta}L_i - \frac{1}{2}\epsilon_{\alpha\beta}(1 - \frac{3}{4}H). \end{aligned} \tag{338}$$

For the other $SU(2|1)$ fundamental representation matrices,

$$
\begin{aligned}
&L_i D_\alpha = -\frac{1}{4}(\sigma_i)_{\beta\alpha}(L_\beta - D_\beta),\\
&L_i H = L_i,\\
&L_\alpha D_\beta = \frac{1}{4}(\epsilon\sigma_i)_{\alpha\beta}L_i - \frac{1}{8}\epsilon_{\alpha\beta}H,\\
&L_\alpha H = -\frac{1}{2}D_\alpha + \frac{3}{2}L_\alpha,\\
&D_\alpha D_\beta = -\frac{1}{4}(\epsilon\sigma_i)_{\alpha\beta}L_i + \frac{1}{2}\epsilon_{\alpha\beta}(1 - \frac{3}{4}H),\\
&D_\alpha H = -\frac{1}{2}L_\alpha + \frac{3}{2}D_\alpha,\\
&H^2 = 3H - 2.
\end{aligned}
\tag{339}
$$

### C.2 $SU(4|1)$

Similar to the $SU(2|1)$ case, with $5\times 5$ unit matrix $1_5$, the $SU(4|1)$ fundamental representation matrices (324) span arbitrary $5\times 5$ matrix. Then, their products can be expressed by their linear combination: for the products of $\Gamma_a$ and $\Gamma_\alpha$,

$$
\begin{aligned}
&\Gamma_a\Gamma_b = 2i\Gamma_{ab} - \frac{1}{3}\delta_{ab}(H-4),\\
&\Gamma_a\Gamma_\alpha = \frac{1}{2}(\gamma_a)_{\beta\alpha}(\Gamma_\beta + D_\beta),\\
&\Gamma_\alpha\Gamma_\beta = -\frac{1}{2}\sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab} - \frac{1}{8}(C\gamma_a)_{\alpha\beta}\Gamma_a - \frac{1}{3}C_{\alpha\beta}(1 - \frac{5}{8}H),
\end{aligned}
\tag{340}
$$

and for the other $SU(4|1)$ matrices,

$$
\begin{aligned}
&\Gamma_a \Gamma_{bc} = \frac{1}{2}(\epsilon_{abcde}\Gamma_{de} + i\Gamma_b \delta_{ac} - i\Gamma_c \delta_{ab}), \\
&\Gamma_a D_\alpha = \frac{1}{2}(\gamma_a)_{\beta\alpha}(\Gamma_\beta + D_\beta), \\
&\Gamma_a \Gamma = \Gamma_a, \\
&\Gamma_{ab}\Gamma_{cd} = 2i(\delta_{ab}\Gamma_{cd} - \delta_{ac}\Gamma_{bd} + \delta_{ad}\Gamma_{bc} - \delta_{bc}\Gamma_{da} + \delta_{bd}\Gamma_{ca} - \delta_{cd}\Gamma_{ba}) - \epsilon_{abcde}\Gamma_e \\
&\qquad\qquad + \frac{1}{3}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})(4 - H), \\
&\Gamma_{ab}\Gamma_\alpha = \frac{1}{2}(\gamma_{ab})_{\beta\alpha}(\Gamma_\beta + D_\beta), \\
&\Gamma_{ab} D_\alpha = \frac{1}{2}(\gamma_{ab})_{\beta\alpha}(\Gamma_\beta + D_\beta), \\
&\Gamma_{ab} H = \Gamma_{ab}, \\
&\Gamma_\alpha D_\beta = -\frac{1}{2}\sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab} + \frac{1}{8}(C\gamma_a)_{\alpha\beta}\Gamma_a + \frac{1}{8}C_{\alpha\beta}H, \\
&\Gamma_\alpha H = -\frac{3}{2}D_\alpha - \frac{5}{2}\Gamma_\alpha, \\
&D_\alpha D_\beta = -\frac{1}{2}\sum_{a<b}(C\gamma_{ab})_{\alpha\beta}\Gamma_{ab} + \frac{1}{8}(C\gamma_a)_{\alpha\beta}\Gamma_a + \frac{1}{3}C_{\alpha\beta}(1 - \frac{5}{8}H), \\
&D_\alpha H = -\frac{5}{2}D_\alpha - \frac{3}{2}\Gamma_\alpha, \\
&H^2 = 5H - 4.
\end{aligned} \tag{341}
$$

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