Extended supersymmetry in massless conformal higher spin theory

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Abstract

We propose superfield equations in tensorial $\mathcal{N}$-extended superspaces to describe the $\mathcal{N} = 2, 4, 8$ supersymmetric generalizations of free conformal higher spin theories. These can be obtained by quantizing a superparticle model in $\mathcal{N}$-extended tensorial superspace. The $\mathcal{N}$-extended higher spin supermultiplets just contain scalar and 'spinor' fields in tensorial space so that, in contrast with the standard (super)space approach, no nontrivial generalizations of the Maxwell or Einstein equations to tensorial space appear when $\mathcal{N} > 2$. For $\mathcal{N} = 4, 8$, the higher spin-tensorial components of the extended tensorial superfields are expressed through additional scalar and spinor fields in tensorial space which obey the same free higher spin equations, but that are axion-like in the sense that they possess Peccei-Quinn-like symmetries.

1 Introduction

It is known [1, 2, 3, 4, 5, 6, 7, 8] that the $D = 4$ free conformal higher spin equations can be formulated as a field theory on a ten-dimensional tensorial space, $\Sigma^{(10|0)}$, parametrized by 10 bosonic coordinates (x^m, y^{mn}),

$$X^{\alpha\beta} = X^{\beta\alpha} = \frac{1}{4} x^m \gamma_m^{\alpha\beta} + \frac{1}{8} y^{mn} \gamma_{mn}^{\alpha\beta} \qquad \alpha, \beta = 1, 2, 3, 4, \quad m, n = 0, 1, 2, 3. \qquad (1.1)$$

and in $\mathcal{N} = 1$ tensorial superspace $\Sigma^{(10|4)}$ with coordinates $(X^{\alpha\beta}, \theta^\alpha)$.

The bosonic tensorial space (1.1) was proposed as a natural basis to build D=4 conformal higher spin theories in [1]. More general tensorial spaces of dimension $\frac{n(n+1)}{2}$ are introduced by means of a symmetric $n \times n$ coordinates matrix $X^{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, n$) which, for even $n = 4, 8, 16$, determine tensorial enlargements of the standard spacetimes of dimension D=4,6,10 ($D = \frac{n}{2} + 2$). Adding n fermionic coordinates θ^α one obtains the $\mathcal{N} = 1$ tensorial superspaces $\Sigma^{(\frac{n(n+1)}{2}|n)}$,

$$\Sigma^{(\frac{n(n+1)}{2}|n)} : \qquad Z^{\mathcal{M}} := (X^{\alpha\beta}, \theta^\alpha), \qquad \begin{cases} \alpha, \beta = 1, ..., n, \\ X^{\alpha\beta} = X^{\beta\alpha}; \end{cases} \qquad (1.2)$$

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rigid tensorial superspaces $\Sigma^{(\frac{n(n+1)}{2}|n)}$ have a supergroup structure. Taking n even is not a restriction if we think of the underlying spinorial origin of the α indices; furthermore this motivates also the restriction to $n = 2^k = 2, 4, 8, 16, ...$ assuming that θ^α are spinors. Although the fermionic coordinates in $\Sigma^{(\frac{n(n+1)}{2}|n)}$ are usually assumed to be real, in the $n = 4$, $D = 4$ case it is convenient to consider θ^α as a Majorana spinor in the Weyl realization of the Dirac matrices so that $\theta^\alpha = (\theta^A, \bar{\theta}_{\dot{A}})$.

For $n = 2$ the spin-tensorial coordinates $X^{\alpha\beta}$ are expressed in terms of the 3-vector spacetime coordinates, $X^{\alpha\beta} \propto x^a \tilde{\gamma}_a^{\alpha\beta}$ so that $\Sigma^{(3|2)}$ is just the standard $D = 3$, $\mathcal{N} = 1$ superspace. The case $n = 32$ gives the tensorial extension of $D{=}11$ superspace $\Sigma^{(528|32)}$, relevant in the context of the BPS preon hypothesis [9, 10] and also in the analysis of the hidden gauge structure of $D{=}11$ supergravity [11, 12]. In this paper we shall restrict ourselves to the $n{=}4,8,16$ cases that are used to describe massless conformal higher spin theories in $D = 4, 6, 10$. Almost all our equations (all but those in Sec. 4.2.1) will be valid for all these dimensions, although we shall make special emphasis on the $n = 4$ case corresponding to $D = 4$.

The first mechanical model in $\mathcal{N}{=}1$, $D{=}4$ tensorial superspace $\Sigma^{(10|4)}$ and in its higher dimensional generalizations $\Sigma^{(\frac{n(n+1)}{2}|n)}$ with $n > 4$ was proposed in [13], where it was noticed that the ground state of such a superparticle model describes BPS states preserving all but one supersymmetry. The possible 'constituent' rôle of such states in string/M-theory was introduced and discussed in general in [9], where they were called 'BPS preons' (see further [10]). Thus, from this viewpoint, the superparticle in [13] may be called 'preonic'. Its quantization was performed in [2], where it was shown that the spectrum of the quantized $n = 4$ preonic superparticle is given by a tower of massless conformal higher spin fields of all possible helicities and where evidence in favor that the $n = 8$ and $n = 16$ models describe conformal higher spin theories in $D = 6$ and $D = 10$ spacetimes was presented.

An elegant form of the bosonic and fermionic higher spin equations in $\Sigma^{(10|0)}$ tensorial space was given and studied in [3, 4]. The explicit form of the conformal higher spin equations in $D = 6, 10$ tensorial spaces was extracted from the their tensorial space version in [8]. The superfield form of the equations for the supermultiplets of $D = 6, 10$ massless conformal fields in the $\mathcal{N} = 1$ tensorial superspaces $\Sigma^{(36|8)}$ and $\Sigma^{(126|16)}$,

$$D = 10 \quad \Sigma^{(126|16)} \quad : \quad Z^{\mathcal{M}} := (X^{\alpha\beta}, \theta^\alpha) , \quad \begin{cases} \alpha, \beta = 1, ..., 16, \\ X^{\alpha\beta} = \frac{1}{16}x^m \tilde{\sigma}_m^{\alpha\beta} + \frac{1}{2\cdot5!}y^{m_1...m_5}\tilde{\sigma}_{m_1...m_5}^{\alpha\beta} , \\ m = 0, 1, \ldots 9 \end{cases}$$

(1.3)

was given in [7]. In particular, the $\mathcal{N} = 1$ supermultiplets of $D{=}4,6,10$ conformal higher spin fields are described by real scalar superfields on the corresponding $n{=}4,8,16$ $\Sigma^{(\frac{n(n+1)}{2}|n)}$ superspaces,

$$\Phi(X^{\alpha\beta}, \theta^\gamma) = b(X) + f_\alpha(X)\theta^\alpha + \sum_{i=2}^{n} \phi_{\alpha_1\cdots\alpha_i}(X)\theta^{\alpha_1}\cdots\theta^{\alpha_i} , \qquad (1.4)$$

obeying the following simple superfield equation [7]

$$D_{[\alpha}D_{\beta]}\Phi(X, \theta) = 0 . \qquad (1.5)$$

Here,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\beta\alpha} , \qquad D_{\alpha\beta} = \partial_{\alpha\beta} := \frac{\partial}{\partial X^{\alpha\beta}} , \qquad (1.6)$$

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are the covariant derivatives on the rigid tensorial superspace $\Sigma^{(\frac{n(n+1)}{2}|n)}$ that satisfy

$$\{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta} \tag{1.7}$$

which, we note in passing, exhibits the central extension structure of the superalgebras of tensorial superspaces [14] (see further [15]).

The tensorial superspace higher spin equations with $\mathcal{N}$-extended supersymmetry, however, have not been studied yet[1]. In this paper we fill this gap by presenting the free $\mathcal{N}$=2, $\mathcal{N}$=4 and $\mathcal{N}$=8 supersymmetric conformal higher spin equations in $\mathcal{N}$-extended tensorial superspaces $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N} n)}$. In particular, we show that the $\mathcal{N} = 2$ supermultiplets of $D = 4, 6$ and 10 conformal higher spin equations are described by scalar, chiral superfields $\Phi(X^{\alpha\beta}, \Theta^\gamma, \bar{\Theta}^\gamma)$ in $\mathcal{N} = 2$ tensorial superspace $\Sigma^{(\frac{n(n+1)}{2}|2n)}$, which obey the following set of linear superfield equations

$$\bar{\mathcal{D}}_\alpha \Phi = 0 , \qquad \mathcal{D}_{[\alpha} \mathcal{D}_{\beta]} \Phi = 0 , \tag{1.8}$$

where

$$
\begin{aligned}
\mathcal{D}_\alpha &= \frac{\partial}{\partial\Theta^\alpha} + i\bar{\Theta}^\beta\partial_{\beta\alpha} = \frac{1}{2}(D_{\alpha 1} + iD_{\alpha 2}) , \\
\bar{\mathcal{D}}_\alpha &= \frac{\partial}{\partial\bar{\Theta}^\alpha} + i\Theta^\beta\partial_{\beta\alpha} = -(\mathcal{D}_\alpha)^* , \qquad \mathcal{D}_{\alpha\beta} = \partial_{\alpha\beta} := \frac{\partial}{\partial X^{\alpha\beta}} ,
\end{aligned}
\tag{1.9}
$$

are the covariant derivatives on the the rigid $\mathcal{N}$=2 extended superspace $\Sigma^{(\frac{n(n+1)}{2}|2n)}$, which obey the superalgebra

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0 , \qquad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = 2i\partial_{\alpha\beta} , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0 . \tag{1.10}$$

The superfield equations in extended tensorial superspaces with even $\mathcal{N} > 2$ are given by a straightforward generalization of (1.8) (Eqs. (3.22), (3.23) in Sec. 3).

We also present the extended supersymmetric $\mathcal{N} > 1$ generalization of the preonic superparticle model of [2] and show how the above superfield equations can be obtained by quantizing the $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N} n)}$ superparticle model for even $\mathcal{N} \geq 2$.

Although our superfield equations make sense for general even $\mathcal{N}$ and n, we elaborate in detail the $\mathcal{N} = 2, 4, 8$ cases which, when $n = 4$, correspond to the supermultiplets of $D = 4$ massless conformal higher spin theories with a clear standard 'lower spin' field theory counterparts. These are the hypermultiplet for $\mathcal{N} = 2$, the supersymmetric Yang-Mills supermultiplet for $\mathcal{N} = 4$, and the maximal supergravity multiplet for $\mathcal{N} = 8$, the linearized versions of which can be described by scalar superfields in the standard extended $D = 4$ superspaces $\Sigma^{(4|4\mathcal{N})}$ with $\mathcal{N} = 2, 4, 8$.

One of the reasons of our interest in $\mathcal{N}$-extended supersymmetric systems of higher spin theories comes from the observation that $\mathcal{N}$-extended supersymmetry with $\mathcal{N} = 4$ unifies the scalar and vector gauge fields. On the other hand, all the higher spin equations have been formulated in terms of scalar and spinor fields in tensorial space, so that the study of $\mathcal{N}$-extended supersymmetries might prove convenient in a search for a sensible generalization of Maxwell and Einstein equations in tensorial superspace.

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[1]See [16] as well as [17] and refs. therein for the description of supersymmetric free higher spin equations in standard superspace and [18, 19] for the superfield form of Vasiliev's interacting higher spin equations [20, 21, 22] in the usual simple and extended superspaces.

Indeed, at some point our study of the superfield equations in $\mathcal{N} = 4$ tensorial superspace $\Sigma^{(4|16)}$ produces a tensorial space counterpart of the Maxwell equations. However, a careful analysis shows that the corresponding (spin-)tensorial fields can be expressed as derivatives of other scalar fields in tensorial superspace so that, for instance, the bosonic fields of $\mathcal{N} = 4$ conformal higher spin multiplet are actually given by two complex scalar fields in tensorial space, ϕ and $\tilde{\phi}$. Similarly, when studying the superfield equations in $\mathcal{N} = 8$ tensorial superspace $\Sigma^{(4|32)}$, although the tensorial space generalizations of the conformal (super)gravity equations do appear at an intermediate stage, it will turn out that they reduce to the scalar (and spinor) field equations in the tensorial space form first presented by Vasiliev [3]. Roughly speaking one can state that the increasing of $\mathcal{N}$ results just in multiplication of the scalar and spinor fields.

However, at $\mathcal{N} = 4$ a new phenomenon does occur. As the new scalar field appears in the theory only through a 'Maxwell-like' field, which is to say under the action of bosonic derivatives $\partial_{\alpha\beta}$, the (free higher spin $\mathcal{N} = 4$) theory becomes invariant under constant shifts of this bosonic field which makes it similar to the axion (for which such a symmetry is called Peccei-Queen [23] symmetry[2]). For the $\mathcal{N} = 8$ multiplet reformulated in terms of scalar and spinor fields, the Peccei-Queen symmetry becomes more complicated for the scalars and is also present for spinor fields entering the model under the action of derivative in the combination simulating the structure of the Rarita-Schwinger fields.

The plan of this paper is as follows. After discussing in Sec. 2 the $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}n)}$ superparticle model and the structure of its constraints, we sketch its quantization in Sec. 3 (using the $\mathcal{N} = 1$ results of [2] to simplify the discussion) and find our superfield equations, Eqs. (3.22) and (3.23), as the condition obeyed by the superparticle wavefunction in the tensorial superspace coordinate representation. In Sec. 4 we study the field content of our superfield equations, which are valid for arbitrary even $\mathcal{N}$ and n (with $n = 4, 8, 16$ corresponding to $D = 4, 6, 10$ free massless conformal higher spin theories) for the particular cases of $\mathcal{N} = 2, 4, 8$. In particular, we show in Sec. 4.1 (where the analysis is not restricted to $n = 4$) that the $\mathcal{N} = 2$ supermultiplets $D = 4, 6$ and 10 of conformal higher spin fields described by complex scalar and complex spinor (s-vector) fields obeying the Vasiliev's tensorial space higher spin equations are encoded in the chiral scalar superfield $\Phi(X^{\alpha\beta}, \Theta^{\gamma}, \bar{\Theta}^{\gamma})$ on $\mathcal{N} = 2$ tensorial superspace $\Sigma^{(\frac{n(n+1)}{2}|2n)}$ which obeys the set of linear superfield equations (1.8). In Sec. (4.2) we study the superfield equations (3.22) and (3.23) in $\mathcal{N} = 4$ tensorial superspace, find a tensorial superspace counterpart of $D = 4$ Maxwell equation and show that its general solution is expressed through the scalar field obeying the tensorial space higher-spin equation of usual type. We also show there that this new scalar field is defined up to a constant shift which resembles the Peccei-Quinn transformation of the axion. Sec. 4.3 analyzes the field content of the complex superfield obeying a $\mathcal{N} = 8$ version of Eqs. (3.22), (3.23) and presents the more complicated Peccei-Quinn-like symmetry characteristic of the $\mathcal{N} = 8$ massless conformal free higher spin theories. Some comments on possible extensions of this work are made in Sec. 5.

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[2]As far as in type IIB string theory and supergravity the axion appears as a member of the family of the RR gauge fields, its Peccei-Queen symmetry can be considered as a counterpart of the gauge symmetries characteristic of higher RR gauge potentials.

2 Superparticle in $\mathcal{N}$-extended tensorial superspace

2.1 An action for the $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$ superparticle

The superparticle action in $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$ has the form

$$S = \int d\tau\, \mathcal{L} = \int d\tau\, [\dot{\hat{X}}^{\alpha\beta}(\tau) - i\dot{\hat{\theta}}^{\alpha I}(\tau)\hat{\theta}^{\beta I}(\tau)]\lambda_\alpha(\tau)\lambda_\beta(\tau)\,, \qquad \begin{cases} \alpha,\beta = 1,...,n\,, \\ I = 1,2,...,\mathcal{N}\,, \end{cases} \quad (2.1)$$

where the $\lambda_\alpha(\tau)$ are auxiliary *commuting* spinor variables, $\hat{X}^{\alpha\beta}(\tau) = \hat{X}^{\beta\alpha}(\tau)$ and $\hat{\theta}^{\alpha I}(\tau)$ are the bosonic and fermionic coordinate functions that define the superparticle worldline $W^1 \in \Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$, $\hat{Z}^{\mathcal{M}}(\tau) = (\hat{X}^{\alpha\beta}(\tau), \hat{\theta}^{\alpha I}(\tau))$, and the dot denotes derivative with respect to proper time τ; due to the time derivative, the term $\dot{\hat{\theta}}^{\alpha I}(\tau)\hat{\theta}^{\beta I}(\tau)$ is also $(\alpha \leftrightarrow \beta)$-symmetric.

It is convenient to write the action (2.1) in the form

$$S = \int_{W^1} \hat{\Pi}^{\alpha\beta}\lambda_\alpha(\tau)\lambda_\beta(\tau)\,, \tag{2.2}$$

where

$$\hat{\Pi}^{\alpha\beta}(\tau) = d\tau\hat{\Pi}^{\alpha\beta}_\tau(\tau) = d\tau(\dot{\hat{X}}^{\alpha\beta} - i\dot{\hat{\theta}}^{I(\alpha}\hat{\theta}^{\beta)I})\,, \qquad \begin{cases} \alpha,\beta = 1,...,n\,, \\ I = 1,2,...,\mathcal{N}\,, \end{cases} \tag{2.3}$$

is the pull-back to W^1 of the vielbein $\Pi^{\alpha\beta}$ of the flat $\mathcal{N}$-extended tensorial superspace $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$ (the bosonic Maurer-Cartan one-form on the $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$ supergroup manifold), namely

$$\Pi^{\alpha\beta} = dX^{\alpha\beta} - id\theta^{I(\alpha}\theta^{\beta)I}\,, \tag{2.4}$$

by the map $\phi : W^1 \to \Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$ $(\phi^*(\Pi^{\alpha\beta}) \equiv \hat{\Pi}^{\alpha\beta}(\tau))$.

The superparticle action is manifestly invariant under the supertranslations on the rigid $\mathcal{N}$-extended tensorial superspace $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}\,n)}$,

$$\delta_a X^{\alpha\beta} = a^{\alpha\beta} \quad,\quad \delta_a \theta^{I\alpha} = 0\,, \tag{2.5}$$

$$\delta_\epsilon X^{\alpha\beta} = i\theta^{I(\alpha}\epsilon^{\beta)I} \quad,\quad \delta_\epsilon \theta^{I\alpha} = \epsilon^{\beta I} \quad, \tag{2.6}$$

which act on the worldline fields as

$$\delta_a \hat{X}^{\alpha\beta}(\tau) = a^{\alpha\beta}\,, \quad \delta_a \hat{\theta}^{I\alpha} = 0\,; \qquad \delta_a \hat{\lambda}_\alpha = 0\,. \tag{2.7}$$

$$\delta_\epsilon \hat{X}^{\alpha\beta} = i\hat{\theta}^{I(\alpha}\epsilon^{\beta)I}\,, \quad \delta_\epsilon \hat{\theta}^{I\alpha} = \epsilon^{\beta I}\,; \quad \delta_\epsilon \hat{\lambda}_\alpha = 0\,. \tag{2.8}$$

The action (2.1) is also manifestly invariant under the $GL(n, \mathbb{R})$ transformations of the $\alpha, \beta = 1,...,n$ indices, which reduce to the n-dimensional representation of $Spin(1, D-1)$ when these indices are thought of as Lorentz-spinorial ones[3].

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[3]In [3, 4] the counterparts of λ_α were called 's-vectors' to avoid their immediate identification as $GL(n,\mathbb{R})$ vectors or $SO(1, D-1)$ spinors.

2.2 Symplectic supertwistor form of the action

Actually, the $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}n)}$ superparticle action (2.1) is invariant under the larger $OSp(\mathcal{N}|2n)$ supergroup. To make this manifest as well as to determine easily the number of physical degrees of freedom it is convenient to use Leibniz's rule[4] to rewrite the action (2.1) in the form

$$S = \int_{W^1} (\lambda_\alpha d\mu^\alpha - \mu^\alpha d\lambda_\alpha - id\chi^I \chi^I) = \int_{W^1} d\Upsilon^\Sigma \Xi_{\Sigma\Omega} \Upsilon^\Omega . \tag{2.9}$$

This action is written in terms of the bosonic spinor $\lambda_\alpha(\tau)$, which is present in (2.1), a second bosonic spinor μ^α and $\mathcal{N}$ real fermionic variables χ^I; these form the $\mathcal{N}$-extended, $(2n + \mathcal{N})$-dimensional orthosymplectic supertwistor (see [13, 2] and also [14] for $\mathcal{N} = 1$)

$$\Upsilon^\Sigma = \begin{pmatrix} \mu^\alpha \\ \lambda_\alpha \\ \chi^I \end{pmatrix} , \qquad \alpha = 1, \ldots, n , \qquad I = 1, \ldots, \mathcal{N} . \tag{2.10}$$

This generalizes the Penrose twistors [24] (or conformal $SU(2, 2)$ spinors) and the Ferber-Schirafuji supertwistors [25, 26] (carrying the basic representation of $D=4$ $SU(\mathcal{N}|2, 2)$). The Υ^Σ's carry the defining representation of the $OSp(\mathcal{N}|2n)$ supergroup (see in this context [13, 2, 14] for $\mathcal{N} = 1$), the transformations of which preserve the $(2n+\mathcal{N}) \times (2n+ \mathcal{N})$ orthosymplectic 'metric' $\Xi_{\Sigma\Omega}$,

$$\Xi_{\Sigma\Omega} = \begin{pmatrix} 0 & \delta_\alpha{}^\beta & 0 \\ -\delta^\alpha{}_\beta & 0 & 0 \\ 0 & 0 & -i\delta^{IJ} \end{pmatrix} , \qquad \alpha = 1, \ldots, n , \quad I = 1, \ldots, \mathcal{N} . \tag{2.11}$$

In fact, $OSp(1|2n)$ may be considered as a supersymmetric generalization of the superconformal group for $D = \frac{n}{2} + 2$ (see [27, 13, 28, 14] and refs. therein).

The relations between the supertwistor components and the variables of the action (2.1) that make the transition between both actions are

$$\mu^\alpha = \hat{X}^{\alpha\beta}\lambda_\beta - \frac{i}{2}\hat{\theta}^{\alpha I}\chi^I , \qquad \chi^I = \hat{\theta}^{\alpha I}\lambda_\alpha , \tag{2.12}$$

which generalize the Penrose and the Ferber-Shirafuji incidence relations [24, 25, 26] (see [13] and [2, 14] for $\mathcal{N} = 1$). Since the action (2.9) does not possess any gauge symmetries, the components of the orthosymplectic supertwistors are the true, physical degrees of freedom ($2n$ bosonic and $\mathcal{N}$ fermionic) of our generalized superparticle model.

2.3 Gauge symmetries

By construction, the actions (2.9) and (2.1) describe the same dynamical system. Thus, since the action (2.1) depends on $\frac{n(n+1)}{2} + n$ bosonic variables and $\mathcal{N}n$ fermionic ones, it should possess $n(n-1)/2$ bosonic gauge symmetries and $\mathcal{N}(n-1)$ fermionic ones to reduce the number of degrees of freedom to those of the supertwistors Υ^Σ appearing in the action (2.9). The simplest way to describe these gauge symmetries, called fermionic

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[4]It is sufficient to use $\lambda_\alpha\lambda_\beta dX^{\alpha\beta} = \lambda_\alpha d(\lambda_\beta X^{\alpha\beta}) - \lambda_\alpha X^{\alpha\beta} d\lambda_\beta$; no integration by parts is needed.

κ-symmetry and bosonic b-symmetry, is to define restrictions on the basic variations of the bosonic and fermionic coordinate functions (see [2, 14] for the $\mathcal{N} = 1$ case)

$$\delta_\kappa \hat{X}^{\alpha\beta} = i\delta_\kappa \hat{\theta}^{I(\alpha} \hat{\theta}^{\beta)I} \, , \qquad \boxed{\delta_\kappa \hat{\theta}^{\alpha I} \lambda_\alpha = 0} \, , \qquad \delta_\kappa \lambda_\alpha = 0 \, , \tag{2.13}$$

$$\boxed{\delta_b \hat{X}^{\alpha\beta} \lambda_\alpha = 0} \, , \qquad \delta_b \hat{\theta}^{I\alpha} \lambda_\alpha = 0 \, , \qquad \delta_b \lambda_\alpha = 0 \, . \tag{2.14}$$

2.4 Constraints and their conversion to first class

In the hamiltonian formalism, the δ_κ and δ_b gauge symmetries in Eqs. (2.13), (2.14) are generated by first class constraints which may be extracted from the following bosonic and fermionic primary constraints of the model (2.1)

$$d_{\alpha I} := \pi_{\alpha I} + iP_{\alpha\beta}\theta^{\beta I} \approx 0 \, , \tag{2.15}$$

$$\mathcal{P}_{\alpha\beta} := P_{\alpha\beta} - \lambda_\alpha \lambda_\beta \approx 0 \, , \qquad P^{\alpha(\lambda)} \approx 0 \, , \tag{2.16}$$

where

$$P_{\alpha\beta} := \frac{1}{2} \frac{\delta\mathcal{L}}{\delta\dot{\hat{X}}^{\alpha\beta}} \, , \qquad P_\alpha^{(\lambda)} := \frac{\delta\mathcal{L}}{\delta\dot\lambda^\alpha} \, , \qquad \pi_{\alpha I} := \frac{\delta\mathcal{L}}{\delta\dot{\hat{\theta}}^\alpha} \, , \tag{2.17}$$

are the canonical momenta conjugated to the coordinate functions and to the auxiliary bosonic spinor (s-vector). Using the canonical Poisson brackets

$$[\hat{X}^{\gamma\delta}, P_{\alpha\beta}]_{PB} = -[P_{\alpha\beta}, \hat{X}^{\gamma\delta}]_{PB} = \delta_\alpha^{(\gamma}\delta_\beta^{\delta)} \, , \quad [\lambda_\beta, P^{\alpha(\lambda)}]_{PB} = -[P^{\alpha(\lambda)}, \lambda_\beta]_{PB} = \delta_\beta^\alpha \tag{2.18}$$

$$\{\pi_\alpha \, , \hat{\theta}^\beta\}_{PB} = \{\hat{\theta}^\beta \, , \pi_\alpha\}_{PB} = -\delta_\alpha^\beta \, , \tag{2.19}$$

it follows that the nonvanishing Poisson brackets of the above constraints are

$$\{d_{\alpha I}, d_{\beta J}\}_{PB} = -2iP_{\alpha\beta}\delta_{IJ} \, , \qquad [\mathcal{P}_{\alpha\beta}, P^{\gamma(\lambda)}]_{PB} = -2\lambda_{(\alpha}\delta_{\beta)}{}^\gamma \, . \tag{2.20}$$

These clearly indicate that the primary constraints above are a mixture of first and second class constraints. Rather than separating them, we use below the so-called 'conversion procedure' (see [2] for references), by which a pair of degrees of freedom is added to each pair of second class constraints to modify Eqs. (2.20) in such a way that they form a closed algebra. In this way, these modified constraints become first class ones generating gauge symmetries in the enlarged phase space. In it, all the constraints of the model are first class and account as well for the original second class constraints. These can be recovered by gauge fixing the additional gauge symmetries/first class constraints of the system in the enlarged phase space. For the $\mathcal{N} = 1$ version of (2.1) this was done in [2].

As the bosonic sector of all the superparticle models is the same irrespective of $\mathcal{N}$, we may use the results of [2] for $\mathcal{N} = 1$ and state that the conversion in the bosonic sector is effectively reduced to ignoring the constraints $P^{\alpha(\lambda)} \approx 0$ in the analysis. An easy way to see that this is indeed consistent is to observe that, as far as $\lambda_\alpha \neq (0, ..., 0)$ (the usual configuration space restriction for twistor-like variables), the second brackets in (2.20) show that $P^{\alpha(\lambda)} = 0$ is a good gauge fixing condition for n of $n(n+1)/2$ gauge symmetries generated by the constraints $\mathcal{P}_{\alpha\beta}$.

To perform the conversion in the fermionic sector, we introduce the $\mathcal{N}$ fermionic variables χ^I and postulate for them the Clifford-type Poisson brackets

$$\{\chi^I, \chi^J\}_{PB} = -2i\delta^{IJ} . \tag{2.21}$$

These χ^I are then used to modify the fermionic constraints to $\mathbb{D}_{\alpha I} = d_{\alpha I} + i\chi^I \lambda_\alpha$. Thus, after conversion, the superparticle model (2.1) is described by the following set of first class constraints

$$\mathbb{D}_{\alpha I} := \pi_{\alpha I} + iP_{\alpha\beta}\theta^{\beta I} + i\chi^I \lambda_\alpha \approx 0 , \qquad \mathcal{P}_{\alpha\beta} := P_{\alpha\beta} - \lambda_\alpha \lambda_\beta \approx 0 , \tag{2.22}$$

which obey the superalgebra of the rigid supersymmetry of $\mathcal{N}$-extended tensorial superspace $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}n)}$ in Eq. (1.10), namely

$$\{\mathbb{D}_{\alpha I}, \mathbb{D}_{\beta J}\}_{PB} = -2i\mathcal{P}_{\alpha\beta}\delta_{IJ} , \qquad [\mathcal{P}_{\alpha\beta}, \mathbb{D}_{\gamma I}]_{PB} = 0 , \qquad [\mathcal{P}_{\alpha\beta}, \mathcal{P}_{\gamma\delta}]_{PB} = 0 . \tag{2.23}$$

3 Quantization of the superparticle in $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}n)}$ with even $\mathcal{N}$ and conformal higher spin equations

Quantizing the model in its orthosymplectic-twistorial formulation (2.9) is straightforward. The canonical hamiltonian is equal to zero and thus the Schrödinger equation simply states that the wavefunction is independent of τ. Following a procedure similar to that in [2] one can show that, in the $n = 4$ tensorial space corresponding to $D = 4$, the wavefunction of the bosonic limit of the superparticle model (2.9) describes the solution of the free higher spin equations. This means that it can be written in terms of an infinite tower of left and right chiral fields $\phi_{A_1...A_{2s}}(p_\mu)$ and $\phi_{\dot{A}_1...\dot{A}_{2s}}(p_\mu)$ for all half-integer values of s with $p_\mu p^\mu = 0$.

Let $\mathcal{N} > 1$ and even (as it will be henceforth). Quantization *à la* Dirac of a dynamical system with first class constraints requires imposing them as equations to be satisfied by its wavefunction. In the case of our superparticle model (2.1) this wavefunction can be chosen to depend on the coordinates of $\mathcal{N}$-extended tensorial superspace ($X^{\alpha\beta} = X^{\beta\alpha}$, $\theta^{\alpha I}$), on the bosonic spinor (s-vector) variable λ_α and on *a half* of the fermionic variables χ^I as far as they are, by (2.21), their own momenta. The separation of a half of the real χ^I coordinates can be achieved by introducing complex[5] variables η^q, $(\eta^q)^* = \bar{\eta}_q$, $q = 1, \ldots, \mathcal{N}/2$, so that $\chi^I = (\chi^q, \chi^{\mathcal{N}/2+q}) = ((\eta^q + \bar{\eta}_q), i(\bar{\eta}_q - \eta^q))$,

$$\eta^q = \frac{\chi^q - i\chi^{\mathcal{N}/2+q}}{2} , \qquad \bar{\eta}_q = \frac{\chi^q + i\chi^{\mathcal{N}/2+q}}{2} , \qquad \{\bar{\eta}_q, \eta^p\}_{PB} = -i\delta_q^{\ p} . \tag{3.1}$$

Then, the wavefunction superfield in the coordinates representation depends only on η^q,

$$\mathcal{W} = \mathcal{W}(X^{\alpha\beta}, \theta^\alpha; \lambda_\alpha; \eta^q) , \tag{3.2}$$

the various momenta are given by the differential operators

$$P_{\alpha\beta} = -i\partial_{\alpha\beta} , \qquad \pi_{\alpha I} = -i\frac{\partial}{\partial\theta^{\alpha I}} , \qquad \bar{\eta}_q = \frac{\partial}{\partial\eta^q} , \tag{3.3}$$

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[5]A separation in pairs of conjugate constraints is used in the Gupta-Bleuler method of quantizing systems with second class constraints as the massive superparticle [29, 30].

and the Poisson brackets become quantum commutators or anticommutators $([\,,\,\}_{PB} \mapsto \frac{1}{i\hbar}[\,,\,\}$; we take $\hbar = 1$). The quantum constraints operators, to be denoted by the same symbol (although having in mind $\mathbb{D}_{\alpha I} \mapsto -i\mathbb{D}_{\alpha I}^{quantum}$, $\mathcal{P}_{\alpha\beta}^{classical} \mapsto -i\mathcal{P}_{\alpha\beta}^{quantum}$) are then

$$\mathbb{D}_{\alpha I} := \frac{\partial}{\partial\theta^{\alpha I}} + i\partial_{\alpha\beta}\theta^{\beta I} - \chi^I\lambda_\alpha \,, \tag{3.4}$$

$$\mathcal{P}_{\alpha\beta} := \partial_{\alpha\beta} - i\lambda_\alpha\lambda_\beta \,, \tag{3.5}$$

and have to be imposed on the wavefunction (3.2).

For even $\mathcal{N}$, it is convenient to introduce complex Grassmann coordinates and complex Grassmann derivatives,

$$\Theta^{\alpha q} = \frac{1}{2}(\theta^{\alpha q} - i\theta^{\alpha(q+\mathcal{N}/2)}) = (\bar\Theta_q^\alpha)^* \quad \Leftrightarrow \quad \partial_{\alpha q} := \frac{\partial}{\partial\Theta^{\alpha q}} = \frac{\partial}{\partial\theta^{\alpha q}} + i\frac{\partial}{\partial\theta^{\alpha(q+\mathcal{N}/2)}} \,, \quad (3.6)$$

$q = 1,\ldots,\mathcal{N}/2$, and conjugate pairs of fermionic constraints

$$\nabla_{\alpha q} := \mathbb{D}_{\alpha q} + i\mathbb{D}_{\alpha(q+\mathcal{N}/2)} = \partial_{\alpha q} + 2i\partial_{\alpha\beta}\bar\Theta_q^\beta - 2\lambda_\alpha\frac{\partial}{\partial\eta^q} =: \mathcal{D}_{\alpha q} - 2\lambda_\alpha\frac{\partial}{\partial\eta^q} \,, \tag{3.7}$$

$$\bar\nabla_\alpha{}^q := \mathbb{D}_{\alpha q} - i\mathbb{D}_{\alpha(q+\mathcal{N}/2)} = \bar\partial_\alpha{}^q + 2i\partial_{\alpha\beta}\Theta^{\beta q} - 2\lambda_\alpha\eta^q =: \bar{\mathcal{D}}_\alpha{}^q - 2\lambda_\alpha\eta^q \,. \tag{3.8}$$

Since $\{\mathcal{D}_{\alpha q}, \bar{\mathcal{D}}_\beta^p\} = 4i\partial_{\alpha\beta}\delta_q^p$, the above $\nabla_{\alpha q}$, $\bar\nabla_\alpha{}^q$ and the bosonic constraint (3.5) determine the superalgebra given by the only nonzero bracket

$$\{\nabla_{\alpha q}, \bar\nabla_\beta^p\} = 4i\mathcal{P}_{\alpha\beta}\delta_q^p \quad . \tag{3.9}$$

This shows that it is sufficient to impose on the superwavefunction (3.2) the fermionic constraints,

$$\nabla_{\alpha q}\mathcal{W} := \mathcal{D}_{\alpha q}\mathcal{W} - 2\lambda_\alpha\frac{\partial}{\partial\eta^q}\mathcal{W} = 0 \,, \tag{3.10}$$

$$\bar\nabla_\beta^p\mathcal{W} := \bar{\mathcal{D}}_\alpha^q\mathcal{W} - 2\lambda_\alpha\eta^q\mathcal{W} = 0 \,, \tag{3.11}$$

since the mass-shell-like bosonic constraint,

$$\mathcal{P}_{\alpha\beta}\mathcal{W} := (\partial_{\alpha\beta} - i\lambda_\alpha\lambda_\beta)\mathcal{W} = 0 \,, \tag{3.12}$$

will follow as a consistency condition for (3.10), (3.11).

Decomposing the superwavefunction in a finite power series in the complex Grassmann variable η^q,

$$\mathcal{W}(X,\Theta^q,\bar\Theta_q,\lambda,\eta^q) = W^{(0)}(X,\Theta^q,\bar\Theta_q,\lambda) + \sum_{k=1}^{\mathcal{N}/2}\frac{1}{k!}\eta^{q_k}\ldots\eta^{q_1} W_{q_1\ldots q_k}^{(k)}(X,\Theta^q,\bar\Theta_q,\lambda) \,, \quad (3.13)$$

we find that Eqs. (3.10), (3.11) imply

$$\mathcal{D}_{\alpha q}W^{(0)} = 2\lambda_\alpha W_q^{(1)} \,, \quad \ldots \,, \quad \mathcal{D}_{\alpha q}W_{q_1\ldots q_k}^{(k)} = 2\lambda_\alpha W_{qq_1\ldots q_k}^{(k+1)} \,, \quad \ldots \,,$$
$$\mathcal{D}_{\alpha q}W_{q_1\ldots q_{\mathcal{N}/2}}^{(\mathcal{N}/2)} = 0 \,, \tag{3.14}$$

and

$$\bar{\mathcal{D}}_\alpha^q W^{(0)} = 0 \,, \quad \bar{\mathcal{D}}_\alpha^q W^{(k)}_{q_1\dots q_k} = 2k\lambda_\alpha W^{(k-1)}_{[q_1\dots q_{k-1}}\delta_{q_k]}{}^q \,, \quad \dots \,,$$

$$\bar{\mathcal{D}}_\alpha^q W^{(\mathcal{N}/2)}_{q_1\dots q_{\mathcal{N}/2}} = \mathcal{N}\lambda_\alpha W^{(\mathcal{N}/2-1)}_{[q_1\dots q_{\mathcal{N}/2-1}}\delta^q_{q_{\mathcal{N}/2}]} \,. \tag{3.15}$$

Eqs. (3.14) show that all the superfields $W^{(k)}_{q_1\dots q_k}$ can be constructed from fermionic derivatives of the superfield $W^{(0)}(X,\Theta^q,\bar{\Theta}_q,\lambda)$ which is chiral as a consequence of the first equation in (3.15),

$$\mathcal{D}_{\alpha_k q_k}\dots\mathcal{D}_{\alpha_1 q_1}W^{(0)} = 2^k\lambda_{\alpha_1}\dots\lambda_{\alpha_k}W^{(k)}_{q_1\dots q_k} \,, \qquad \bar{\mathcal{D}}_\alpha^q W^{(0)} = 0 \,. \tag{3.16}$$

Then, the wavefunction $\mathcal{W}$ is completely characterized by the chiral superfield $W^{(0)}(X,\Theta^q,\bar{\Theta}_q,\lambda)$. As a consequence of (3.12), $W^{(0)}$ obeys

$$\mathcal{P}_{\alpha\beta}W^{(0)} := (\partial_{\alpha\beta} - i\lambda_\alpha\lambda_\beta)W^{(0)} = 0 \,, \tag{3.17}$$

which is solved by a planewave in tensorial space,

$$W^{(0)}(X,\Theta^q,\bar{\Theta}_q,\lambda) = \tilde{W}(\lambda,\Theta^q,\bar{\Theta}_q)\exp\{i\lambda_\alpha\lambda_\beta X^{\alpha\beta}\} \,. \tag{3.18}$$

Then, the chirality of $W^{(0)}$ and the first equation in (3.14), which now implies $\partial_{\alpha q}W^{(0)} \propto \lambda_\alpha$, show that the general solution for the superparticle wavefunction is determined by the following chiral plane wave superfield

$$W^{(0)}(X,\Theta^q,\bar{\Theta}_q,\lambda) = w(\lambda,\Theta^q\lambda)\exp\{i\lambda_\alpha\lambda_\beta(X+2i\Theta^p\bar{\Theta}_p)^{\alpha\beta}\} \,, \tag{3.19}$$

where $\Theta^q\lambda = \Theta^{\alpha q}\lambda_\alpha$ and

$$w(\lambda,\Theta^q\lambda) = w^{(0)}(\lambda) + \sum_{k=1}^{\mathcal{N}/2}\frac{1}{k!}(\lambda\Theta^{q_k})\dots(\lambda\Theta^{q_1})\,w^{(k)}_{q_1\dots q_k}(\lambda) \,. \tag{3.20}$$

We refer to [2] for a discussion on how the arbitrary function $w^{(0)}(\lambda_\alpha)$ with $\alpha = 1,2,3,4$ encodes all the solutions of the massless higher spin equations in $D = 4$ and to [2, 8] for the $D = 6, 10$ cases. The key point is that λ_α carries the degrees of freedom of a light-like momentum ($\lambda\gamma_a\lambda$ is light-like in $D=4,6,10$ which corresponds to $n=4,8,16$) plus those of spin. The $d.o.f.$ of λ_α and those of the lightlike momenta are encoded, both up to a scale factor, in the coordinates of the compact manifolds $S^{n-1} = S^{2D-5}$ and $S^{\frac{n}{2}} = S^{D-2}$, respectively. The spheres[6] $S^{\frac{n}{2}-1}$ are related to helicity in the $n = 4$, $D = 4$ case and to its multidimensional generalizations for $D = 6, 10$ [2, 8].

To obtain a superfield on tensorial superspace describing massless conformal higher spin theories with extended supersymmetry, the wavefunction (3.19) has to be integrated over $\mathbb{R}^n - \{0\} \sim \mathbb{S}^{n-1}\times\mathbb{R}_+$, parametrized by λ_α, with an appropriate measure that we denote by $d^n\lambda$,

$$\Phi(X,\Theta^q,\bar{\Theta}_q) = \int d^n\lambda W^{(0)}(X,\Theta^q,\bar{\Theta}_q,\lambda) = \int d^n\lambda w(\lambda,\Theta^q\lambda)e^{i\lambda_\alpha\lambda_\beta(X+2i\Theta^p\bar{\Theta}_p)^{\alpha\beta}} \,. \tag{3.21}$$

---

[6]The 'celestial spheres' S^{D-2} are the bases $S^{2,4,8}$ of the Hopf fibrations $S^{2D-5} \to S^{D-2}$ ($S^{n-1} \to S^{\frac{n}{2}}$) of $S^{3,7,15}$, $(n,D)=(4,4)$, (8,6), (16,10). The fibres $S^{D-3} = S^{1,3,7}$ of these bundles correspond to the complex, quaternion and octonion numbers of unit modulus. The remaining $n=2$, $D=3$ case corresponds to the first of the four Hopf fibrations, $S^1 \to \mathbb{R}P^1$; its fibre is determined by the reals of unit modulus, Z_2, and there are no extra coordinates.

One can easily check that the superfield Φ is chiral

$$\bar{\mathcal{D}}_\alpha^q \Phi(X, \Theta^{q'}, \bar{\Theta}_{p'}) = 0 \qquad (3.22)$$

and satisfies the equation

$$\mathcal{D}_{q[\beta} \mathcal{D}_{\gamma]p} \Phi(X, \Theta^{q'}, \bar{\Theta}_{p'}) = 0 . \qquad (3.23)$$

These are the superfield equations for the wavefunction of the superparticle in $\mathcal{N}$-extended tensorial superspace for even $\mathcal{N}$.

4 From the superfield to the component form of the higher spin equations in tensorial space

4.1 $\mathcal{N} = 2$

When $\mathcal{N} = 2$, Eqs. (3.22) and (3.23) coincide with Eqs. (1.8). It is easy to check that Eq. (3.23) with $\mathcal{N} = 2$ implies the vanishing of all the components of the 'chiral' superfield $\Phi(X, \Theta, \bar{\Theta}) = \Phi(X + 2i\Theta \cdot \bar{\Theta}, \Theta)$, except the first two,

$$\Phi(X, \Theta, \bar{\Theta}) = \phi(X_L) + i\Theta^\alpha \psi_\alpha(X_L) \quad , \quad X_L^{\alpha\beta} = X^{\alpha\beta} + 2i\Theta^{(\alpha} \bar{\Theta}^{\beta)} = X_L^{\beta\alpha} \quad , \qquad (4.1)$$

where $X_L^{\alpha\beta}$ is the analogue of the bosonic coordinates for the chiral basis of standard $D = 4$ superspace. The above components are the *complex* bosonic scalar and the complex fermionic spinor fields obeying the free higher spin equations in tensorial space form [3],

$$\partial_{\alpha[\gamma} \partial_{\delta]\beta} \phi(X) = 0 , \qquad \partial_{\alpha[\beta} \psi_{\gamma]}(X) = 0 . \qquad (4.2)$$

Let us recall that the $\mathcal{N} = 1$ supermultiplet contains a real bosonic scalar and a real fermionic spinor field that obey the same equations (4.2). Hence, the $\mathcal{N} = 2$ supermultiplet of the conformal fields in tensorial superspace is given by the complexification of the $\mathcal{N} = 1$ supermultiplet.

Clearly, the above results are n-independent and thus, besides $n = 4$, they are also valid for the $n = 8$ and $n = 16$ cases corresponding to the $D = 6$ and $D = 10$ multiplets of massless conformal higher spin fields.

4.2 $\mathcal{N} = 4$

In contrast with the $\mathcal{N} = 2$ case, spin-tensorial components are present when $\mathcal{N} > 2$. For $\mathcal{N} = 4$, the general solution of the superfield equations (3.22) and (3.23) is given by

$$\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i\Theta^{\alpha q} \psi_{\alpha q}(X_L) + \epsilon_{pq} \Theta^{\alpha q} \Theta^{\beta p} \mathcal{F}_{\alpha\beta}(X_L) , \qquad (4.3)$$

$$X_L^{\alpha\beta} = X^{\alpha\beta} + 2i\Theta^{q(\alpha} \bar{\Theta}_q^{\beta)} , \quad q = 1, 2 , \qquad (4.4)$$

where, again, the complex scalar and spinor fields obey the standard higher spin equations in their tensorial superspace form,

$$\partial_{\alpha[\gamma} \partial_{\delta]\beta} \phi(X) = 0 , \qquad \partial_{\alpha[\beta} \psi_{\gamma]q}(X) = 0 , \qquad (4.5)$$

while the complex symmetric bi-spinor (or 's-tensor' [3]) $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\beta\alpha}$ satisfies the tensorial counterpart of the $D = 4$ Maxwell equations (when these are written in spinorial notation [24], see also below),

$$\partial_{\alpha[\gamma}\mathcal{F}_{\delta]\beta}(X) = 0 , \qquad \mathcal{F}_{\alpha\beta} = \mathcal{F}_{\beta\alpha} . \tag{4.6}$$

However, one can easily show that the general solution of Eq. (4.6) is expressed through a new complex scalar superfield $\tilde{\phi}(X)$ satisfying the bosonic tensorial space equation in (4.5),

$$\mathcal{F}_{\alpha\beta} = \partial_{\alpha\beta}\tilde{\phi}(X) , \tag{4.7}$$
$$\partial_{\alpha[\gamma}\partial_{\delta]\beta}\tilde{\phi}(X) = 0 . \tag{4.8}$$

4.2.1 $n = 4 , D = 4$

To prove this when $n = 4$ ($\alpha, \beta = 1, 2, 3, 4$), we begin by decomposing the complex symmetric $GL(4)$ tensor $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\beta\alpha}$ in 2×2 blocks, thus keeping only the $GL(2,\mathbb{C})$ manifest symmetry,

$$n = 4: \quad \mathcal{F}_{\alpha\beta} = \begin{pmatrix} F_{AB} & V_{A\dot{B}} \\ V_{B\dot{A}} & F_{\dot{A}\dot{B}} \end{pmatrix} , \qquad A, B = 1, 2 , \quad \dot{A}, \dot{B} = 1, 2 . \tag{4.9}$$

Let us first notice that the block components of Eq. (4.6) which contain the antisymmetric tensors (encoded in the symmetric spin-tensors F_{AB} and $F_{\dot{A}\dot{B}}$) only,

$$\partial_{\dot{A}[B}F_{C]D} = 0 , \qquad \partial_{A[\dot{B}}F_{\dot{C}]\dot{D}} = 0 , \tag{4.10}$$

are equivalent to the Maxwell equations for the complex selfdual field $F_{ab} = \frac{i}{2}\epsilon_{abcd}F^{cd} \propto \sigma_{ab}{}^{CD}F_{CD}$ i.e., they imply $\partial^a F_{ab} = 0$ and $\partial_{[a}F_{bc]} = 0$.

Consider now the components of Eq. (4.6) which contain the complex vector $V_{A\dot{B}} = \sigma^a_{A\dot{B}}V_a$ only, namely

$$\partial_{A[\dot{B}}V_{\dot{C}]D} = 0 , \qquad \partial_{\dot{B}[A}V_{D]\dot{C}} = 0 \tag{4.11}$$

and

$$\partial_{A[B}V_{C]\dot{D}} = 0 , \qquad \partial_{\dot{A}[\dot{B}}V_{\dot{C}]D} = 0 . \tag{4.12}$$

Eqs. (4.11) imply $\partial_{[a}V_{b]} = 0$ and $\partial^a V_a = 0$. The first is solved by $V_a = \partial_a\tilde{\phi}$ and the second implies that the scalar field $\tilde{\phi}$ obeys the Klein-Gordon equation $\partial^a\partial_a\tilde{\phi} = 0$. In the spin-tensor notation these read

$$V_{A\dot{B}} = \partial_{A\dot{B}}\tilde{\phi} , \qquad \partial_{A[\dot{B}}\partial_{\dot{C}]D}\tilde{\phi} = 0 . \tag{4.13}$$

Next, the components of Eq. (4.6) which contain both vector and antisymmetric tensor components, $\partial_{A\dot{B}}F_{CD} - \partial_{AC}V_{D\dot{B}} = 0$ and $\partial_{A\dot{B}}F_{\dot{C}\dot{D}} - \partial_{\dot{C}\dot{D}}V_{A\dot{B}} = 0$, can be written in the form

$$\partial_{A\dot{B}}(F_{CD} - \partial_{CD}\tilde{\phi}) = 0 , \qquad \partial_{A\dot{B}}(F_{\dot{C}\dot{D}} - \partial_{\dot{C}\dot{D}}\tilde{\phi}) = 0 , \tag{4.14}$$

the only covariant solution of which is given by

$$F_{CD} = \partial_{CD}\tilde{\phi} \, , \qquad F_{\dot{C}\dot{D}} = \partial_{\dot{C}\dot{D}}\tilde{\phi} \, . \tag{4.15}$$

Keeping in mind the Maxwell equations (4.10), one finds that the scalar field $\tilde{\phi}(X)$ satisfies, besides the Klein-Gordon equation in (4.13), also the remaining components of Eq. (4.8),

$$\partial_{A[B}\partial_{C]D}\tilde{\phi} = 0 \, , \qquad \partial_{\dot{A}[\dot{B}}\partial_{\dot{C}]\dot{D}}\tilde{\phi} = 0 \, . \tag{4.16}$$

4.2.2 Proof for arbitrary n

We now prove that Eqs. (4.7), (4.8) provide the general solution of the Maxwell-like equation in tensorial space, Eq. (4.6), for any n. The Fourier transform of Eq. (4.6) is

$$p_{\alpha[\gamma}\mathcal{F}_{\delta]\beta}(p) = 0 \, . \tag{4.17}$$

The solution of this equation is nontrivial iff the matrix of the generalized momentum has rank one, this is to say when $p_{\alpha\beta} = \lambda_\alpha\lambda_\beta$ for arbitrary $\lambda_\alpha \neq (0,...,0)$ or, equivalently, when this matrix obeys $p_{\alpha[\gamma}p_{\delta]\beta} = 0$. The general solution is characterized by $\mathcal{F}_{\alpha\beta}(\lambda) = \lambda_\alpha\lambda_\beta\phi(\lambda)$ and can be equivalently written in the form $\mathcal{F}_{\alpha\beta}(p) = p_{\alpha\beta}\tilde{\phi}(p)$ if $p_{\alpha[\gamma}p_{\delta]\beta}\tilde{\phi}(p) = 0$ [7] of Eqs. (4.6). As far as set of equations

$$\mathcal{F}_{\alpha\beta}(p) = p_{\alpha\beta}\tilde{\phi}(p) \, , \qquad p_{\alpha[\gamma}p_{\delta]\beta}\tilde{\phi}(p) = 0 \tag{4.18}$$

provide the Fourier transforms of Eqs. (4.7), (4.8), these describe the general solution.

4.2.3 Peccei-Quinn-like symmetry

Thus, the $\mathcal{N} = 4$ higher spin supermultiplet actually contains two complex scalar fields and two Dirac spinor fields in tensorial space, $\phi(X), \psi_\alpha^1(X), \psi_\alpha^2(X), \tilde{\phi}(X)$, which satisfy the free bosonic and fermionic higher spin equations, Eqs. (4.5), (4.8). They appear in the on-shell scalar superfield decomposition as

$$\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i\Theta^{\alpha q}\psi_{\alpha q}(X_L) + \epsilon_{pq}\Theta^{\alpha q}\Theta^{\beta p}\partial_{\alpha\beta}\tilde{\phi}(X_L) \, , \qquad q, p = 1, 2 \, . \tag{4.19}$$

However, as the second complex scalar field $\tilde{\phi}$ enters the original superfield with a derivative, its zero mode is not fixed. In other words, this scalar is axion-like: it possesses the Peccei-Quinn-like symmetry

$$\tilde{\phi}(X) \mapsto \tilde{\phi}(X) + const \, . \tag{4.20}$$

4.3 $\mathcal{N} = 8$

For higher $\mathcal{N} > 4$ the general solution of the set of superfield equations (3.22) and (3.23) is given by

$$\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i\Theta^{\alpha q}\psi_{\alpha q}(X_L) + \sum_{k=2}^{\mathcal{N}/2} \frac{1}{k!}\Theta^{\alpha_k q_k}\ldots\Theta^{\alpha_1 q_1}\mathcal{F}_{\alpha_1\ldots\alpha_k \, q_1\ldots q_k}(X_L) \, , \tag{4.21}$$

$$\mathcal{F}_{\alpha_1\ldots\alpha_k \, q_1\ldots q_k}(X_L) = \mathcal{F}_{(\alpha_1\ldots\alpha_k) \, [q_1\ldots q_k]}(X_L) \, , \quad X_L^{\alpha\beta} = X^{\alpha\beta} + 2i\Theta^{q(\alpha}\bar{\Theta}_q^{\beta)} \, , \quad q = 1,...,4, \tag{4.22}$$

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[7] More formally, the solution of this equation is given by a distribution with support on the subspace of tensorial momentum space defined by the condition $p_{\alpha[\gamma}p_{\delta]\beta} = 0$, so that $\tilde{\phi}(p) \propto \delta(p_{\alpha[\gamma}p_{\delta]\beta})$.

where $\phi(X_L)$ and $\psi_{\alpha q}(X_L)$ obey the standard higher spin equations (4.2) while the higher components satisfy

$$\partial_{\alpha[\gamma}\mathcal{F}_{\delta]\beta_2...\beta_k\,q_1...q_k}(X_L) = 0 \, , \qquad \mathcal{F}_{\alpha_1...\alpha_q} = \mathcal{F}_{(\alpha_1...\alpha_q)} \, . \tag{4.23}$$

For instance, for $\mathcal{N} = 8$ the superfield solution of the higher spin equations (3.22) and (3.23) reads

$$\Phi(X, \Theta^q, \bar{\Theta}_q) = \phi(X_L) + i\Theta^{\alpha q}\psi_{\alpha q}(X_L) + \frac{1}{2}\Theta^{\alpha_2 q_2}\Theta^{\alpha_1 q_1}\mathcal{F}_{\alpha_1\alpha_2\,q_1 q_2}(X_L) +$$

$$+\frac{i}{3!}\Theta^{\alpha_3 q_3}\Theta^{\alpha_2 q_2}\Theta^{\alpha_1 q_1}\epsilon_{q_1 q_2 q_3 q}\psi^q_{\alpha_1\alpha_2\alpha_3}(X_L) +$$

$$+\frac{1}{4!}\epsilon_{q_1 q_2 q_3 q_4}\Theta^{\alpha_4 q_4}\ldots\Theta^{\alpha_1 q_1}\mathcal{F}_{\alpha_1...\alpha_4}(X_L) \, . \tag{4.24}$$

Its two lowest components obey Eqs. (4.2), while its higher order nonvanishing field components satisfy Eqs. (4.23),

$$\partial_{\alpha[\gamma}\mathcal{F}_{\delta]\beta\,q_1 q_2}(X) = 0 \, , \tag{4.25}$$

$$\partial_{\alpha[\gamma}\psi_{\delta]\beta_2\beta_3}{}^q(X) = 0 \, , \tag{4.26}$$

$$\partial_{\alpha[\gamma}\mathcal{F}_{\delta]\beta_2\beta_3\beta_4}(X) = 0 \, . \tag{4.27}$$

It is tempting to identify (4.26) with the tensorial space generalization of the Rarita-Scwinger equations and Eq. (4.27) with that of the linearized conformal gravity equation imposed on Weyl tensor. However, similarly to the $\mathcal{N} = 2$ case in Sec. 4.1, it is possible to show that the general solutions of Eqs. (4.25), (4.26) and (4.27) are expressed in terms of a sextuplet of scalar fields $\phi_{q_1 q_2}(X) = \phi_{[q_1 q_2]}(X)$, a quadruplet of spinorial fields $\tilde{\psi}^q_{\alpha_3}$ and a singlet of scalar field $\tilde{\phi}(X)$ obeying the standard tensorial space fermionic and bosonic higher spin equations (4.2),

$$\mathcal{F}_{\alpha\beta\,q_1 q_2}(X) = \partial_{\alpha\beta}\phi_{q_1 q_2}(X) \, , \tag{4.28}$$

$$\psi^q_{\alpha_1\alpha_2\alpha_3}(X) = \partial_{\alpha_1\alpha_2}\tilde{\psi}^q_{\alpha_3}(X) \, , \tag{4.29}$$

$$\mathcal{F}_{\alpha_1...\alpha_4}(X) = \partial_{\alpha_1\alpha_2}\partial_{\alpha_3\alpha_4}\tilde{\phi}(X) \, . \tag{4.30}$$

Summarizing, the $\mathcal{N} = 8$ supermultiplet of free higher spin fields is described by a set of two scalar fields, a sextuplet of scalar fields, a spinor field and a quadruplet of spinorial fields, all in tensorial superspace, which obey the usual type free higher spin equations

$$\partial_{\alpha[\gamma}\partial_{\delta]\beta}\phi(X) = 0 \, , \qquad \partial_{\alpha[\beta}\psi_{\gamma]}(X) = 0 \, ,$$

$$\partial_{\alpha[\gamma}\partial_{\delta]\beta}\phi_{qp}(X) = 0 \, ,$$

$$\partial_{\alpha[\beta}\tilde{\psi}_{\gamma]}{}^q(X) = 0 \, , \qquad \partial_{\alpha[\gamma}\partial_{\delta]\beta}\tilde{\phi}(X) = 0 \, . \tag{4.31}$$

Thus, we conclude that, in tensorial superspace, at least all the free field dynamics is carried by the scalar and spinor fields. However, the 'higher' scalar and spinor fields appear in the basic superfield under the action of one or two derivatives and, hence, the model is invariant under the following generalized bosonic and fermionic Peccei-Quinn-like symmetries

$$\phi_{qp}(X) \;\mapsto\; \phi_{qp}(X) + a_{qp} \, ,$$

$$\tilde{\psi}_\alpha{}^q(X) \;\mapsto\; \tilde{\psi}_\alpha{}^q(X) + \beta_\alpha{}^q \, ,$$

$$\tilde{\phi}(X) \;\mapsto\; \tilde{\phi}(X) + a + X^{\alpha\beta}a_{\alpha\beta} \, , \tag{4.32}$$

with constant bosonic parameters $a_{qp} = -a_{pq}$, a, $a_{\alpha\beta}$ and constant fermionic parameter $\beta_\alpha{}^q$. Note that the non-constant shift in $\tilde{\phi}(X)$ is allowed by the presence of two derivatives in Eq. (4.30).

5   Conclusion and discussion

In this paper we have obtained the superfield equations in $\mathcal{N} = 2, 4$ and 8 extended tensorial superspaces $\Sigma^{(10|\mathcal{N}4)}$, which describe the supermultiplets of the $D = 4$ massless conformal free higher spin field theory with $\mathcal{N}$-extended supersymmetry. Actually our results are valid for $\Sigma^{(\frac{n(n+1)}{2}|\mathcal{N}n)}$ with arbitrary n and hence for other D's; we have elaborated the cases for $\mathcal{N} = 2, 4, 8$ since for $n = 4$ these have clear 'lower spin' $D = 4$ counterparts (*e.g.* SYM for $\mathcal{N} = 4$, supergravity for $\mathcal{N} = 8$)

The $\mathcal{N} = 2$ supermultiplet of massless conformal higher spin equations is simply given by the complexification of the $\mathcal{N} = 1$ supermultiplet. For $\mathcal{N} = 4, 8$ the $\mathcal{N}$-extended superfields contain higher components carrying symmetric tensor representations of $GL(n, \mathbb{R})$ that satisfy first order equations in tensorial superspace. It is tempting to identify these equations with a tensorial superspace generalization of the Maxwell equations and, in the $\mathcal{N} = 8$ case, also with the Rarita-Schwinger and linearized conformal gravity equations. However, the general solutions of these equations are expressed through scalar and spinor fields in tensorial space which obey the standard higher spin equations in their tensorial space version. As these additional scalar and spinor fields of the $\mathcal{N}$-extended supermultiplets appear in the basic superfield under derivatives, the theory is invariant under Peccei-Quinn–like symmetries shifting these fields.

The superfield equations (3.22) and (3.23) were found for even $\mathcal{N}$. It would be interesting to consider as well the case of extended tensorial superspaces with odd $\mathcal{N} > 1$ and to look for any specific properties of the $\mathcal{N}$-extended supermultiplets of the massless conformal higher spin fields for $\mathcal{N}$ odd.

Another possible direction for future study is to generalize the construction presented here to the $OSp(\mathcal{N}|2n)$ supergroup manifolds, which provide [1, 2, 28, 7] the AdS generalization of the tensorial (super)spaces, and to compare the results with the free field limit of [18, 19].

References

[1] C. Fronsdal, *Massless particles, ortosymplectic symmetry and another type of Kaluza-Klein theory*, UCLA preprint UCLA/85/TEP/10, in *Essays on Supersymmetry*, C. Fronsdal ed., Reidel, 1986 (Mathematical Physics Studies, vol. 8), p. 164-262.

[2] I. Bandos, J. Lukierski and D. Sorokin, *Superparticle models with tensorial central charges*, Phys. Rev. **D61**, 045002 (2000) [hep-th/9904109].

[3] M.A. Vasiliev, *Conformal higher spin symmetries of 4d massless supermultiplets and osp(L, 2M) invariant equations in generalized (super)space*, Phys. Rev. **D66** 066006 (2002) [hep-th/0106149].

[4] M.A. Vasiliev, *Relativity, causality, locality, quantization and duality in the $Sp(2M)$ invariant generalized space-time*, in *Multiple facets of quantization and supersymmetry. Marinov's Memorial Volume* (M. Olshanetsky and A. Vainshtein eds.) p. 826-872 [hep-th/0111119].

[5] M. Plyushchay, D. Sorokin and M. Tsulaia, *Higher spins from tensorial charges and $OSp(N|2n)$ symmetry*, JHEP **0304**, 013 (2003) [arXiv:hep-th/0301067]; *GL Flatness of $OSp(1|2n)$ and higher spin field theory from dynamics in tensorial spaces*, arXiv:hep-th/0310297.

[6] O. A. Gelfond and M. A. Vasiliev, *Higher rank conformal fields in the $Sp(2M)$ symmetric generalized space-time*, Theor. Math. Phys. **145**, 1400-1424 (2005) [Teor. Mat. Fiz. **145**, 35-65 (2005)] [arXiv:hep-th/0304020]; *$Sp(8)$ invariant higher spin theory, twistors and geometric BRST formulation of unfolded field equations*, JHEP **0912**, 021-0-26 (2009) [arXiv:0901.2176 [hep-th]].

[7] I. Bandos, P. Pasti, D. Sorokin and M. Tonin, *Superfield theories in tensorial superspaces and the dynamics of higher spin fields*, JHEP **0411**, 023 (2004) [arXiv:hep-th/0407180].

[8] I. Bandos, X. Bekaert, J. A. de Azcárraga, D. Sorokin and M. Tsulaia, *Dynamics of higher spin fields and tensorial space*, JHEP **0505**, 031 (2005) [arXiv:hep-th/0501113].

[9] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, *BPS states in M-theory and twistorial constituents*, Phys. Rev. Lett. **86**, 4451-4454 (2001) [hep-th/0101113].

[10] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo, M. Picón and O. Varela, *On BPS preons, generalized holonomies and $D=11$ supergravities*, Phys. Rev. **D69**, 105010-1-11 (2004) [arXiv:hep-th/0312266].

[11] R. D'Auria and P. Fré, *Geometric supergravity in $D = 11$ and its hidden supergroup*, Nucl. Phys. **B201**, 101-140 (1982) [Erratum-ibid. **B206**, 496 (1982)].

[12] I. A. Bandos, J. A. de Azcárraga, M. Picón and O. Varela, *On the underlying gauge group structure of $D=11$ supergravity*, Phys. Lett. **B596**, 145-155 (2004) [arXiv:hep-th/0406020]; *On the formulation of $D = 11$ supergravity and the composite nature of its three-form field*, Ann. Phys. **317**, 238-279 (2005) [hep-th/0409100].

[13] I. Bandos and J. Lukierski, *Tensorial central charges and new superparticle models with fundamental spinor coordinates*, Mod. Phys. Lett. **14**, 1257-1272 (1999) [hep-th/9811022].

[14] I. A. Bandos, J. A. de Azcárraga, M. Picón and O. Varela, *$D = 11$ superstring model with 30 kappa-symmetries and 30/32 BPS states in an extended superspace*, Phys. Rev. D **69**, 085007-1-23 (2004) [arXiv:hep-th/0307106].

[15] C. Chryssomalakos, J.A. de Azcárraga, J.M. Izquierdo and J.C. Pérez Bueno, *The geometry of branes and extended superspaces*, Nucl. Phys. **B567**, 293-330 (2000) [arXiv:hep-th/9904137];

J. A. de Azcárraga and J. M. Izquierdo, *Superalgebra cohomology, the geometry of extended superspaces and superbranes*, AIP Conf. Proc. **589**, 3-17 (2001) [arXiv:hep-th/0105125].

[16] S. M. Kuzenko and A. G. Sibiryakov, *Massless gauge superfields of higher integer superspins*, JETP Lett. **57** 539 (1993) [Pisma Zh. Eksp. Teor. Fiz. **57**, 526 (1993)]; S. M. Kuzenko, A. G. Sibiryakov and V. V. Postnikov, *Massless gauge superfields of higher half integer superspins*, JETP Lett. **57**, 534 (1993) [Pisma Zh. Eksp. Teor. Fiz. **57**, 521 (1993)].

[17] S. J. J. Gates and K. Koutrolikos, *A unified spinorial superfield treatment of the higher superspin superfield formalism*, arXiv:1004.3572 [hep-th]; *A codicil to massless gauge superfields of higher integer superspins*, arXiv:1103.3565 [hep-th].

[18] E. Sezgin and P. Sundell, *Higher spin N=8 supergravity*, JHEP **9811**, 016 (1998) [arXiv:hep-th/9805125].

[19] J. Engquist, E. Sezgin and P. Sundell, *On N = 1,2,4 higher spin gauge theories in four dimensions*, Class. Quant. Grav. **19**, 6175-6196 (2002) [arXiv:hep-th/0207101]; *Superspace formulation of 4D higher spin gauge theory*, Nucl. Phys. B **664**, 439-456 (2003) [hep-th/0211113].

[20] M. A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, Phys. Lett. **B209**, 491 (1988); *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, Ann. Phys. (*NY*) **190**, 59 (1989).

[21] M. A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions*, Phys. Lett. **B285**, 225-234 (1992).

[22] K. B. Alkalaev and M. A. Vasiliev, *N = 1 supersymmetric theory of higher spin gauge fields in AdS(5) at the cubic level*, Nucl. Phys. **B655**, 57-92 (2003) [hep-th/0206068].

[23] R. D. Peccei, H. R. Quinn, *Constraints imposed by CP conservation in the presence of instantons*, Phys. Rev. **D16** , 1791-1797 (1977).

[24] R. Penrose, *Twistor quantization and curved space-time*, Int. Journ. Theor. Phys. **1**, 61-69 (1968); *Twistor theory, its aims and achievements*, in *Quantum gravity*, C.J. Isham *et al.* (eds.), Clarendon, Oxford 1975, p. 268-407, and references therein; R. Penrose and W. Rindler, *Spinors and spacetime*, CUP (1986); R. Penrose and M. A. H. MacCallum, *Twistor theory: an approach to the quantization of fields and space-Time*, Phys. Rept. **6**, 241 (1972) and refs. therein.

[25] A. Ferber, *Supertwistors and conformal supersymmetry*, Nucl. Phys. **B132**, 55-64 (1978).

[26] T. Shirafuji, *Lagrangian mechanics of massless particles with spin*, Prog. Theor. Phys. **70**, 18-35 (1983).

[27] J.W. van Holten and A. van Proeyen, *N=1 supersymmetry algebras in D = 2, D = 3, D = 4 mod 8*, J. Phys. **A15**, 3763-3783 (1982).

[28] I. Bandos, J. Lukierski, C. Preitschopf and D. Sorokin, *OSp supergroup manifolds, superparticles and supertwistors*, Phys. Rev. **D61** 065009-1-13 (2000) [hep-th/9907113].

[29] J. A. de Azcárraga, J. Lukierski, *Supersymmetric particles with internal symmetries and central charges*, Phys. Lett. **B113**, 170-174 (1982); *Gupta-Bleuler quantization of massive superparticle models in $D = 6$, $D = 8$ and $D = 10$*, Phys. Rev. **D38**, 509-513 (1988).

[30] A. Frydryszak, *N-Extended free superfields (N=2, N=4, N=6, N=8) from quantization of supersymmetric particle model*, Phys. Rev. **D30**, 2172-2180 (1984).