CONTINUITY OF THE LYAPUNOV EXPONENT FOR ANALYTIC QUASI-PERODIC COCYCLES WITH SINGULARITIES

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ABSTRACT. We prove that the Lyapunov exponent of quasi-periodic cocyles with singularities behaves continuously over the analytic category. We thereby generalize earlier results, where singularities were either excluded completely or constrained by additional hypotheses. Applications are one-parameter families of analytic Jacobi operators, such as extended Harper's model describing crystals subject to external magnetic fields.

Dedicated to Richard S. Palais on the occasion of his 85th birthday.

1. Introduction

Denote by $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ the torus equipped with its Haar measure μ, $\mu(\mathbb{T}) = 1$. Given β irrational and a measurable $D : \mathbb{T} \to M_2(\mathbb{C})$ satisfying $\log|\det D| \in L^1(\mathbb{T}, \mathrm{d}\mu)$, a *cocycle* is a pair $(\beta, D(x))$, understood as linear skew-product acting on $\mathbb{T} \times \mathbb{C}^2$ by $(x, v) \mapsto (x + \beta, D(x)v)$.

Using the sub-additive ergodic theorem, for any cocycle (β, D) one can define the Lyapunov-exponent (LE) by

$$L(\beta, D) = \lim_{n\to\infty} \frac{1}{n} \log\|D(x + (n-1)\beta) \dots D(x)\| = \lim_{n\to\infty} \frac{1}{n} \int \|D(x)\| \mathrm{d}\mu(x) \ . \tag{1.1}$$

In this paper we would like to analyze the dependence of the LE on the matrix valued function D upon variation over the analytic category. In view of the following definition, given a Banach space X and $\delta > 0$, we denote by $\mathcal{C}^\omega_\delta(\mathbb{T}, X)$ the analytic X-valued functions on $\mathbb{T}$ with extension to a neighborhood of $\mathbb{T}_\delta := \{|\mathrm{Im}\, z| \leq \delta\}$ ("the δ-strip of $\mathbb{T}$").

Definition 1.1. Let $(\beta, D(x))$ be a cocyle. If $D \in \mathcal{C}^\omega_\delta(\mathbb{T}, M_2(\mathbb{C}))$ for some $\delta > 0$, we call $(\beta, D(x))$ an *analytic cocyle*. An analytic cocycle $(\beta, D(x))$ is called *singular* if $\det(D(x_0)) = 0$ for some $x_0 \in \mathbb{T}$, in which case x_0 is referred to as singularity of the cocycle $(\beta, D(x))$.

We amend that analyticity automatically guarantees $\left|\int \log|\det D(x)|\mathrm{d}\mu(x)\right| < \infty$ (for a simple argument see the proof of Lemma 2.9).

For $\delta > 0$, let $D \in \mathcal{C}^\omega_\delta(\mathbb{T}, M_2(\mathbb{C}))$. Setting $\det D(x) =: d(x)$, for $d(x)$ not vanishing identically, analyticity allows for only finitely many zeros. In particular, for our analysis it will then prove useful to define the renormalization $D'(x)$,

$$D'(x) := \frac{1}{\sqrt{|d(x)|}} D(x) \ . \tag{1.2}$$

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To simplify notation, we write

$$L'(\beta, D) := L(\beta, D') \tag{1.3}$$

Finally, the space $\mathcal{C}^\omega_\delta(\mathbb{T}, M_2(\mathbb{C}))$ is naturally equipped with a topolgy induced by the norm $||D(z)||_\delta := \sup_{|\mathrm{Im}\, z|\leq\delta}\|D(z)\|$, where $\|.\|$ denotes the usual matrix norm. As our main result, we establish the following:

Theorem 1.2. *Let $\delta > 0$. For fixed Diophantine β, the Lyapunov exponents $L(\beta, .)$ and $L'(\beta, .)$ are continuous on $\mathcal{C}^\omega_\delta(\mathbb{T}, M_2(\mathbb{C}))$ with respect to the topology induced by $||.||_\delta$.*

We recall that β is called Diophantine, if there exists $0 < b(\beta)$ and $1 < r(\beta) < +\infty$ such that for all $j \in \mathbb{Z} \setminus \{0\}$

$$|\sin(2\pi j\beta)| > \frac{b(\beta)}{|j|^{r(\beta)}} \ . \tag{1.4}$$

Besides from being a natural question to ask, our motivation for Theorem 1.2 comes from the spectral theory of quasi-periodic analytic Jacobi operators on $l^2(\mathbb{Z})$,

$$(H_{\theta;\beta}\psi)_k := v(\theta + \beta k)\psi_k + c(\theta + \beta k)\psi_{k+1} + \overline{c}(\theta + \beta(k-1))\psi_{k-1} \ . \tag{1.5}$$

Here, β is a fixed irrational and v, c are $\mathcal{C}^\omega_\delta(\mathbb{T}, \mathbb{C})$ for some $\delta > 0$. Moreover, v is taken to be a real-valued function on $\mathbb{T}$, which makes (1.5) a bounded self-adjoint operator for each $\theta \in \mathbb{T}$. An important special case of (1.5) is given by $c(x) = 1$ (Schrödinger operators).

Spectral analysis of (1.5) amounts to the study of solutions to the finite difference equation $H_{\theta;\beta}\psi = E\psi$ over $\mathbb{C}^\mathbb{Z}$. It is well known that this problem can be tackled from a dynamical point of view, defining the transfer matrix,

$$B^E(x) := \frac{1}{c(x)} \begin{pmatrix} E - v(x) & -\overline{c}(x-\beta) \\ c(x) & 0 \end{pmatrix} \ . \tag{1.6}$$

We are particularly interested in operators (1.5) where $c(x)$ is not bounded away from zero. In this case, the transfer matrix $B^E(x)$ is well defined except for finitely many points determined by the zeros of $c(x)$.

The dynamical system relevant to the spectral analysis of (1.5) is then given by the cocyle (β, B^E). In addition, there is also an associated analytic cocyle (β, A^E) given by

$$A^E(x) := \begin{pmatrix} E - v(x) & -\overline{c}(x-\beta) \\ c(x) & 0 \end{pmatrix} \ . \tag{1.7}$$

Obviously, for μ a.e. x the two relevant cocyles are related by $B^E(x) = \dfrac{1}{c(x)}A^E(x)$. In particular, for a given Jacobi operator, we obtain the following relation between the LE of its associated cocyles (β, B^E) and (β, A^E),

$$L(\beta, B^E) = L'(\beta, A^E) \ . \tag{1.8}$$

Oftentimes one studies one-paramter families of quasi-periodic Jacobi matrices. For instance, a two dimensional crystal layer subject to an external magnetic field of flux β perpendicular to the lattice plane may be described by an operator of the form (1.5) with functions c, v given by

$$c(x) := \lambda_3 \mathrm{e}^{-2\pi i(x+\frac{\beta}{2})} + \lambda_2 + \lambda_1 \mathrm{e}^{2\pi i(x+\frac{\beta}{2})} \ , \ v(x) := 2\cos(2\pi x) \ . \tag{1.9}$$

Here, the parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ models the lattice geometry as well as the interactions between the nuclei situated at the lattice points of $\mathbb{Z}^2$. The operator associated with (1.9) is known as extended Harper's operator [3, 4, 7, 8]. We mention that a prominent special case of (1.9) arises for $\lambda_1 = \lambda_3 = 0$, the associated operator being known as almost Mathieu operator (or Harper's operator in physics literature).

For such one-parameter families, it is a natural conjecture to expect continuity of the LE upon variation of the parameter λ. For extended Harper's model this question constitutes an important ingredient for the spectral analysis, which so far is only known for the almost Mathieu case. Theorem 1.2 answers this question from a general point of view.

Continuity of the Lyapunov exponent for analytic cocycles has been the subject of earlier studies. These considerations however imposed restrictions on the determinant $d(x)$.

In [14], Hölder continuity of the LE was established for Schrödinger cocyles (thus c=d=1) under a strong Diophantine condition with $|j|^{r(\beta)}$ in (1.4) replaced by $|j| \log |j|^{r(\beta)}$.

An analogue of Theorem 1.2 for $d(x) = 1$ (or more generally for $d(x)$ bounded away from zero) was proven in [2]. This statement in particular implies continuous dependence of the LE on the coupling constant for the almost Mathieu operator.

Later, in [1], $d(x)$ was allowed to vanish, however, in order to deal with these zeros, the space of interest was restricted to analytic cocycles having the *same* $d(x)$ (see Theorem 1 in [1]). Applied to extended Harper's equation, the latter result already implied continuity of $L(\beta, A^E)$ in the energy, however, since variations of λ change $\det A^E$, it does not yield continuity in the coupling [1].

At this point we mention that when allowing $d(x)$ to vanish, details of the number theoretic nature of the frequency β come into play. Whereas the earlier result in [2] is valid for any irrational β, the result in [1] could only be proven for Diophantine β.

The achievement here is to deal with *all* non-trivial singular analytic cocycles (β, D), thus removing the above mentioned constraints on $d(x)$ imposed by previous studies. Allowing for zeros in $d(x)$ however, results in signatures of the arithmetic properties of β which manifest themselves in a Diophantine condition on β.

Following, we employ a similar general strategy as in [1] to prove Theorem 1.2. However, allowing the determinant to vary requires changes in the heart of the proof of [1] where the authors provide a large deviation bound for analytic non-SL(2,$\mathbb{C}$) cocycles (Lemma 1 in [1]).

The key to Theorem 1.2 is to appropriately generalize this large deviation bound to also incorporate a variation of $d(x)$. With this new, *uniform*, large deviation bound at hand, the remainder of the proof given in [1] carries over more or less literally to imply Theorem 1.2.

The paper is organized as follows. As preparation, in Sec. 2 we establish a uniform version of the Lojasiewicz inequality, Theorem 2.4, allowing us to deal with

[1]We correct a false remark in [1] which asserted joint continuity of the Lyapunov exponent for extended Harper's equation based on Theorem 1 therein. In [3] (based on [1]), the authors, however, only used continuity with respect to E which, as mentioned, does indeed follow from Theorem 1 proven in [1].

zeros of $d(x)$ upon continuous variation of the cocycle. This enters as crucial ingredient in the proof of the uniform large deviation bound, Theorem 3.1, established in Sec. 3. Finally, we conclude with some corollaries to the main theorem in Sec. 4.

2. Openness of α-transversality

We start with some preparations exploring basic properties of complex analytic functions. To deal with possible zeros of $d(x)$, in [1] the authors made use of the following basic fact valid for every *real* analytic function $f(x)$:

Theorem 2.1 (Lojasiewicz inequality [5])**.** *Given a real analytic function $f(x)$ on $\mathbb{T}$, there exist constants $0 < \alpha, \epsilon_0 \leq 1$ such that*

$$\mu\{|f(x)| < \epsilon\} < \epsilon^{\alpha} \ , \tag{2.1}$$

for every $0 < \epsilon < \epsilon_0$.

Note that the exponent α as well as ϵ_0 depend on the function f. For our purposes we would like to be able to choose these constants *uniformly* over functions sufficiently close in a suitable topology.

Since in our situation the functions of interest are not only real but even complex analytic with holomorphic extensions to a neighborhood of some strip $|\mathrm{Im}\, z| \leq \delta$, for $\mathcal{C}_\delta^\omega(\mathbb{T}, \mathbb{C})$ we choose the topology induced by the norm $||f||_\delta := \sup_{z \in |\mathrm{Im}\, z| \leq \delta} |f(z)|$.

More generally, if $K \subset \mathbb{C}$ compact, we equip the space of functions holomorphic on a neighborhood of K with the norm $||f||_K := \sup_{z \in K} |f(z)|$; the resulting topological space shall be denoted by $\mathfrak{A}(K)$.

We amend that in [1] the authors chose a weaker topology for $\mathcal{C}_\delta^\omega(\mathbb{T}, \mathbb{C})$, induced by $||f||_\mathbb{T} := \sup_{z \in \mathbb{T}} |f(z)|$ resulting however in the need to fix the determinant $d(x)$ of the analytic cocycles under consideration.

As we will show for *complex* analytic functions, (2.1) together with the desired uniformity of the constants will follow from basic properties of holomorphic functions.

Suggested by (2.1), we introduce:

Definition 2.2. Fix $\delta > 0$. We say that $g \in \mathcal{C}_\delta^\omega(\mathbb{T}, \mathbb{C})$ satisfies an (α, ϵ_0)-transversality condition if (2.1) holds for given exponent $0 < \alpha \leq 1$ and $\epsilon_0 > 0$. We denote the class of such functions in $\mathcal{C}_\delta^\omega(\mathbb{T}, \mathbb{C})$ by $\mathcal{T}_\alpha^{\epsilon_0}$.

Remark 2.3.
(i) Clearly, $\mathcal{T}_\alpha^{\epsilon_0} \subseteq \mathcal{T}_\beta^{\epsilon_1}$ if $\beta < \alpha$ and $\epsilon_1 \leq \epsilon_0$.
(ii) If $f \in \mathcal{C}_\delta^\omega(\mathbb{T}, \mathbb{C})$ has no zeros on $\mathbb{T}$, $f \in \mathcal{T}_\alpha^{\epsilon_0}$ for all $0 < \alpha, \leq 1$, some $\epsilon_0(\alpha)$.

For $g \in \mathfrak{A}(K)$ not identically zero, let $\mathcal{N}(g; K) \in \mathbb{N}_0$ denote the number of zeros of g on K counting multiplicity. Here, $\mathbb{N}_0$ is the non-negative integers. We note the following simple fact about holomorphic functions.

Proposition 2.1. *Let $K \subseteq \mathbb{C}$ compact with piecewise $\mathcal{C}^1$-boundary.*
(i) *$\mathcal{N}(.; K) : \mathfrak{A}(K) \setminus \{0\} \to \mathbb{N}_0$ is upper-semicontinuous. Moreover, if f has no zeros on ∂K, then $\mathcal{N}(g; K)$ is constant in a neighborhood of f.*
(ii) *For $j \in \mathbb{N}_0$, $g \in \mathfrak{A}(K)$ and $z \in K^\circ$ let*

$$a_j(g, z) := \frac{1}{2\pi i} \int_{\partial K} \frac{g(\zeta)}{(\zeta - z)^{j+1}} \mathrm{d}\zeta \ . \tag{2.2}$$

Then, $a_j(g, z)$ is jointly continuous on $\mathfrak{A}(K) \times K^\circ$.

Proof. Part (ii) follows trivially from

$$|a_j(g,z_0) - a_j(f,z_1)| = j!|g^{(j)}(z_0) - f^{(j)}(z_1)| \ . \tag{2.3}$$

To prove (i), let $f \in \mathfrak{A}(K) \setminus \{0\}$. It suffices to show $\mathcal{N}(g;K) \le \mathcal{N}(f;K)$ for g in a neighborhood of f. We distinguish the following two cases.

Case 1: If f has *no* zeros on ∂K, it is well known that

$$\mathcal{N}(f;K) = \frac{1}{2\pi i}\int_{\partial K} \frac{f'(z)}{f(z)}\mathrm{d}z \ . \tag{2.4}$$

Using holomorphicity, if $g_\alpha \to f$ also $g'_\alpha \to f'$. Moreover, since f has no zeros on ∂K, the same will eventually hold for g_α. In particular, the analogue of (2.4) eventually expresses $\mathcal{N}(g_\alpha, K)$. Thus by bounded convergence we obtain, $\mathcal{N}(g_\alpha;K) = \mathcal{N}(f;K)$ eventually.

Case 2: If f does have zeros on ∂K, there exists a compact neighborhood U of ∂K such that f is holomorphic on a neighborhood of U and has no zeros on $U \setminus \partial K$. Applying above considerations separately to $K \setminus U^\circ$ and U we obtain $\mathcal{N}(g;K) \le \mathcal{N}(f;K)$ for every g sufficiently close to f.

□

For later use we mention the following simple consequence, easily obtained by separating zeros by arbitrarily small closed balls.

Corollary 2.1. *Let $K \subset \mathbb{C}$ compact. For $f \in \mathfrak{A}(K) \setminus \{0\}$ let $\mathfrak{Z}(f;K)$ denote the set of zeros of f on K. Then, $\mathfrak{Z}(.;K)$ is continuous in the Hausdorff metric.*

For $f \in \mathcal{C}^\omega_\delta(\mathbb{T},\mathbb{C})$ let $l(f)$ denote the maximal multiplicity of the distinct zeros of f on $\mathbb{T}$. As we shall argue, Proposition 2.1 implies openness of α-transversality:

Theorem 2.4. *For fixed $\delta > 0$,*

(i) *Suppose $f \in \mathcal{C}^\omega_\delta(\mathbb{T},\mathbb{C})$ does not vanish identically but possesses zeros on $\mathbb{T}$. Then, f satisfies an (α,ϵ_0)-transversality condition with $\alpha = \big(l(f)^{-1}\big)^-$.*

(ii) *Let $f \in \mathcal{C}^\omega_\delta(\mathbb{T},\mathbb{C})$ then for any $0 \le \alpha < l(f)^{-1}$, there is $\epsilon_0(f,\alpha)$ such that $f \in \mathrm{Int}\mathcal{T}^{\epsilon_0}_\alpha$ wrt $\mathcal{C}^\omega_\delta(\mathbb{T},\mathbb{C})$.*

Proof. Let $f \in \mathcal{C}^\omega_\delta(\mathbb{T},\mathbb{C}) \setminus \{0\}$ be fixed. Clearly, if $l(f) = 0$ so is $l(g)$ for any $g \in \mathcal{C}^\omega_\delta(\mathbb{T},\mathbb{C})$ sufficiently close to f in which case the theorem becomes trivial. Hence, without loss, we may assume $l(f) \ge 1$.

Let $x_1, \dots, x_n$ be the distinct zeros of f with multiplicities respectively, $l_1, \dots, l_n$. Choose K, a compact neighborhood of $\mathbb{T}$, such that $f(z) \ne 0$ on $K \setminus \mathbb{T}$. Separating the zeros by closed balls of some appropriate radius r, there exists $\eta > 0$ such that $||g - f||_\delta < \eta$ guarantees $|g(z)| > \frac{1}{2}\min_{K\setminus\cup_{j=1}^n B(x_j,r)}|f(z)| =: m > 0$ on $K \setminus \cup_{j=1}^n B(x_j,r)$. In particular, possible zeros on K of any such function g will lie in $\cup_{j=1}^n B(x_j,r)$. Using Proposition 2.1(ii), η can be chosen small enough to also ensure $\mathcal{N}(g;K) = \mathcal{N}(f;K)$.

We claim

Lemma 2.5. *There exist $0 < \eta' < \eta, 0 < \eta'' < r$ and $\kappa > 0$ such that uniformly over $||g - f||_\delta < \eta'$,*

$$|g(z)| \ge \kappa |p_{g,\tilde{z}}(z)| \ , \ |z - x_j| < \eta'' \ , \tag{2.5}$$

for every zero $\tilde{z}$ of g on K, $\tilde{z} \in B(x_j, r)$, and some monic polynomial $p_{g,\tilde{z}}(z)$ of degree at most l_j, with $p_{g,\tilde{z}}(\tilde{z}) = 0$. Moreover, as $\|g - f\|_\delta \to 0$, $p_{g,\tilde{z}}(z) \to (z - x_j)^{l_j}$ uniformly on compact subsets of $\mathbb{C}$.

Proof. For g with $||g - f|| < \eta$ using holomorphicity we can write

$$g(z) = (1 + h_{g,\tilde{z}}(z)) \sum_{k=l(g,\tilde{z})}^{l_j} a_k(g, \tilde{z})(z - \tilde{z})^k \ , \tag{2.6}$$

locally about a zero $\tilde{z}$ of g, $\tilde{z} \in B(x_j, r)$, and some holomorphic $h_{g,\tilde{z}}$ with $h_{g,\tilde{z}} = o(1)$ uniformly as $z \to \tilde{z}$, where a_k are as in (2.2).

In fact, using Cauchy estimates and Proposition 2.1(i), there is $0 < \eta' < \eta$ and $0 < \eta^{''} < r$ such that uniformly over $||g - f||_\delta < \eta'$ and for every zero $\tilde{z}$ of g on K with $\tilde{z} \in B(x_j, r)$, $|h_{g,\tilde{z}}(z)| < 1/2$ if $|z - x_j| < \eta^{''}$. By Proposition 2.1(ii) (note that all zeros of g are in K°), the polynomial in (2.6) is uniformly close on compact subsets of $\mathbb{C}$ to $a_{l_j}(f, x_j)(z - x_j)^{l_j}$. In particular, this implies that η' can be chosen small enough so that $|a_{l_j}(g, \tilde{z})| > 1/2 \min_{1 \le j \le n} |a_{l_j}(f, x_j)| > 0$, which yields the claim. □

To complete the proof of Theorem 2.4 we use the following well-known theorem due to Pólya [11].

Theorem 2.6 (Pólya)**.** *Let $p_n(z)$ be a complex monic polynomial of degree at most $n \ge 1$. Then, for $\epsilon > 0$*

$$\mu_L\left(\{x \in \mathbb{R} : |p_n(x + iy)| \le \epsilon\}\right) \le 4\epsilon^{1/n} \ . \tag{2.7}$$

Here, μ_L denotes the Lebesgue mesure.

Remark 2.7. The proof in [11] actually shows that under the hypotheses of Theorem 2.6 one has

$$\mu_L\left(\{x \in \mathbb{R} : |p_n(x + iy)| \le \epsilon\}\right) \le 2^{2-1/n}\epsilon^{1/n} \ . \tag{2.8}$$

Let $\epsilon_0 := \frac{1}{2} \min_{z \in K \setminus \cup_{j=1}^n B(x_j, \eta'')} |f(z)| > 0$. If $\|g - f\|_\delta < \eta'$, using Lemma 2.5 we conclude for $0 < \epsilon < \epsilon_0$:

$$\{x \in \mathbb{T} : |g(x)| < \epsilon\} \subseteq \bigcup_{\tilde{z} \in \mathfrak{Z}(g;K)} \{x \in \mathbb{T} : |p_{g,\tilde{z}}(x)| < \epsilon/\kappa\} \ . \tag{2.9}$$

Applying Theorem 2.6 we thus obtain

$$\mu_L\left(\{x \in \mathbb{T} : |g(x)| \le \epsilon\}\right) \le 4\mathcal{N}(f; K)\left(\frac{\epsilon}{\kappa}\right)^{1/l(f)} \ , \tag{2.10}$$

for $0 < \epsilon < \epsilon_0$ for all g with $\|g - f\|_\delta < \eta'$.

In particular, for $0 < \gamma < 1/l(f)$ and $0 < \alpha_\gamma = 1/l(f) - \gamma$, we conclude that $g \in \mathcal{T}_{\alpha_\gamma}^{\epsilon_\gamma}$ with $\epsilon_\gamma = \min\left\{\epsilon_0, \left(\frac{4\mathcal{N}(f;K)}{\kappa^{1/l(f)}}\right)^{1/\gamma}\right\}$. □

Remark 2.8. (1) We mention that Pólya's Theorem 2.6 was used to deal with a possible "collapse of zeros" as $g \to f$. If we restrict to functions g having the same number of *distinct* zeros as f in *some* neighborhood of $\mathbb{T}$, then (2.10) can be obtained directly from Lemma 2.5 since in this case (2.5) simplifies to

$$|g(z)| \ge \kappa |z|^{l(f)} \ , \ |z - x_j| < \eta^{''} \ . \tag{2.11}$$

(2) An alternative proof can be obtained using the Cartan's estimate.

We conclude this section with the following Lemma closely related to Proposition 2.1, which will come handy in the proof of the uniform large deviation bound:

Lemma 2.9. *For $f \in \mathcal{C}^\omega_\delta(\mathbb{T}, \mathbb{C})$, not vanishing identically let*

$$I(f) := \frac{1}{2\pi} \int \log|f(x)| \mathrm{d}\mu(x) \ . \tag{2.12}$$

Then I is continuous on $\mathcal{C}^\omega_\delta(\mathbb{T}, \mathbb{C}) \setminus \{0\}$ w.r.t. $||.||_\delta$.

Proof. Fixing $f \in \mathcal{C}^\omega_\delta(\mathbb{T}, \mathbb{C})$, let $x_1, \dots x_n$ denote the zeros of f on $\mathbb{T}$ counting multiplicities. Fix a compact neighborhood U of $\mathbb{T}$ such that f extends holomorphically to a neighborhood of U and $f(z) \neq 0$ for $z \in U \setminus \mathbb{T}$. Letting

$$g(z) = \frac{f(z)}{\prod_{j=1}^n (z - \mathrm{e}^{2\pi i x_j})} \ , \tag{2.13}$$

we obtain $I(g) = I(f)$ (which is a proof of $\log|f| \in L^1(\mathbb{T})$). Here, we made use of the identity,

$$\frac{1}{2\pi} \int_0^1 \log|1 - \mathrm{e}^{2\pi i x_j}| \mathrm{d}x = 0 \ . \tag{2.14}$$

If $f_\alpha \to f$ then eventually f_α will have no zeros on ∂U; hence letting $w_1, \dots, w_m$, denote the zeros of f_α on U° counting mulitplicities, in analogy to g we can define g_α, dividing out these zeros.

By Proposition 2.1(i), $m = n$ eventually as $f_\alpha \to f$; hence, also making use of Corollary 2.1 and Proposition 2.1(ii) when treating small neighborhoods of x_j, we deduce $g_\alpha \to g$ uniformly on U. Finally, note that by Jensen's formula and (2.14), $I(g_\alpha) = I(f_\alpha)$. □

3. Uniform large deviation bound

We first fix some notation. Given an analytic cocycle $(\beta, D(x))$ we define its iterates

$$D_n(x) := D(x + (n-1)\beta) \dots D(x) \ , \ n \in \mathbb{N} \ . \tag{3.1}$$

Moreover, for $n \in \mathbb{N}$ let

$$L_n(\beta, D) := \frac{1}{n} \int \log\|D_n(x)\| \mathrm{d}\mu(x) \ , \tag{3.2}$$

denote the nth approximate of $L(\beta, D)$.

We then claim the following uniform version of the crucial Lemma 1 in [1]:

Theorem 3.1 (Uniform large deviation bound for analytic cocycles (ULDB))**.** *Fix β Diophantine and $D(x) \in \mathcal{C}^\omega_\delta(\mathbb{T}, M_2(\mathbb{C}))$ with $d(x)$ not vanishing identically. Let p/q denote an approximant of β: $|\beta - \frac{p}{q}| < \frac{1}{q^2}$ with $(p, q) = 1$.*

There exist $\gamma(D) > 0$ and constants $0 < c, C < \infty$ such that for $0 < \kappa < 1$, $n > (C\kappa^{-2}q)^\eta$ with $\eta = \eta(\beta) > 1$ and for q sufficiently large, uniformly over $||\tilde{D} - D||_\delta < \gamma$,

$$\mu\left\{ \left| \frac{1}{n} \log\|\tilde{D}_n(x)\| - L_n(\beta, \tilde{D}) \right| > \kappa \right\} < \mathrm{e}^{-c\kappa q} \ . \tag{3.3}$$

Proof. For $||\tilde{D} - D||_\delta < \gamma$ we set $\tilde{u}_n = \tilde{u}_n(\beta, \tilde{D}; x) := \frac{1}{n} \log\|\tilde{D}_n(x)\|$, $n \in \mathbb{N}$. By hypotheses, $\tilde{u}_n$ extends to a subharmonic function on the δ-strip about $\mathbb{T}$. Notice that due to possible zeros of $\tilde{d}(x)$, in general, $\tilde{u}_n$ will not be bounded.

We shall deal with the unboundedness of $\tilde{u}_n$, introducing appropriate cut-offs. To this end choose $0 < A < \infty$ such that $\inf_{||\tilde{D}-D||_\delta<\gamma} I(\tilde{d}) > -2A$ (which is finite by a compactness argument). Here, $I(.)$ is defined as in Lemma 2.9. For $n \in \mathbb{N}$, let $\tilde{w}_n(z) := \max\{\tilde{u}_n(z), -A\}$; thereby we obtain a family $\{\tilde{w}_n, n \in \mathbb{N}\}$ of subharmonic functions on the δ-strip of $\mathbb{T}$, uniformly bounded in n and over $||\tilde{D} - D||_\delta < \gamma$.

The strategy to prove the ULDB is to estimate deviations of the individual terms in

$$\begin{aligned} |\frac{1}{n} \log\|\tilde{D}_n(x)\| - L_n(\beta, \tilde{D})| < |\tilde{u}_n(x) - \tilde{w}_n(x)| + \\ |\tilde{w}_n(x) - \langle\tilde{w}_n\rangle| + |\langle\tilde{w}_n\rangle - L_n(\beta, \tilde{D})| \ . \end{aligned} \tag{3.4}$$

Here, and following we use the notation $\langle f\rangle$ to denote the 0th Fourier coefficient of a function $f \in L^1(\mathbb{T})$.

We start by estimating the 2nd contribution in (3.4). For any $R > 0$ we can write

$$\begin{aligned} |\tilde{w}_n(x) - \langle\tilde{w}_n\rangle| \leq \left|\tilde{w}_n(x) - \sum_{|j|<R} \frac{R-|j|}{R^2} \tilde{w}_n(x+j\beta)\right| + \\ \left|\sum_{|j|<R} \frac{R-|j|}{R^2} \tilde{w}_n(x+j\beta) - \langle\tilde{w}_n\rangle\right| =: |(\mathrm{I})| + |(\mathrm{II})| \ . \end{aligned} \tag{3.5}$$

R will be suitably chosen later.

Contribution (II) is readily controlled using the following result [2],

Lemma 3.2 (Large deviation bound for bounded subharmonic functions [2]; see also[1], p. 1888)**.** *Let $v(x)$ be a bounded 1-periodic subharmonic function defined on a neighborhood of $\mathbb{R}$. Let $|\beta - \frac{p}{q}| < \frac{1}{q^2}$, $(p,q) = 1$ and $0 < \kappa < 1$. Then for appropriate $0 < C_1, c_1 < \infty$ and for $R > C_1\kappa^{-1}q$ we have,*

$$\mu\left\{\left|\sum_{|j|<R} \frac{R-|j|}{R^2} v(x+j\beta) - \langle v\rangle\right| > \kappa\right\} < \mathrm{e}^{-c_1\kappa q} \ . \tag{3.6}$$

Remark 3.3. For a uniformly bounded family of subharmonic functions, the constants c_1, C_1 can be chosen uniformly over this family [1]. This in particular, applies to the family $\{\tilde{w}_n, n \in \mathbb{N}$ and $||\tilde{D} - D||_\delta < \gamma\}$.

To estimate contribution (I) in (3.5), we establish the following

Proposition 3.1. *Uniformly over $||\tilde{D} - D||_\delta < \gamma$, there exists $0 < c_2$ such that for any $0 < \epsilon < 1$ there is $0 < C_2 < \infty$ with*

$$\mu\left\{|\tilde{w}_n(x) - \tilde{w}_n(x+\beta)| > \frac{C_2}{n^{1-\epsilon}}\right\} < \mathrm{e}^{-c_2 n^\epsilon} \ , \tag{3.7}$$

for sufficiently large n (only depending on $d(x)$ and ϵ).

Proof. Let $B := \gamma + ||D||_\delta$ whence $\sup_{||\tilde{D}-D||_\delta<\gamma} ||\tilde{D}||_\delta < B$. By definition of $\tilde{w}_n$ we have

$$|\tilde{w}_n(x) - \tilde{w}_n(x+\beta)| < \frac{1}{n}\left|\log \frac{\|\tilde{D}_n(x)\|}{\|\tilde{D}_n(x+\beta)\|}\right| \ . \tag{3.8}$$

For any $M \in GL_2(\mathbb{C})$,

$$\|M^{-1}\| = \frac{\|M\|}{|\det M|} \ , \tag{3.9}$$

whence

$$\max\left\{\frac{\|\tilde{D}_n(x)\|}{\|\tilde{D}_n(x+\beta)\|}, \frac{\|\tilde{D}_n(x+\beta)\|}{\|\tilde{D}_n(x)\|}\right\} < B^2 \max\left\{\frac{1}{|\tilde{d}(x+n\beta)|}, \frac{1}{|\tilde{d}(x)|}\right\} \ . \tag{3.10}$$

Let $0 < \epsilon < 1$. If $|\tilde{d}(x+j\beta)| \geq \mathrm{e}^{-n^\epsilon}$ for *both* $j = 0, n$ using (3.10) we obtain

$$|\tilde{w}_n(x) - \tilde{w}_n(x+\beta)| < \frac{2\log B}{n} + n^{\epsilon-1} < \frac{C_2}{n^{1-\epsilon}} \ . \tag{3.11}$$

Hence, using Theorem 2.4 we estimate

$$\begin{aligned} &\mu\left\{|\tilde{w}_n(x) - \tilde{w}_n(x+\beta)| > \frac{C_2}{n^{1-\epsilon}}\right\} \leq \\ &\mu\left\{|\tilde{d}(x+j\beta)| < \mathrm{e}^{-n^\epsilon} \ , \text{some } j \in \{0, n\}\right\} \leq 2\mathrm{e}^{-\alpha n^\epsilon} \ , \end{aligned} \tag{3.12}$$

for n sufficiently large uniformly over $||\tilde{D} - D||_\delta < \gamma$. Here and in the following, α is the exponent for $d(x)$ determined by Theorem 2.4 by the maximal multiplicity of the zeros of $d(x)$ on $\mathbb{T}$. Finally choosing $c_2 < \alpha$ we obtain the claim of the Proposition. □

We are now ready to estimate $\mu\{x : |\tilde{w}_n(x) - \langle\tilde{w}\rangle| > \kappa\}$: Let $\mathfrak{X} := \{x : |\tilde{w}_n(x) - \tilde{w}_n(x+\beta)| > \frac{C_2}{n^{1-\epsilon}}\}$, where C_2, ϵ are as in Proposition 3.1. Denote by T the rotation by β on $\mathbb{T}$.

If $x \in \mathbb{T}$ is such that $\cup_{j=-R+1}^{R+1} T^j x \subseteq \mathbb{T} \setminus \mathfrak{X}$, then referring to (3.5) we obtain

$$|(I)| = |\tilde{w}_n - \sum_{|j|<R} \frac{R-|j|}{R^2}\tilde{w}_n(x+j\beta)| < \frac{C_2}{n^{1-\epsilon}}R \ . \tag{3.13}$$

In particular, choosing $R < \frac{\kappa n^{1-\epsilon}}{2C_2}$ implies that for such x we have $|(\mathrm{I})| < \kappa/2$.

The largeness condition on R from Lemma 3.2 will also be taken care of when letting $C_1\kappa^{-1}q < R < \frac{\kappa n^{1-\epsilon}}{2C_2}$; this is accommodated choosing $n > N$ with

$$N := \left(2C_1C_2\kappa^{-2}q\right)^{\frac{1}{1-\epsilon}} \ . \tag{3.14}$$

Thus fixing

$$R := \frac{1}{2}\left(\frac{\kappa N^{1-\epsilon}}{2C_2} + C_1\kappa^{-1}q\right) \ , \tag{3.15}$$

and using Lemma 3.2 and Proposition 3.1, we have for $n > N$

$$\begin{aligned} &\mu\{x : |\tilde{w}_n(x) - \langle\tilde{w}_n\rangle| > \kappa\} \leq \mu_L\{x : |(\mathrm{I})| > \kappa/2, \cup_{j=-R+1}^{R} T^j x \subseteq \mathbb{T}\setminus\mathfrak{X}\} + \\ &\mu\{x : |(\mathrm{I})| > \kappa/2, T^j x \in \mathfrak{X} \text{ some } -R+1 \leq j \leq R\} + \mu_L\{x : |(\mathrm{II})| > \kappa/2\} \leq \\ &\mathrm{e}^{-c_1\frac{\kappa}{2}q} + 2R\mathrm{e}^{-c_2\frac{\kappa}{2}q} \leq \mathrm{e}^{-c_3\kappa q} \ , \end{aligned} \tag{3.16}$$

for suitable $c_3 > 0$. This completes the estimate of the 2nd contribution in (3.4).

Consider now the third term in in (3.4). Set

$$\tilde{\mathfrak{Y}}_n := \{x : \tilde{w}_n(x) \neq \tilde{u}_n(x)\} = \{x : \|\tilde{D}_n(x)\| < \mathrm{e}^{-nA}\} \tag{3.17}$$

$$\subseteq \{x : (\prod_{j=0}^{n-1} |\tilde{d}(x + j\beta)|)^{\frac{1}{2}} < \mathrm{e}^{-nA}\} \ . \tag{3.18}$$

In order to analyze the product of analytic functions occurring in (3.18) we establish the following:

Proposition 3.2. *Let $f \in \mathcal{C}_\delta^\omega$ not vanishing identically and let β be fixed satisfying the Diophantine condition (1.4). Then, there exist $0 < C_3 = C_3(\beta)$ and $0 < C_4 = C_4(f)$ such that for $n \in \mathbb{N}$ we have*

(3.19)

$$|\frac{1}{n}\sum_{j=1}^{n} \log|f(x+j\beta)| - \langle \log|f|\rangle| \leq C_3 \mathcal{N}(f;\mathbb{T}) n^{-1/r(\beta)} \log^2(n) \left| \min_{1\leq j\leq n} \log|f(x+j\beta)|\right| + \frac{C_4}{n} \ .$$

The constant $C_4(\tilde{f})$ can be chosen uniformly over $||\tilde{f} - f||_\delta < \epsilon$ for $\epsilon > 0$ sufficiently small.

Remark 3.4.
(i) It is through this Proposition that a Diophantine condition is imposed on β in Theorem 3.1.
(ii) Using upper-semicontinuity of $\mathcal{N}(.;\mathbb{T})$ established in Proposition 2.1, $\mathcal{N}(\tilde{f};\mathbb{T})$ can be chosen uniformly over $||\tilde{f} - f||_\delta < \epsilon$ for sufficiently small $\epsilon > 0$.

We mention that Proposition 3.2 improves on and provides a uniform version of Proposition C from [1], originally proven in [12]. The statement given here was inspired by estimates on trigonometric products which played an important role in [13] (see Sec. 9.2 therein).

Proof. First, write f as

$$f(z) = g(z) \prod_{j=1}^{\mathcal{N}(f;\mathbb{T})} (\mathrm{e}^{2\pi i x_j} - z) \tag{3.20}$$

on a compact neighborhood U where f is holomorphic and exhibits its *only* zeros $\{x_j, 1 \leq j \leq \mathcal{N}(f;\mathbb{T})\}$ on $\mathbb{T}$ (counting multiplicities). In particular g is holomorphic on a neighborhood of U and $\mathcal{N}(g;U) = 0$.

We first establish the zero free version of Proposition 3.2.

Lemma 3.5 (Zero-free version of Proposition 3.2)**.** *Let $\beta \in [0, 1)$ be a fixed Diophantine number satisfying condition (1.4) and g a zero free function on a compact neighborhood U of $\mathbb{T}$. Then for $n \in \mathbb{N}$,*

$$|\frac{1}{n}\sum_{j=1}^{n} \log|g(x + j\beta)| - \langle \log|g|\rangle| \leq \frac{C_4}{n} \ . \tag{3.21}$$

Here, $C_4 = C_4(g)$ can be chosen uniformly over $||\tilde{g} - g||_U < \epsilon$ for $\epsilon > 0$ sufficiently small (only depending on $\min_{z\in U} |g(z)|$).

Remark 3.6.
(i) The main purpose here is to convince the reader that C_4 can be chosen *uniformly*; the mere rate of convergence for the zero free situation is a standard fact from harmonic analysis.

(ii) Uniformity of $C_4(\tilde{f})$ follows from Lemma 3.5 since $f_\alpha \to f$ uniformly on U implies uniform convergence of the respective zero-free functions (for a simple argument see the proof of Lemma 2.9).

Proof. Find $\epsilon_0 > 0$ such that $|\tilde{g}(z)| > 1/2 \min_{z \in U} |g(z)| > 0$ for $||\tilde{g} - g||_U < \epsilon_0$. In particular, $||\tilde{g}-g||_U < \epsilon_0$ implies $\mathcal{N}(\tilde{g}; U) = \mathcal{N}(g; U) = 0$ whence letting $\tilde{G} := \log|\tilde{g}|$ we obtain that $\tilde{G}$ is harmonic on a neighborhood of U. Hence, $\tilde{G} = \sum_{k\in\mathbb{Z}} \tilde{G}_k \mathrm{e}^{2\pi i k x}$ converges absolutely and uniformly on $\mathbb{T}$ and for $n \in \mathbb{N}$

$$\frac{1}{n}\sum_{j=0}^{n-1} \tilde{G}(x + j\beta) - \tilde{G}_0 = \frac{1}{n} \sum_{k\in\mathbb{Z}\setminus\{0\}} \tilde{G}_k \mathrm{e}^{2\pi i k x} \frac{1 - \mathrm{e}^{2\pi i k n \beta}}{1 - \mathrm{e}^{2\pi i k \beta}} \ . \tag{3.22}$$

Making use of Eq. (1.4), results in

$$\left| \frac{1}{n}\sum_{j=0}^{n-1} \tilde{G}(x + j\beta) - \tilde{G}_0 \right| \leq \frac{2b(\beta)}{n} \sum_{k\in\mathbb{Z}} |\tilde{G}_k| k^M \ , \tag{3.23}$$

where $M = M(\beta) := \lceil r(\beta) \rceil$.

As in Katznelson [9], let $A(\mathbb{T}) \subset L^1(\mathbb{T})$ denote the class of 1-periodic functions with absolutely converging Fourier series equipped with the norm $\|f\|_{A(\mathbb{T})} := \sum_{k\in\mathbb{Z}} |f_k|$. $A(\mathbb{T})$ is a homogeneous Banach space of $L^1(\mathbb{T})$ isomorphic to $l^1(\mathbb{Z})$.

We employ the following standard fact :

Proposition 3.3. [9] *Let $f \in L^1(\mathbb{T})$ be absolutely continuous with $f' \in L^2(\mathbb{T})$. Then $f \in A(\mathbb{T})$ and*

$$\|f\|_{A(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} + \left(2 \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \|f'\|_{L^2(\mathbb{T})} \ . \tag{3.24}$$

In summary, Proposition 3.3 and (3.23) imply

$$\left| \frac{1}{n}\sum_{j=0}^{n-1} \tilde{G}(x + j\beta) - \tilde{G}_0 \right| \leq \frac{4b(\beta)}{n} \left(\|\tilde{G}^{(M)}\|_{\mathbb{T}} + \|\tilde{G}^{(M+1)}\|_{\mathbb{T}} \right) \ . \tag{3.25}$$

Finally, it is a basic property of harmonic functions that $\tilde{G} \to G$ uniformly on U implies $\tilde{G}^{(m)} \to G^{(m)}$ for any $m \in \mathbb{N}$ [10], which yields the claim. □

Thus we are left with analyzing the rate of convergence of the terms $\log|\mathrm{e}^{2\pi i x_j} - z|$ in a Caesaro mean. We employ the following Lemma proven in [13]

Lemma 3.7. *Let β be irrational. For $n \in \mathbb{N}$, let p_n/q_n denote the nth approximant of β. Then, if $1 \leq k_0 \leq q_n$ is determined by*

$$\left| \sin\left(\frac{2\pi(x - x_0) + k_0\beta}{2} \right) \right| := \min_{1\leq k\leq q_n} \left| \sin\left(\frac{2\pi(x - x_0) + k\beta}{2} \right) \right| \ , \tag{3.26}$$

we have:

$$\left| \sum_{\substack{k=1 \\ k \neq k_0}}^{q_n} \log \left| \sin \left(\frac{2\pi(x - x_0) + k\beta}{2} \right) \right| + (q_n - 1)\log(2) \right| = \tag{3.27}$$

$$\left| \sum_{\substack{k=1 \\ k \neq k_0}}^{q_n} \log \left| \mathrm{e}^{2\pi i(x+k\beta)} - \mathrm{e}^{2\pi i x_0} \right| \right| < C_5 \log(q_n) \ . \tag{3.28}$$

Here, $C_5 = C_5(\beta)$.

Fix $1 \leq j \leq \mathcal{N}(f;\mathbb{T})$. For $n \in \mathbb{N}$ arbitrary let $s = s(n) \geq 0$ such that $q_s \leq n < q_{s+1}$. By successive division represent n as $n = \sum_{k=0}^{s} l_k q_k$. Recall that by (1.4) [14],

$$q_{k+1} \leq \frac{2\pi}{b(\beta)} q_k^{r(\beta)} \ , \ k \in \mathbb{N}_0 \ . \tag{3.29}$$

In particular, this allows to control the divisors

$$l_k \leq \frac{q_{k+1}}{q_k} \leq \left(\frac{2\pi}{b}\right)^{1/r} q_{k+1} \ , \ k < s \ , \text{ and } l_s \leq \frac{n}{q_s} \leq \left(\frac{2\pi}{b}\right)^{1/r} n^{1-1/r} \ . \tag{3.30}$$

Let $1 \leq k_0 \leq n$ such that

$$\left| \sin \left(\frac{2\pi(x - x_j) + k_0\beta}{2} \right) \right| := \min_{1 \leq k \leq q_n} \left| \sin \left(\frac{2\pi(x - x_j) + k\beta}{2} \right) \right| \ . \tag{3.31}$$

Making use of Lemma 3.7 and (3.30) yields

$$\begin{aligned} \left| \sum_{\substack{k=1 \\ k \neq k_0}}^{n} \log \left| \mathrm{e}^{2\pi i(x+k\beta)} - \mathrm{e}^{2\pi i x_j} \right| \right| &\leq C_5 \sum_{k=0}^{s} \log q_k \frac{q_{k+1}}{q_k} \leq \\ C_5 \log q_s \left(\left(\frac{2\pi}{b}\right)^{1/r} q_s^{1-1/r} \frac{2}{\log 2} \log q_s + \log q_s \left(\frac{2\pi}{b}\right)^{1/r} n^{1-1/r} \right) &\leq \\ C_6(\beta) \log^2(n) n^{1-1/r} \ . \end{aligned} \tag{3.32}$$

Here, we also used a general fact that allows to control s by $s(n) \leq \frac{2\log q_s}{\log 2}$ (valid for any irrational β) [14].

Finally, combining (3.20), (3.21), and (3.32) implies the claim of Proposition 3.2. □

We can now estimate $\mu\{\tilde{\mathfrak{Y}}_n\}$ (see (3.17)). To this end, suppose $x \in \tilde{\mathfrak{Y}}_n$, then employing Proposition 3.2 results in

$$\min_{0 \leq j \leq n-1} |\tilde{d}(x + j\beta)| < \mathrm{e}^{-n^{\epsilon}} \tag{3.33}$$

for n sufficiently large and any $0 < \epsilon < r(\beta)^{-1} \leq 1$. Hence, by theorem 2.4

$$\mu\{\tilde{\mathfrak{Y}}_n\} \leq \mu\{x : \min_{0 \leq j \leq n-1} |\tilde{d}(x + j\beta)| < \mathrm{e}^{-n^{\epsilon}}\} \leq n\mathrm{e}^{-n^{\epsilon}\alpha} \leq \mathrm{e}^{-n^{\epsilon}c_2} \ , \tag{3.34}$$

for α determined by d, $\gamma = \gamma(d)$ sufficiently small and n sufficiently large, uniformly over $||\tilde{D} - D||_\delta < \gamma$.

Since $\|M\|^2 \geq |\det M|$ for $M \in M_2(\mathbb{C})$,

(3.35)
$$|\langle \tilde{w}_n\rangle - L_n(\beta,\tilde{D})| \leq \frac{1}{n}\int_{\tilde{\mathfrak{Y}}_n} \log\left|\frac{\mathrm{e}^{-nA}}{\|\tilde{D}_n(x)\|}\right| \mathrm{d}x \leq \frac{1}{n}\int_{\tilde{\mathfrak{Y}}_n} \log\left(\frac{\mathrm{e}^{-nA}}{\prod_{j=0}^{n-1}|\tilde{d}(x+j\beta)|^{\frac{1}{2}}}\right)\mathrm{d}x \,.$$

Lemma 3.8. *Uniformly over $||\tilde{D}-D||_\delta < \gamma$ there exists $\alpha = \alpha(d)$ and $0 < C_7 = C_7(\alpha) < \infty$ such that for $i, j \in \{0, \ldots, n-1\}$,*

$$\left|\int_{|\tilde{d}_i(x)|<\epsilon} \log|\tilde{d}_j(x)|\mathrm{d}x\right| \leq C_7\epsilon^{\alpha}|\log\epsilon| \;, \tag{3.36}$$

for sufficiently small $\epsilon > 0$.

Proof. Take α, ϵ_0 as in Theorem 2.4 with $f = d$. Take $\gamma(d)$ such that the $\gamma(d)$-neighborhood of d is contained in $\operatorname{Int}\mathcal{T}_\alpha^{\epsilon_0}$. Set $d_i(x) = d(x+i\beta)$ and define $\tilde{A}_i := \{x : |\tilde{d}_i(x)| < \epsilon\}$ and $\tilde{B}_k^j := \{x : \frac{1}{2^k}\epsilon < |\tilde{d}_j(x)| < \frac{1}{2^{k-1}}\epsilon\}$ for $k \in \mathbb{N}$.

Then, for $\gamma < \gamma(d)$ sufficiently small and ϵ sufficiently small determined uniformly for $||\tilde{D}-D||_\delta < \gamma$, by Theorem 2.4,

$$\begin{aligned}\left|\int_{|\tilde{d}_i(x)|<\epsilon} \log|\tilde{d}_j(x)|\mathrm{d}x\right| &\leq \sum_{k\in\mathbb{N}}\left|\int_{\tilde{A}_i\cap\tilde{B}_k^j} \log|\tilde{d}_j(x)|\mathrm{d}x\right| + \\ &\left|\int_{\tilde{A}_i\setminus\cup_{k\in\mathbb{N}}\tilde{B}_k^j} \log|\tilde{d}_j(x)|\mathrm{d}x\right| \leq C_7\epsilon^{\alpha}|\log\epsilon| \;.\end{aligned} \tag{3.37}$$

□

Fix $\epsilon < r(\beta)^{-1}$. Equation (3.35) together with Lemma 3.8 , (3.17) and (3.34) imply, for uniformly large n:

$$\begin{aligned}|\langle \tilde{w}_n\rangle - L_n(\beta,\tilde{D})| &\leq \frac{1}{n}\sum_{i=0}^{n-1}\int_{|\tilde{d}_i(x)|<\mathrm{e}^{-n^\epsilon}} \log\left(\frac{\mathrm{e}^{-nA}}{\prod_{j=0}^{n-1}|\tilde{d}(x+j\beta)|^{\frac{1}{2}}}\right)\mathrm{d}x \leq \\ &An\mathrm{e}^{-\alpha n^\epsilon} + \frac{1}{2}C_7 n^\epsilon \mathrm{e}^{-\alpha n^\epsilon} < \mathrm{e}^{-c_4 n^\epsilon} \;,\; 0 < c_4 < \alpha \;.\end{aligned} \tag{3.38}$$

Thus,

$$\begin{aligned}&\mu\{x\in\mathbb{T} : |\tilde{u}_n(x) - L_n(\beta,\tilde{D})| > \kappa\} \leq \mu_L\{x : |\tilde{u}_n(x) - \tilde{w}_n(x)| > \frac{\kappa}{2} - \frac{\mathrm{e}^{-c_4 n^\epsilon}}{2}\} + \\ &\mu\{x : |\tilde{w}_n(x) - \langle\tilde{w}\rangle_n(x)| > \frac{\kappa}{2} - \frac{\mathrm{e}^{-c_4 n^\epsilon}}{2}\} \leq \mu_L\{\tilde{\mathfrak{Y}}_n\} + \mu_L\{x : |\tilde{w}_n(x) - \langle\tilde{w}_n\rangle| > \kappa/3\}\end{aligned} \tag{3.39}$$

for n sufficiently large.

Finally, making use of (3.16) and (3.34) we obtain Theorem 3.1. □

4. Concluding remarks

In [1] the authors also obtain a continuity statement in the frequency (Theorem 2, [1]). Repeating their proof we obtain an analogous statement here: To this end we denote the set of numbers satisfying the Diophantine condition (1.4) for a given $r > 1$ by $DC(r)$.

Theorem 4.1. *Both $L'(.,.), L(.,.) : DC(r) \times \mathcal{C}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C})) \to \mathbb{R}$ are jointly continuous.*

Note that Theorem 4.1 does *not* give continuity of the LE under rational approximants of the frequency due to the Diophantine condition imposed.

If however the $\det D(x)$ is bounded away from zero Theorem 4.1 can be strengthened using the continuity statement from [2]. For $\delta >$ fixed, define the space

$$\mathcal{B}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C})) := \{D \in \mathcal{C}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C})) : \mathcal{N}(\det D; |\mathrm{Im}(z)| \leq \delta) = 0\} \ . \tag{4.1}$$

By Proposition 2.1 $\mathcal{B}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C}))$ is open in $\mathcal{C}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C}))$.

Theorem 4.2. *If β_0 is irrational, $L'(.,.), L(.,.) : \mathbb{R}\times\mathcal{B}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C})) \to \mathbb{R}$ are jointly continuous at $(\beta_0, .)$.*

Proof. The statement is known for $L(.,.)$ [2]. To prove continuity for L', let β be irrational and fix a $D \in \mathcal{B}^{\omega}_{\delta}(\mathbb{T}, M_2(\mathbb{C}))$. It suffices to show that if $r_n = \frac{p_n}{q_n}$ is a sequence of *rational* approximants of β, $(p_n, q_n) = 1$, and $\|D_n - D\|_{\delta} \to 0$ we have $L'(r_n, D_n) \to L'(\beta, D)$.

First, notice that for any n, $r_n \in \mathbb{Q}$ implies

$$L(r_n, D'_n) = L(r_n, D) - \frac{1}{2q_n} \sum_{j=0}^{q_n - 1} \log|\det D(x + jr_n)| \ . \tag{4.2}$$

Thus, the claimed continuity property of $L'(.,.)$ is reduced to establishing

$$\frac{1}{q_n} \sum_{j=0}^{q_n-1} \log|\det D(x + jr_n)| \to \int_{\mathbb{T}} \log|\det D(x)| \mathrm{d}x \ , \tag{4.3}$$

as $n \to \infty$.

Using harmonicity of $\log \det D(x)$ on $|\mathrm{Im}|(z) \leq \delta$,

$$\frac{1}{q_n} \sum_{j=0}^{q_n-1} \log|\det D(x + j\beta)| \to \int_{\mathbb{T}} \log|\det D(x)| \mathrm{d}x \ , \tag{4.4}$$

uniformly on $\mathbb{T}$ as $n \to \infty$.

Hence, (4.3) follows by successive approximation also making use of Lemma 2.9:

$$\begin{aligned} &\left| \frac{1}{q_n} \sum_{j=0}^{q_n-1} \log|\det D_n(x + jr_n)| - \int_{\mathbb{T}} \log|\det D_n(x)| \mathrm{d}x \right| \leq \frac{1}{q_n} \sum_{j=0}^{q_n-1} \big| \log|\det D_n(x + jr_n)| \\ &\qquad - \log|\det D(x + j\beta)| \big| + \left| \frac{1}{q_n} \sum_{j=0}^{q_n-1} \log|\det D(x + j\beta)| - \int_{\mathbb{T}} \log|\det D(x)| \mathrm{d}x \right| \\ &\qquad + \left| \int_{\mathbb{T}} \log|\det D(x)| \mathrm{d}x - \int_{\mathbb{T}} \log|\det D_n(x)| \mathrm{d}x \right| \ . \end{aligned} \tag{4.5}$$

□

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