EQUIDISTRIBUTION OF ZEROS OF HOLOMORPHIC SECTIONS IN THE NON COMPACT SETTING

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ABSTRACT. We consider tensor powers L^N of a positive Hermitian line bundle (L, h^L) over a non-compact complex manifold X. In the compact case, B. Shiffman and S. Zelditch proved in [32] that the zeros of random sections become asymptotically uniformly distributed as $N \to \infty$ with respect to the natural measure coming from the curvature of L. Under certain boundedness assumptions on the curvature of the canonical line bundle of X and on the Chern form of L we prove a non-compact version of this result. We give various applications, including the limiting distribution of zeros of cusp forms with respect to the principal congruence subgroups of $SL_2(\mathbb{Z})$ and to the hyperbolic measure, the higher dimensional case of arithmetic quotients and the case of orthogonal polynomials with weights at infinity. We also give estimates for the speed of convergence of the currents of integration on the zero-divisors.

CONTENTS

1. INTRODUCTION AND RELATED RESULTS

This note is concerned with the asymptotic distribution of zeros of random holomorphic sections in the high tensor powers L^N of a positive Hermitian line bundle (L, h^L) over a non-compact complex manifold X. Distribution of zeros of random polynomials is a classical subject, starting with the papers of Bloch-Pólya, Littlewood-Offord, Hammersley, Kac and Erdös-Turán, see e.g. Bleher-Di [6] and Shepp-Vanderbei [31] for a review and complete references.

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Shiffman and Zelditch [32] obtained a far-reaching generalization by proving that the zeros of random sections of powers L^N of a positive line bundle L over a projective manifold become asymptotically uniformly distributed as $N \to \infty$ with respect to the natural measure coming from L. In a long series of papers these authors further considered the correlations between zeros and their variance (see e.g. [7, 33]). Berman [5] generalized some of these results in the context of pseudoconcave domains.

A different method to study the distribution of zeros was introduced by Sibony and the first-named author [14] using the formalism of meromorphic transforms. They also gave bounds for the convergence speed in the compact case, improving on the ones in [32].

There is an interesting connection between equidistribution of zeros and Quantum Unique Ergodicity related to a conjecture of Rudnik and Sarnak [29] about the behaviour of high energy Laplace eigenfunctions on a Riemannian manifold. By replacing Laplace eigenfunctions with modular forms one is lead to study of the equidistribution of zeros of Hecke modular forms. This was done by Rudnick [28], Holowinsky and Soundararajan [17] and generalized by Marshall [21] and Nelson [25].

Another area where random polynomials and holomorphic sections play a role is statistical physics. Holomorphic random sections provide a model for quantum chaos and the distribution of their zeros was intensively studied by physicists e.g. [6, 9, 26].

The proof of the equidistribution in [14, 32] involves the asymptotic expansion of the Bergman kernel. In [19, 20] we obtained the asymptotic expansion of the L^2-Bergman kernel for positive line bundles over complete Hermitian manifolds under some natural curvature conditions, see (1.4) and Remark 1.3. In this paper we regain the asymptotic equidistribution of random zeros of holomorphic L^2-sections under the additional assumption, that the spaces of holomorphic L^2-sections of L^N are of finite dimension. We give more explanations after we have stated the theorem precisely.

Let us consider an n-dimensional complex Hermitian manifold (X, J, Θ), where J is the complex structure and Θ a positive $(1,1)$-form. The manifold (X, J, Θ) is called Kähler if $d\Theta = 0$. To Θ we associate a J-invariant Riemannian metric g^{TX} given by $g^{TX}(u, v) = \Theta(u, Jv)$ for all $u, v \in T_xX$, $x \in X$.

We consider further a Hermitian holomorphic line bundle (L, h^L) on X. The curvature form of L is denoted by R^L. We denote by $L^N := L^{\otimes N}$ the N-th tensor power of L. The Hermitian metrics Θ and h^L provide an L^2 Hermitian inner product on the space of sections of L^N and we can introduce the space of holomorphic L^2-sections $H^0_{(2)}(X, L^N)$, cf. (2.3).

For a section $s \in H^0_{(2)}(X, L^N)$ we denote by $\mathrm{Div}(s)$ the divisor defined by s; then $\mathrm{Div}(s)$ can be written as a locally finite linear combination $\sum c_i V_i$, where V_i are irreducible analytic hypersurfaces and $c_i \in \mathbb{Z}$ are the multiplicities of s along the V_i's.

Recall here the notion of current: let $\Omega^{p,q}_0(X)$ denote the space of smooth compactly supported (p, q)-forms on X, and we let $\Omega'^{p,q}(X) = \Omega^{n-p,n-q}_0(X)'$ denote the space of (p, q)-currents on X; $(T, \varphi) = T(\varphi)$ denotes the pairing of $T \in \Omega'^{p,q}(X)$ and $\varphi \in \Omega^{n-p,n-q}_0(X)$.

We denote by $[\mathrm{Div}(s)]$ the current of integration on $\mathrm{Div}(s)$. If $\mathrm{Div}(s) = \sum c_i V_i$ then

$$\Big([\mathrm{Div}(s)], \varphi\Big) := \sum_i c_i \int_{V_i} \varphi\,, \quad \varphi \in \Omega^{n-1,n-1}_0(X)\,,$$

where the integrals are well-defined by a theorem of Lelong (cf. [13, III-2.6], [16, p. 32]).

Assume that the spaces $H^0_{(2)}(X, L^N)$ have finite dimension d_N for all N. We endow $H^0_{(2)}(X, L^N)$ with the natural L^2-metric (cf. (2.2)) and this induces a Fubini-Study metric $\omega_{\rm FS}$ on the projective space $\mathbb{P}H^0_{(2)}(X, L^N)$. The volume form $\omega_{\rm FS}^{d_N-1}$ defines by normalization a probability measure $\sigma_{\rm FS}$ on $\mathbb{P}H^0_{(2)}(X, L^N)$. We consider the probability space

$$(\Omega, \mu) = \prod_{N=1}^{\infty} \left(\mathbb{P}H^0_{(2)}(X, L^N), \sigma_{\rm FS}\right). \tag{1.1}$$

Note that for two elements $s, s' \in H^0_{(2)}(X, L^N)$ which are in the same equivalence class in $\mathbb{P}H^0_{(2)}(X, L^N)$ we have $\operatorname{Div}(s) = \operatorname{Div}(s')$, so Div is well defined on $\mathbb{P}H^0_{(2)}(X, L^N)$.

Definition 1.1. We say that the zero-divisors of generic random sequences (s_N) with $s_N \in \mathbb{P}H^0_{(2)}(X, L^N)$ are equidistributed with respect to a Hermitian metric ω on X, if for μ-almost all sequences $(s_N) \in \prod_{N=1}^{\infty} \mathbb{P}H^0_{(2)}(X, L^N)$ we have

$$\frac{1}{N}\Big[\operatorname{Div}(s_N)\Big] \to \omega\,, \quad N \to \infty\,, \tag{1.2}$$

in the sense of currents, that is, for any test $(n-1, n-1)$-form $\varphi \in \Omega_0^{n-1,n-1}(X)$ there holds

$$\lim_{N\to\infty} \Big(\frac{1}{N}\Big[\operatorname{Div}(s_N)\Big], \varphi\Big) = \int_X \omega \wedge \varphi. \tag{1.3}$$

Our first result is a generalization to non-compact manifolds of a seminal result of Shiffman-Zelditch [32, Th. 1.1].

Let K_X denote the canonical bundle of X, i.e., $K_X = \Lambda^{n,0}T^*X$. If Θ is a Hermitian metric on X we consider the induced metric on K_X with curvature R^{K_X}. If Θ is Kähler, then $\sqrt{-1}R^{K_X} = -\operatorname{Ric}_\Theta$, where $\operatorname{Ric}_\Theta$ is the Ricci curvature of the Riemannian metric associated to Θ (cf. (4.1)-(4.2)).

Theorem 1.2. *Let (X, Θ) be an n-dimensional complete Hermitian manifold. Let (L, h^L) be a Hermitian holomorphic line bundle over X. Assume that there exist constants $\varepsilon > 0$, $C > 0$ such that*

$$\sqrt{-1}\,R^L > \varepsilon\Theta, \quad \sqrt{-1}R^{K_X} < C\Theta, \quad |\partial\Theta|_{g^{TX}} < C\,. \tag{1.4}$$

Furthermore, assume that the spaces $H^0_{(2)}(X, L^N)$ have finite dimension for all N. Then the zero-divisors of generic random sequences $(s_N) \in \prod_{N=1}^{\infty} \mathbb{P}H^0_{(2)}(X, L^N)$ are equidistributed with respect to $\frac{\sqrt{-1}}{2\pi}R^L$.

Remark 1.3. Some comments about the hypothesis of the theorem are in order. If (X, Θ) is Kähler ($d\Theta = 0$, equivalently, $\partial\Theta = 0$) they are the natural hypotheses to apply Hörmander's L^2 $\overline{\partial}$-method (with singular weights) in order to find L^2 holomorphic sections of L^N which separate points and give local coordinates. If (X, Θ) is not Kähler, then we have to apply the expansion of the Bergman kernel to achieve this goal (see Corollary 2.2 (i)). Indeed, (1.4) implies that the Kodaira-Laplacian on L^N has a spectral gap (via the Bochner-Kodaira-Nakano formula) and the Bergman kernel of L^N has an asymptotic expansion (cf. [20], [19, Th. 6.1.1], see Theorem 2.1).

Let us remark that:

(i) if $L = K_X$, the first two conditions in (1.4) can be replaced by the following simpler condition: h^{K_X} is induced by Θ and $\sqrt{-1}R^{K_X} > \varepsilon\Theta$, for some $\varepsilon > 0$ (see also (iii)).
(ii) If (X, Θ) is Kähler, the condition $\partial\Theta = 0$ is trivially satisfied.
(iii) If we consider the equidistribution of sections of $L^N \otimes K_X$ of n-holomorphic forms with values in L^N we can skip the condition $\sqrt{-1}R^{K_X} < C\Theta$ in (1.4). In fact, we can state the following result. Assume that

$$\sqrt{-1}\, R^L > \varepsilon\Theta, \quad |\partial\Theta|_{g^{TX}} < C\,. \tag{1.5}$$

Furthermore, assume that the spaces $H^0_{(2)}(X, L^N \otimes K_X)$ have finite dimension for all N. Then the zero-divisors of generic random sequences $(s_N) \in \prod_{N=1}^{\infty} \mathbb{P}H^0_{(2)}(X, L^N \otimes K_X)$ are equidistributed with respect to $\frac{\sqrt{-1}}{2\pi}R^L$.

Using these remarks we can obtain various versions of Theorem 1.2.

Remark 1.4. Note that on a compact manifold equipped with a positive line bundle one can always realize condition (1.4) by choosing $\Theta = \frac{\sqrt{-1}}{2\pi}R^L$. This is not always possible in general; if X is non-compact, the metric associated to $\Theta = \frac{\sqrt{-1}}{2\pi}R^L$ might be non-complete. The existence of a complete metric Θ with $\sqrt{-1}R^L > \varepsilon\Theta$ is equivalent to saying that $\sqrt{-1}R^L$ defines a complete Kähler metric.

Another interesting fact is that by changing Θ we change the L^2-product (2.2) on the spaces of sections of L^N. However, as long as (1.4) holds, the L^2-holomorphic sections are equidistributed with respect to $\frac{\sqrt{-1}}{2\pi}R^L$.

Theorem 1.2 is a consequence of the following equidistribution result on relatively compact open sets. We also address here the problem of the convergence speed of the currents $[\frac{1}{N}\operatorname{Div}(s_N)]$ towards $\frac{\sqrt{-1}}{2\pi}R^L$. For a compact manifold X the estimates of the convergence speed from Theorem 1.5 were obtained in [14] and we will adapt the method of [14] in the present context.

Theorem 1.5. *Let (X, Θ) and (L, h^L) be as in Theorem 1.2. Then for any relatively compact open subset U of X there exist a constant $c = c(U) > 0$ and an integer $N(U)$ with the following property. For any real sequence (λ_N) with $\lim_{N\to\infty}(\lambda_N/\log N) = \infty$ and for any $N \geqslant N(U)$ there exists a set $E_N \subset \mathbb{P}H^0_{(2)}(X, L^N)$ such that:*

(a) $\sigma_{\rm FS}(E_N) \leqslant c\, N^{2n}\, e^{-\lambda_N/c}$,
(b) *For any $s_N \in \mathbb{P}H^0_{(2)}(X, L^N) \setminus E_N$ we have the following estimate*

$$\Big|\Big(\frac{1}{N}\Big[\operatorname{Div}(s_N)\Big] - \frac{\sqrt{-1}}{2\pi}R^L, \varphi\Big)\Big| \leqslant \frac{\lambda_N}{N}\|\varphi\|_{\mathscr{C}^2}\,, \quad \varphi \in \Omega_0^{n-1,n-1}(U)\,. \tag{1.6}$$

In particular, for a generic sequence $(s_N) \in \prod_{N=1}^{\infty} \mathbb{P}H^0_{(2)}(X, L^N)$ the estimate (1.6) *holds for $s = s_N$, for N large enough.*

By choosing (λ_N) such that $\lim_{N\to\infty}(\lambda_N/N) = 0$ we obtain that zero-divisors of generic random sequences $(s_N) \in \prod_{N=1}^{\infty} \mathbb{P}H^0_{(2)}(X, L^N)$ are equidistributed with respect to $\frac{\sqrt{-1}}{2\pi}R^L$ on U. From this observation, Theorem 1.2 follows immediately.

Of course, we can use different values of (λ_N) depending of our purpose: if we want a better estimate on E_N, the speed of equidistribution will be worse; if we want a better convergence speed, we have to allow larger sets E_N. For example, we can take $\lambda_N =$

$(\log N)^{1+\beta}$, $0 < \beta \ll 1$. Then (λ_N/N) in (1.6) converges very fast to 0. Note that in [32, p. 671] it is observed that in the compact case the convergence speed is bounded above by $N^{\varepsilon-\frac{1}{2}}$, for any $\varepsilon > 0$.

Let X be an n-dimensional, $n \geqslant 2$, irreducible quotient of a bounded symmetric domain D by a torsion-free arithmetic group $\Gamma \subset \mathrm{Aut}(D)$. We call such manifolds arithmetic quotients. The Bergman metric $\omega_D^{\mathcal{B}}$ on D (cf. (4.5)) descends to a Hermitian metric on X, called Bergman metric on X and denoted by $\omega_X^{\mathcal{B}}$. This metric induces a volume form on X and a Hermitian metric on K_X and thus a L^2 Hermitian inner product on the sections of K_X^N.

Corollary 1.6. *Let X be an n-dimensional, $n \geqslant 2$, arithmetic quotient. Let $H^0_{(2)}(X, K_X^N)$ be the space of holomorphic sections of K_X^N which are square-integrable with respect to the L^2 Hermitian inner product induced by the Bergman metric. Then the zero-divisors of generic random sequences $(s_N) \in \prod_{N=1}^{\infty} \mathbb{P}H^0_{(2)}(X, K_X^N)$ are equidistributed with respect to the Bergman metric $\frac{1}{2\pi}\omega_X^{\mathcal{B}}$ on X. Moreover, we have an estimate of the convergence speed on compact sets as in Theorem 1.5.*

Theorem 1.2 has the following application to the equidistribution of zeros of modular forms.

Corollary 1.7. *Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index that acts freely on the hyperbolic plane $\mathbb{H}$. Consider the spaces of cusp forms $\mathcal{S}_{2N}$ as Gaussian probability spaces with the measure induced by the Petterson inner product. Then for almost all random sequences (f_{2N}) in the product space $\prod_{N=1}^{\infty} \mathcal{S}_{2N}$ the associated sequence of zeros becomes asymptotically uniformly distributed, i.e. for piecewise smooth open sets U contained in one fundamental domain we have*

$$\lim_{N\to\infty} \frac{1}{N} \sharp\Big\{ z \in U ; f_{2N}(z) = 0 \Big\} = \frac{1}{2\pi}\,\mathrm{Vol}\,(U),$$

where Vol *denotes the hyperbolic volume. Moreover, we have an estimate of the convergence speed on compact sets as in Theorem 1.5.*

This result concerns *typical* sequences of cusp forms. If one considers Hecke modular forms, by a result of Rudnick [28] and its generalization by Marshall [21], the zeros of *all* sequences of Hecke modular forms are equidistributed. The method used in [21, 28] follows the seminal paper of Nonnenmacher-Voros [26] and consists in showing the equidistribution of masses of Hecke forms. Rudnick [28] invoked for this purpose the Generalized Riemann Hypothesis and this hypothesis was later removed by Holowinsky and Soundararajan [17]. Marshall [21] extended their methods to the higher-dimensional setting. Nelson [25] removed later some of the hypotheses in [21].

This lay-out of this paper is as follows. In Section 2, we collect the necessary ingredients about the asymptotic expansion of Bergman kernel. In Section 3, we prove the main results, Theorems 1.2 and 1.5, about the equidistribution on compact sets together with the estimate of the convergence speed. In the next sections we give applications of our main result in several geometric contexts and prove equidistribution: for sections of the pluricanonical bundles over pseudoconcave manifolds and arithmethic quotients in dimension greater than two (Section 4), for modular forms over Riemann surfaces (Section 5), for sections of positive line bundles over quasi-projective manifolds (Section 6) and finally for polynomials over $\mathbb{C}$ endowed with the Poincaré metric at infinity (Section 7).

2. BACKGROUND ON THE BERGMAN KERNEL

Let (X, J, Θ) be a complex Hermitian manifold of dimension n, where J is the complex structure and Θ is the $(1,1)$-form associated to a Riemannian metric g^{TX} compatible with J, i.e.

$$\Theta(u,v) = g^{TX}(Ju,v)\,,\ g^{TX}(Ju,Jv) = g^{TX}(u,v)\,,\quad \text{for all } u,v\in T_xX,\, x\in X\,. \tag{2.1}$$

The volume form of the metric g^{TX} is given by $dv_X = \Theta^n/n!$. The Hermitian manifold (X, J, Θ) is called complete if g^{TX} is a complete Riemannian metric.

Let (E, h^E) be a Hermitian holomorphic line bundle. For $v, w \in E_x$, $x \in X$, we denote by $\langle v, w\rangle_E$ the inner product given by h^E and by $|v|_E = \langle v, v\rangle_E^{1/2}$ the induced norm. The L^2–Hermitian product on the space $\mathscr{C}_0^\infty(X, E)$ of compactly supported smooth sections of E, is given by

$$(s_1, s_2) = \int_X \langle s_1(x), s_2(x)\rangle_E\, dv_X(x)\,. \tag{2.2}$$

We denote by $L^2(X, E)$ the completion of $\mathscr{C}_0^\infty(X, E)$ with respect to (2.2). Consider further the space of holomorphic L^2-sections of E:

$$H^0_{(2)}(X,E) := \left\{ s\in L^2(X,E) : \overline{\partial}^E s = 0\right\}. \tag{2.3}$$

Here the condition $\overline{\partial}^E s = 0$ is taken in the sense of distributions. Elementary continuity properties of differential operators on distributions show that $H^0_{(2)}(X, E)$ is a closed subspace of $L^2(X, E)$. Moreover, the hypoellipticity of $\overline{\partial}^E$ implies that elements of $H^0_{(2)}(X, E)$ are smooth and indeed holomorphic.

For a Hermitian holomorphic line bundle (E, h^E) we denote by R^E its curvature, which is a $(1,1)$-form on X. Given the Riemannian g^{TX} metric on X associated to Θ as in (2.1), we can identify R^E to a Hermitian matrix $\dot{R}^E \in \mathrm{End}(T^{(1,0)}X)$ such that for $u, v \in T_x^{(1,0)}X$,

$$R^E(u,\overline{v}) = \langle \dot{R}^E(u), v\rangle. \tag{2.4}$$

There exists an orthonormal basis of $(T_x^{(1,0)}X, g^{TX})$ such that $\dot{R}^E = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$. The real numbers $\alpha_1, \dots, \alpha_n$ are called the eigenvalues of R^E with respect to Θ at $x \in X$. We denote

$$\det(\dot{R}^E) = \prod_{k=1}^n \alpha_k\,. \tag{2.5}$$

For a holomorphic section s in a holomorphic line bundle, let $\mathrm{Div}(s)$ be the zero divisor of s and we denote by the same symbol $\mathrm{Div}(s)$ the current of integration on $\mathrm{Div}(s)$. We denote by K_X the canonical line bundle over X.

Let $(L, h^L) \to X$ be a Hermitian holomorphic vector bundle. As usual, we will write L^N instead of $L^{\otimes N}$ for the $N-$th tensor power of L and omit the $\otimes-$symbol in all similar expressions. On L^N we consider the induced Hermitian metric $h^{L^N} = (h^L)^{\otimes N}$. An inner product on the spaces of holomorphic sections $H^0(X, L^N)$ is defined by (2.2). Denote by

$H^0_{(2)}(X,L^N)$ the subspace of holomorphic L^2-sections. The Bergman kernel $P_N(x,y)$ is the Schwartz kernel of the orthogonal projection

$$P_N : L^2(X,L^N) \to H^0_{(2)}(X,L^N)\,.$$

If $H^0_{(2)}(X,L^N) = 0$, we have of course $P_N(x,x) = 0$ for all $x \in X$. If $H^0_{(2)}(X,L^N) \neq 0$, consider an orthonormal basis $(S^N_j)_{j=1}^{d_N}$ of $H^0_{(2)}(X,L^N)$ (where $1 \leqslant d_N \leqslant \infty$). Then

$$P_N(x,x) = \sum_{j=1}^{d_N} |S^N_j(x)|^2_{L^N} \quad \text{in } \mathscr{C}^\infty_{loc}(X)\,. \tag{2.6}$$

The following proposition is a special case of [19, Th. 6.1.1]. The case when X is compact is due to Catlin [11] and Zelditch [38].

Theorem 2.1. *Let (X,Θ) be an n-dimensional complete Hermitian manifold. Let (L,h^L) be a Hermitian holomorphic line bundle over X. Assume that there exist constants $\varepsilon > 0$, $C > 0$ such that*

$$\sqrt{-1}R^L > \varepsilon\Theta, \quad \sqrt{-1}R^{K_X} < C\Theta, \quad |\partial\Theta|_{g^{TX}} < C.$$

Then there exist coefficients $b_r \in \mathscr{C}^\infty(X)$, for $r \in \mathbb{N}$, such that the following asymptotic expansion

$$P_N(x,x) = \sum_{r=0}^{\infty} b_r(x) N^{n-r} \tag{2.7}$$

holds in any $\mathscr{C}^\ell$-topology on compact sets of X. Moreover, $b_0 = \det\left(\frac{\dot{R}^L}{2\pi}\right)$.

To be more precise, the asymptotic expansion (2.7) means that for any compact set $K \subset X$ and any $k,\ell \in \mathbb{N}$ there exists a constant $C_{k,\ell,K} > 0$, such that for any $N \in \mathbb{N}$,

$$\left| P_N(x,x) - \sum_{r=0}^{k} b_r(x) N^{n-r} \right|_{\mathscr{C}^\ell(K)} \leq C_{k,\ell,K} N^{n-k-1}\,. \tag{2.8}$$

Let

$$Bs_N := \{x \in X : s(x) = 0 \ \text{ for all } s \in H^0_{(2)}(X,L^N)\} \tag{2.9}$$

be the base locus of $H^0_{(2)}(X,L^N)$, which is an analytic set. Assume now that for N large enough

$$d_N := \dim H^0_{(2)}(X,L^N) < \infty\,. \tag{2.10}$$

The Kodaira map is the holomorphic map

$$\begin{aligned} \Phi_N :\ & X \setminus Bs_N \to \mathbb{P}H^0_{(2)}(X,L^N)^* \\ & x \longmapsto \left\{ s \in H^0_{(2)}(X,L^N) :\ s(x) = 0 \right\} \end{aligned} \tag{2.11}$$

In this definition we identify the projective space $\mathbb{P}H^0_{(2)}(X,L^N)^*$ of lines in $H^0_{(2)}(X,L^N)^*$ to the Grassmannian manifold of hyperplanes in $H^0_{(2)}(X,L^N)$. To be precise, for a section $s \in H^0_{(2)}(X,L^N)$ and a local holomorphic frame $e_L : U \to L$ of L, there exists a holomorphic function $f \in \mathcal{O}(U)$ such that $s = f e_L^{\otimes N}$. We denote f by $s/e_L^{\otimes N}$. To $x \in X$ and a choice of local holomorphic frame e_L of L, we assign the element $s \mapsto (s/e_L^{\otimes N})(x)$ in $H^0_{(2)}(X,L^N)^*$. If $x \notin Bs_N$, this defines a line in $H^0_{(2)}(X,L^N)^*$ which is by definition $\Phi_N(x)$.

By a choice of basis $\{S_i^N\}$ of $H^0_{(2)}(X, L^N)$ it is easy to see that Φ_N is holomorphic and has the coordinate representation

$$x \mapsto [(S_0^N/e_L^{\otimes N})(x), \dots, (S_{d_N}^N/e_L^{\otimes N})(x)].$$

As a consequence of the asymptotic expansion (2.7) of the Bergman kernel we obtain:

Corollary 2.2. *(i) Under the assumption of Theorem 2.1 the ring $\oplus_N H^0_{(2)}(X, L^N)$ separates points and gives local coordinates at each point of X.*

(ii) Let $K \subset X$ be compact. There exists an integer $N(K)$ such that for every $N > N(K)$ we have $Bs_N \cap K = \emptyset$ and Φ_N is an embedding in a neighbourhood of K.

Proof. Item (i) follows by applying the analytic proof of the Kodaira embedding theorem by the Bergman kernel expansion (see e.g. [19, §8.3.5], where symplectic manifolds are considered; the arguments therein extend to the non-compact case due to [19, Th. 6.1.1]). Item (ii) follows immediately from (i). □

Let us note a case when the Kodaira map (2.11) gives a global embedding.

Proposition 2.3. *Under the assumption of Theorem 2.1 suppose further that the base manifold X is* 1*-concave (cf. Definition 4.1). Then there exists N_0 such that for all $N \geqslant N_0$ we have $Bs_N = \emptyset$ and the Kodaira map* (2.11) *is an embedding of X.*

Proof. By Corollary 2.2 (i) the ring $\oplus_N H^0_{(2)}(X, L^N)$ separates points and gives local coordinates at each point of X. Applying the proof of the Andreotti-Tomassini embedding theorem [3] to the ring $\oplus_N H^0_{(2)}(X, L^N)$ we obtained the desired conclusion. □

Let V be a finite dimensional Hermitian vector space and let V^* be its dual. Let $\mathcal{O}(-1)$ be the universal (tautological) line bundle on $\mathbb{P}(V^*)$. Let us denote by $\mathcal{O}(1) = \mathcal{O}(-1)^*$ the hyperplane line bundle over the projective space $\mathbb{P}(V^*)$.

A Hermitian metric h^V on V induces naturally a Hermitian metric h^{V^*} on V^*, thus it induces a Hermitian metric $h^{\mathcal{O}(-1)}$ on $\mathcal{O}(-1)$, as a sub-bundle of the trivial bundle V^* on $\mathbb{P}(V^*)$. Let $h^{\mathcal{O}(1)}$ be the Hermitian metric on $\mathcal{O}(1)$ induced by $h^{\mathcal{O}(-1)}$.

For any $v \in V$, the linear map $V^* \ni f \to (f, v) \in \mathbb{C}$ defines naturally a holomorphic section σ_v of $\mathcal{O}(1)$ on $\mathbb{P}(V^*)$. By the definition, for $f \in V^* \setminus \{0\}$, at $[f] \in \mathbb{P}(V^*)$, we have

$$|\sigma_v([f])|^2_{h^{\mathcal{O}(1)}} = |(f, v)|^2 / |f|^2_{h^{V^*}}. \tag{2.12}$$

For $N \geqslant N(K)$, $\Phi_N : K \longrightarrow \mathbb{P}H^0(X, L^N)^*$ is holomorphic and the map

$$\begin{aligned} &\Psi_N : \Phi_N^* \mathcal{O}(1) \to L^N, \\ &\Psi_N((\Phi_N^* \sigma_s)(x)) = s(x), \quad \text{for any } s \in H^0(X, L^N) \end{aligned} \tag{2.13}$$

defines a canonical isomorphism from $\Phi_N^* \mathcal{O}(1)$ to L^N on X, and under this isomorphism, we have

$$h^{\Phi_N^* \mathcal{O}(1)}(x) = P_N(x, x)^{-1} h^{L^N}(x) \tag{2.14}$$

on K (see e.g. [19, (5.1.15)]). Here $h^{\Phi_N^* \mathcal{O}(1)}$ is the metric on $\Phi_N^* \mathcal{O}(1)$ induced by the Fubini-Study metric $h^{\mathcal{O}(1)}$ on $\mathcal{O}(1) \to \mathbb{P}H^0(X, L^N)^*$.

The Fubini-Study form $\omega_{\rm FS}$ is the Kähler form on $\mathbb{P}(V^*)$, and is defined as follows: for any $0 \neq v \in V$ we set

$$\omega_{\rm FS} = \frac{\sqrt{-1}}{2\pi} R^{\mathcal{O}(1)} = \frac{\sqrt{-1}}{2\pi} \overline{\partial}\partial \log |\sigma_v|^2_{h^{\mathcal{O}(1)}} \quad \text{on } \left\{x \in \mathbb{P}(V^*), \sigma_v(x) \neq 0\right\}. \tag{2.15}$$

Identity (2.14) immediately implies

$$\begin{aligned} \Phi_N^* \omega_{\rm FS} &= \frac{\sqrt{-1}}{2\pi} R^{L^N} + \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log P_N(x,x) \\ &= N \frac{\sqrt{-1}}{2\pi} R^L + \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log P_N(x,x). \end{aligned} \tag{2.16}$$

We obtain as a consequence the analogue of the Tian-Ruan convergence of the Fubini-Study metric on compact subsets of X.

Corollary 2.4 ([19, Corollary 6.1.2])**.** *In the conditions of Theorem 2.1 for any $\ell \in \mathbb{N}$ there exists $C_{\ell,K}$ such that*

$$\left| \frac{1}{N} \Phi_N^* \omega_{\rm FS} - \frac{\sqrt{-1}}{2\pi} R^L \right|_{\mathscr{C}^\ell(K)} \leqslant \frac{C_{\ell,K}}{N}. \tag{2.17}$$

Thus the induced Fubini-Study form $\frac{1}{N}\Phi_N^*\omega_{\rm FS}$ converges to ω in the $\mathscr{C}^\infty_{loc}$ topology as $N \to \infty$.

Proof. By (2.16),

$$\frac{1}{N} \Phi_N^* \omega_{\rm FS} - \frac{\sqrt{-1}}{2\pi} R^L = \frac{\sqrt{-1}}{2\pi N} \partial\overline{\partial} \log P_N(x,x)$$

and by (2.8)

$$\left| \partial\overline{\partial} \log P_N(x,x) \right|_{\mathscr{C}^\ell(K)} = \mathrm{O}(1), \quad N \to \infty,$$

where $\mathrm{O}(1)$ is the Landau symbol. □

Tian obtained the estimate (2.17), with the bound $\mathrm{O}(1/\sqrt{N})$ and for $\ell = 2$, for compact manifolds [34, Th. A] and for complete Kähler manifolds (X, ω) with $\mathrm{Ric}_\omega \leqslant -k\,\omega$ for some constant $k > 0$, [34, Th. 4.1]. Ruan [27] proved the $\mathscr{C}^\infty$-convergence and improved the bound to $\mathrm{O}(1/N)$.

Remark 2.5. Assume that in Theorem 1.2 we have $\Theta = \frac{\sqrt{-1}}{2\pi} R^L$. Then $b_0(x) = 1$. Thus from (2.7) and (2.16), we can improve the estimate (2.17) by replacing the left-hand side by $C_{\ell,K}/N^2$ (cf. [19, Rem. 5.1.5]).

3. Equidistribution on compact sets and speed of convergence

Let (X, ω) be a Hermitian complex manifold of dimension n. Let U be a relatively compact open subset of X. If S is a real current of bidegree (p,p) and of order 0 on X and $\alpha \geqslant 0$, define

$$||S||_{U,-\alpha} := \sup_{\varphi} |(S, \varphi)|$$

where the supremum is taken over all smooth real $(k-p, k-p)$-forms φ with compact support in U such that $||\varphi||_{\mathscr{C}^\alpha} \leqslant 1$. When $\alpha = 0$, we obtain the mass of S on U that is denoted by $||S||_U$.

It is clear that if $\beta \geq \alpha$, then

$$||S||_{U,-\alpha} \geqslant ||S||_{U,-\beta}.$$

If W is an open set such that $U \Subset W \Subset X$, by theory of interpolation between Banach spaces [36], we have

$$||S||_{U,-\alpha} \leqslant c||S||_{W}^{1-\alpha/\beta}||S||_{W,-\beta}^{\alpha/\beta},$$

where $c > 0$ is a constant independent of S, see [15].

For an arbitrary complex vector space V we denote by $\mathbb{P}(V)$ the projective space of 1-dimensional subspaces of V. Fix now a vector space V of complex dimension $d+1$. Recall that there is a canonical identification of $\mathbb{P}(V^*)$ with the Grassmannian $G_{d-1}(V)$ of hyperplanes in V, given by $\mathbb{P}(V^*) \ni [\xi] \mapsto H_\xi := \ker \xi \in G_{d-1}(V)$, for $\xi \in V^* \setminus \{0\}$.

Once we fix a Hermitian product on V, we endow the various projective spaces with the Fubini-Study metric ω_{FS}, normalized such that the induced measure $\sigma_{\text{FS}} := \omega_{\text{FS}}^d$ is a probability measure.

Theorem 3.1. *Let (X, ω) be a Hermitian complex manifold of dimension n and let U be a relatively compact open subset of X. Let V be a Hermitian complex vector space of complex dimension $d+1$. There is a constant $c > 0$ independent of d such that for every $\gamma > 0$ and every holomorphic map $\Phi : X \to \mathbb{P}(V)$ of generic rank n we can find a subset E of $\mathbb{P}(V^*)$ satisfying the following properties:*

(a) $\sigma_{\text{FS}}(E) \leqslant c\, d^2\, e^{-\gamma/c}$;

(b) *If ξ is outside E, the current $\Phi^*[H_\xi]$ is well-defined and we have*

$$\left\| \Phi^*[H_\xi] - \Phi^*(\omega_{\text{FS}}) \right\|_{U,-2} \leqslant \gamma.$$

The proof of the above result uses some properties of quasi-psh functions that we will recall here. For the details, see [14]. For simplicity, we will state these properties for $\mathbb{P}^d$ but we will use them for $\mathbb{P}(V^*)$.

A function $u : \mathbb{P}^d \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-psh* if it is locally the difference of a psh function and a smooth function. For such a function, there is a constant $c > 0$ such that $dd^c u + c\omega_{\text{FS}}$ is a positive closed $(1,1)$-current. Following Proposition A.3 and Corollary A.4 in [14] (in these results we can choose $\alpha = 1$), we have:

Proposition 3.2. *There is a universal constant $c > 0$ independent of d such that for any quasi-psh function u with $\max u = 0$ and $dd^c u \geqslant -\omega_{\text{FS}}$ we have*

$$||u||_{L^1} \leqslant \frac{1}{2}(1 + \log d) \quad \textit{and} \quad ||e^{-u}||_{L^1} \leqslant cd,$$

where the norm L^1 is with respect to σ_{FS}.

Quasi-psh functions are quasi-potentials of positive closed $(1,1)$-currents. For such a current, we will use the following notion of mass

$$||S|| := (S, \omega_{\text{FS}}^d)$$

which is equivalent to the classical mass norm. More precisely, we have the following result.

Lemma 3.3 ($\partial\overline{\partial}$-Lemma for currents)**.** *Let S be a positive closed $(1,1)$-current on $\mathbb{P}^d$. Assume that the mass of S is equal to 1, that is, S is cohomologous to ω_{FS}. Then there is a unique quasi-psh function v such that*

$$\max v = 0 \quad \textit{and} \quad \sqrt{-1}\partial\overline{\partial}v = S - \omega_{\text{FS}}.$$

Finally, we will need the following lemma.

Lemma 3.4. *Let Σ be a closed subset of $\mathbb{P}^d$ and u an L^1 function which is continuous on $\mathbb{P}^d \setminus \Sigma$. Let γ be a positive constant. Suppose there is a positive closed $(1,1)$-current S of mass 1 such that $-S \leqslant \sqrt{-1}\partial\overline{\partial}u \leqslant S$ and $\int u d\sigma_{\text{FS}} = 0$. Then, there is a universal constant $c > 0$ and a Borel set $E \subset \mathbb{P}^d$ depending only on S and γ such that*

$$\sigma_{\text{FS}}(E) \leqslant cd^2 e^{-\gamma} \quad \textit{and} \quad |u(a)| \leqslant \gamma$$

for $a \notin \Sigma \cup E$.

Proof. Let v be as in Lemma 3.3. Define $m := \int v d\sigma_{\text{FS}}$. By Proposition 3.2, we have $-\frac{1}{2}(1+\log d) \leqslant m \leqslant 0$. Define $w := u + v$. We have $\sqrt{-1}\partial\overline{\partial}w \geqslant -\omega_{\text{FS}}$. Since u is continuous outside Σ, the last property implies that w is equal to a quasi-psh function outside Σ. We still denote this quasi-psh function by w. Define $l := \max w$. Applying Proposition 3.2 to $w - l$, we obtain that

$$m - l = \int (w - l) d\sigma_{\text{FS}} \geqslant -\frac{1}{2}(1+\log d).$$

It follows that

$$l \leqslant \frac{1}{2}(1+\log d).$$

We have

$$u = w - v \leqslant l - v \leqslant \frac{1}{2}(1+\log d) - v.$$

Let E denote the set $\{v < -\gamma + \frac{1}{2}(1+\log d)\}$. This set depends only on γ and on S. We have $u \leqslant \gamma$ outside $\Sigma \cup E$. The same property applied to $-u$ implies that $|u| \leqslant \gamma$ outside $\Sigma \cup E$. It remains to bound the size of E. The last estimate in Proposition 3.2 yields

$$\sigma_{\text{FS}}(E) \lesssim d \exp\Big(-\gamma + \tfrac{1}{2}(1+\log d)\Big) \lesssim d^2 e^{-\gamma}.$$

This is the desired inequality. □

Proof of Theorem 3.1. Fix an open set W such that $U \Subset W \Subset X$. Observe that when $\Phi^{-1}(H_\xi)$ does not contain any open subset of X then $\Phi^*[H_\xi]$ is well-defined. Indeed, we can write locally $[H_\xi] = \sqrt{-1}\partial\overline{\partial}u$ for some psh function u and define $\Phi^*[H_\xi] := \sqrt{-1}\partial\overline{\partial}(u \circ \Phi)$. The function u is smooth outside H_ξ and equal to $-\infty$ on H_ξ. So, the expression $\sqrt{-1}\partial\overline{\partial}(u \circ \Phi)$ is meaningful since $u \circ \Phi$ is not identically $-\infty$, see [22] for details. Let Σ denote the closure of the set of ξ which do not satisfy the above condition. By shrinking X we can assume that Σ is an analytic set with boundary in $\mathbb{P}(V^*)$. In particular, its volume is equal to 0. The currents $\Phi^*[H_\xi]$ depend continuously on $\xi \in \mathbb{P}(V^*) \setminus \Sigma$.

Fix a smooth positive (n,n)-form ν with compact support in W such that for any $\mathscr{C}^2$ real form φ of bidegree $(n-1,n-1)$ with compact support in U and $\|\varphi\|_{\mathscr{C}^2} \leqslant 1$ we have $-\nu \leqslant \sqrt{-1}\partial\overline{\partial}\varphi \leqslant \nu$. Let M be the analytic subset of points (x,ξ) in $X \times \mathbb{P}(V^*)$ such that

$x \in H_\xi$. It is of dimension $n+d-1$. Let π_1 and π_2 denote the natural projections from M onto X and $\mathbb{P}(V^*)$ respectively. Define $v := (\pi_2)_*(\pi_1)^*(\varphi)$. This is a function on $\mathbb{P}(V^*)$ whose value at $\xi \in \mathbb{P}(V^*) \setminus \Sigma$ is the integration of $\pi_1^*(\varphi)$ on the fiber $\pi_2^{-1}(\xi)$. So, we have

$$v(\xi) = (\Phi^*[H_\xi], \varphi)\,.$$

Hence, v is continuous on $\mathbb{P}(V^*) \setminus \Sigma$. Since the form $\omega_{\rm FS}$ on $\mathbb{P}(V)$ is the average of $[H_\xi]$ with respect to the measure $\sigma_{\rm FS}$ on ξ, we also have

$$\int v d\sigma_{\rm FS} = (\Phi^*(\omega_{\rm FS}), \varphi)\,.$$

Define also $T := (\pi_2)_*(\pi_1)^*(\nu)$. Recall that ν is positive. It is closed since it is of maximal bidegree. It follows that T is a positive closed $(1,1)$-current on $\mathbb{P}(V^*)$. Since $-\nu \leqslant \sqrt{-1}\partial\overline{\partial}\varphi \leqslant \nu$, we have $-T \leqslant \sqrt{-1}\partial\overline{\partial}v \leqslant T$. Let m denote the mass of ν considered as a positive measure. If a is a generic point in X, then T is cohomologous to $m(\pi_2)_*(\pi_1)^*(\delta_a)$ where δ_a is the Dirac mass a, because ν is cohomologous to $m\delta_a$. The last expression is m times the current of integration of the hyperplane H of points ξ such that $\Phi(a) \in H_\xi$. So, the mass of T is equal to m. In particular, it is independent of Φ and φ.

Define the function u on $\mathbb{P}(V^*)$ by

$$u := \frac{1}{m}\Big(v - (\Phi^*(\omega_{\rm FS}), \varphi)\Big)\,.$$

We deduce from the above discussion that u satisfies the hypothesis of Lemma 3.4 for $S := \frac{1}{m}T$. Applying this lemma to γ/m instead of γ, we find a set E' independent of φ such that $\sigma_{\rm FS}(E') \leqslant cd^2e^{-\gamma/m}$ and $|u| \leqslant \gamma/m$ outside $\Sigma \cup E'$. It follows that

$$\Big\|\Phi^*[H_\xi] - \Phi^*(\omega_{\rm FS})\Big\|_{U,-2} \leqslant \gamma$$

for ξ out of $E := \Sigma \cup E'$. It is enough to replace c by $\max(c,m)$ in order to obtain the theorem. $\square$

We show now how to apply Theorem 3.1 to prove Theorem 1.5. Consider the Kodaira map associated to a line bundle over X, i.e. the map Φ_N from X to $\mathbb{P}H^0(X, L^N)^*$ defined in (2.11). So, $d \simeq N^n$ and $\log d \simeq \log N$. For $\xi \in \mathbb{P}H^0_{(2)}(X, L^N)$, the hyperplane $H_\xi \subset \mathbb{P}H^0_{(2)}(X, L^N)^*$ determines a current of integration $\Phi_N^*[H_\xi]$ on the zero-divisor of a section $s_\xi \in \xi$. This section is unique up to a multiplicative constant.

Proof of Theorem 1.5. We know by Corollary 2.2 that there exists $N'(U)$ such that the Kodaira map

$$\Phi_N:\ X \setminus Bs_N \to \mathbb{P}H^0_{(2)}(X, L^N)^*$$

is an embedding in the neighbourhood of U for $N \geqslant N'(U)$. By (2.17), $N^{-1}\Phi_N^*(\omega_{\rm FS})$ differs from $\frac{\sqrt{-1}}{2\pi}R^L$ on U by a form of norm bounded by $C_{2,U}/N$. We apply Theorem 3.1 for Φ_N, $N \geqslant N'(U)$ and choose $\gamma = \lambda_N/2$ therein. Thus, for $N \geqslant N'(U)$ there exist $E_N \subset \mathbb{P}H^0_{(2)}(X, L^N)$ such that $\sigma_{\rm FS}(E_N) \leqslant c\, N^{2n}\, e^{-\lambda_N/c}$ and for all $\xi_N \in \mathbb{P}H^0_{(2)}(X, L^N) \setminus E_N$ we have

$$\Big|\Big(\frac{1}{N}\Phi_N^*[H_{\xi_N}] - \frac{1}{N}\Phi_N^*(\omega_{\rm FS}), \varphi\Big)\Big| \leqslant \frac{\lambda_N}{2N}\|\varphi\|_{\mathscr{C}^2}\,, \quad \varphi \in \Omega_0^{n-1,n-1}(U)\,.$$

Hence

$$\Big|\Big(\frac{1}{N}\Phi_N^*[H_{\xi_N}] - \frac{\sqrt{-1}}{2\pi}R^L, \varphi\Big)\Big| \leqslant \Big(\frac{C_{2,U}}{N} + \frac{\lambda_N}{2N}\Big)\|\varphi\|_{\mathscr{C}^2}, \quad \varphi \in \Omega_0^{n-1,n-1}(U).$$

Choose now $N''(U)$ such that $\lambda_N \geqslant 2C_{2,U}$ for all $N \geqslant N''(U)$. We obtain the items (a) and (b) of Theorem 1.5 by setting $N(U) = \max\{N'(U), N''(U)\}$.

Finally, the last property holds for a generic sequence $(\xi_N) \in \Omega$, since $\sum \sigma_{\rm FS}(E_N) < \infty$. □

Recall that by Remark 2.5 we can replace $C_{2,U}/N$ by $C_{2,U}/N^2$ if $\Theta = \frac{\sqrt{-1}}{2\pi}R^L$, so we can in this case improve the estimate on $N(U)$.

Remark 3.5. Let $1 \leqslant p \leqslant n$ be an integer. Suppose there is a positive, closed $(n-p+1, n-p+1)$-form ν with compact support in X, which is strictly positive on $\overline{U}$. Then we can extend Theorem 1.5 to projective subspaces of codimension p instead of hyperplanes H_ξ. This can be applied to obtain the equidistribution on U of common zeros of p random holomorphic sections i.e. of currents of the form $[s_N^{(1)} = \ldots = s_N^{(p)} = 0]$.

Proof of Theorem 1.2. Take an exhaustion $(U_j)_{j\in\mathbb{N}}$ of X by open relatively compact sets. By Theorem 1.5 there exist sets $\mathcal{N}_j \subset \Omega$, $j \in \mathbb{N}$, of μ-measure zero such that (1.3) holds for all $\varphi \in \Omega_0^{n-1,n-1}(U_j)$ and all sequences $\mathrm{s} \in \Omega \setminus \mathcal{N}_j$. Now, since $\Omega_0^{n-1,n-1}(X) = \cup_{j\in\mathbb{N}}\Omega_0^{n-1,n-1}(U_j)$, (1.3) holds for all $\mathrm{s} \in \Omega \setminus \mathcal{N}$ where $\mathcal{N} = \cup_{j\in\mathbb{N}}\mathcal{N}_j$. □

4. ZEROS OF PLURICANONICAL SECTIONS ON PSEUDOCONCAVE AND ARITHMETIC QUOTIENTS

We recall the definition of pseudoconcavity in the sense of Andreotti and Grauert.

Definition 4.1 (Andreotti-Grauert [2])**.** Let X be a complex manifold of complex dimension n and $1 \leqslant q \leqslant n$. X is called *q-concave* if there exists a smooth function $\varphi : X \longrightarrow (a, b]$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$, such that for all $c \in]a, b]$ the superlevel sets $X_c = \{\varphi > c\}$ are relatively compact in X, and $\partial\overline{\partial}\varphi$ has at least $n - q + 1$ positive eigenvalues outside a compact set.

Example 4.2. Let Y be a compact complex space and let $A \subset Y$ be an analytic subset of dimension q which contains the singular locus of Y. Then $Y \setminus A$ is a $(q+1)$-concave manifold (see e.g. [37, Prop. 9]).

We can formulate now the following consequence of Theorem 1.2. We first recall some terminology. Let (X, J, ω) be a Hermitian manifold, let g^{TX} be the Riemannian metric associated to ω by (2.1) and let Ric the Ricci curvature of g^{TX}. The Ricci form Ric_ω is defined as the $(1,1)$-form associated to Ric by

$$\mathrm{Ric}_\omega(u, v) = \mathrm{Ric}(Ju, v), \quad \text{for any } u, v \in T_xX,\ x \in X. \tag{4.1}$$

If the metric g^{TX} is Kähler, then

$$\mathrm{Ric}_\omega = \sqrt{-1}R^{K_X^*} = -\sqrt{-1}R^{K_X} \tag{4.2}$$

where K_X^* and K_X are endowed with the metric induced by g^{TX} (see e.g. [19, Prob. 1.7]). The metric g^{TX}, or the associated Kähler form ω, are called *Kähler-Einstein* if there exists a real constant k such that

$$\mathrm{Ric} = k\, g^{TX} \tag{4.3}$$

or, equivalently

$$\mathrm{Ric}_\omega = k\,\omega\,. \tag{4.4}$$

Theorem 4.3. *Let (X,ω) be an n-dimensional complete Kähler manifold with $\mathrm{Ric}_\omega \leqslant -k\,\omega$ for some constant $k>0$. Assume that X is $(n-1)$-concave. Then the zero-divisors of generic random sequences $(s_N)\in\prod_{N=1}^{\infty}\mathbb{P}H^0_{(2)}(X,K_X^N)$ are equidistributed with respect to $-\frac{1}{2\pi}\,\mathrm{Ric}_\omega$. If (X,ω) is Kähler-Einstein with $\mathrm{Ric}_\omega=-k\,\omega$, $k>0$, then we have equidistribution with respect to $\frac{k}{2\pi}\omega$.*

The L^2 inner product here is constructed with respect to the volume form of ω and the Hermitian metric on K_X induced by ω.

Proof. The proof follows immediately from Theorem 1.2, since $\sqrt{-1}R^{K_X} = -\,\mathrm{Ric}_\omega$. Moreover, an important feature of any q-concave manifold X is that for every holomorphic line bundle F on X the space of holomorphic sections $H^0(X,F)$ is of finite dimension. This is a consequence of the finiteness theorem of Andreotti-Grauert [2] (cf. also [19, Th. 3.4.5]). ☐

Example 4.4. Assume that X is an n-dimensional Zariski open set in a compact complex space X^* such that the codimension of the analytic set $X^*\setminus X$ is at least two. Then X is $(n-1)$-concave. Assume moreover that (e.g. after desingularization of X^*) X is biholomorphic to $\widetilde{X}\setminus D$, where $\widetilde{X}$ is a compact manifold, D is an effective divisor with only normal crossings such that $K_{\widetilde{X}}\otimes[D]$ is ample. Then X admits a (unique up to constant multiple) Kähler-Einstein metric with negative Ricci curvature, by a theorem due to R. Kobayashi [18] and Tian-Yau [35].

Example 4.5. Another class of examples is the following. By a well-known theorem of Cheng-Yau [12] and Mok-Yau [24], any Riemann domain $\pi: D\to\mathbb{C}^n$ with $\pi(D)$ bounded (e.g. a bounded domain of holomorphy in $\mathbb{C}^n$) carries a unique complete Kähler-Einstein metric ω of constant negative Ricci curvature, $\mathrm{Ric}_\omega=-\omega$.

Assume that X is an n-dimensional Zariski open set in a compact complex space X^* such that the codimension of the analytic set $X^*\setminus X$ is at least two and X is covered by a bounded domain of holomorphy in $\mathbb{C}^n$. Then X is $(n-1)$-concave and it carries a complete Kähler-Einstein metric ω of constant negative Ricci curvature, $\mathrm{Ric}_\omega=-\omega$.

Other examples where Theorem 4.3 applies are provided by arithmetic quotients. Let us quote the following fundamental result about compactification of arithmetic quotients.

Theorem 4.6 (Satake [30], Baily-Borel [4])**.** *Let X be an n-dimensional, $n\geqslant 2$, irreducible quotient of a bounded symmetric domain D by a torsion-free arithmetic group $\Gamma\subset\mathrm{Aut}(D)$. Then there exists a compactification X^* of X such that: (i) X^* is a normal projective variety and (ii) $X^*\setminus X$ is a complex analytic variety is of codimension $\geqslant 2$ in X^*.*

We are now in the position to prove Corollary 1.6.

Proof of Corollary 1.6. By Example 4.2 it follows that an n-dimensional arithmetic quotient X, with $n \geqslant 2$, is $(n-1)$-concave, since the singular locus of X^* has dimension at most $n-2$. (The proof of pseudoconcavity (initially in a weaker sense) of arithmetic quotients by Andreotti-Grauert [1] and Borel [10] actually also shows that X is $(n-1)$-concave.)

Assume that $X = \Gamma\backslash D$, where D is a bounded symmetric domain and Γ an arithmetically defined discontinuous group that acts freely on D. The domain D has a canonical $\mathrm{Aut}(D)$-invariant Kähler metric, namely the Bergman metric. Pick a basis $\{S_i\}_{i\geqslant 1}$ of the Hilbert space $H^0_{(2)}(D)$ of square-integrable holomorphic functions. The Bergman kernel is defined as the locally uniformly convergent sum

$$P(z,w) = \sum S_i(z)\,\overline{S}_i(w)$$

which is in fact independent of the choice of the basis. The Bergman metric

$$\omega^{\mathcal{B}}_D := \sqrt{-1}\partial\bar{\partial}\log P(z,z) \tag{4.5}$$

is invariant under $\mathrm{Aut}(D)$ ([23, Ch. 4, §1, Prop. 2]) and obviously Kähler. Therefore it descends to a Kähler metric on X, denoted $\omega^{\mathcal{B}}_X$. Since $(D, \omega^{\mathcal{B}}_D)$ is a Hermitian symmetric manifold, the Bergman metric is complete. Therefore $(X, \omega^{\mathcal{B}}_X)$ is complete with respect to the Bergman metric.

The volume form of $\omega^{\mathcal{B}}_D$ defines a Hermitian metric in the canonical line bundle K_D and there holds $\sqrt{-1}R^{K_D} = \omega^{\mathcal{B}}_D$. In fact, since $\omega^{\mathcal{B}}_D$ is invariant, the forms $(\omega^{\mathcal{B}}_D)^n$ and $P(z,z)dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$ are both invariant (n,n) forms ([23, Ch. 4, §1, Prop. 2]). Since D is homogeneous, an invariant object is determined by its value in a point. Hence, two $(n,n)-$forms only differ by a constant. Writing

$$(\omega^{\mathcal{B}}_D)^n = c\,P(z,z)dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n$$

shows that $\sqrt{-1}R^{K_D} = \sqrt{-1}\partial\bar{\partial}\log P(z,z) = \omega^{\mathcal{B}}_D$ ([23, Ch. 4, §1, Prop. 3]). Since $\omega^{\mathcal{B}}_D$ descends to X, we also have $\sqrt{-1}R^{K_X} = \omega^{\mathcal{B}}_X$. The result follows therefore from Theorems 1.2 and 1.5. $\square$

Note that in the above examples the base manifold X turns out to be actually quasi-projective. In fact, we can replace the hypothesis that X is $(n-1)$-concave in Theorem 4.3 by the hypothesis that X is quasi-projective and obtain the same conclusions. More precisely we have the following.

Theorem 4.7. *Let (X,ω) be an n-dimensional complete Kähler manifold with $\mathrm{Ric}_\omega \leqslant -k\,\omega$ for some constant $k>0$. Assume that X is is quasi-projective. Then the zero-divisors of generic random sequences $(s_N)\in\prod_{N=1}^\infty \mathbb{P}H^0_{(2)}(X,K^N_X)$ are equidistributed with respect to $-\frac{1}{2\pi}\mathrm{Ric}_\omega$. If (X,ω) is Kähler-Einstein with $\mathrm{Ric}_\omega = -k\,\omega$, $k>0$, then we have equidistribution with respect to $\frac{k}{2\pi}\omega$. Moreover, we have an estimate of the convergence speed on compact sets as in Theorem 1.5.*

Proof. Let $\overline{X}$ be a smooth compactification of X such that $D = \overline{X}\setminus X$ is a divisor with simple normal crossings. By [34, Lemma 5.1] any holomorphic section of $H^0_{(2)}(X,K^N_X)$ extends to a meromorphic section of $K^N_{\overline{X}}$ with poles along of order at most N along D, i.e., $H^0_{(2)}(X, K_X^N) \subset H^0(\overline{X}, K_{\overline{X}}^N \otimes [D]^N)$. It follows that $H^0_{(2)}(X, K_X^N)$ are finite dimensional. By applying Theorems 1.2 and 1.5 we obtain the result. $\square$

5. Equidistribution of zeros of modular forms

Consider the group $SL_2(\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}$ via linear fractional transformations. Let Γ be any subgroup of finite index that acts freely and properly discontinously. Then the quotient space is naturally endowed with a smooth manifold structure and we want to consider holomorphic line bundles on $Y = \Gamma\backslash\mathbb{H}$.

Denote by p the projection map $\mathbb{H} \to \Gamma\backslash\mathbb{H}$. For any line bundle $L \to Y$ there exists a global trivialization

$$\varphi : p^*L \to \mathbb{H} \times \mathbb{C}.$$

By invariance, $(p^*L)_\tau = (p^*L)_{\gamma\tau}$ for any $\gamma \in \Gamma$ and we can form $j_\gamma(\tau) := \varphi_{\gamma\tau} \circ \varphi_\tau^{-1}$. Clearly, $j_\gamma \in \mathcal{O}^*(\mathbb{H})$ and satisfies $j_{\gamma\gamma'}(\tau) = j_\gamma(\gamma'\tau) j_{\gamma'}(\tau)$. The map $j : \Gamma \times \mathbb{H} \to \mathbb{C}^\times$ is called an automorphy factor for Γ. Conversely, any automorphy factor induces a $\Gamma-$ action on the trivial bundle $\mathbb{H} \times \mathbb{C}$ via $(\tau, z) \mapsto (\gamma\tau, j_\gamma(\tau)z)$ and the quotient becomes a holomorphic line bundle on Y with transition functions given by the j_γ.

Holomorphic sections of L can be identified with holomorphic functions on $\mathbb{H}$ that satisfy $f(\gamma\tau) = j_\gamma(\tau)f(\tau)$. Sections of the tensor powers L^N satisfy $f(\gamma\tau) = j_\gamma(\tau)^N f(\tau)$. Hermitian metrics on L are identified with smooth (real) functions that satisfy $h(\gamma\tau) = |j_\gamma(\tau)|^{-2}h(\tau)$, the induced metric on L^N corresponds to $h^N(\gamma\tau) = |j_\gamma(\tau)|^{-2N}h^N(\tau)$.

We consider a canonical map $\Gamma \times \mathbb{H} \to \mathbb{C}^\times$,

$$(\gamma, \tau) \mapsto \left.\frac{d\gamma}{dz}\right|_\tau$$

where $\frac{d\gamma}{dz}$ is the complex differential. The chain rule implies $\frac{d\gamma\gamma'}{dz}(\tau) = \frac{d\gamma}{dz}(\gamma'\tau)\frac{d\gamma'}{dz}(\tau)$. We call this map the canonical automorphy factor. Recall that $\gamma \in \Gamma$ are the transition maps of the coordinate charts on Y. Therefore the $\frac{d\gamma}{dz}$ are the transition functions of the tangent bundle TY. Explicitly,

$$\gamma\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}, \qquad \frac{d\gamma}{dz}(\tau) = \frac{1}{(c\tau + d)^2}.$$

The transition functions of the dual bundle $T^*Y = K_Y$ are then $j_\gamma(\tau) = (c\tau + d)^2$.

Summarizing, holomorphic functions on $\mathbb{H}$ that satisfy the transformation law $f(\gamma\tau) = (c\tau+d)^{2N}f(\tau)$ are in correspondence with holomorphic sections of the bundle $K_Y^{\otimes N} \to Y$.

Definition 5.1. A modular form for Γ of weight $2N$ is a function on $\mathbb{H}$ such that:
(1) $f(\gamma\tau) = (c\tau + d)^{2N} f(\tau)$, all $\tau \in \mathbb{H}$,
(2) f is holomorphic on $\mathbb{H}$,
(3) f is "holomorphic at the cusps".
A modular form f that is zero at every cusp of Γ is called a cusp form. We write $\mathcal{M}_{2N}(\Gamma)$ for the space of modular forms of weight $2N$ and $\mathcal{S}_{2N}(\Gamma)$ for the subspace of cusp forms.

The last condition means the following. If the index $[SL_2(\mathbb{Z}) : \Gamma]$ is finite, then for some natural number ℓ the transformation $z \mapsto z + \ell$ is contained in Γ and by (1), $f(z + \ell) = f(z)$. We identify a neighbourhood $\{0 < \Re z \leqslant \ell, \Im z > c\}$ of ∞ with the punctured disk $\{0 < |w| < \exp(-2\pi c/\ell)\}$ via the map $q : z \mapsto \exp(2\pi\sqrt{-1}z/\ell)$ and define the q-expansion $\hat{f}$ of f at the cusp ∞ by $q^*\hat{f} = f$. By definition, f is holomorphic (resp. 0) at ∞ if $\hat{f}$ is holomorphic (resp. 0) at 0. Now any other cusp σ of Γ is of the form $\sigma = \alpha\infty$ with some $\alpha \in SL_2(\mathbb{Z})$ and we say that f is holomorphic (resp. 0) at σ if $j_\alpha^{-1} f \circ \alpha$ is is holomorphic (resp. 0) at ∞.

The space $\mathcal{M}_{2N}(\Gamma)$ is finite dimensional for any N, in case $N = 0$ the dimension is 1 and in case $N < 0$ the dimension is 0. In the following, we consider $N \geqslant 1$.

The bundle K_Y inherits a positively curved metric from the hyperbolic space. Namely, we endow the canonical bundle of $\mathbb{H}$ (which is trivial) with the metric h described in terms of the length of the section 1 by $|1|_h(z) := |\Im z| = |y|$. This metric descends to a metric on K_Y and we have

$$\begin{aligned}
p^*(\sqrt{-1}R^{K_Y}) &= -\sqrt{-1}\partial\bar{\partial}\log y^2 \\
&= -\sqrt{-1}\partial\bar{\partial}\log(-\frac{1}{4}(z-\bar{z})^2) \\
&= -\sqrt{-1}\frac{2}{(z-\bar{z})^2}dz \wedge d\bar{z} \\
&= \sqrt{-1}\frac{1}{2y^2}dz \wedge d\bar{z} \\
&= \frac{dx \wedge dy}{y^2}\,.
\end{aligned}$$

Let D be a fundamental domain for Γ. If f is any modular form of weight $2N$, g a cusp form of the same degree, then the integral

$$\int_D f(z)\bar{g}(z)y^{2N-2}dxdy$$

converges and defines a Hermitian product on $\mathcal{S}_{2N}(\Gamma)$, called the Petersson inner product. This is just the induced L^2 product on $H^0(Y, K_Y^N)$. We wish to describe the subspace $H^0_{(2)}(Y, K^N)$ of square integrable sections, i.e. $f \in H^0_{(2)}(Y, K^N)$ is a function on $\mathbb{H}$ satisying conditions (1),(2) from Definition 5.1 and such that

$$\int_D |f|^2 y^{2N-2}dxdy < \infty.$$

Since the cusps of Γ are all of the form $\sigma_i\infty$ with $\sigma_i \in SL_2(\mathbb{Z}), 1 \leqslant i \leqslant |SL_2(\mathbb{Z}) : \Gamma|$, it suffices to consider the integral in a neighbourhood $U(\infty)$ of ∞. Using $\hat{f}$ defined on $A := \{0 < |w| < \exp(-2\pi c/\ell)\}$ we compute

$$\begin{aligned}
\int_{U(\infty)\cap D} |f|^2 y^{2N-2}dxdy &\sim \int_c^\infty \int_0^\ell |f|^2 y^{2N-2}dxdy \\
&\geqslant \int_{q^{-1}(A)} |f|^2 dx \wedge dy \qquad\qquad \text{(since } N \geqslant 1) \\
&= \int_A |\hat{f}(w)|^2 \frac{\sqrt{-1}}{2}\frac{dw \wedge d\bar{w}}{4\pi^2|w|^2} \\
&\gtrsim \int_A |\hat{f}(w)|^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w}\,.
\end{aligned}$$

Hence, at each cusp the q-expansion of f is locally square integrable.

Lemma 5.2. *Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\widehat{f}$ be holomorphic in $\mathbb{D} \setminus \{0\}$. Then $\widehat{f} \in L^2_{loc}(\mathbb{D})$ if and only if $\widehat{f}$ can be holomorphically extended to the whole disk.*

Proof. Consider the Laurent-expansion $\widehat{f} = \sum_{\nu\in\mathbb{Z}} a_\nu w^\nu$. In the annuli $R_{r_1,r_2} = \{r_1 \leqslant |w| \leqslant r_2\}$ the series is normally convergent. We compute

$$\begin{aligned}\int_{R_{r_1,r_2}} |\widehat{f}|^2 \tfrac{\sqrt{-1}}{2} dw \wedge d\bar{w} &= \sum_{\mu,\nu} a_\mu \bar{a}_\nu \Big(\int w^\mu \bar{w}^\nu \tfrac{\sqrt{-1}}{2} dw \wedge d\bar{w} \Big) \\ &= \sum_{\nu\in\mathbb{Z}} |a_\nu|^2 2\pi \int_{r_1}^{r_2} r^{2\nu+1} dr.\end{aligned}$$

Either the principal part of $\widehat{f}$ at 0 is zero or there exists $\nu < 0$ such that $|a_\nu|^2 > 0$. In this case,

$$||\widehat{f}||^2_{L^2(\mathbb{D})} \geqslant 2\pi |a_\nu|^2 \int_{r_1}^{r_2} r^{2\nu+1} dr$$

and this is unbounded as $r_1 \to 0$. □

Thus $H^0_{(2)}(Y, K_Y^N)$ is of finite dimension. More precisely there holds:

Lemma 5.3. *With the above notations we have $H^0_{(2)}(Y, K_Y^N) \simeq \mathcal{S}_{2N}(\Gamma)$.*

Proof. Since the Petersson inner product is the L^2 product we have $S_{2N}(\Gamma) \subset H^0_{(2)}(Y, K_Y^N)$. Let $f \in H^0_{(2)}(Y, K_Y^N)$. By the preceding lemma, f corresponds to a modular form of weight $2N$. We have to show that this modular forms vanishes at the cusps. It suffices to consider the cusp at ∞. Assume that f does not vanish at ∞ and $\widehat{f}$ is the $q-$expansion of f around ∞. Then $|\widehat{f}(0)|^2 = b > 0$ and $|f|^2 \geqslant b/2$ in a neighbourhood of ∞. Then

$$\int_c^\infty \int_0^\ell |f|^2 y^{2N-2} dx dy \geqslant \frac{\ell b}{2} \int_c^\infty y^{2N-2} dy = \infty, \quad N \geqslant 1.$$

This contracdiction shows that f corresponds to a cusp form. □

Theorems 1.2, 1.5 and the above discussion immediately imply Corollary 1.7.

6. Equidistribution on quasiprojective manifolds

In the previous sections we considered the eqidistribution with respect to some canonical Kähler metrics. We turn now to the case of quasiprojective manifolds and construct adapted metrics using a method of Cornalba and Griffiths. They depend of some choices but have the advantage of being very general.

Let $X \subset \mathbb{P}^k$ be a quasiprojective manifold, denote by $\overline{X} \subset \mathbb{P}^k$ its projective closure and let $\Sigma = \overline{X} \setminus X$. Denote by $L = \mathcal{O}(1)|_{\overline{X}}$ the restriction of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{P}^k$.

We consider a resolution of singularities $\pi : \widetilde{X} \to \overline{X}$ in order to construct appropriate metrics on X and $L|_X$. More precisely, there exists a finite sequence of blow-ups

$$\widetilde{X} = X_m \xrightarrow{\tau_m} X_{m-1} \xrightarrow{\tau_{m-1}} \cdots \xrightarrow{\tau_1} X_0 = \overline{X}$$

along smooth centers Y_j such that

(1) Y_j is contained in the strict transform $\Sigma_j = \overline{\tau_j^{-1}(\Sigma_{j-1} \backslash Y_{j-1})}$, $\Sigma_0 = \Sigma$,
(2) the strict transform of Σ through $\pi = \tau_m \circ \tau_{m-1} \circ \cdots \tau_1$ is smooth and $\pi^{-1}(\Sigma)$ is a divisor with simple normal crossings in $\widetilde{X}$,

(3) $X = \overline{X}\backslash\Sigma \simeq \widetilde{X}\backslash\pi^{-1}(\Sigma)$ are biholomorphic.

If $\pi^{-1}(\Sigma) = \cup S_j$ is a decomposition into smooth irreducible components, we denote for each j the associated holomorphic line bundle by $\mathcal{O}_{\widetilde{X}}(S_j)$ and by σ_j a holomorphic section vanishing to first order along S_j. Let Θ be the fundamental form of any smooth Hermitian metric on $\widetilde{X}$. Since $\widetilde{X}$ is projective, we can choose a Kähler form Θ, but the construction works in general. The generalized Poincaré metric on $\widetilde{X}\backslash\pi^{-1}(\Sigma) \simeq X$ is defined by the Hermitian form

$$\Theta_\varepsilon = \Theta - \sqrt{-1}\,\varepsilon \sum_j \partial\overline{\partial}\log(-\log\|\sigma_j\|_j^2)^2\,, \quad 0 < \varepsilon \ll 1 \text{ fixed}, \tag{6.1}$$

where we have chosen smooth Hermitian metrics $\|\cdot\|_j$ on $\mathcal{O}_{\widetilde{X}}(S_j)$ such that $\|\sigma_j\|_j < 1$. The generalized Poincaré metric (6.1) is a complete Hermitian metric on $\widetilde{X}\backslash\pi^{-1}(\Sigma) \simeq X$ and satisfies the curvature estimates

$$-C\Theta_\varepsilon < \sqrt{-1}R^{K_X} < C\Theta_\varepsilon, \qquad |\partial\Theta_\varepsilon|_{\Theta_\varepsilon} < C$$

with some positive constant C (where the metric on K_X is the induced metric by Θ_ε). A proof of this fact can be found in [19, Lemma 6.2.1]. Next we construct a metric on L that dominates the Poincaré metric. By [19, Lemma 6.2.2], there exists a Hermitian line bundle $(\widetilde{L}, h^{\widetilde{L}})$ on $\widetilde{X}$ with positive curvature $R^{\widetilde{L}}$ on $\widetilde{X}$, and such that

$$\widetilde{L}|_{\widetilde{X}\backslash\pi^{-1}(\Sigma)} \simeq \pi^*(L^m)|_X\,,$$

with some $m \in \mathbb{N}$. If we equip $L|_X$ with the metric

$$h^L_\delta = (h^{\widetilde{L}})^{\frac{1}{m}} \prod_j(-\log\|\sigma_j\|_j^2)^{2\delta}, \qquad \text{for } 0 < \delta \ll 1, \tag{6.2}$$

we obtain the estimate

$$\sqrt{-1}R^{h^L_\delta} > \eta\Theta_\varepsilon\,, \quad \text{for } 0 < \delta, \eta \ll 1,$$

as $R^{\widetilde{L}}$ extends to a strictly positive $(1,1)$-form dominating a small positive multiple of Θ on $\widetilde{X}$. Thus, the expansion of the Bergman kernel holds as in Theorem 2.1. Moreover, the space of holomorphic L^2-sections

$$H^0_{(2)}(X, L^N, \Theta_\varepsilon, h^{L^N}_\delta) := \Big\{ s \in \mathcal{O}_X(L^N) : \int_X |s|^2_{h^{L^N}_\delta}\Theta^n_\varepsilon/n! < \infty \Big\}$$

is finite dimensional since the holomorphic L^2-sections extend holomorphically to all of $\widetilde{X}$, more precisely, $H^0_{(2)}(X, L^N) \subset H^0(\widetilde{X}, \pi^*L^N)$, see [19, (6.2.7.)]. In view of the previous discussion, Theorem 1.2 yields the following.

Corollary 6.1. *Let $X \subset \mathbb{P}^k$ be a quasiprojective manifold. Denote by $L = \mathcal{O}(1)|_X$ the restriction of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{P}^k$. Fix metrics Θ_ε as in* (6.1) *and h^L_δ as in* (6.2). *Then the zero-divisors of generic random sequences $(s_N) \in \prod_{N=1}^\infty \mathbb{P}H^0_{(2)}(X, L^N, \Theta_\varepsilon, h^{L^N}_\delta)$ are equidistributed with respect to $\frac{\sqrt{-1}}{2\pi}R^{(L,h^L_\delta)}$, where Θ_ε is the generalized Poincaré metric* (6.1) *and h^L_δ is defined by* (6.2).

7. Equdistribution of zeros of orthogonal polynomials

We wish to illustrate the result of the previous Section in the case of polynomials. As mentioned in the Introduction the distribution of zeros of random polynomials is a classical subject. Recent results were obtained by Bloom-Shiffman [8] (see also [5]) concerning the equilibrium measure $\mu_{\rm eq}$ of a compact set K endowed with a measure μ satisfying the Bernstein-Markov inequality. In this case the zeros of polynomials in $L^2(\mu)$ tend to concentrate around the Silov boundary of K. In the following we consider the equidistribution of the zeros of polynomials with respect to the Poincaré metric at infinity on $\mathbb{C}$.

Of course $\mathbb{C}$ is a special case of a quasi-projective variety, its complement in $\mathbb{P}^1$ is the hyperplane at infinity $H_\infty = \{z_0 = 0\}$, via the embedding $\mathbb{C} \ni \zeta \mapsto [1 : \zeta] \in \mathbb{P}^1$. We denote as usual $U_j = \{[z] \in \mathbb{P}^1 : z_j \neq 0\}$. The hyperplane section bundle $\mathcal{O}(1) \to \mathbb{P}^1$ comes along with the canonical (defining) section s_0 locally given as $s_0|_{U_0} = 1 \cdot e_0$, $s_0|_{U_i} = z_0/z_i \cdot e_i = \xi_i^0 \cdot e_i$, where e_j are the canonical frames of $\mathcal{O}(1)|_{U_j}$, $j = 0, 1$. Now $\cup\{\xi_i^0 < R\}$ is an open neighbourhood of H_∞.

Let us consider the charts:

$$\mathbb{P}^1 = U_0 \cup \{\infty\} = U_0 \cup U_1, \quad z : U_0 \to \mathbb{C}, \quad w : U_1 \to \mathbb{C}.$$

Consider the divisor D given by the following cover together with meromorphic functions $\{(U_0, 1), (U_1, w)\}$. The associated line bundle $[D]$ is defined by the cocycle $U_0 \cap U_1, g_{01} = 1/w$ and we have $[D] = \mathcal{O}(1)$. A metric in $[D]$ corresponds to functions $h_i \in \mathscr{C}^\infty(U_i, \mathbb{R}_{>0})$ that satisfy $h_1 = |g_{01}|^2 h_0$. In $\{|w| < R\}$ set $h_1 = 1$ and extend it to a smooth metric over $\mathbb{P}^1$. Then $s_0 = 0$ precisely at ∞. To determine the L^2-condition in the Poincaré metric it suffices to investigate the integrals in a neighbourhood of ∞.

It is well known that the holomorphic sections of $[D]^{\otimes N}$ are identified with complex polynomials of degree $\leqslant N$ in the chart U_0. We denote this space by $\mathcal{H}_N$. The Poincaré metric on $\mathbb{C}$ is

$$\Theta_\varepsilon = \omega_{\rm FS} - \sqrt{-1}\varepsilon\partial\overline{\partial}\log(-\log\|s_0\|^2)^2,$$

the metric on $\mathcal{O}(1)$ is

$$h_\delta^{\mathcal{O}(1)} = h^{\mathcal{O}(1)} \cdot (-\log\|s_0\|^2)^{2\delta}.$$

Note that choosing $\delta = 2\pi\varepsilon$ provides $\sqrt{-1}R^{(\mathcal{O}(1), h_\delta^{\mathcal{O}(1)})} = 2\pi\Theta_\varepsilon$.
A polynomial $P \in \mathcal{H}_N$ lies in $L^2(\mathbb{C}, \Theta_\varepsilon, h_\delta^N)$ if and only if the integral

$$\int_{|z|>R} |P(z)|^2 \underbrace{(1+|z|^2)^{-N}}_{=h^{\mathcal{O}(N)}} \Big(-\log\Big|\frac{1}{z}\Big|^2\Big)^{2N\delta} \underbrace{\Big\{\frac{\sqrt{-1}}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} - \sqrt{-1}\varepsilon\partial\overline{\partial}\log\Big(-\log\Big|\frac{1}{z}\Big|^2\Big)^2\Big\}}_{=\Theta_\varepsilon}$$

is finite. If $\deg P = d$, then $|P(z)|^2(1+|z|^2)^{-N} = \mathrm{O}((1+|z|^2)^{-n})$, with $-2N \leqslant n = 2d - 2N \leqslant 0$ and in particular bounded.
First consider

$$I = \int_{|z|>R} (\log|z|^2)^{2N\delta}\frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

In polar coordinates

$$\begin{aligned} I &= C\int_R^\infty (\log r^2)^{2N\delta}\frac{r\,dr}{(1+r^2)^2} \qquad \text{substitute } r^2=e^x,\, 2r\,dr=e^x dx \\ &= C'\int_{2\log R}^\infty x^{2N\delta}\frac{e^x dx}{(1+e^x)^2} \\ &\leqslant C'\int \frac{x^{2N\delta}dx}{(1+e^x)} \\ &\leqslant C'\int x^{2N\delta}e^{-x}dx < \infty\,. \end{aligned}$$

To estimate the second integral compute

$$\partial\overline{\partial}\log\Big(-\log\big|\tfrac{1}{z}\big|^2\Big)^2 = \partial\overline{\partial}\log(\log|z|^2)^2 = \frac{-2dz\wedge d\bar{z}}{|z|^2(\log|z|^2)^2}\,.$$

Therefore

$$\begin{aligned} \int_{|z|>R}(1+|z|^2)^{-n}(\log|z|^2)^{2N\delta-2}\frac{\sqrt{-1}dz\wedge d\bar{z}}{2|z|^2} &= C\int_R^\infty(1+r^2)^{-n}(\log r^2)^{2N\delta-2}\frac{r\,dr}{r^2} \\ &= C'\int_{2\log R}^\infty (1+e^x)^{-n}x^{2N\delta-2}dx \end{aligned}$$

is finite ($N\to\infty$), only if $n<0$. This shows that

$$\mathcal{H}^N\cap L^2(\mathbb{C},\Theta_\varepsilon,h_\delta^N) = \mathcal{H}^{N-1}$$

as sets.

Corollary 7.1. *Denote by $\mathbb{P}\mathcal{H}^{N-1}$ the projective space associated to $\mathcal{H}^{N-1}$. Then zero-divisors of generic random sequences $(s_N)\in\prod_{N=1}^\infty \mathbb{P}\mathcal{H}^{N-1}$ are equidistributed with respect to Θ_ε.*

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