First-order transition in Potts models with "invisible" states
—— *Rigorous proofs* ——

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In some recent papers by Tamura, Tanaka and Kawashima,[20)–22)] a class of Potts models with "invisible" states was introduced, for which the authors argued by numerical arguments and by a mean-field analysis that a first-order transition occurs. Here we show that the existence of this first-order transition can be proven rigorously, by relatively minor adaptations of existing proofs for ordinary Potts models. In our argument we present a random-cluster representation for the model, which might be of independent interest.

§1. Introduction

In Refs. 20), 21), the authors introduced a class of Potts models, in which next to q ordinary -visible- colours (the Potts states), between which a standard ferromagnetic nearest-neighbour Potts interaction exists, r "invisible" colours (states) are possible, which have zero interaction energy with any neighbour, whatever state that neighbour is in.

Although the number of ground states and low-temperature states equals q, and there is at low temperatures spontaneous symmetry breaking of the q-fold permutation symmetry just as in the standard q-state Potts model, the transition for low $q = 2, 3, 4$ and high r is different from the second-order transition of the ordinary two-dimensional q-state Potts model. In fact a first-order transition in the temperature-parameter appears.

The occurrence of such a first-order transition contradicts a simple form of universality which would predict that all systems with the same broken symmetry in the same dimension with short-range interactions have the same type of transition.

However, such a universality property is known to be too strong to be true. The question of first-order versus second-order is *not* a universal question. Some counterexamples illustrating this point are the two-dimensional 3-state Kac-Potts model,[12)] in which a first-order transition occurs in presence of a broken 3-fold rotation (= permutation) symmetry, or the three-dimensional versions of the nonlinear $O(n)$-models treated in Ref. 1), 4), 7)–9), in which a first-order transition occurs in presence of a broken continuous rotation symmetry. In both cases the same type of symmetry-breaking is also known to be possible with a second-order transition. This occurs for the standard nearest-neighbour 3-state two-dimensional Potts model, or for the standard three-dimensional classical Heisenberg or XY models respectively.

In fact, the model with many invisible states has a first-order transition for the

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same reason the high-q Potts model has a first-order transition. At the transition temperature there is coexistence between high-energy phases and a high-entropy phase, and between them "free-energy barriers" exist. Such a coexistence can be proven by a form of a Peierls-type free-energy-contour argument. For the standard Potts model, by now there exists a variety of such proofs,[3),16)–18)] whether by Reflection Positivity and Chessboard Estimates, or by a Pirogov-Sinai argument, either within a spin description or in a random-cluster version. Typically those proofs can be adapted without too much effort to include the model described above.

Inside standard Potts "free-energy contours", as were described above, sites exist on the border between the ordered and disordered phases and hence they are neither ordered nor disordered themselves. They have neither all neighbours different, nor all neighbours equal, and as a consequence, in dimension d they lose a free-energy fraction per site of order $\frac{1}{2d}$ with respect to the free energy of either a disordered site (where free energy is purely entropic) or with respect to the free energy of an ordered site (whose free energy is purely energetic). At the transition temperature the entropy of a disordered site approximately equals the energy of an ordered site.

In the model we will consider below, the ordered sites need to be in a visible colour, whereas disordered sites can have all neighbours either different or invisible. It is enough to consider only the two-dimensional version, but this is not essential, and the arguments directly generalize to higher dimensions. As the presence of a first-order transition in higher-dimensional Potts models is less surprising, the main interest seems to be in two dimensions.

As a further comment we mention that the term "invisible" is actually a bit of a misnomer, as at high temperatures the density of "invisible" colours is higher than those of the "visible" ones when $r \geq q$. Thus most of the colours which appear would be the "invisible" ones.

§2. Main result

At each site there is a discrete-valued spin variable which can take one out of $q+r$ colours, q of which "visible" and r "invisible". The (q,r)-model then is defined by the following (formal) Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) \sum_{\alpha=1}^{q} \delta(\sigma_i, \alpha)$$

Here the pairs of nearest-neighbour sites $\langle i,j \rangle$ live on a lattice of dimension at least two. Now we have the following result.

Theorem 1. *For $q+r$ large enough the above model undergoes a first-order transition in temperature. At the transition temperature q ordered extremal Gibbs states coexist with a disordered extremal Gibbs state.*

Proofs. There are various ways in which one may adapt existing proofs. For example, the proof originally due to Kotecký and Shlosman,[16)] later also treated in Refs. 11), 19), could be adapted by observing that

- Our model has a C-potential (in Georgii's terms), so reflection positivity holds.
- An ordered bond now will be a bond whose two sites have the same *visible* colour.
- The "restricted ensemble" for the disordered phase is formed by all configurations having disordered bonds only, which has an approximate entropy density $\ln(q+r)$, when $q+r$ is large.

Then the arguments used in section 19.3 of Ref. 11) or in Ref. 19), using chessboard estimates to provide a contour estimate, apply.

Here we will sketch in some more detail an alternative proof based on the Fortuin-Kasteleyn random-cluster description, first derived in Ref. 17), and later treated e.g. in Ref. 13). This has the advantage that it extends to values of q and r which need not correspond to a spin model interpretation, e.g. $q = 1$, or q and/or r non-integer. To do this we will adapt the ordinary random-cluster representation to include the model with invisible states.

In analogy with the standard Potts model,[10),15)] it is possible to rewrite the partition function for the (q, r)-Potts model in terms of the partition function for a variant of the random-cluster model, which we will call the "r-biased" random cluster model. Just as the standard random-cluster model, the r-biased model is a correlated bond-percolation model.

Let $\mathbb{G} = (S, B)$ be a finite graph, where S denotes the set of sites, and B the set of bonds in the graph. The *r-biased random-cluster* model on $\mathbb{G}$ is given by a probability distribution on the sets $X \subseteq B$. The distribution has three parameters $0 \leq p \leq 1$, $q > 0$ and $r \geq 0$ and is defined by

$$\phi_{p,q,r}(X) = \frac{1}{Z^{RC}_{p,q,r}(\mathbb{G})} \left[\prod_{b \in B} p^{\delta(b \in X)} (1-p)^{\delta(b \notin X)} \right] (q+r)^{\kappa_0(S,X)} q^{\kappa_1(S,X)} ,$$

in which $\kappa_0(S, X)$ denotes the number of isolated vertices of the graph (S, X), $\kappa_1(S, X)$ the number of non-singleton connected components of (S, X) and $Z^{RC}_{p,q,r}(\mathbb{G})$ the partition function. Notice that for $r = 0$, the model reduces to the standard random-cluster model, in which both singleton and non-singleton connected components have weight q. For $r > 0$, the above model induces a bias towards singleton connected components. Namely, the singleton connected components have weight $(q + r)$ whereas the non-singleton connected components have weight q.

Let us now see how the (q, r)-Potts model is related to the r-biased random-cluster model. Let Ω be the set of (q, r)-Potts configurations on $\mathbb{G}$. The partition function of this model can be rewritten as

$$\begin{aligned} Z_\beta(\mathbb{G}) &= \sum_{\sigma \in \Omega} e^{\beta \sum_{\{i,j\} \in B} \delta(\sigma_i = \sigma_j \leq q)} \\ &= \sum_{\sigma \in \Omega} \prod_{\{i,j\} \in B} e^{\beta \delta(\sigma_i = \sigma_j \leq q)} \\ &= \sum_{\sigma \in \Omega} \prod_{\{i,j\} \in B} \left[1 + \delta(\sigma_i = \sigma_j \leq q)(e^\beta - 1) \right] \end{aligned}$$

$$= \sum_{\sigma\in\Omega} \sum_{X\subseteq B} \prod_{\{i,j\}\in X} \delta(\sigma_i = \sigma_j \leq q)(e^{\beta} - 1)^{|X|}$$
$$= \sum_{\sigma\in\Omega} \sum_{X\subseteq B} \pi(\sigma, X) \ ,$$

where

$$\pi(\sigma, X) = e^{\beta|B|} \prod_{\{i,j\}\in B} \left[\delta(\{i,j\} \in X)\delta(\sigma_i = \sigma_j \leq q)(1 - e^{-\beta}) + \delta(\{i,j\} \notin X)e^{-\beta}\right] \ .$$

The expression above describes a coupling of the (q, r)-Potts distribution on $\Omega = \{1, ...q + r\}^S$ and a probability distribution on the space $\{0, 1\}^B$ (compare Ref. 5)). The marginal of this coupling on the space $\{0, 1\}^B$ is simply the r-biased random-cluster distribution $\phi_{p_\beta,q,r}$ with $p_\beta = 1 - e^{-\beta}$. In particular, the weight $\pi(\sigma, X)$ can also be expressed as

$$\pi(\sigma, X) = e^{\beta|B|} \cdot 1_{F_r}(\sigma, X) \cdot \prod_{\{i,j\}\in B} [p_\beta\, \delta(\{i, j\} \in X) + (1 - p_\beta)\, \delta(\{i, j\} \notin X)] \ ,$$

where

$$F_r \triangleq \{(\sigma, X) : \sigma_i = \sigma_j \leq q \text{ for all } \{i, j\} \in X\} \ .$$

This expression gives us the model with free boundary conditions. The wired boundary conditions in the (q, r)-model correspond to all spins on the boundary having the same visible colour.

The Peierls (free-energy) contour estimate is almost unchanged in comparison with the standard case. The only difference is that, (except for the degeneracy term of the free contours) the contours of Ref. 17) satisfy the same Peierls estimate with q replaced by $q+r$. More precisely, in Ref. 13), in equation (7.59), the adapted Peierls estimate for wired contours is of the form

$$\Phi_w(\gamma_w) \leq (q + r)^{-\frac{\|\gamma_w\|}{2d}}\ e^{5\|\gamma_w\|} \ ,$$

whereas the corresponding estimate for free contours, below equation (7.61) of Ref. 13), takes the form

$$\Phi_f(\gamma_f) \leq q\ (q + r)^{-\frac{\|\gamma_f\|}{2d}}\ e^{5\|\gamma_f\|} \ .$$

Convergence of the Peierls estimate will hold once $q + r$ is large enough, as we then have (both for free and wired contours) an estimate of the form

$$\Phi(\gamma) \leq e^{-\tau\|\gamma\|} \ ,$$

for a sufficiently large τ.

For a more detailed treatment we refer to Refs. 6), 14). □

§3. Comments and conclusions

In this note we have shown how the Potts model with many invisible states, introduced by Tamura, Tanaka and Kawashima, can be proven to have a first-order phase transition, similarly as occurs for the standard high-q Potts models. The transition temperature is asymptotically given by $\beta \approx \ln(q+r)$. The proofs, as usual, apply for quite high values of q and/or r, the numerical approach of Refs. 20), 21) might give a better indication of the values at which the first-order transition first occurs.

We conjecture that the dynamical properties of the Potts model with r invisible states, which were considered in Ref. 22), have a similar corresponding behaviour as occurs in the ordinary Potts model, for which they were rigorously analysed in Ref. 2).

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