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Toroidal p-branes, anharmonic oscillators and (hyper)elliptic solutions

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Abstract

Exact solvability of brane equations is studied, and a new $U(1) \times U(1) \times \ldots U(1)$ invariant anzats for the solution of p-brane equations in $D = (2p+1)$-dimensional Minkowski space is proposed. The reduction of the p-brane Hamiltonian to the Hamiltonian of p-dimensional relativistic anharmonic oscillator with the monomial potential of the degree equal to $2p$ is revealed. For the case of degenerate p-torus with equal radii it is shown that the p-brane solutions are expressed in terms of elliptic ($p = 2$) or hyperelliptic ($p > 2$) functions.

1 Introduction

Membranes and p-branes play an important role in M/string theory [1], and their quantization is one of current hot problems. Its solution is complicated by non-linearity of the classical brane equations in contrast to the string (p=1) case [2]. The question of the membrane (p=2) and p-brane dynamics and its integrability attracts much attention (see e.g. [3-15] , etc.) One of

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the ways to answer the question lies in study of various classical solutions of brane equations. However, not so much is known about such solutions. Studying this problem Hoppe in [16] proposed the $U(1)$ invariant anzats for closed p-branes and reformulated the membrane equations in D=5 into the system of 2-dim non-linear equations. The elliptic solution of these equations, describing a family of closed contracting 2d tori together with the solution corresponding to a spinning 2d torus were found in [17] as an example of time-dependent particular solutions. The exact solvability of the static equations of $U(1)$ invariant membranes in $D = (2N + 1)$-dimensional Minkowski space was revealed in [17], and their general solution for any $N > 1$ was constructed. A geometric approach to these invariant membranes in D=5 was developed in [18], where their connection with a two-dimensional generalization of the nonlinear Abel equation and with the pendulum equations was found.

Here we generalize this approach and propose a new anzats for the solutions of p-branes evolving in $D = (2p+1)$-dimensional Minkowski space with any integer $p > 1$. This rigidly fixed (D, p)-correlation between the space-time and brane dimensions covers, in particular, interesting cases of globally invariant 5-branes of M/string theory in D=11 space-time, membranes in D=5 and 3-branes in D=7. The proposed anzats corresponds to closed compact p-branes with the global rotational symmetry $U(1) \times U(1) \times \ldots U(1)$ (with p multipliers) of their p-dimensional hypersurfaces Σ_p. These hypersurfaces turn out to be isomorphic to flat p-tori with zero curvatures. The Hamiltonians and equations of invariant p-branes are constructed, and it is shown that they describe p-dimensional anharmonic oscillators with the quartic potential for membranes in D=5 and the monomial potential of the degree $2p$ for the p-brane in D=2p+1. A characteristic feature of these Hamiltonians is the absence of the quadratic terms which are contained in the Hamiltonian of the harmonic oscillator [1]. The p-brane equations are reformulated into the equations for elastic relativistic media with the symmetric stress tensor corresponding to isotropic pressure. For the case of degenerate p-torus with all equal radii it is found that the solutions of the corresponding nonlinear equations are expressed in terms of elliptic cosine for $p = 2$ or hyperelliptic functions for higher $p > 2$.

[1] Usually the notion of anharmonic oscillator is used in the case when the quartic and higher terms in the potential energy are small in comparison with its quadratic terms. Here we use this term despite the absence of the quadratic term in the p-brane Hamiltonians and do not assume the smallness of the higher monomials.

2 P-brane dynamics for any (D, p)

The Dirac action for a p-brane without boundaries is defined by the integral in the dimensionless world-volume parameters ξ^α $(\alpha = 0, \ldots, p)$ [2]

$$S = T \int \sqrt{|G|} d^{p+1}\xi,$$

where G is the determinant of the induced metric $G_{\alpha\beta} := \partial_\alpha x_m \partial_\beta x^m$ and T is the p-brane tension with the dimension $L^{-(p+1)}$, because x^m has the dimension of length. After splitting of the world $x^m = (x^0, x^i) = (t, \vec{x})$ and internal coordinates $\xi^\alpha = (\tau, \sigma^r)$, the Euler-Lagrange equations and $p+1$ primary constraints generated by S take the form

$$\partial_\tau \mathcal{P}^m = -T\partial_r(\sqrt{|G|} G^{r\alpha} \partial_\alpha x^m), \quad \mathcal{P}^m = T\sqrt{|G|} G^{\tau\beta} \partial_\beta x^m, \tag{1}$$

$$\tilde{T}_r := \mathcal{P}^m \partial_r x_m \approx 0, \quad \tilde{U} := \mathcal{P}^m \mathcal{P}_m - T^2 |\det G_{rs}| \approx 0, \tag{2}$$

where $\mathcal{P}^m$ is the energy-momentum density. Then we use the orthogonal gauge simplifying the metric $G_{\alpha\beta}$

$$L\tau = x^0 \equiv t, \quad G_{\tau r} = -L(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0, \tag{3}$$

$$g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad G_{\alpha\beta} = \begin{pmatrix} L^2(1 - \dot{\vec{x}}^2) & 0 \\ 0 & -g_{rs} \end{pmatrix}$$

with $\dot{\vec{x}} := \partial_t \vec{x} = L^{-1}\partial_\tau \vec{x}$. The solution of the constraint $\tilde{U}$ (2) takes the form

$$\mathcal{P}_0 = \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}, \quad g = \det(g_{rs}) \tag{4}$$

and becomes the Hamiltonian density $\mathcal{H}_0$ of the p-brane since $\dot{\mathcal{P}}_0 = 0$ in view of Eq. (1). Using the definition of $\mathcal{P}_0$ (1) and $G^{\tau\tau} = 1/L^2(1 - \dot{\vec{x}}^2)$ we find the expression of $\mathcal{P}_0$ as a function of the p-brane velocity $\dot{\vec{x}}$

$$\mathcal{P}_0 := TL\sqrt{|detG|} G^{\tau\tau} = T\sqrt{\frac{|g|}{1 - \dot{\vec{x}}^2}} \ . \tag{5}$$

Taking into account this expression for $\mathcal{P}_0$ and the definition (1) one can present $\vec{\mathcal{P}}$ and its evolution equation (1) as

$$\vec{\mathcal{P}} = \mathcal{P}_0 \dot{\vec{x}}, \quad \dot{\vec{\mathcal{P}}} = T^2 \partial_r \left(\frac{|g|}{\mathcal{P}_0} g^{rs} \partial_s \vec{x} \right). \tag{6}$$

[2] Here the D-dimensional Minkowski space has the signature $\eta_{mn} = (+, -, \ldots, -)$.

Then Eqs. (6) yield the second-order PDE for $\vec{x}$

$$\ddot{\vec{x}} = \frac{T}{\mathcal{P}_0}\partial_r \left(\frac{T}{\mathcal{P}_0}|g|g^{rs}\partial_s \vec{x} \right). \tag{7}$$

These equations may be presented in the canonical Hamiltonian form

$$\dot{\vec{x}} = \{H_0, \vec{x}\}, \qquad \dot{\vec{\mathcal{P}}} = \{H_0, \vec{\mathcal{P}}\}, \qquad \{\mathcal{P}_i(\sigma), x_j(\tilde{\sigma})\} = \delta_{ij}\delta^{(2)}(\sigma^r - \tilde{\sigma}^r)$$

using the integrated Hamiltonian density (4) $\mathcal{H}_0 (= \mathcal{P}_0)$

$$H_0 = \int d^p\sigma \sqrt{\vec{\mathcal{P}}^2 + T^2|g|}. \tag{8}$$

The presence of square rout in (8) shows that the orthonormal gauge (3) has a residual symmetry

$$\tilde{t} = t, \qquad \tilde{\sigma}^r = f^r(\sigma^s) \tag{9}$$

generated by the constraints $\tilde{T}_r$ reduced to the form

$$T_r := \vec{\mathcal{P}}\partial_r \vec{x} = 0 \quad \Leftrightarrow \quad \dot{\vec{x}}\partial_r \vec{x} = 0, \quad (r = 1, 2, ., p). \tag{10}$$

The freedom allows to put p additional time-independent conditions on $\vec{x}$ and its space-like derivatives. The above description of brane dynamics is valid for any space-time and brane world-volume dimensions (D, p) (with $D > p$).

3 $U(1) \times U(1) \times \ldots U(1)$ invariant p-branes

Here we consider p-branes evolving in $D = (2p+1)$-dimensional Minkowski space-time and assume that their shape is invariant under the direct product $\mathcal{U} := \prod_{a=1}^{p} U_a(1)$. Each of these $U(1)$ symmetries is locally isomorphic to one of the $O(2)$ subgroups of the $SO(2p, 1)$ group of rotations. Thus the p-dimensional hypersurface Σ_p of $\mathcal{U}$ invariant p-brane has the group $\mathcal{U}$ as its isometry with the p Killing vectors. This points to the existence of a parametrization of the p-brane hypersurface Σ with the metric tensor g_{rs} independent of σ^r. Our analysis will be restricted by the case of $\mathcal{U}$ invariant p-branes without boundaries. The invariant p-branes with boundaries are treated similarly taking into account additional boundary terms.

To construct $\mathcal{U}$ invariant p-brane hypersurface we introduce the following anzats for its vector $\vec{x}$

$$\vec{x}^T = (q_1 \cos\theta_1, q_1 \sin\theta_1, q_2 \cos\theta_2, q_2 \sin\theta_2, \ldots, q_p \cos\theta_p, q_p \sin\theta_p), \qquad (11)$$
$$q_a = q_a(t), \;\; (a = 1, \ldots, p); \quad \theta_r = \theta_r(\sigma^s), \;\; (r, s = 1, \ldots, p),$$

where T is the transposition of the column usually used for vector components. This space vector lies in the $2p$-dimensional Euclidean subspace of $(2p+1)$-dimensional Minkowski space and automatically satisfies the orthogonality constraints (10): $\dot{\vec{x}}\partial_r\vec{x} = 0$. The anzats (11) originates from the realization of a $2p$-dimensional vector $\vec{x}$ describing any p-brane, and is defined by p pairs of its "polar" coordinates

$$\vec{x}^T(t, \sigma^r) = (q_1 \cos\theta_1, q_1 \sin\theta_1, \ldots, q_p \cos\theta_p, q_p \sin\theta_p) \qquad (12)$$

with the coordinates q_a, θ_a depending on all parameters $(t, \sigma^1, ..., \sigma^p)$ of the p-brane world volume: $q_a = q_a(t, \sigma^r)$, $\theta_a = \theta_a(t, \sigma^r)$. Thus, the proposed anzats (11) is obtained from the general representation (12) by excluding the time dependence for all "polar" angles $\theta_a = \theta_a(\sigma^r)$ as well as the σ^r dependence for all "radial" coordinates $m_a = m_a(t)$. As a result, at any fixed moment t the vector $\vec{x}^T$ (11) is produced from the $\vec{x_0}^T = (q_1, 0, q_2, 0, \ldots, q_p, 0)$ by the rotations of the diagonal subgroup $\mathcal{U} \in SO(2p)$, parametrized by the angles θ_a and rotating the planes x_1x_2, x_3x_4 ,...,$x_{2p-1}x_{2p}$.

So, the anzats (11) describes one of the representatives of the family of $\mathcal{U}$ invariant hypersurfaces embedded in the $2N$ dimensional Euclidean space. Each of the members of the family has the $\mathcal{U}$ symmetry as its inherent symmetry. The membrane world-volume metric $G_{\alpha\beta}$ corresponding to (11) has the form similar to (3) with the non-zero components

$$G_{tt} = 1 - \dot{\mathbf{q}}^2, \quad \mathbf{q} := (q_1, .., q_p), \quad g_{rs} = \sum_{a=1}^{p} q_a^2 \theta_{a,r}\theta_{a,s}, \qquad (13)$$

where $\theta_{s,r} := \partial_r\theta_s$ and $\dot{\mathbf{q}} \equiv \partial_t\mathbf{q}$ and yields the following interval ds^2 on Σ

$$ds^2_{2p+1} = (1 - \dot{\mathbf{q}}^2) - \sum_{a=1}^{p} q_a^2(t) d\theta_a d\theta_a \qquad (14)$$

independent of σ^r. We find that the change of σ^r by the new parameters $\theta_a(\sigma^r)$ of Σ_p makes the transformed metric independent of σ^r. The above mentioned Killing vectors on Σ take the form of the derivatives $\frac{\partial}{\partial\theta_a}$ in the θ_a parametrization. All that shows that $\mathcal{U}$ invariant hypersurface (11) has zero curvature and is isomorphic to a flat p-dimensional torus $S^1 \times S^1 \times \dots S^1$ (with p cirles S^1) at any fixed moment of time t.

The canonical momentum components $\pi_a = \frac{\partial\mathcal{L}}{\partial\dot{q_a}} = \vec{\mathcal{P}}\frac{\partial\dot{\vec{x}}}{\partial\dot{q_a}}$, $(a = 1, 2, .., p)$, $\boldsymbol{\pi} := (\pi_1, ..., \pi_p)$ conjugate to the coordinates $\mathbf{q}$ (11) may be presented in the explicit form as

$$\boldsymbol{\pi} = \mathcal{P}_0\dot{\mathbf{q}}, \quad \mathcal{P}_0 = \sqrt{\frac{|g|}{1-\dot{\mathbf{q}}^2}} \ , \tag{15}$$

after using (6) and the relations

$$\vec{x}^2 = \mathbf{q}^2, \quad \dot{\vec{x}}^2 = \dot{\mathbf{q}}^2, \qquad g = det(\sum_{a=1}^{p} q_a^2\theta_{a,r}\theta_{a,s}). \tag{16}$$

Then the Hamiltonian density (4) in the $(\mathbf{q}, \boldsymbol{\pi})$ phase space takes the form

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\boldsymbol{\pi}^2 + T^2|g|}, \quad \dot{\mathcal{P}}_0 = 0 \tag{17}$$

and yields the following representation for $\mathcal{P}_0$ through the velocities $\dot{\mathbf{q}}$

$$\mathcal{P}_0 = T\sqrt{\frac{|g|}{1-\dot{\mathbf{q}}^2}}. \tag{18}$$

The corresponding Hamiltonian equations of motion are transformed in Eqs.

$$\dot{\mathbf{q}} = \{H_0, \mathbf{q}\} = \frac{1}{\mathcal{P}_0}\boldsymbol{\pi}, \qquad \dot{\boldsymbol{\pi}} = \{H_0, \boldsymbol{\pi}\}, \tag{19}$$

where the non-zero canonical Poisson bracket and the Hamiltonian are defined by

$$\{\pi_a, q_b\} = \delta_{ab}, \qquad H_0 = \int d^p\sigma\sqrt{\boldsymbol{\pi}^2 + T^2|g|}. \tag{20}$$

In the next section we study the equations of motion for the $\mathbf{q}$ coordinates.

4 Equations of $\mathcal{U}$ invariant p-branes

To simplify the equations of motion of $\mathcal{U}$ invariant p-brane we can simplify the representation (11) for its $\mathbf{q}$ coordinates. This can be done by fixing the residual gauge symmetry (9) with the help of the conditions: $\theta_a = \delta_{ar}\sigma^r$ [3]

$$\theta_1(\sigma^r) = \sigma^1, \quad \theta_2(\sigma^r) = \sigma^2, \ldots, \theta_p(\sigma^r) = \sigma^p. \tag{21}$$

In the gauge (21) the anzats (11) and g_{rs} (13) are expressed as follows

$$\vec{x}^T(t) = (q_1 \cos\sigma^1, q_1 \sin\sigma^1, \ldots, q_p \cos\sigma^p, q_p \sin\sigma^p), \tag{22}$$

$$g_{rs}(t) = q_r^2(t)\delta_{rs}, \quad g = (q_1 q_2 ... q_p)^2 \tag{23}$$

with the diagonalized metric $g_{rs}(t)$ depending on only the time t. In the gauge (21) the interval (14) and the inverse metric tensor g^{rs} take the form

$$ds^2 = \sum_{r=1}^{p} q_r^2(t)(d\sigma^r)^2, \quad g^{rs}(t) = \frac{1}{q_r^2}\delta_{rs}. \tag{24}$$

This gauge clarifies the physical sense of the coordinates $\mathbf{q}(t) = (q_1, q_2, \ldots, q_p)$ as the time dependent radii $\mathbf{R}(t) = (R_1, R_2, \ldots, R_p)$ of the flat hypertorus Σ. Moreover, in the gauge (21) the Hamiltonian density $\mathcal{H}_0$ becomes independent of the hypertorus parameters σ^r and reduces to a constant C

$$\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\boldsymbol{\pi}^2 + T^2 \prod_{r=1}^{p} q_r^2} = C, \quad \dot{\mathcal{P}}_0 = 0, \quad \frac{\partial}{\partial\sigma^r}\mathcal{P}_0 = 0. \tag{25}$$

This property yields the condition for the initial data generated by the representation (18)

$$\mathcal{P}_0 = C \quad \to \quad T\sqrt{\frac{|g|}{1 - \dot{\mathbf{q}}^2}} \equiv T\sqrt{\frac{(q_1 q_2 \ldots q_p)^2}{1 - \dot{\mathbf{q}}^2}} = C. \tag{26}$$

Then Eqs. (6) for the vector $\vec{x}$ are simplified to the form

$$\ddot{\vec{x}} - (\frac{T}{C})^2 g g^{rs} \partial_{rs}\vec{x} = 0. \tag{27}$$

Taking into account the relation

$$g g^{rs} = \frac{\delta_{rs}}{q_r^2}\prod_{t=1}^{p} q_t^2, \tag{28}$$

[3] To cover the case of p-brane with windings one can fix the gauge conditions as $\theta_a = n_a\delta_{ar}\sigma^r$, where $(n_1, n_2, ..., n_p)$ are the integer numbers corresponding to the winding numbers on the circles $0 \leq \sigma^r \leq 2\pi$ parametrized by σ^r.

following from (23), we present the system (27), equivalent to the Hamiltonian Eqs. (19), in terms of the $\mathbf{q}$ components

$$\ddot{q}_r + (\frac{T}{C})^2(q_1 \dots q_{r-1}q_{r+1} \dots q_p)^2 q_r = 0, \quad r = (1, 2, \dots, p). \tag{29}$$

Multiplication of the r-th equation of the system (29) by q_r and subsequent summing in r results in the first integral of (29)

$$\dot{\mathbf{q}}^2 + (\frac{T}{C})^2(q_1 q_2 ... q_p)^2 = c \tag{30}$$

which coincides with the initial data (26) if the integration constant $c = 1$. It is easily seen that Eqs. (29) may be present in a condenced form as

$$C\ddot{\mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}}, \tag{31}$$

where the elastic energy density $V(\mathbf{q})$ turns out to be proportional to the determinant g of the $\mathcal{U}$ invariant hypersurface of p-brane

$$V(\mathbf{q}) := \frac{T^2}{2C} g \equiv \frac{T^2}{2C}(q_1 ... q_p)^2. \tag{32}$$

Below we shall use this representation to explain the physics described by the $U(1) \times U(1) \times \dots U(1)(:\equiv \mathcal{U})$ invariant p-branes.

5 $\mathcal{U}$ invariant p-branes and p-dimensional anharmonic oscillators

To clarify the physical sense of Eqs. (31) let us remind the general equations of motion of elastic non relativistic media [19] with the mass density ρ

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ik}}{\partial x_k}, \tag{33}$$

where $\ddot{u}_i$ and σ_{ik} are the media acceleration and stress tensor, respectively. Then we observe that Eqs. (31) have a form of relativistic generalization of Eqs. (33)

$$C\ddot{q}_r = -\frac{T^2}{2C}\delta_{rs}\frac{\partial g}{\partial q_s} \tag{34}$$

with the symmetric stress tensor σ_{rs} given by

$$\sigma_{rs} := -p\delta_{rs}, \quad p := \frac{T^2}{2C}g \equiv \frac{T^2}{2C}\prod_{s=1}^{p} q_s^2, \quad r = (1, 2, \ldots, p). \tag{35}$$

The representation (35) shows that p is an isotropic pressure per unit area of its p-brane hypersurface, and C is a relativistic generalization of the mass density ρ in accordance with (18). The pressure p is created by the internal forces F_r

$$F_r(t) := -\frac{\partial V}{\partial q_r} \equiv -\frac{T^2}{C}(q_1 \ldots q_{r-1}q_{r+1} \ldots q_p)^2 q_r \tag{36}$$

associated with the p-brane elastic potential V (32).

It is instructive to note that the discussed equations may be generated by a Hamiltonian H free from the square root present in H_0 (20). Such a possibility is a consequence of our fixing the residual gauge symmetry that reduces $\mathcal{P}_0$ to the constant C. Then C can be used to write the square root free Hamiltonian H accompanied with the standard PB's

$$H := \int d^p\sigma \mathcal{H}, \quad \mathcal{H} = \frac{1}{2C}(\boldsymbol{\pi}^2 + T^2 \prod_{s=1}^{p} q_s^2), \tag{37}$$
$$\{\pi_a, q_b\} = \delta_{ab}, \quad \{q_a, q_b\} = 0, \quad \{\pi_a, \pi_b\} = 0.$$

The Hamiltonian (37) and equations (29) that it generates describe p-dimensional anharmonic oscillators for every $p > 1$, but without the quadratic terms typical of harmonic oscillators. These Hamiltonians contain the quartic (for p=2) potential energy and higher degree monomials in the componets of $\mathbf{q}$ for $p > 2$. A characteristic feature of the non-linear system (29) is that the r-th coordinate q_r under the force F_r evolves with the cyclic "frequency" ω_r proportional to the products of the remaining coordinates at any moment t_0

$$\omega_r(t) = \frac{T}{C}|q_1 \ldots q_{r-1}q_{r+1} \ldots q_p|, \quad r = (1, 2, \ldots, p). \tag{38}$$

as it follows from (29). These frequences ω_r can not be infinitely large because of the initial data constraint (30) with $c = 0$

$$\sqrt{1 - \dot{\mathbf{q}}^2} = \frac{T}{C}|q_1 q_2 \ldots q_p| \tag{39}$$

Taking into account the condition (39) one can transform (38) to the relations

$$\omega_r(t) = \frac{\sqrt{1 - \dot{\mathbf{q}}^2}}{|q_r|} \tag{40}$$

which shows finiteness of $\omega_r(t$. On the other hand if $\omega_r(t)$ goes to zero then $|\dot{\mathbf{q}}| \to 1$ which means that the p-brane velocity goes to the velocity of light. This limit corresponds to the case of tensionless p-branes [9, 11], i.e. the brane tension $T = 0$, as it follows from (40) or, equivalently from the demand of $\mathcal{P}_0$ (18) finiteness. In the limit $T = 0$ the nonlinear system (29) reduces to the equations of p massless particles

$$T = 0: \quad \Rightarrow \quad \ddot{\mathbf{q}} = 0, \quad |\dot{\mathbf{q}}| = 1. \tag{41}$$

In the general case the system (29) is rather complicate because of the monomial character of the interaction potential V (37). However, there is a special solvable case that we discuss below.

6 Elliptic and hyperelliptic solutions

In the degenerate case characterized by the coincidence of all components $q_1 = q_2 = ... = q_p \equiv q$ the system (29) reduces to the following nonlinear differential equation

$$\ddot{q} + (\frac{T}{C})^2 q^{(2p-1)} = 0 \tag{42}$$

equivalent to the initial data constraint (39) for the single function $q(t)$

$$p\dot{q}^2 + (\frac{T}{C})^2 q^{2p} = 1. \tag{43}$$

After the change of q by the new variable $y = \Omega^{\frac{1}{p}}\sqrt{p}q$, and the introduction of a dependent frequency $\Omega := \frac{T}{C}p^{-\frac{p}{2}}$, equation (43) takes the form

$$(\frac{dy}{d\tilde{t}})^2 = \frac{1}{2}(1 - y^p)(1 + y^p), \tag{44}$$

where a new variable $\tilde{t} := \sqrt{2}\Omega^{\frac{1}{p}}t$ is used.

In the membrane case ($p = 2$) Eq. (44) coincides with the equation defining the Jacobi elliptic cosine $cn(x;k)$

$$(\frac{dy}{dx})^2 = (1-y^2)(1-k^2+k^2y^2), \tag{45}$$

if the elliptic modulus $k = \frac{1}{\sqrt{2}}$. Thus, $y(t) = cn(\sqrt{2\omega}t; \frac{1}{\sqrt{2}})$ with $2\omega = T/C$. Using the relation $q \equiv y/\sqrt{2\omega}$ we obtain the elliptic cosine solution for the desired coordinate $q(t)$

$$q(t) = \frac{1}{\sqrt{2\omega}} cn(\sqrt{2\omega}(t+t_0); \frac{1}{\sqrt{2}}) \equiv \sqrt{\frac{C}{T}} cn(\sqrt{\frac{T}{C}}(t+t_0); \frac{1}{\sqrt{2}}) \tag{46}$$

that is similar to the elliptic solution for the $U(1)$ invariant membrane earlier obtained in [17]. If the initial data are $\dot{q}(t_0) > 0$ then the solution (46) describes an expanding torus which at some point reaches the maximal size $q_{max} = \sqrt{\frac{C}{T}}$ and then shrinks to a point after a finite time $\mathbf{K}(1/\sqrt{2})\sqrt{\frac{C}{T}}$ (where $\mathbf{K}(1/\sqrt{2}) = 1,8451$ is the quarter period of the elliptic cosine).

The explicit equation of surface $\Sigma_2(t)$ of the contracting torus (46) in $D = 5$ is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4\frac{C}{T} cn\left(\sqrt{\frac{T}{C}}(t+t_0), \frac{1}{\sqrt{2}}\right)^2, \quad x_1x_4 = x_2x_3.$$

For the case $p > 2$ integration of Eq. (44) defines the implicit dependence of q on time

$$\tilde{t} = \pm\sqrt{2}\int \frac{dy}{\sqrt{1-y^{2p}}} + const. \tag{47}$$

The integral (47) is an hyperelliptic integral, thus the general solution of Eq. (43) is expressed in terms of hyperelliptic functions which are well known generalizations of elliptic functions. So, the study of the degenerate p-torus with the coinciding radii reveals connection of the p-brane equations with hyperelliptic curves.

7 Summary

A new anzats describing a set of closed p-brane hypersurfaces Σ_p immersed in the $D = (2p+1)$-dimensional Minkowski spaces and invariant under

$U(1) \times U(1) \times \ldots U(1)$, which is a subgroup of the rotational symmetry $SO(2p)$, is studied. It is shown that each of the compact hypersurfaces Σ_p is isomorphic to a flat p-torus with zero curvature immersed into the $(2p+1)$-dimensional Minkowski space. The Hamiltonians and equations of these toroidal p-branes are derived. It is shown that they coincide with the Hamiltonian and equations of p-dimensional relativistic anharmonic oscillator with the monomial potential of the degree $2p$ and without the quadratic terms. The p-brane equations are presented as the equations of an elastic relativistic media subjected to isotropic pressure dependent of time. It is found that the solutions of the equations of degenerate p-torus with all coinciding radii are given by elliptic cosine (for p=2) and hyperelliptic functions for higher $p > 2$. The obtained results shed a new light on the characteristic nonlinearities associated with the p-brane dynamics and, in particular, with the 5-branes of 11-dimensional M/string theory.

The considered anzats may be generalized to the case of curved spaces for obtaining new information on AdS/CFT correspondence.

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References

[1] P. K. Townsend, *The eleven-dimensional supermembrane revisited*, Phys. Lett. B350 (1995) 184; E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl. Phys. B443 (1995) 85; T. Banks, W. Fishler, S.H. Shenker, L. Susskind, *M theory as a Matrix Mode: A conjecture* Phys. Rev. D55(8):5112; M. J. Duff, *The world in Eleven Dimensions: Supergravity, Supermembranes and M-theory* (IOP, Bristol 1999).

[2] C. Rebbi, *Dual models and relativistic strings*, Phys. Rep. 12 (1974) 1.

[3] R. W. Tucker, *Extended Particles and the Exterior Calculus*, Lectures given at the Rutherford Laboratory, Feb 1976.

[4] J. Hoppe, MIT PhD Thesis,1982, and in Proc. Int. Workshop on Constraints Theory and Relativistic Dynamics. G. Longhi, L. Lusanna (eds), Singapore: World Scientific 1987.

[5] E. Berqshoeff, E. Sezgin, P.K. Townsend, Phys. Lett. B189 (1987) 75.

[6] M. Duff, P. Howe, T. Inami, K. Stelle, Phys. Lett. B191 (1987) 70.

[7] E. Floratos, J. Illipoulos, Phys. Lett. B201 (1988) 237.

[8] B. de Witt, J. Hoppe, G. Nicolai, Nucl. Phys. B305 [FS23] (1988) 545.

[9] A.A. Zheltukhin, Sov. J. Nucl. Phys. 48 (1988) 375.

[10] B. de Witt, M. Lusher, G. Nicolai, Nucl. Phys. B320 (1989) 135.

[11] I.A. Bandos and A.A. Zheltukhin, Fortsch. Phys. 41 (1993) 619.

[12] M. Bordemann, J. Hoppe, Phys. Lett. B317 (1993) 315 ; ibid. B325 (1994) 359.

[13] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4729.

[14] D.V. Uvarov and A.A. Zheltukhin, Phys. Lett. B545 (2002) 183.

[15] J. Arnlind, J. Hoppe, S. Theisen, Phys. Lett. B 599 (2004) 118.

[16] J. Hoppe, Compl. Anal. Oper. Theory 3 (2009) 419.

[17] M. Trzetrzelewski and A.A. Zheltukhin, Phys. Lett. B679 (2009) 523.

[18] A.A. Zheltukhin and M. Trzetrzelewski, J. Math. Phys. 51 (2010) 062303.

[19] L.D. Landau and E.M. Lifshits, *Theory of Elasticity* Moscow, "Science", 1987.