Archimedean Survival Processes

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Abstract

We introduce a class of multivariate Markov processes that we call 'Archimedean survival processes' (ASPs) and present some of their properties. An ASP is defined over a finite time horizon and, *a priori*, its terminal value has an ℓ_1-norm symmetric distribution and an Archimedean survival copula. Indeed, there is a bijection between the class of Archimedean copulas and the class of ASPs. The finite-dimensional distributions of an ASP are of multivariate Liouville-type. The one-dimensional marginal processes are so-called gamma random bridges (that is, each marginal is the product of gamma bridge with an independent positive random variable). These marginal processes are increasing and, in general, not independent, but they are identical in law. The law of an n-dimensional ASP is equivalent to that of an n-dimensional gamma process, and we provide details of the associated change of measure. The law an n-dimensional ASP is identical to the law of a positive random variable multiplied by the Hadamard product of an n-dimensional Dirichlet random variable and a vector of n independent gamma bridges. We generalise ASPs to a family of stochastic processes that we call 'Liouville processes.' A Liouville process is also a multivariate Markov process whose finite-dimensional distributions are multivariate Liouville. However, in general, the terminal value of a Liouville process does not have an Archimedean survival copula, and its one-dimensional marginal processes are not identical in law.

Keywords: Archimedean copula, gamma process, Liouville distribution, Lévy random bridge, dependence modelling.

1 Introduction

The use of copulas has become commonplace for dependence modelling in finance, insurance, and risk management (see, for example, Cherubini *et al.* [3], Frees & Valdez

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[6], and McNeil *et al.* [13]). The Archimedean copulas—a subclass of copulas—have received particular attention in the literature for both their tractability and practical convenience. An n-dimensional Archimedean copula $C : [0,1]^n \to [0,1]$ can be written as

$$C(\mathbf{u}) = h(h^{-1}(u_1) + \cdots + h^{-1}(u_n)), \tag{1}$$

where h is the *generator function* of C. We introduce a family of multivariate stochastic processes that we call *Archimedean survival processes* (ASPs). ASPs are constructed in such a way that they are naturally linked to Archimedean copulas. An ASP is defined over a finite time horizon, and, *a priori*, its terminal value has an ℓ_1-norm symmetric distribution. This implies that the terminal value of an ASP has an Archimedean survival copula. Indeed, there is a bijection from the class of Archimedean copulas to the class of ASPs.

Schönbucher & Schubert [18] and Rogge & Schönbucher [16] describe continuous-time processes that have Archimedean copulas at all times, and they are used to model default times in credit-risk applications. By construction, these processes are limited to have copulas with completely-monotone generating functions. Although they bear the link to stochastic processes with Archimedean copulas, these processes are otherwise not closely related to the present work.

A random vector $\mathbf{X}$ has a multivariate Liouville distribution if

$$\mathbf{X} \overset{\text{law}}{=} R \frac{\mathbf{G}}{\sum_{i=1}^n G_i}, \tag{2}$$

where R is a non-negative random variable, and $\mathbf{G}$ is a vector of n independent gamma random variables with identical scale parameters (see, for example, Fang *et al.* [5]). In the special case where $\mathbf{G}$ is a vector of identical exponential random variables, $\mathbf{X}$ has an ℓ_1-norm symmetric distribution. McNeil & Nešlehová [11] give a account of how Archimedean copulas coincide with survival copulas of ℓ_1-norm symmetric distributions which have no point-mass at the origin. This particular relationship relies on the characterization of n-monotone functions through an integral transform introduced by Williamson [20], which is analogous to the Laplace transform characterisation of completely monotone functions (see, for example, Widder [19]). Then in [12], McNeil & Nešlehová generalise Archimedean copulas to so-called Liouville copulas, which are defined as the survival copulas of multivariate Liouville distributions.

Brody *et al.* [2] studied an increasing process $\{\xi_{tT}\}_{0 \le t \le T}$ constructed as

$$\xi_{tT} = R\gamma_{tT}, \tag{3}$$

where R is a positive random variable and $\{\gamma_{tT}\}$ is an independent gamma bridge satisfying $\gamma_{0T} = 0$ and $\gamma_{TT} = 1$, for some $T \in (0, \infty)$. Such a process is useful in modelling of cumulative gains or losses. The random variable R is the total, final gain. We can interpret R as a signal and the gamma bridge $\{\gamma_{tT}\}$ as independent multiplicative noise. Brody *et al.* showed that $\{\xi_{tT}\}$ was a Markov process, and that

$$\mathbb{E}[X \mid \xi_{tT} = x] = \frac{\int_x^\infty z^{2-mT}(z-x)^{m(T-t)-1}\,\nu(\mathrm{d}z)}{\int_x^\infty z^{1-mT}(z-x)^{m(T-t)-1}\,\nu(\mathrm{d}z)}, \tag{4}$$

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where ν is the law of R, and $m > 0$ is a parameter. The process $\{\xi_{tT}\}$ can be considered to be a gamma process conditioned to have the marginal law ν at time T, and so belongs to the class of Lévy random bridges (see Hoyle *et al.* [9]). As such, we call a process that can be decomposed as in (3) a 'gamma random bridge' (GRB).

Archimedean survival processes are an n-dimensional extension of gamma random bridges. Let the process $\{(\xi_t^{(1)}, \xi_t^{(2)}, \ldots, \xi_t^{(n)})^\top\}_{0 \le t \le T}$ be an ASP. Then each one-dimensional marginal process $\{\xi_t^{(i)}\}$ is a GRB. Thus we can write

$$\xi_t^{(i)} = X_i \gamma_{tT}^{(i)}, \tag{5}$$

for some gamma bridge $\{\gamma_{tT}^{(i)}\}$ and some independent $X_i > 0$. The X_i's are identical but in general not independent, and the $\{\gamma_{tT}^{(i)}\}$'s are identical but in general not independent. We shall construct each $\{\xi_t^{(i)}\}$ by splitting a 'master' GRB into n non-overlapping subprocesses. This method of splitting a Lévy random bridge into subprocesses (which are themselves Lévy random bridges) was used by Hoyle *et al.* [8] to develop a bivariate insurance reserving model based on random bridges of the stable-1/2 subordinator. A remarkable feature of the proposed construction is that the terminal vector $(\xi_T^{(1)}, \xi_T^{(2)}, \ldots, \xi_T^{(n)})^\top$ has an ℓ_1-norm symmetric distribution, and hence an Archimedean survival copula. In particular, we shall show that

$$\mathbb{P}\left[\bar{F}(\xi_T^{(1)}) > u_1, \bar{F}(\xi_T^{(2)}) > u_2, \ldots, \bar{F}(\xi_T^{(n)}) > u_n\right] = \bar{F}\left(\sum_{i=1}^n \bar{F}^{-1}(u_i)\right), \tag{6}$$

where

$$\bar{F}(u) = \mathbb{P}\left[\xi_T^{(i)} > u\right], \qquad \text{for } i = 1, 2, \ldots, n. \tag{7}$$

Here $\bar{F}(x)$ is the marginal survival function of the $\xi_T^{(i)}$'s, and $\bar{F}^{-1}(u)$ is its (generalised) inverse. The right-hand side of (6) is an Archimedean copula with the generator function $\bar{F}(x)$.

We also construct Liouville processes by splitting a GRB into n pieces. By allowing more flexibility in the splitting mechanism and by employing some deterministic time changes, a broader range of behaviour can be achieved by Liouville processes than ASPs. For example, the one-dimensional marginal processes of Liouville process are in general not identical.

A direct application of ASPs and Liouville processes is to the modelling of multivariate cumulative gain (or loss) processes. Consider, for example, an insurance company that underwrites several lines of motor business (such as personal motor, fleet motor or private-hire vehicles) for a given accident year. A substantial payment made on one line of business is unlikely to coincide with a substantial payment made on another line of business (e.g. a large payment is unlikely to be made on a personal motor claim at the same time as a large payment is made on a fleet motor claim). However, the total sums of claims arising from the lines of business will depend on certain common factors such as prolonged periods of adverse weather or the quality of the underwriting process

at the company. Such common factors will produce dependence across the lines. An ASP or a Liouville process might be a suitable model for the cumulative paid-claims processes of the lines of motor business. The one-dimensional marginal processes of a Liouville process are increasing and do not exhibit simultaneous large jumps, but they can display strong correlation.

ASPs can be used to interpolate the dependence structure when using Archimedean copulas in discrete-time models. Consider a risk model where the marginal distributions of the returns on n assets are fitted for the future dates $t_1 < \cdots < t_n < T < \infty$. An Archimedean copula C is used to model the dependence of the returns to time T. At this stage we have a model for the joint distribution of returns to time T, but we have only the one-dimensional marginal distributions at the intertemporal times $t_1, \ldots, t_n$. The problem then is to choose copulas to complete the joint distributions of the returns to the times $t_1, \ldots, t_n$ in a way that is consistent with the time-T joint distribution. For each time t_i, this can achieved by using the the time-t_i survival copula implied by the Archimedean survival process with survival copula C at time terminal time T.

This paper is organized as follows: In Section 2, we review multivariate ℓ_1-norm symmetric distributions, multivariate Liouville distributions, Archimedean copulas and gamma random bridges. In Section 3, we define ASPs and provide various characterisations of their law. We also detail how to construct a multivariate process such that each one-dimensional marginal is *a priori* uniform. In Section 4, we generalise ASPs to Liouville processes.

2   Preliminaries

This work draws together ideas from mathematical statistics and the theory of stochastic processes. This extended preliminary section gives relevant background results from both of these subjects.

We fix a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and assume that all processes and filtrations under consideration are càdlàg. We let f^{-1} denote the generalised inverse of a monotonic function f, i.e.

$$f^{-1}(y) = \begin{cases} \inf\{x : f(x) \geq y\}, & f \text{ increasing,} \\ \inf\{x : f(x) \leq y\}, & f \text{ decreasing.} \end{cases} \tag{8}$$

We denote the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$ by $\|\mathbf{x}\|$, i.e.

$$\|\mathbf{x}\| = \sum_{i=1}^{n} |x_i|. \tag{9}$$

2.1   Multivariate distributions

In this subsection we present some definitions and results from the theory of multivariate distributions. We refer the reader to the thorough exposition of Fang *et al.* [5] for further details.

2.1.1 Multivariate ℓ_1-norm symmetric distributions

The multivariate ℓ_1-norm symmetric distributions form a family of distributions that are closely related to Archimedean copulas. The definition of the n-dimensional ℓ_1-norm symmetric distribution is in terms of a random variable uniformly distributed on the simplex

$$S = \left\{ \mathbf{u} \in [0,1]^n : \|\mathbf{u}\| = 1 \right\}. \tag{10}$$

Such a random variable $\mathbf{U}$ has the stochastic representation

$$\mathbf{U} \stackrel{\text{law}}{=} \frac{\mathbf{E}}{\|\mathbf{E}\|}, \tag{11}$$

where $\mathbf{E}$ is a vector of n independent, identically-distributed, exponential random variables. Note that this representation holds for any value of the rate parameter $\lambda > 0$ of the exponential random variables, and that the random variable $\|\mathbf{E}\|$ has a gamma distribution with shape parameter n, and scale parameter λ^{-1}. Each marginal variable U_i has a beta distribution with parameters $\alpha = 1$ and $\beta = n - 1$; thus the survival funciton of U_i is

$$\mathbb{P}[U_i > u] = (1 - u)^{n-1}, \tag{12}$$

for $0 \leq u \leq 1$.

Definition 2.1. *A random variable $\mathbf{X}$ taking values in $\mathbb{R}^n$ has a* multivariate ℓ_1-norm symmetric distribution *if*

$$\mathbf{X} \stackrel{\text{law}}{=} R\mathbf{U}, \tag{13}$$

where R is a non-negative random variable, and $\mathbf{U}$ is a random vector uniformly distributed on the simplex S. We say that the law of R is the generating law *of the distribution.*

Remark 2.2. *The construction of multivariate ℓ_1-norm symmetric random variables is similar to the construction of elliptical random variables. To be precise, in (13) if $\mathbf{U}$ was uniformly distributed on the unit sphere in $\mathbb{R}^n$, then $\mathbf{X}$ would have an elliptical distribution.*

Note that if R admits a density, then $\mathbf{X}$ satisfying (13) admits a density, and this density is simplectly contoured. This is analogous to the elliptical contours of elliptical distributions.

If $\mathbf{X}$ is a multivariate ℓ_1-norm symmetric random variable with generating law ν, then the survival function of each one-dimensional marginal of $\mathbf{X}$ is

$$\bar{F}(x) = \mathbb{P}[X_i > x]$$
$$= \int_x^\infty (1 - x/r)^{n-1}\, \nu(\mathrm{d}r), \tag{14}$$

for $x \geq 0$. The survival function $\bar{F}$ determines the law ν. Indeed, using the results of Williamson [20], McNeil & Nešlehová [11] showed that

$$\nu([0, x]) = 1 - \sum_{k=0}^{n-2} \frac{(-1)^k x^k \bar{F}_0^{(k)}(x)}{k!} - \frac{(-1)^{n-1} x^{n-1} \max[0, \bar{F}_0^{(n-1)}(x)]}{(n-1)!}, \qquad (15)$$

where $\bar{F}^{(k)}$ is the kth derivative of $\bar{F}$, and

$$\bar{F}_0(x) = \begin{cases} \bar{F}(x), & x > 0 \\ 1 - \bar{F}(0), & x = 0. \end{cases} \qquad (16)$$

The following theorem provides the multivariate version of (14); the proof can be found in Fang *et al.* [5, Theorem 5.4].

Theorem 2.3. *If* $\mathbf{X}$ *has a multivariate* ℓ_1*-norm symmetric distribution with generating law* ν*, then the joint survival function of* $\mathbf{X}$ *is*

$$\mathbb{P}[X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n] = \int_{\|\mathbf{x}\|}^{\infty} (1 - \|\mathbf{x}\|/r)^{n-1} \, \nu(\mathrm{d}r)$$
$$= \bar{F}(\|\mathbf{x}\|),$$

for $\mathbf{x} \in \mathbb{R}_+^n$.

2.1.2 Multivariate Liouville distributions

The multivariate Liouville distribution is an extension of the multivariate ℓ_1-norm symmetric distribution. Before defining the multivariate Liouville distribution, it is convenient to first define the Dirichlet distribution. The n-dimensional Dirichlet distribution is a distribution on the simplex S defined in (10).

Definition 2.4. *Let* $\mathbf{G}$ *be vector of independent random variables such that* G_i *is a gamma random variable with shape parameter* $\alpha_i > 0$ *and scale parameter unity. Then the random vector*

$$\mathbf{D} = \frac{\mathbf{G}}{\|\mathbf{G}\|}, \qquad (17)$$

has a Dirichlet distribution *with* parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^\top$.

Remark 2.5. *The scaling property of the gamma distribution implies that* $\kappa\mathbf{G}$, $\kappa > 0$, *is a vector of gamma random variables each with scale parameter* κ*. Since (17) holds, if we replace* $\mathbf{G}$ *with* $\kappa\mathbf{G}$*, we could have used an arbitrary positive scale parameter in Definition 2.4.*

In two dimensions, a Dirichlet random variable can be written as $(B, 1 - B)^\top$, where B is a beta random variable. If all the elements of the parameter vector $\boldsymbol{\alpha}$ are identical, then $\mathbf{D}$ is said to have a *symmetric* Dirichlet distribution. Notice that

if $\alpha_i = 1$ for $i = 1, 2, \ldots, n$, then $\mathbf{D}$ is uniformly distributed on the simplex S. The density of $(D_1, D_2, \ldots, D_{n-1})^\top$ is

$$\mathbf{x} \mapsto \frac{\prod_{i=1}^{n} \Gamma[\alpha_i]}{\Gamma[\|\boldsymbol{\alpha}\|]} \prod_{i=1}^{n} x_i^{\alpha_i - 1}, \tag{18}$$

for $\mathbf{x} \in [0, 1]^{n-1}$, $\|\mathbf{x}\| \leq 1$, where $x_n = 1 - \sum_{i=1}^{n-1} x_i$, and $\Gamma[z]$ is the gamma function, defined as usual for $x > 0$ by

$$\Gamma[x] = \int_0^\infty u^{x-1} \mathrm{e}^{-u} \, \mathrm{d}u. \tag{19}$$

The first- and second-order moments of the Dirichlet distribution are given by

$$\mathbb{E}[D_i] = \frac{\alpha_i}{\|\boldsymbol{\alpha}\|}, \tag{20}$$

$$\mathrm{Var}[D_i] = \frac{\alpha_i(\|\boldsymbol{\alpha}\| - \alpha_i)}{\|\boldsymbol{\alpha}\|^2(\|\boldsymbol{\alpha}\| + 1)}, \tag{21}$$

$$\mathrm{Cov}[D_i, D_j] = -\frac{\alpha_i \alpha_j}{\|\boldsymbol{\alpha}\|^2(\|\boldsymbol{\alpha}\| + 1)}, \qquad \text{for } i \neq j. \tag{22}$$

The Dirichlet distribution is an extension of a random variable uniformly distributed on a simplex. The multivariate Liouville distribution is a similar extension of the multivariate ℓ_1-norm symmetric distribution.

Definition 2.6. *A random variable* $\mathbf{X}$ *has a* multivariate Liouville distribution *if*

$$\mathbf{X} \stackrel{\mathrm{law}}{=} R\mathbf{D}, \tag{23}$$

for $R \geq 0$ *a random variable, and* $\mathbf{D}$ *a Dirichlet random variable with parameter vector* $\boldsymbol{\alpha}$. *We call the law of* R *the* generating law, *and* $\boldsymbol{\alpha}$ *the* parameter vector *of the distribution.*

In the case where R has a density p, the density of $\mathbf{X}$ exists and can be written as

$$\mathbf{x} \mapsto \Gamma[\|\boldsymbol{\alpha}\|] \frac{p(\|\mathbf{x}\|)}{(\|\mathbf{x}\|)^{\|\boldsymbol{\alpha}\| - 1}} \prod_{i=1}^{n} \frac{x_i^{\alpha_i - 1}}{\Gamma[\alpha_i]}, \tag{24}$$

for $\mathbf{x} \in \mathbb{R}^n$. Writing $\mu_1 = \mathbb{E}[R]$ and $\mu_2 = \mathbb{E}[R^2]$ (when these moments exist), the first- and second-order moments of $\mathbf{X}$ are given by

$$\mathbb{E}[X_i] = \mu_1 \frac{\alpha_i}{\|\boldsymbol{\alpha}\|}, \tag{25}$$

$$\mathrm{Var}[X_i] = \frac{\alpha_i}{\|\boldsymbol{\alpha}\|} \left(\mu_2 \frac{\alpha_i + 1}{\|\boldsymbol{\alpha}\| + 1} - \mu_1^2 \frac{\alpha_i}{\|\boldsymbol{\alpha}\|} \right), \tag{26}$$

$$\mathrm{Cov}[X_i, X_j] = \frac{\alpha_i \alpha_j}{\|\boldsymbol{\alpha}\|} \left(\frac{\mu_2}{\|\boldsymbol{\alpha}\| + 1} - \frac{\mu_1^2}{\|\boldsymbol{\alpha}\|} \right), \qquad \text{for } i \neq j. \tag{27}$$

2.2 Archimedean copulas

A copula is a distribution function on the unit hypercube with the added property that each one-dimensional marginal distribution is uniform. For further details, we refer to Nelsen [15]. We define a copula as follows:

Definition 2.7. *An n-copula defined on the n-dimensional unit hypercube $[0,1]^n$ is a function $C : [0,1]^n \to [0,1]$, which satisfies the following:*

1. $C(\boldsymbol{u}) = 0$ whenever $u_j = 0$ for at least one $j = 1, 2, .., n$.

2. $C(\boldsymbol{u}) = u_j$ if $u_i = 1$ for all $i \neq j$.

3. C is n-increasing on $[0,1]^n$, that is

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1,i_1}, \ldots, u_{n,i_n}) \geq 0, \tag{28}$$

for all $(u_{1,1}, u_{2,1}, \ldots, u_{n,1})^\top$ and $(u_{1,2}, u_{2,2}, \ldots, u_{n,2})^\top$ in $[0,1]^n$ with $u_{j,1} \leq u_{j,2}$.

In the definition above, condition 3 is necessary to ensure that the function C is a well-defined distribution function. The theory of copulas is founded upon a theorem of Sklar. This theorem was reformulated in terms of survival functions by McNeil & Nešlehová [11] as follows:

Theorem 2.8. *Let $\bar{H}$ be an n-dimensional survival function with margins $\bar{F}_i$, $i = 1, 2, \ldots, n$. Then there exists a copula C, referred to as the* survival copula *of $\bar{H}$, such that, for any $\mathbf{x} \in \mathbb{R}^n$,*

$$\bar{H}(\mathbf{x}) = C(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)). \tag{29}$$

Furthermore, C is uniquely determined on

$$D = \left\{ \mathbf{u} \in [0,1]^n : u \in \mathrm{ran}\bar{F}_1 \times \cdots \times \mathrm{ran}\bar{F}_n \right\},$$

where $\mathrm{ran}f$ denotes the range of f. In addition, for any $\mathbf{u} \in D$,

$$C(\mathbf{u}) = \bar{H}(\bar{F}_1^{-1}(u_1), \ldots, \bar{F}_n^{-1}(u_n)).$$

Conversely, given a copula C and univariate survival functions $\bar{F}_i$, $i = 1, \ldots, n$, $\bar{H}$ defined by (29) is an n-dimensional survival function with marginals $F_1, \ldots, F_n$ and survival copula C.

From a modelling perspective, one of the attractive features of copulas is that they allow the fitting of one-dimensional marginal distributions to be performed separately from the fitting of cross-sectional dependence. Although, this two-step approach of modelling multivariate phenomena by first specifying marginals, and then choosing a copula is not suited to all situations (for criticism see, for example, Mikosch [14]).

Archimedean copulas are copulas that take a particular functional form. The following definition given in [11] is convenient for the present work:

Definition 2.9. *A decreasing and continuous function* $h : [0, \infty) \to [0, 1]$ *which satisfies the conditions* $h(0) = 1$ *and* $\lim_{x \to \infty} h(x) = 0$, *and is strictly decreasing on* $[0, \inf\{x : h(x) = 0\}]$ *is called an* Archimedean generator. *An n-dimensional copula* C *is called an* Archimedean copula *if it permits the representation*

$$C(\mathbf{u}) = h(h^{-1}(u_1) + \cdots + h^{-1}(u_n)), \qquad \mathbf{u} \in [0, 1]^n,$$

for some Archimedean generator h *with inverse* $h^{-1} : [0, 1] \to [0, \infty)$, *where we set* $h(\infty) = 0$ *and* $h^{-1}(0) = \inf\{u : h(u) = 0\}$.

If $\mathbf{X}$ is a random vector with a multivariate ℓ_1-norm symmetric distribution such that $\mathbb{P}[\mathbf{X} = \mathbf{0}] = 0$, then its marginal survival function $\bar{F}$ given in (14) is continuous. Hence it follows from Theorem 2.3 that

$$\mathbb{P}[\bar{F}(X_1) > u_1, \bar{F}(X_2) > u_2, \ldots, \bar{F}(X_n) > u_n] = \bar{F}\left(\sum_{i=1}^n \bar{F}^{-1}(u_i)\right). \qquad (30)$$

In other words, $\mathbf{X}$ has an Archimedean survival copula with generating function $h(x) = \bar{F}(x)$. McNeil & Nešlehová [11] showed that the converse is also true:

Theorem 2.10. *Let* $\mathbf{U}$ *be a random vector whose distribution function is an n-dimensional Archimedean copula* C *with generator* h. *Then* $(h^{-1}(U_1), h^{-1}(U_2), \ldots, h^{-1}(U_n))^\top$ *has a multivariate* ℓ_1-*norm distribution with survival copula* C *and generating law* ν. *Furthermore,* ν *is uniquely determined by*

$$\nu([0, x]) = 1 - \sum_{k=0}^{n-2} \frac{(-1)^k x^k h^{(k)}(x)}{k!} - \frac{(-1)^{n-1} x^{n-1} \max[0, h^{(n-1)}(x)]}{(n-1)!}.$$

Remark 2.11. *There is one-to-one mapping from distribution functions on the positive half-line to the class of n-dimensional Archimedean copulas through the invertible transformation* $\nu \leftrightarrow h$.

2.3 Gamma random bridges

A gamma random bridge is an increasing stochastic process, and both the gamma process and gamma bridge are special cases.

2.3.1 Gamma process

A gamma process is a subordinator (an increasing Lévy process) with gamma distributed increments (see, for example, Sato [17]). The law of a gamma process is uniquely determined by its mean and variance at time 1, which are both positive. Let $\{\gamma_t\}$ be a gamma process with mean and variance $m > 0$ at time 1; then

$$\mathbb{E}[\gamma_t] = mt, \quad \text{and} \quad \text{Var}[\gamma_t] = mt. \qquad (31)$$

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The density of γ_t is

$$f_t(x) = \mathbb{1}_{\{x>0\}} \frac{x^{mt-1}}{\Gamma[mt]} e^{-x}. \tag{32}$$

Due to the scaling property of the gamma distribution, if $\kappa > 0$ then the process $\{\kappa\gamma_t\}$ is a gamma process with mean $m\kappa$, and variance $m\kappa^2$ at $t = 1$. The characteristic function of γ_t is

$$\mathbb{E}[e^{i\lambda\gamma_t}] = (1 - i\lambda)^{-mt}, \qquad \text{for } \lambda \in \mathbb{C}. \tag{33}$$

As noted in Brody *et al.* [2], the parameter m has units of inverse time, and so $\{\gamma_t\}$ is dimensionless. Taking $\kappa = 1/m$, the scaled process $\{\kappa\gamma_t\}$ has units of time, making this alternative parameterisation suitable as a basis for a stochastic time change (see, for example, Madan & Seneta [10]). The characteristic function of $\kappa\gamma_t$ is then

$$\mathbb{E}[e^{i\lambda\kappa\gamma_t}] = (1 - i\lambda/m)^{-mt}. \tag{34}$$

In the limit $m \to \infty$ this characteristic function is $e^{i\lambda t}$, which is the characteristic function of the Dirac measure centred at t. It follows that $\{\kappa\gamma_t\} \xrightarrow{\text{law}} \{t\}$ as $m \to \infty$.

2.3.2 Gamma bridge

A gamma bridge is a gamma process conditioned to have a fixed value at a fixed future time. A gamma bridge is a Lévy bridge, and hence a Markov process (see, for example, Hoyle [7]). Gamma bridges exhibit a number of remarkable similarities to Brownian bridges, some of which have been presented by Émery & Yor [4]. Let $\{\gamma_{tT}\}_{0 \leq t \leq T}$ be a gamma bridge identical in law to the gamma process $\{\gamma_t\}$ pinned to the value 1 at time T. Using the Bayes theorem, the transition law of $\{\gamma_{tT}\}$ is given by

$$\begin{aligned}
\mathbb{P}[\gamma_{tT} \in dy \,|\, \gamma_{sT} = x] &= \mathbb{P}[\gamma_t \in dy \,|\, \gamma_s = x, \gamma_T = 1] \\
&= \frac{f_{t-s}(y-x)f_{T-t}(1-y)}{f_{T-s}(1-x)} \\
&= \mathbb{1}_{\{x<y<1\}} \frac{\left(\frac{y-x}{1-x}\right)^{m(t-s)-1}\left(\frac{1-y}{1-x}\right)^{m(T-t)-1}}{(1-x)\mathrm{B}[m(t-s), m(T-t)]} \, dy,
\end{aligned} \tag{35}$$

for $0 \leq s < t \leq T$ and $x \geq 0$. Here $\mathrm{B}[\alpha, \beta]$ is the beta function, defined for $\alpha, \beta > 0$ by

$$\mathrm{B}[\alpha, \beta] = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, dx = \frac{\Gamma[\alpha]\Gamma[\beta]}{\Gamma[\alpha+\beta]}. \tag{36}$$

We say that m is the *activity parameter* of $\{\gamma_{tT}\}$. If the gamma bridge $\{\gamma_{tT}\}$ has reached the value x at time s, then it must yet travel a distance $1 - x$ over the time period $(s, T]$. Equation (35) shows that the proportion of this distance that the gamma bridge will cover over $(s, t]$ is a random variable with a beta distribution (with parameters $\alpha = m(t-s)$ and $\beta = m(T-t)$). The conditional characteristic function of γ_{tT} is

$$\mathbb{E}\left[e^{i\lambda\gamma_{tT}} \,\middle|\, \gamma_{sT} = x\right] = M[m(t-s), m(T-s), i(1-x)\lambda], \tag{37}$$

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where $M[\alpha, \beta, z]$ is Kummer's confluent hypergeometric function of the first kind, which can be expanded as the power series [1, 13.1.2]

$$M[\alpha, \beta, z] = 1 + \frac{\alpha}{\beta} z + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \frac{z^3}{3!} + \cdots. \tag{38}$$

Taking the limit as $m \to \infty$ in (37), we have

$$\mathbb{E}\left[e^{i\lambda\gamma_{tT}} \,\middle|\, \gamma_{sT} = x \right] \to \sum_{k=0}^{\infty} \left(\frac{t-s}{T-s} \right)^k \frac{(i(1-x)\lambda)^k}{k!}$$

$$= \exp\left(i\frac{t-s}{T-s}(1-x)\lambda \right), \tag{39}$$

which is the characteristic function of the Dirac measure centered at $(1-x)(t-s)/(T-s)$. It then follows from the Markov property of gamma bridges that $\{\gamma_{tT}\} \xrightarrow{\text{law}} \{t/T\}$ as $m \to \infty$.

It is a property of gamma processes that the renormalised process $\{\gamma_t/\gamma_T\}_{0 \le t \le T}$ is independent of γ_T (indeed, this independence property characterises the gamma process among Lévy processes). This leads to the remarkable identity

$$\left\{ \frac{\gamma_t}{\gamma_T} \right\} \overset{\text{law}}{=} \{\gamma_{tT}\}. \tag{40}$$

The identity (40) can be proved by showing that the process on the left-hand side is Markov, and then verifying that its transition law is the same as (35). This can be done using the results in Brody *et al.* [2]. We note three properties of gamma bridges that follow from (40). The first is that the bridge of the scaled gamma process $\{\kappa\gamma_t\}$ is, for any $\kappa > 0$, identical in law to the bridge of the unscaled process. The second property is that a $\{\gamma_t\}$-bridge to the value $z > 0$ at time T is identical in law to the process $\{z\gamma_{tT}\}$. The third is that the joint distribution of increments of a gamma bridge is Dirichlet. To see this last fact, fix times $0 = t_0 < t_1 < \cdots < t_n = T$ and define

$$\bar{\Delta}_i = \gamma_{t_i} - \gamma_{t_{i-1}}, \tag{41}$$

$$\Delta_i = \gamma_{t_i,T} - \gamma_{t_{i-1},T}. \tag{42}$$

Then $\bar{\Delta}_i$ has a gamma distribution with shape parameter $\alpha_i = m(t_i - t_{i-1})$ and scale parameter unity. Hence

$$(\Delta_1, \Delta_2, \ldots, \Delta_n) \overset{\text{law}}{=} \frac{(\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_n)}{\|(\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_n)\|}. \tag{43}$$

The fact then follows from Definition 2.4.

2.3.3 Gamma random bridge

The gamma random bridge was introduced as a model for cumulative gains processes by Brody *et al.* [2]. We define a gamma random bridge as follows:

Definition 2.12. *The process* $\{\Gamma_t\}_{0 \leq t \leq T}$ *is a* gamma random bridge *if*

$$\{\Gamma_t\} \stackrel{\text{law}}{=} \{R\gamma_{tT}\}, \tag{44}$$

for $R > 0$ *a random variable, and* $\{\gamma_{tT}\}$ *a gamma bridge. We say that* $\{\Gamma_t\}$ *has* generating law ν *and* activity parameter m, *where of* ν *is the law of* R *and* m *is the activity parameter of* $\{\gamma_{tT}\}$.

Remark 2.13. *Suppose that* $\{\Gamma_t\}$ *is a GRB satisfying (44). If* $\mathbb{P}[R = z] = 1$ *for some* $z > 0$, *then* $\{\Gamma_t\}$ *is a gamma bridge. If* R *is gamma random variable with shape parameter* mT *and scale parameter* κ, *then* $\{\Gamma_t\}$ *is a gamma process such that* $\mathbb{E}[\Gamma_t] = m\kappa t$ *and* $\mathrm{Var}[\Gamma_t] = m\kappa^2 t$, *for* $t < T$.

Gamma random bridges (GRBs) fall within the class of Lévy random bridges described by Hoyle *et al.* [9]. The process $\{\Gamma_t\}$ is identical in law to a gamma process defined over $[0, T]$, and conditioned to have the law of R at time T. The bridges of a GRB are gamma bridges. GRBs are Markov processes, and the transition law of $\{\Gamma_t\}$ is given by (see [9])

$$\mathbb{P}[\Gamma_t \in \mathrm{d}y \,|\, \Gamma_s = x]$$
$$= \frac{\mathbb{1}_{\{y > x\}}}{\mathrm{B}[m(T-t), m(t-s)]} \frac{\int_y^\infty (z-y)^{m(T-t)-1} z^{1-mT} \,\nu(\mathrm{d}z)}{\int_x^\infty (z-x)^{m(T-s)-1} z^{1-mT} \,\nu(\mathrm{d}z)} (y-x)^{m(t-s)-1} \,\mathrm{d}y, \tag{45}$$

and

$$\mathbb{P}[\Gamma_T \in \mathrm{d}y \,|\, \Gamma_s = x] = \frac{\mathbb{1}_{\{y > x\}} (y-x)^{m(T-s)-1} y^{1-mT} \,\nu(\mathrm{d}y)}{\int_x^\infty (z-x)^{m(T-s)-1} z^{1-mT} \,\nu(\mathrm{d}z)}, \tag{46}$$

where $\mathrm{B}[\alpha, \beta]$ is the Beta function. Since increments of a gamma bridge have a Dirichlet distribution, it follows from Definition 2.6 that the increments of a gamma random bridge have a multivariate Liouville distribution. The following lemma, stated as a corollary in [9] for a general Lévy random bridge, is a key result for the construction of Archimedean survival processes:

Lemma 2.14. *Let* $\{\Gamma_t\}$ *be a GRB with terminal law* ν *and activity parameter* m.

(A) *Fix times* s_1, T_1 *satisfying* $0 < T_1 \leq T - s_1$. *The time-shifted, space-shifted partial process*
$$\xi_t^{(1)} = \Gamma_{s_1+t} - \Gamma_{s_1}, \qquad (0 \leq t \leq T_1),$$
is a gamma random bridge with activity parameter m, *and with generating law*

$$\nu^{(1)}(\mathrm{d}x) = \frac{x^{mT_1-1}}{\mathrm{B}[mT_1, m(T-T_1)]} \int_{z=x}^\infty z^{mT-1}(z-x)^{m(T-T_1)-1} \nu(\mathrm{d}z)\,\mathrm{d}x.$$

(**B**) *Construct partial processes $\{\xi_t^{(i)}\}_{0 \leq t \leq T_i}$, $i = 1, \ldots, n$, from non-overlapping portions of $\{\Gamma_t\}$, in a similar way to that above. The intervals $[s_i, s_i + T_i]$, $i = 1, \ldots, n$, are non-overlapping except possibly at the endpoints. Set $\xi_t^{(i)} = \xi_{T_i}^{(i)}$ when $t > T_i$. If $u > t$, then*

$$\mathbb{P}\left[\xi_u^{(1)} - \xi_t^{(1)} \leq x_1, \ldots, \xi_u^{(n)} - \xi_t^{(n)} \leq x_n \,\middle|\, \mathcal{F}_t\right] =$$
$$\mathbb{P}\left[\xi_u^{(1)} - \xi_t^{(1)} \leq x_1, \ldots, \xi_u^{(n)} - \xi_t^{(n)} \leq x_n \,\middle|\, \sum_{i=1}^n \xi_t^{(i)}\right],$$

where the filtration $\{\mathcal{F}_t\}$ is is given by

$$\mathcal{F}_t = \sigma\left(\{\xi_s^{(i)}\}_{0 \leq s \leq t}, i = 1, 2, \ldots, n\right).$$

Remark 2.15. *Define the process $\{R_t\}$ by*

$$R_t = \sum_{i=1}^n \xi_t^{(i)}, \tag{47}$$

for $t \in [0, \max_i T_i]$. Then $\{R_t\}$ is a GRB with terminal law ν, and time-dependent activity parameter

$$M(t) = m \sum_{i=1}^n \mathbb{1}_{\{t \leq T_i\}}. \tag{48}$$

The proof of this result is similar to the proof of the special case that appears later in Proposition 3.6.

We can construct an n-dimensional Markov process $\{\boldsymbol{\xi}_t\}$ from the partial processes of Lemma 2.14, part (B), by setting

$$\boldsymbol{\xi}_t = (\xi_t^{(1)}, \ldots, \xi_t^{(n)})^\top. \tag{49}$$

The Markov property means that, for any fixed time $s \geq 0$, the $\mathcal{F}_s$-conditional law of $\{\boldsymbol{\xi}_t\}_{s \leq t}$ is identical to the $\boldsymbol{\xi}_s$-conditional law of $\{\boldsymbol{\xi}_t\}_{s \leq t}$. The remarkable feature of Lemma 2.14, part (B), is that the $\mathcal{F}_s$-conditional law of $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \leq t}$ is identical to the R_s-conditional law of $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \leq t}$. Hence the increment probabilities of the n-dimensional process $\{\boldsymbol{\xi}_t\}$ can be described by the one-dimensional state process $\{R_t\}$.

3   Archimedean survival process

We construct an Archimedean survival process (ASP) by splitting a gamma random bridge into n non-overlapping subprocesses. We start with a 'master' GRB $\{\Gamma_t\}_{0 \leq t \leq n}$ with activity parameter $m = 1$ and generating law ν, where $n \in \mathbb{N}_+$, $n \geq 2$. In

this section, we write f_t for the gamma density with shape parameter unity and scale parameter unity (in (32) we set $m = 1$). That is

$$f_t(x) = \frac{x^{t-1} \mathrm{e}^{-x}}{\Gamma[t]}. \tag{50}$$

Definition 3.1. *The process $\{\boldsymbol{\xi}_t\}_{0 \leq t \leq 1}$ is an n-dimensional Archimedean survival process if*

$$\{\boldsymbol{\xi}_t\}_{0 \leq t \leq 1} \stackrel{\mathrm{law}}{=} \left\{ \begin{bmatrix} \Gamma_t - \Gamma_0 \\ \vdots \\ \Gamma_{(i-1)+t} - \Gamma_{i-1} \\ \vdots \\ \Gamma_{(n-1)+t} - \Gamma_{n-1} \end{bmatrix} \right\}_{0 \leq t \leq 1}$$

for $\{\Gamma_t\}_{0 \leq t \leq n}$ is a gamma random bridge with activity parameter $m = 1$. We say that the generating law of $\{\Gamma_t\}$ is the generating law of $\{\boldsymbol{\xi}_t\}$.

Note that, from Definition 2.12, $\mathbb{P}[\Gamma_n = 0] = 0$, and so $\mathbb{P}[\boldsymbol{\xi}_t = \mathbf{0}] = 0$. Each one-dimensional marginal process of an ASP is a subprocess of a GRB, and hence a GRB. Hence ASPs are a multivariate generalisation of GRBs. We defined ASPs over the time interval $[0, 1]$; it is straightforward to restate the definition to cover an arbitrary closed interval.

Proposition 3.2. *The terminal value of an Archimedean survival process has an Archimedean survival copula.*

Proof. Let $\{\boldsymbol{\xi}_t\}$ be an n-dimensional ASP with generating law ν. Then we have

$$\mathbb{P}[\boldsymbol{\xi}_1 \in \mathrm{d}\mathbf{x}] = \mathbb{P}\left[\Gamma_1 \in \mathrm{d}x_1, \Gamma_2 - \Gamma_1 \in \mathrm{d}x_2, \ldots, \Gamma_n - \Gamma_{n-1} \in \mathrm{d}x_n\right]$$

$$= \mathbb{P}\left[R\frac{\gamma_1}{\gamma_n} \in \mathrm{d}x_1, R\frac{\gamma_2 - \gamma_1}{\gamma_n} \in \mathrm{d}x_2, \ldots, R\frac{\gamma_n - \gamma_{n-1}}{\gamma_n} \in \mathrm{d}x_n\right], \tag{51}$$

for $\mathbf{x} \in \mathbb{R}^n$, R a random variable with law ν, and $\{\gamma_t\}$ a gamma process such that γ_t has the density (50). Each increment $\gamma_i - \gamma_{i-1}$ has an exponential distribution (with unit rate). Thus

$$\mathbb{P}[\boldsymbol{\xi}_1 \in \mathrm{d}\mathbf{x}] = \mathbb{P}\left[R\frac{\mathbf{E}}{\|\mathbf{E}\|} \in \mathrm{d}\mathbf{x}\right], \tag{52}$$

for $\mathbf{E}$ an n-vector of independent, identically-distributed, exponential random variables. From Definition 2.1, $\boldsymbol{\xi}_1$ has a multivariate ℓ_1-norm symmetric distribution. Therefore, it has an Archimedean survival copula. $\square$

Remark 3.3. *Let $g_i : \mathbb{R}_+ \to \mathbb{R}$ be strictly decreasing for $i = 1, \ldots, n$, and let $\{\boldsymbol{\xi}_t\}$ be an ASP. Then, a priori, the vector-valued process*

$$\left\{ \left(g_1(\xi_t^{(1)}), \ldots, g_i(\xi_t^{(i)}), \ldots, g_n(\xi_t^{(n)}) \right)^\top \right\}_{0 \leq t \leq 1}$$

has an Archimedean copula at time $t = 1$.

3.1 Characterisations

In this subsection we shall characterize ASPs first through their finite-dimensional distributions, and then through their transition probabilities.

3.1.1 Finite-dimensional distributions

The finite-dimensional distributions of the master process $\{\Gamma_t\}$ are given by

$$\mathbb{P}[\Gamma_{t_1} \in \mathrm{d}x_1, \ldots, \Gamma_{t_k} \in \mathrm{d}x_k, \Gamma_n \in \mathrm{d}z] = \mathbb{P}[\Gamma_{t_1} \in \mathrm{d}x_1, \ldots, \Gamma_{t_k} \in \mathrm{d}x_k \mid \Gamma_n = z]\,\nu(\mathrm{d}z), \tag{53}$$

where $x_0 = 0$, for all $k \in \mathbb{N}_+$, all partitions $0 = t_0 < t_1 < \cdots < t_k < n$, all $z \in \mathbb{R}_+$, and all $(x_1, \ldots, x_k)^\top = \mathbf{x} \in \mathbb{R}_+^k$. It was mentioned earlier that the bridges of a GRB are gamma bridges. (In fact, this is the basis of the definition of Lévy random bridges given in Hoyle *et al.* [9].) Hence, for $\{\gamma_t\}$ a gamma process such that $\mathbb{E}[\gamma_1] = 1$ and $\mathrm{Var}[\gamma_1] = 1$, we have

$$\begin{aligned} \mathbb{P}[\Gamma_{t_1} \in \mathrm{d}x_1, \ldots, &\Gamma_{t_k} \in \mathrm{d}x_k, \Gamma_n \in \mathrm{d}z] \\ &= \mathbb{P}[\gamma_{t_1} \in \mathrm{d}x_1, \ldots, \gamma_{t_k} \in \mathrm{d}x_k \mid \gamma_n = z]\,\nu(\mathrm{d}z). \end{aligned} \tag{54}$$

From (40) and (44), we have

$$(\Gamma_{t_1} - \Gamma_{t_0}, \ldots, \Gamma_{t_k} - \Gamma_{t_{k-1}}, \Gamma_n - \Gamma_{t_k}) \overset{\text{law}}{=} \frac{R}{\gamma_n}(\gamma_{t_1} - \gamma_{t_0}, \ldots, \gamma_{t_k} - \gamma_{t_{k-1}}, \gamma_n - \gamma_{t_k}). \tag{55}$$

Hence, from Definition 2.6, $(\Gamma_{t_1} - \Gamma_{t_0}, \ldots, \Gamma_{t_k} - \Gamma_{t_{k-1}}, \Gamma_n - \Gamma_{t_k})^\top$ has a multivariate Liouville distribution with generating law ν and parameter vector $(t_1 - t_0, \ldots, t_k - t_{k-1}, n - t_k)^\top$.

We can use these results to characterise the law of the ASP $\{\boldsymbol{\xi}_t\}$ through the joint distribution of its increments. Fix $k_i \geq 1$ and the partitions

$$0 = t_0^i < t_1^i < \cdots < t_{k_i}^i = 1, \tag{56}$$

for $i = 1, \ldots, n$. Then define the non-overlapping increments $\{\Delta_{ij}\}$ by

$$\Delta_{ij} = \xi_{t_j^i}^{(i)} - \xi_{t_{j-1}^i}^{(i)}, \tag{57}$$

for $j = 1, \ldots, k_i$ and $i = 1, \ldots, n$. The distribution of the vector

$$\begin{aligned} \boldsymbol{\Delta} = (\Delta_{11}, \Delta_{12}, &\ldots, \Delta_{1k_1}, \\ &\Delta_{21}, \Delta_{22}, \ldots, \Delta_{2k_2}, \\ &\vdots \\ &\Delta_{n1}, \Delta_{n2}, \ldots, \Delta_{nk_n})^\top \end{aligned} \tag{58}$$

characterises the finite-dimensional distributions of the ASP $\{\boldsymbol{\xi}_t\}$. Thus it follows from the Kolmogorov extension theorem that the distribution of $\boldsymbol{\Delta}$ characterises the law of

$\{\boldsymbol{\xi}_t\}$. Note that $\boldsymbol{\Delta}$ contains non-overlapping increments of the master GRB $\{\Gamma_t\}$ such that $\|\boldsymbol{\Delta}\| = \Gamma_n$. Hence $\boldsymbol{\Delta}$ has a multivariate Liouville distribution with parameter vector

$$\begin{aligned}
\boldsymbol{\alpha} = (&t_1^1 - t_0^1, t_2^1 - t_1^1, \ldots, t_{k_1}^1 - t_{k_1-1}^1, \\
&t_1^2 - t_0^2, t_2^2 - t_1^2, \ldots, t_{k_2}^2 - t_{k_2-1}^2, \\
&\vdots \\
&t_1^n - t_0^n, t_2^n - t_1^n, \ldots, t_{k_n}^n - t_{k_n-1}^n)^\top,
\end{aligned} \tag{59}$$

and the generating law ν.

3.1.2 Transition law

We denote the filtration generated by $\{\boldsymbol{\xi}_t\}_{0 \le t \le 1}$ by $\{\mathcal{F}_t\}$. From Lemma 2.14, $\{\boldsymbol{\xi}_t\}$ is a Markov process with respect to $\{\mathcal{F}_t\}$. We shall calculate the transition probabilities of $\{\boldsymbol{\xi}_t\}$ after introducing some further notation.

For a set $B \subset \mathbb{R}$ and a constant $x \in \mathbb{R}$, we write $B + x$ for the shifted set

$$B + x = \{y \in \mathbb{R} : y - x \in B\}. \tag{60}$$

In what follows, we assume that $\{\boldsymbol{\xi}_t\}$ is an n-dimensional ASP with generating law ν, and that $\{\Gamma_t\}$ is a master process of $\{\boldsymbol{\xi}_t\}$. We define the process $\{R_t\}_{0 \le t \le 1}$ by setting

$$R_t = \sum_{i=1}^n \xi_t^{(i)} = \|\boldsymbol{\xi}_t\|. \tag{61}$$

Note that the terminal value of $\{R_t\}$ is the terminal value of the master process $\{\Gamma_t\}$, i.e. $R_1 = \Gamma_n$. We define a family of unnormalised measures, indexed by $t \in [0, 1)$ and $x \in \mathbb{R}_+$, as follows:

$$\psi_0(B; x) = \nu(B), \tag{62}$$

$$\begin{aligned}
\psi_t(B; x) &= \int_B \frac{f_{n(1-t)}(z - x)}{f_n(z)} \nu(\mathrm{d}z) \\
&= \frac{\Gamma[n]\mathrm{e}^x}{\Gamma[n(1-t)]} \int_B \mathbb{1}_{\{z>x\}} z^{1-n} (z-x)^{n(1-t)-1} \nu(\mathrm{d}z)
\end{aligned} \tag{63}$$

for $B \in \mathcal{B}(\mathbb{R})$. We also write

$$\Psi_t(x) = \psi_t([0, \infty); x). \tag{64}$$

It follows from (54) and the independent increments of gamma processes that

$$\begin{aligned}
\mathbb{P}[\Gamma_{t_1} \in \mathrm{d}x_1, \ldots, \Gamma_{t_k} \in \mathrm{d}x_k, \Gamma_n \in \mathrm{d}z] &= \prod_{i=1}^k [f_{t_i - t_{i-1}}(x_i - x_{i-1}) \, \mathrm{d}x_i] \frac{f_{n-t_k}(z - x_n)}{f_n(z)} \nu(\mathrm{d}z) \\
&= \prod_{i=1}^k [f_{t_i - t_{i-1}}(x_i - x_{i-1}) \, \mathrm{d}x_i] \psi_{t_k/n}(\mathrm{d}z; x_n).
\end{aligned} \tag{65}$$

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Proposition 3.4. *The ASP $\{\boldsymbol{\xi}_t\}$ is a Markov process with the transition law given by*

$$\mathbb{P}\left[\xi_1^{(1)} \in dz_1, \dots, \xi_1^{(n-1)} \in dz_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right] =$$

$$\frac{\psi_{\tau(s)}(B + \sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n-1} \frac{(z_i - x_i)^{-s} e^{-(z_i - x_i)}}{\Gamma[1-s]} \, dz_i, \quad (66)$$

and

$$\mathbb{P}[\boldsymbol{\xi}_t \in d\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}] = \frac{\Psi_t(\|\mathbf{y}\|)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n} \frac{(y_i - x_i)^{(t-s)-1} e^{-(y_i - x_i)}}{\Gamma[t-s]} \, dy_i, \quad (67)$$

where $\tau(t) = 1 - (1-t)/n$, $0 \le s < t < 1$, and $B \in \mathcal{B}(\mathbb{R})$.

Proof. We begin by verifying (66). From the Bayes theorem we have

$$\mathbb{P}\left[\xi_1^{(1)} \in dz_1, \dots, \xi_1^{(n-1)} \in dz_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right] =$$

$$= \frac{\mathbb{P}\left[\xi_1^{(1)} \in dz_1, \dots, \xi_1^{(n-1)} \in dz_{n-1}, \|\boldsymbol{\xi}_1\| \in B + \sum_{i=1}^{n-1} z_i, \boldsymbol{\xi}_s \in d\mathbf{x}\right]}{\mathbb{P}[\boldsymbol{\xi}_s \in d\mathbf{x}]}. \quad (68)$$

The *a priori* law of $R_1 = \|\boldsymbol{\xi}_1\|$ is ν; hence using (65) the numerator of (68) is

$$\int_{u \in B + \sum_{i=1}^{n-1} z_i} \mathbb{P}\left[\xi_1^{(1)} \in dz_1, \dots, \xi_1^{(n-1)} \in dz_{n-1}, \boldsymbol{\xi}_s \in d\mathbf{x} \,\middle|\, R_1 = u\right] \nu(du) =$$

$$\prod_{i=1}^{n}[f_s(x_i) \, dx_i] \prod_{i=1}^{n-1}[f_{1-s}(z_i - x_i) \, dz_i] \int_{u \in B + \sum_{i=1}^{n-1} z_i} \frac{f_{1-s}(u - \sum_{i=1}^{n-1} z_i)}{f_n(u)} \nu(du), \quad (69)$$

and the denominator is

$$\int_{u=0}^{\infty} \mathbb{P}[\boldsymbol{\xi}_s \in d\mathbf{x} \,|\, R_1 = u] \nu(du) = \prod_{i=1}^{n}[f_s(x_i) \, dx_i] \int_{u=0}^{\infty} \frac{f_{n(1-s)}(u - \|\mathbf{x}\|)}{f_n(u)} \nu(du). \quad (70)$$

In equations (69) and (70) we have used the fact that, given $\|\boldsymbol{\xi}_1\| = R_1$, $\{\boldsymbol{\xi}_t\}$ is a vector of subprocesses of a gamma bridge. Dividing (69) by (70) yields

$$\frac{\int_{u \in B + \sum_{i=1}^{n-1} z_i} \frac{1}{f_n(u)} f_{1-s}(u - \sum_{i=1}^{n-1} z_i) \nu(du)}{\int_{u=0}^{\infty} \frac{1}{f_n(u)} f_{n(1-s)}(u - \|\mathbf{x}\|) \nu(du)} \prod_{i=1}^{n-1}[f_{1-s}(z_i - x_i) \, dz_i] =$$

$$\frac{\psi_{\tau(s)}(B + \sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\psi_s([0,\infty); \|\mathbf{x}\|)} \prod_{i=1}^{n-1} \frac{(z_i - x_i)^{-s} e^{-(z_i - x_i)}}{\Gamma[1-s]} \, dz_i, \quad (71)$$

as required.

We shall now verify (67) following similar steps. From the Bayes theorem we have

$$\mathbb{P}[\boldsymbol{\xi}_t \in d\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}] = \frac{\mathbb{P}[\boldsymbol{\xi}_t \in d\mathbf{y}, \boldsymbol{\xi}_s \in d\mathbf{x}]}{\mathbb{P}[\boldsymbol{\xi}_s \in d\mathbf{x}]}. \quad (72)$$

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The numerator of (72) is

$$\int_{z=0}^{\infty} \mathbb{P}\left[\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y}, \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \mid R_1 = z\right] \nu(\mathrm{d}z) =$$

$$\prod_{i=1}^{n} [f_s(x_i) \, \mathrm{d}x_i] \prod_{i=1}^{n} [f_{t-s}(y_i - x_i) \, \mathrm{d}y_i] \int_{z=0}^{\infty} \frac{f_{n(1-t)}(z - \|\mathbf{y}\|)}{f_n(z)} \nu(\mathrm{d}z), \quad (73)$$

and the denominator is given in (70). Dividing (73) by (70) yields

$$\frac{\int_{z=0}^{\infty} \frac{1}{f_n(z)} f_{n(1-t)}(z - \|\mathbf{y}\|) \, \nu(\mathrm{d}z)}{\int_{z=0}^{\infty} \frac{1}{f_n(z)} f_{n(1-s)}(z - \|\mathbf{x}\|) \, \nu(\mathrm{d}z)} \prod_{i=1}^{n} [f_{t-s}(y_i - x_i) \, \mathrm{d}y_i] =$$

$$\frac{\psi_t([0, \infty); \|\mathbf{y}\|)}{\psi_s([0, \infty); \|\mathbf{x}\|)} \prod_{i=1}^{n} \frac{(y_i - x_i)^{(t-s)-1} \mathrm{e}^{-(y_i - x_i)}}{\Gamma[t - s]} \, \mathrm{d}y_i, \quad (74)$$

which completes the proof. $\qquad\square$

Remark 3.5. *When the generating law ν admits a density p, (68) is equivalent to*

$$\mathbb{P}\left[\boldsymbol{\xi}_1 \in \mathrm{d}\mathbf{z} \mid \boldsymbol{\xi}_s = \mathbf{x}\right] = \frac{\Gamma[n]\mathrm{e}^{\|\mathbf{x}\|} p(\|\mathbf{z}\|)}{\Psi_s(\|\mathbf{x}\|)\|\mathbf{z}\|^{n-1}} \prod_{i=1}^{n} \frac{(z_i - x_i)^{-s}}{\Gamma[1 - s]} \, \mathrm{d}z_i. \quad (75)$$

3.1.3 Increments of ASPs

We shall show that increments of an ASP have n-dimensional Liouville distributions. Indeed, at time $s \in [0, 1)$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$, $t \in (s, 1]$, has a multivariate Liouville distribution with a generating law that can be expressed in terms of the $\boldsymbol{\xi}_s$-conditional law of the norm variable $R_t = \|\boldsymbol{\xi}_t\|$. Before we show this, we first examine the law of the process $\{R_t\}$.

Proposition 3.6. *The process $\{R_t\}_{0 \leq t \leq T}$ is a GRB with generating law ν and activity parameter n. That is,*

$$\mathbb{P}[R_t \in \mathrm{d}r \mid \boldsymbol{\xi}_s = \mathbf{x}] = \frac{\Psi_t(r)}{\Psi_s(\|\mathbf{x}\|)} \frac{(r - \|\mathbf{x}\|)^{n(t-s)-1} \exp(-(r - \|\mathbf{x}\|))}{\Gamma[n(t - s)]} \mathrm{d}r, \quad (76)$$

and

$$\mathbb{P}[R_1 \in \mathrm{d}r \mid \boldsymbol{\xi}_s = \mathbf{x}] = \frac{\psi_s(\mathrm{d}r; \|\mathbf{x}\|)}{\Psi_s(\|\mathbf{x}\|)}, \quad (77)$$

for $0 < s < t < 1$.

Before proceeding the proof, note that, after simplification, (76) and (77) are consistent with (45) and (46).

Proof. Since $\{\boldsymbol{\xi}_t\}$ is a Markov process with respect to $\{\mathcal{F}_t\}$, $\{R_t\}$ is a Markov process with respect to $\{\mathcal{F}_t\}$. Thus to prove the proposition we need only verify that the transition probabilities $\{R_t\}$ match those given in (76) and (77).

We first verify the $\boldsymbol{\xi}_s$-conditional law of R_1. We can calculate this using the Bayes theorem;

$$\mathbb{P}[R_1 \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s = \mathbf{x}] = \frac{\mathbb{P}[\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,|\, R_1 = r]\mathbb{P}[R_1 \in \mathrm{d}r]}{\int_{r=0}^{\infty} \mathbb{P}[\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x} \,|\, R_1 = r]\mathbb{P}[R_1 \in \mathrm{d}r]}$$

$$= \frac{\frac{1}{f_n(r)} f_{n(1-s)}(r - \|\mathbf{x}\|)\, \nu(\mathrm{d}r)}{\int_{r=0}^{\infty} \frac{1}{f_n(r)} f_{n(1-s)}(r - \|\mathbf{x}\|)\, \nu(\mathrm{d}r)}$$

$$= \frac{\psi_s(\mathrm{d}r; \|\mathbf{x}\|)}{\Psi_s(\|\mathbf{x}\|)}. \tag{78}$$

Similarly, the $\boldsymbol{\xi}_s$-conditional law of R_t for $t \in (s, 1)$ can be derived by the use of the Bayes theorem;

$$\mathbb{P}[R_t \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s = \mathbf{x}] = \frac{\int_{z=0}^{\infty} \mathbb{P}[\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}, R_t \in \mathrm{d}r \,|\, R_1 = z]\mathbb{P}[R_1 \in \mathrm{d}z]}{\int_{z=0}^{\infty} \int_{r=0}^{\infty} \mathbb{P}[\boldsymbol{\xi}_s \in \mathrm{d}\mathbf{x}, R_t \in \mathrm{d}r \,|\, R_1 = z]\mathrm{d}r\mathbb{P}[R_1 \in \mathrm{d}z]}$$

$$= \frac{\int_{z=0}^{\infty} \frac{1}{f_n(z)} f_{n(t-s)}(r - \|\mathbf{x}\|) f_{n(1-t)}(z - r)\mathrm{d}r\, \nu(\mathrm{d}z)}{\int_{z=0}^{\infty} \frac{1}{f_n(z)} \int_{r=\|\mathbf{x}\|}^{z} f_{n(t-s)}(r - \|\mathbf{x}\|) f_{n(1-t)}(z - r)\mathrm{d}r\, \nu(\mathrm{d}z)} \tag{79}$$

$$= \frac{\Psi_t(r)}{\Psi_s(\|\mathbf{x}\|)} f_{n(t-s)}(r - \|\mathbf{x}\|)\mathrm{d}r. \tag{80}$$

The denominator of (79) was simplified using the fact that gamma densities with common scale parameter are closed under convolution. $\qquad\square$

We define the measure ν_{st}, $0 \le s < t \le 1$, by

$$\nu_{st}(B) = \mathbb{P}[R_t \in B \,|\, \boldsymbol{\xi}_s]. \tag{81}$$

Thus we have

$$\nu_{s1}(\mathrm{d}r) = \frac{\psi_s(\mathrm{d}r; R_s)}{\Psi_s(R_s)}, \tag{82}$$

and

$$\nu_{st}(\mathrm{d}r) = \frac{\Psi_t(r)}{\Psi_s(R_s)} \frac{(r - R_s)^{n(t-s)-1} \exp(-(r - R_s))}{\Gamma[n(t-s)]} \mathrm{d}r, \qquad (t < 1). \tag{83}$$

When ν_{st} admits a density, we denote it by $p_{st}(r) = \nu_{st}(\mathrm{d}r)/\mathrm{d}r$. We see from (83) that p_{st} exists for $t < 1$. When $t = 1$, it follows from the definition of ψ_t that p_{st} only exists if ν admits a density.

Note that $\mathbb{P}[R_t \in \mathrm{d}r \,|\, \boldsymbol{\xi}_s] = \mathbb{P}[R_t \in \mathrm{d}r \,|\, R_s]$ for $t \in (s, 1]$. This is not surprising since $\{R_s\}$ is a GRB, and hence a Markov process with respect to its natural filtration.

We now show that the increments of an ASP are of multivariate Liouville-type.

Proposition 3.7. *Fix $s \in [0,1)$. Given $\boldsymbol{\xi}_s$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$, $t \in (s, 1]$, has an n-variate Liouville distribution with generating law*

$$\nu^*(B) = \nu_{st}(B + R_s), \tag{84}$$

and parameter vector $\alpha = (t - s, \ldots, t - s)^\top$.

Proof. First we prove the case $t < 1$. Thus, the density p_{st} exists. From (67) and (83), we have

$$\mathbb{P}[\boldsymbol{\xi}_t - \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s] = \frac{\Psi_t(\|\mathbf{y}\| + R_s)}{\Psi_s(R_s)} \prod_{i=1}^n \frac{y_i^{(t-s)-1} \mathrm{e}^{-y_i}}{\Gamma[t-s]} \, \mathrm{d}y_i$$

$$= \frac{p_{st}(\|\mathbf{y}\| + R_s)\Gamma[n(t-s)]}{\|\mathbf{y}\|^{n(t-s)-1}} \prod_{i=1}^n \frac{y_i^{(t-s)-1}}{\Gamma[t-s]} \, \mathrm{d}y_i. \tag{85}$$

Comparing (85) to (24) shows it to be the law of Liouville distribution with generating law $p_{st}(x + R_s)\mathrm{d}x$ and parameter vector $(t - s, \ldots, t - s)^\top$. Noting that $p_{st}(x + R_s)\mathrm{d}x = \nu^*(\mathrm{d}x)$, where ν^* is given by (84), yields the required result.

We now consider the case $t = 1$ when ν admits a density p. Thus, the density p_{s1} exists. From (75) and (82), we have

$$\mathbb{P}[\boldsymbol{\xi}_1 - \boldsymbol{\xi}_s \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s] = \frac{\Gamma[n]\mathrm{e}^{R_s}p(\|\mathbf{y}\| + R_s)}{\Psi_s(R_s)(\|\mathbf{y}\| + R_s)^{n-1}} \prod_{i=1}^n \frac{y_i^{-s}}{\Gamma[1-s]} \, \mathrm{d}y_i$$

$$= \frac{\Gamma[n(1-t)]p_{s1}(\|\mathbf{y}\| + R_s)}{\|\mathbf{y}\|^{n(1-t)-1}} \prod_{i=1}^n \frac{y_i^{-s}}{\Gamma[1-s]} \, \mathrm{d}y_i. \tag{86}$$

Hence $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$ has the required density.

For the final case where $t = 1$ and ν has no density we only outline the proof since the details are far from illuminating. Given $\boldsymbol{\xi}_s$, the law of $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_s$ is characterised by (66). We then need to show that this law is equal to the law of $X\mathbf{D}$, where X is a random variable with law ν^* given by (84), and $\mathbf{D}$ is a Dirichlet random variable, independent of X, with parameter vector $(1 - s, \ldots, 1 - s)^\top$. This is possible by mixing the Dirichlet density (18) with random scale parameter X. $\qquad\square$

3.2   Moments

In this subsection we fix a time $s \in [0, 1)$, and we assume that the first two moments of ν exist and are finite.

Proposition 3.8. *The first- and second-order moments of $\boldsymbol{\xi}_t$, $t \in (s, 1]$, are*

1. $\quad \mathbb{E}\left[\xi_t^{(i)} \,\Big|\, \boldsymbol{\xi}_s\right] = \dfrac{1}{n}\mu_1 + \xi_s^{(i)},$

2. $\quad \mathrm{Var}\left[\xi_t^{(i)} \,\Big|\, \boldsymbol{\xi}_s\right] = \dfrac{1}{n}\left[\left(\dfrac{t - s + 1}{n(t - s) + 1}\right)\mu_2 - \dfrac{1}{n}\mu_1^2\right],$

3. $\quad \mathrm{Cov}\left[\xi_t^{(i)}, \xi_t^{(j)} \,\Big|\, \boldsymbol{\xi}_s\right] = \dfrac{t - s}{n}\left[\dfrac{\mu_2}{n(t - s) + 1} - \dfrac{\mu_1^2}{n(t - s)}\right], \quad (i \neq j),$

where

$$\mu_1 = \frac{t - s}{1 - s}\left(\mathbb{E}[R_1 \mid R_s] - R_s\right),$$

$$\mu_2 = \frac{(t - s)(1 + n(t - s))}{(1 - s)(1 + n(1 - s))}\mathbb{E}[(R_1 - R_s)^2 \mid R_s].$$

Proof. Given $\boldsymbol{\xi}_s$, the increment $\boldsymbol{\xi}_t - \boldsymbol{\xi}_s$ has an n-dimensional Liouville distribution with generating law

$$\nu^*(A) = \nu_{st}(A + R_s) \tag{87}$$

for $t \in (s, 1]$, and with parameter vector $(t - s, \ldots, t - s)^\top$. We have

$$\mu_1 = \int_0^\infty y\,\nu^*(\mathrm{d}y) = \int_{R_s}^\infty y\,\nu_{st}(\mathrm{d}y) - R_s = \mathbb{E}[R_t \mid \boldsymbol{\xi}_s] - R_s, \tag{88}$$

and

$$\mu_2 = \int_0^\infty y^2\,\nu^*(\mathrm{d}y) = \int_{R_s}^\infty (y - R_s)^2\,\nu_{st}(\mathrm{d}y) = \mathbb{E}[(R_t - R_s)^2 \mid \boldsymbol{\xi}_s]. \tag{89}$$

It then follows from equations (25)-(27) that

1. $\mathbb{E}\left[\xi_t^{(i)} \,\Big|\, \boldsymbol{\xi}_s\right] = \dfrac{1}{n}\left(\mathbb{E}[R_t \mid \boldsymbol{\xi}_s] - R_s\right) + \xi_s^{(i)},$

2. $\mathrm{Var}\left[\xi_t^{(i)} \,\Big|\, \boldsymbol{\xi}_s\right] = \dfrac{1}{n}\left[\left(\dfrac{t - s + 1}{n(t - s) + 1}\right)\mathbb{E}[(R_t - R_s)^2 \mid \boldsymbol{\xi}_s] - \dfrac{1}{n}\left(\mathbb{E}[R_t \mid \boldsymbol{\xi}_s] - R_s\right)^2\right],$

3. $\mathrm{Cov}\left[\xi_t^{(i)}, \xi_t^{(j)} \,\Big|\, \boldsymbol{\xi}_s\right] = \dfrac{t - s}{n}\left[\dfrac{(\mathbb{E}[(R_t - R_s)^2 \mid \boldsymbol{\xi}_s])}{n(t - s) + 1} - \dfrac{(\mathbb{E}[R_t \mid \boldsymbol{\xi}_s] - R_s)^2}{n(t - s)}\right], \quad (i \neq j).$

To compute $\mathbb{E}[R_t \mid \boldsymbol{\xi}_s]$ and $\mathbb{E}[(R_t - R_s)^2 \mid \boldsymbol{\xi}_s]$, we use two results results about Lévy random bridges found in [9]. First, we can write

$$\mathbb{E}[R_t \mid R_s] = \frac{t - s}{1 - s}\mathbb{E}[R_1 \mid R_s] + \frac{1 - t}{1 - s}R_s.$$

The expression for μ_1 then follows directly. Second, given R_s, the process $\{R_t - R_s\}_{s \leq t \leq 1}$ is a GRB with terminal law $\bar{\nu}(B) = \nu_{s1}(B + R_s)$ and activity parameter n. Hence, given R_s,

$$\{R_t - R_s\}_{s \leq t \leq 1} \overset{\text{law}}{=} \{X\gamma_{t1}\}_{s \leq t \leq 1}, \tag{90}$$

where X is a random variable with law $\bar{\nu}$, and $\{\gamma_{t1}\}_{s \leq t \leq 1}$ is a gamma bridge with activity parameter n, independent of X, satisfying $\gamma_{s1} = 0$ and $\gamma_{11} = 1$. Note that γ_{t1}, $t \in (s, 1)$, is a beta random variable with parameters $\alpha = n(t - s)$ and $\beta = n(1 - t)$. Thus

$$\begin{aligned}
\mathbb{E}[(R_t - R_s)^2 \mid R_s] &= \mathbb{E}[\gamma_{t1}]\mathbb{E}[X^2] \\
&= \mathbb{E}[\gamma_{t1}] \int_0^\infty x^2 \, \bar{\nu}(\mathrm{d}x) \\
&= \mathbb{E}[\gamma_{t1}] \int_{R_s}^\infty (y - R_s)^2 \, \nu_{s1}(\mathrm{d}x) \\
&= \frac{(t - s)(1 + n(t - s))}{(1 - s)(1 + n(1 - s))}\mathbb{E}[(R_1 - R_s)^2 \mid R_s].
\end{aligned} \tag{91}$$

The expression for μ_2 follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3 Measure change

In this section we shall show that the law of an n-dimensional ASP is equivalent to an n-dimensional gamma process. To demonstrate this result it is convenient to begin by assuming that under some measure $\mathbb{Q}$ the process $\{\boldsymbol{\xi}_t\}$ is an n-dimensional gamma process, and then show that $\{\boldsymbol{\xi}_t\}$ is an ASP under an equivalent measure $\mathbb{P}$.

Under $\mathbb{Q}$, we assume that $\{\boldsymbol{\xi}_t\}$ is an n-dimensional gamma process such that

$$\mathbb{Q}[\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}] = \prod_{i=1}^n \frac{x_i^{t-1}}{\Gamma[t]}\mathrm{e}^{-x_i}\,\mathrm{d}x_i. \tag{92}$$

Hence the gamma processes $\{\xi_t^{(i)}\}$, $i = 1, 2, \ldots, m$, are independent and identical in law. The process $\{R_t\}_{0 \leq t \leq 1}$, defined as above by $R_t = \|\boldsymbol{\xi}_t\|$, is a one-dimensional gamma process and satisfies

$$\mathbb{Q}[R_t \in \mathrm{d}x] = \frac{x^{nt-1}}{\Gamma[nt]}\mathrm{e}^{-x}\,\mathrm{d}x. \tag{93}$$

As before, the filtration $\{\mathcal{F}_t\}$ is that generated by $\{\boldsymbol{\xi}_t\}$.

We shall show that the process $\{\Psi_t(R_t)\}_{0 \leq t < 1}$ is a martingale, where

$$\begin{aligned}
\Psi_t(R_t) &= \int_{R_t}^\infty \frac{f_{n(1-t)}(z - R_t)}{f_n(z)}\,\nu(\mathrm{d}z) \\
&= \frac{\Gamma[n]\exp(R_t)}{\Gamma[n(1-t)]}\int_{R_t}^\infty z^{n-1}(z - R_t)^{n(1-t)-1}\,\nu(\mathrm{d}z).
\end{aligned} \tag{94}$$

For times $0 \le s < t < 1$, we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\Psi_t(R_t) \,|\, \mathcal{F}_s\right] &= \mathbb{E}_{\mathbb{Q}}\left[\int_{R_t}^{\infty} \frac{f_{n(1-t)}(z - R_t)}{f_n(z)}\, \nu(\mathrm{d}z) \,\bigg|\, \mathcal{F}_s\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\int_{R_t}^{\infty} \frac{f_{n(1-t)}(z - R_s - (R_t - R_s))}{f_n(z)}\, \nu(\mathrm{d}z) \,\bigg|\, \boldsymbol{\xi}_s\right] \\
&= \int_{y=0}^{\infty} \int_{z=R_s+y}^{\infty} \frac{f_{n(1-t)}(z - R_s - y)}{f_n(z)}\, \nu(\mathrm{d}z)\, f_{n(t-s)}(y)\, \mathrm{d}y \\
&= \int_{z=R_s}^{\infty} \frac{1}{f_n(z)} \int_{y=0}^{z-R_s} f_{n(1-t)}(z - R_s - y) f_{n(t-s)}(y)\, \mathrm{d}y\, \nu(\mathrm{d}z) \\
&= \int_{R_s}^{\infty} \frac{f_{n(1-s)}(z - R_s)}{f_n(z)}\, \nu(\mathrm{d}z) \\
&= \Psi_s(R_s).
\end{aligned}
\tag{95}
$$

Since $\Psi_0(R_0) = 1$ and $\Psi_t(R_t) > 0$, the process $\{\Psi_t(R_t)\}_{0 \le t < 1}$ is a Radon-Nikodym density process.

Proposition 3.9. *Define a measure $\mathbb{P}$ by*

$$
\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t} = \Psi_t(R_t).
\tag{96}
$$

Under $\mathbb{P}$, $\{\boldsymbol{\xi}_t\}_{0 \le t < 1}$ is an ASP with generating law ν.

Proof. We prove the proposition by verifying that the transition law of $\{\boldsymbol{\xi}_t\}$ under $\mathbb{P}$ is that of an ASP.

$$
\begin{aligned}
\mathbb{P}\left[\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x} \,|\, \mathcal{F}_s\right] &= \mathbb{E}_{\mathbb{P}}[\mathbb{1}\{\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}\} \,|\, \mathcal{F}_s] \\
&= \frac{1}{\Psi_s(R_s)} \mathbb{E}_{\mathbb{Q}}[\Psi_t(R_t)\mathbb{1}\{\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}\} \,|\, \boldsymbol{\xi}_s] \\
&= \frac{\Psi_t(R_t)}{\Psi_s(R_s)} \prod_{i=1}^{n} f_{t-s}(x_i - \xi_s^{(i)})\, \mathrm{d}x_i \\
&= \frac{\Psi_t(R_t)}{\Psi_s(R_s)} \prod_{i=1}^{n} \frac{(x_i - \xi_s^{(i)})^{(t-s)-1} \mathrm{e}^{-(x_i - \xi_s^{(i)})}}{\Gamma[t-s]}\, \mathrm{d}x_i.
\end{aligned}
\tag{97}
$$

Comparing equations (97) and (67) completes the proof. $\qquad\square$

We can restate the results of this subsection as the following:

Proposition 3.10. *Suppose that $\{\boldsymbol{\xi}_t\}_{0 \le t \le 1}$ is an ASP with generating law ν under some measure $\mathbb{P}$. Then*

$$
\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \Psi_t(R_t)^{-1}
\tag{98}
$$

defines a probability measure $\mathbb{Q}$ for $t \in [0, 1)$. Furthermore, under $\mathbb{Q}$ $\{\boldsymbol{\xi}_t\}_{0 \leq t < 1}$ is an n-dimensional gamma process such that

$$\mathbb{Q}[\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{x}] = \prod_{i=1}^{n} \frac{x_i^{t-1}}{\Gamma[t]} \mathrm{e}^{-x_i} \, \mathrm{d}x_i.$$

3.4 Independent gamma bridges representation

The increments of an n-dimensional ASP are identical in law to a positive random variable multiplied by the Hadamard product of an n-dimensional Dirichlet random variable and a vector of n independent gamma bridges. For notational convenience, in this subsection we denote a gamma bridge defined over $[0, 1]$ as $\{\gamma(t)\}$ (instead of $\{\gamma_{t1}\}$).

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we denote their Hadamard product as $\mathbf{x} \circ \mathbf{y}$. That is,

$$\mathbf{x} \circ \mathbf{y} = (x_1 y_1, \ldots, x_n y_n)^\top. \tag{99}$$

Proposition 3.11. *Given the value of $\boldsymbol{\xi}_s$, the ASP process $\{\boldsymbol{\xi}_t\}$ satisfies the following identity in law:*

$$\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \leq t \leq 1} \stackrel{\mathrm{law}}{=} \{R^* \, \mathbf{D} \circ \boldsymbol{\gamma}_t\}_{s \leq t \leq 1},$$

where

1. *$\mathbf{D} \in [0, 1]^n$ is a symmetric Dirichlet random variable with parameter vector $(1 - s, \ldots, 1 - s)^\top$;*

2. *$\{\boldsymbol{\gamma}_t\}$ is a vector of n independent gamma bridges, each with activity parameter $m = 1$, starting at the value 0 at time s, and terminating with unit value at time 1;*

3. *$R^* > 0$ is a random variable with law ν^* given by*

$$\nu^*(A) = \nu_{st}(A + R_s);$$

4. *R^*, $\mathbf{D}$, and $\{\boldsymbol{\gamma}_t\}$ are mutually independent.*

Proof. Fix $k_i \geq 1$ and the partition

$$s = t_0^i < t_1^i < \cdots < t_{k_i}^i = 1, \tag{100}$$

for $i = 1, \ldots, n$. Define the non-overlapping increments $\{\Delta_{ij}\}$ by

$$\Delta_{ij} = \xi_{t_j^i}^{(i)} - \xi_{t_{j-1}^i}^{(i)}, \tag{101}$$

for $j = 1, \ldots, k_i$ and $i = 1, \ldots, n$. The distribution of the vector

$$\boldsymbol{\Delta} = (\Delta_{11}, \Delta_{12}, \ldots, \Delta_{1k_1},$$
$$\Delta_{21}, \Delta_{22}, \ldots, \Delta_{2k_2},$$
$$\vdots$$
$$\Delta_{n1}, \Delta_{n2}, \ldots, \Delta_{nk_n})^\top \tag{102}$$

characterises the finite-dimensional distributions of the process $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \le t \le 1}$. It follows from the Kolmogorov extension theorem that the distribution of $\boldsymbol{\Delta}$ characterises the law of $\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}$. Note that $\boldsymbol{\Delta}$ are non-overlapping increments of the master GRB $\{\Gamma_t\}$. Thus, given $\boldsymbol{\xi}_s$, $\boldsymbol{\Delta}$ has a multivariate Liouville distribution with parameter vector

$$\boldsymbol{\alpha} = (t_1^1 - t_0^1, t_2^1 - t_1^1, \ldots, t_{k_1}^1 - t_{k_1-1}^1,$$
$$t_1^2 - t_0^2, t_2^2 - t_1^2, \ldots, t_{k_2}^2 - t_{k_2-1}^2,$$
$$\vdots$$
$$t_1^n - t_0^n, t_2^n - t_1^n, \ldots, t_{k_n}^n - t_{k_n-1}^n)^\top, \tag{103}$$

and generating law

$$\nu^*(A) = \nu_{st}(A + R_s) \tag{104}$$

for $t \in (s, 1]$. It follows from Fang $et\ al.$ [5, theorem 6.9] that

$$(\Delta_{i1}, \ldots, \Delta_{ik_i})^\top \stackrel{\text{law}}{=} R^* D_i \mathbf{Y}_i, \qquad \text{for } i = 1, \ldots, n, \tag{105}$$

where (i) R^* has law ν^*, (ii) $\mathbf{D} = (D_1, \ldots, D_n)^\top$ has a Dirichlet distribution with parameter vector $(1 - s, \ldots, 1 - s)^\top$, (iii) $\mathbf{Y}_i \in [0, 1]^{k_i}$ has a Dirichlet distribution with parameter vector $(t_1^i - t_0^i, \ldots, t_{k_i}^i - t_{k_i-1}^i)^\top$, (iv) $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$, R^*, and $\mathbf{D}$ are mutually independent.

Let $\{\gamma(t)\}_{s \le t \le 1}$ be a gamma bridge with activity parameter $m = 1$ such that $\gamma(s) = 0$ and $\gamma(1) = 1$. Then the increment vector

$$(\gamma(t_1^i) - \gamma(t_0^i), \ldots, \gamma(t_{k_i}^i) - \gamma(t_{k_i-1}^i))^\top \tag{106}$$

has a Dirichlet distribution with parameter vector $(t_1^i - t_0^i, \ldots, t_{k_i}^i - t_{k_i-1}^i)^\top$. Hence the increment vector (106) is identical in law to $\mathbf{Y}_i$. From the Kolmogorov extension theorem, this identity characterises the law of $\{\gamma(t)\}$. It follows that

$$\{\xi_t^{(i)} - \xi_s^{(i)}\}_{s \le t \le 1} \stackrel{\text{law}}{=} \{R^* D_i \gamma_t\}_{s \le t \le 1}, \qquad \text{for } i = 1, \ldots, n, \tag{107}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.5 Uniform process

We construct a multivariate process from the ASP $\{\boldsymbol{\xi}_t\}$ such that each one-dimensional marginal is *a priori* uniformly distributed for every time $t \in (0, 1]$.

Fix a time $t \in (0, 1]$. Each $\xi_t^{(i)}$ is a scale-mixed beta random variable with survival function

$$
\begin{aligned}
\bar{F}_t(x) &= \int_x^\infty \left(1 - I_{x/y}[t, n - t]\right) \nu(\mathrm{d}y) \\
&= \int_x^\infty I_{1-x/y}[n - t, t]\, \nu(\mathrm{d}y),
\end{aligned}
\tag{108}
$$

where $I_z[\alpha, \beta]$ is the regularized incomplete Beta function, defined as usual for $z \in [0, 1]$ by

$$
I_z[\alpha, \beta] = \frac{\int_0^z u^{\alpha-1}(1 - \beta)^{\beta-1}\, \mathrm{d}u}{\int_0^1 u^{\alpha-1}(1 - \beta)^{\beta-1}\, \mathrm{d}u} \qquad (\alpha, \beta > 0).
\tag{109}
$$

The random variables

$$
Y_t^{(i)} = \bar{F}_t(\xi_t^{(i)}), \qquad i = 1, \ldots, n,
\tag{110}
$$

are then uniformly distributed.

We now define a process $\{\mathbf{Y}\}_{0 \le t \le 1}$ by

$$
\mathbf{Y}_t = \left(\bar{F}_t(\xi_t^{(1)}), \ldots, \bar{F}_t(\xi_t^{(n)})\right)^\top.
\tag{111}
$$

By construction, each one-dimensional marginal $Y_t^{(i)}$ is uniform for $t > 0$.

4 Liouville process

We generalise ASPs to a family of stochastic processes that we call *Liouville processes*. A Liouville process is a Markov process whose finite-dimensional distributions are multivariate Liouville. Liouville processes display a broader range of dynamics than ASPs. The one-dimensional marginal processes of a Liouville process are in general not identical. This generalisation comes at the expense of losing the direct connection to Archimedean copulas.

Definition 4.1. *Fix* $n \in \mathbb{N}_+$, $n \ge 2$, *and the vector* $\mathbf{m} \in \mathbb{R}_+^n$ *satisfying* $m_i > 0$, $i = 1, \ldots, n$. *Define the strictly increasing sequence* $\{u_i\}_{i=1}^n$ *by*

$$
\begin{aligned}
u_0 &= 0 \\
u_i &= u_{i-1} + m_i \qquad \text{for } i = 1, \ldots, n.
\end{aligned}
\tag{112}
$$

Then a process $\{\boldsymbol{\xi}_t\}_{0 \le t \le 1}$ satisfying

$$\{\boldsymbol{\xi}_t\}_{0 \le t \le 1} \overset{\text{law}}{=} \left\{ \begin{bmatrix} \Gamma_{t(u_1)} - \Gamma_0 \\ \vdots \\ \Gamma_{t(u_i - u_{i-1}) + u_{i-1}} - \Gamma_{u_{i-1}} \\ \vdots \\ \Gamma_{t(u_n - u_{n-1}) + u_{n-1}} - \Gamma_{u_{n-1}} \end{bmatrix} \right\}_{0 \le t \le 1}$$

for $\{\Gamma_t\}_{0 \le t \le u_n}$ a GRB with activity parameter $m = 1$, is an n-dimensional Liouville process. We say that the generating law of $\{\Gamma_t\}$ is the generating law of $\{\boldsymbol{\xi}_t\}$ and the activity parameter of $\{\boldsymbol{\xi}_t\}$ is $\boldsymbol{m}$.

Let $\{\boldsymbol{\xi}_t\}$ be a Liouville process with generating law ν and parameter vector $\mathbf{m}$. Each one-dimensional marginal process of $\{\boldsymbol{\xi}_t\}$ is a GRB with activity parameter m_i, and Definition 4.1 ensures that $\boldsymbol{\xi}_t$ is well-defined for each $t \in [0, 1]$. It is straightforward to adjust the definition so that a Liouville process is defined over an arbitrary closed interval.

In the language of McNeil & Nešlehová [12], $\boldsymbol{\xi}_1$ has a *Liouville copula*, which is defined as a survival copula of a Liouville distribution. We shall provide the transition law, moments and an independent gamma bridge representation of a Liouville process. We present the results as propositions. Proofs are omitted since they are similar to the proofs in Section 3.

We define a family of unnormalised measures, indexed by $t \in [0, 1)$ and $x \in \mathbb{R}_+$, as

$$\psi_0(B; x) = \nu(B), \tag{113}$$

$$\psi_t(B; x) = \int_B \frac{f_{T(1-t)}(z - x)}{f_T(z)} \, \nu(\mathrm{d}z)$$

$$= \frac{\Gamma[T] \mathrm{e}^x}{\Gamma[T(1-t)]} \int_B \mathbb{1}_{\{z > x\}} z^{T-1}(z - x)^{T(1-t)-1} \, \nu(\mathrm{d}z), \tag{114}$$

for $B \in \mathcal{B}(\mathbb{R})$ where $T = \|\mathbf{m}\|$. Again we write

$$\Psi_t(x) = \psi_t([0, \infty); x), \tag{115}$$

and

$$R_t = \|\boldsymbol{\xi}_t\|. \tag{116}$$

The process $\{R_t\}$ is a GRB with activity parameter T. Given $\boldsymbol{\xi}_s$, the law of R_1 is

$$\nu_{s1}(\mathrm{d}r) = \frac{\psi_s(\mathrm{d}r; R_s)}{\Psi_s(R_s)}, \tag{117}$$

and law of R_t is

$$\nu_{st}(\mathrm{d}r) = \frac{\Psi_t(r)}{\Psi_s(\|\mathbf{x}\|)} \frac{(r - \|\mathbf{x}\|)^{T(t-s)-1} \exp(-(r - \|\mathbf{x}\|))}{\Gamma[T(t-s)]} \mathrm{d}r \tag{118}$$

for $t \in (s, 1)$.

Proposition 4.2. *The Liouville process $\{\boldsymbol{\xi}_t\}$ is a Markov process with the transition law given by*

$$\mathbb{P}\left[\xi_1^{(1)} \in \mathrm{d}z_1, \ldots, \xi_1^{(n-1)} \in \mathrm{d}z_{n-1}, \xi_1^{(n)} \in B \,\middle|\, \boldsymbol{\xi}_s = \mathbf{x}\right] =$$

$$\frac{\psi_{\tau(s)}(B + \sum_{i=1}^{n-1} z_i; x_n + \sum_{i=1}^{n-1} z_i)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n-1} \frac{(z_i - x_i)^{m_i(1-s)-1} \mathrm{e}^{-(z_i-x_i)}}{\Gamma[m_i(1-s)]} \, \mathrm{d}z_i, \quad (119)$$

and

$$\mathbb{P}\left[\boldsymbol{\xi}_t \in \mathrm{d}\mathbf{y} \,|\, \boldsymbol{\xi}_s = \mathbf{x}\right] = \frac{\Psi_t(\|\mathbf{y}\|)}{\Psi_s(\|\mathbf{x}\|)} \prod_{i=1}^{n} \frac{(y_i - x_i)^{m_i(t-s)-1} \mathrm{e}^{-(y_i-x_i)}}{\Gamma[m_i(t-s)]} \, \mathrm{d}y_i, \quad (120)$$

where $\tau(t) = 1 - (1-t)/T$, $0 \le s < t < 1$, and $B \in \mathcal{B}(\mathbb{R})$.

Proposition 4.3. *Fix $s \in [0,1)$. The first- and second-order moments of $\boldsymbol{\xi}_t$, $t \in (s,1]$, are*

1. $\mathbb{E}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \dfrac{m_i}{T}\mu_1 + \xi_s^{(i)},$

2. $\mathrm{Var}\left[\xi_t^{(i)} \,\middle|\, \boldsymbol{\xi}_s\right] = \dfrac{m_i}{T}\left[\left(\dfrac{m_i(t-s)+1}{T(t-s)+1}\right)\mu_2 - \dfrac{m_i}{T}\mu_1^2\right],$

3. $\mathrm{Cov}\left[\xi_t^{(i)}, \xi_t^{(j)} \,\middle|\, \boldsymbol{\xi}_s\right] = \dfrac{m_i m_j(t-s)}{T}\left[\dfrac{\mu_2}{T(t-s)+1} - \dfrac{\mu_1^2}{T(t-s)}\right], \quad (i \ne j),$

where

$$\mu_1 = \frac{t-s}{1-s}(\mathbb{E}[R_1 \,|\, R_s] - R_s),$$

$$\mu_2 = \frac{(t-s)(1+T(t-s))}{(1-s)(1+T(1-s))}\mathbb{E}[(R_1 - R_s)^2 \,|\, R_s].$$

The law of the increments of an n-dimensional Liouville process can be characterised by a positive random variable multiplied by the Hadamard product of an n-dimensional Dirichlet random variable and a vector of n independent gamma bridges.

Proposition 4.4. *Given the value of $\boldsymbol{\xi}_s$, the Liouville process $\{\boldsymbol{\xi}_t\}$ satisfies the following identity in law:*

$$\{\boldsymbol{\xi}_t - \boldsymbol{\xi}_s\}_{s \le t \le 1} \stackrel{\mathrm{law}}{=} \{R^* \mathbf{D} \circ \boldsymbol{\gamma}_t\}_{s \le t \le 1},$$

where

1. $\mathbf{D} \in [0,1]^n$ *has a Dirichlet distribution with parameter vector $(1-s)\boldsymbol{m}$;*

2. $\{\boldsymbol{\gamma}_t\}$ *is a vector of n independent gamma bridges, such that the ith marginal process is a gamma bridge with activity parameter m_i, starting at the value 0 at time s, and terminating with unit value at time 1;*

3. $R^* > 0$ *is a random variable with law ν^* given by*

$$\nu^*(A) = \nu_{st}(A + R_s);$$

4. R^*, $\mathbf{D}$, *and $\{\boldsymbol{\gamma}_t\}$ are mutually independent.*

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