EQUILIBRIUM WITH EXPONENTIAL UTILITY AND NON-NEGATIVE CONSUMPTION

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Abstract. We study a multi-period Arrow-Debreu equilibrium in a heterogeneous economy populated by agents trading in a complete market. Each agent is represented by an exponential utility function, where additionally no negative level of consumption is permitted. We derive an explicit formula for the optimal consumption policies involving a put option depending on the state price density. We exploit this formula to prove the existence of an equilibrium and then provide a characterization of all possible equilibria, under the assumption of positive endowments. Via particular examples, we demonstrate that uniqueness is not always guaranteed. Finally, we discover the presence of infinitely many equilibria when endowments are vanishing.

1. Introduction

The approach dedicated to asset pricing by equilibrium analysis has gained an extensive attention over the past decades from both a theoretical and a practical perspective. This theory carries ambitious objectives such as complete derivation of consumption allocations and pricing kernels in terms of primitives of a given economy. The key questions in this field are mainly concerned with the existence, uniqueness and a description of market equilibria. We refer to Chapter 12 in Cvitanic and Zapatero (2004) and Chapter 4 in Karatzas and Shreve (1990) for a detailed exposition of these issues. For more applied aspects of this theory, we refer the reader to Shoven and Whalley (1992).

The study of Arrow-Debreu equilibrium with exponential preferences in one period models was first introduced in Bühlmann (1980). A subsequent paper, Bühlmann (1984), extends the results to existence for general utility functions. Further works by Mas-Colell (1986), Duffie (1986), Karatazas, Lechovsky and Shreve (1990), Karatazas, Lechovsky and Shreve (1991) Mas-Colell and Zame (1991), Dana (1993a) and Dana (1993b) are devoted to the study of existence and uniqueness issues for

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rather abstract multi-period models including both discrete and continuous time settings. Malamud and Trubowitz (2006) study existence and provide examples of non-unique equilibria in an infinite time horizon context.

We revisit and explore a variation of the classical problem of equilibrium asset pricing for heterogeneous investors represented by exponential utility functions. We mend a prominent drawback in the classical paradigm of exponential utility by restricting it to the positive real half line. The economic significance of this modification is obvious, it prevents the individual from the possession of a negative level of consumption.

Roughly speaking, an Arrow-Debreu equilibrium in a complete market is a situation where the supply is equal to the demand under an optimal performance of each individual. The solution to the classical problem with exponential utility functions is rather simple and enjoys properties as uniqueness and a full characterization of the equilibrium state price density. Furthermore, all the corresponding parameters in equilibrium are, in a certain sense, smooth. In contrast to the classical setting, assuming that no negative consumption is allowed, the situation becomes much more delicate, due to the absence of certain regularity conditions (e.g. Inada's condition is no longer valid). Nonetheless, existence and a comprehensive description of the equilibrium is still available. This is in effect one of the few examples where exact formulas can be obtained, unlike the case of power utility functions, where usually only a rather abstract description of the pricing kernel can be established. On the other hand, the lack of certain regularity conditions in our context is crucial and gives a rise to non-uniqueness.

We briefly outline the contents of the paper. We consider heterogeneous agents represented by exponential utility functions (defined on the positive half real line) and endowment streams. In the first stage, we consider the corresponding individual's utility maximization problem in a complete market setting, and derive a formula that describes the optimal consumption stream. This formula involves a put option on the logarithm of the state price density, and some strike that depends on the risk aversion, time scaling and the endowment stream of the agent. We then note that this problem is similar to the unconstrained one, for large levels of endowments. We turn then to conducting an equilibrium analysis by exploiting the latter formula combined with some standard arguments involving the "excess demand" function. Under the assumption of positive endowments, we prove the existence of equilibrium and characterize explicitly the associated state price densities and consumption allocations. A simple corollary is that in a homogeneous economy the equilibrium is unique. Uniqueness does not hold in general. This is illustrated in a two-agent economy and a deterministic market model consisting only of a risk-less security. We extend the existence result to the more general case of possibly vanishing endowments. In this case, we show that there exist infinitely many equilibria, all are of the same canonical form.

2. The Model

We fix a final time horizon $T \in \mathbb{N}$. The uncertainty in our model is captured by a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_0 = \{\phi, \Omega\} \subseteq \mathcal{F}_1 \subseteq ... \subseteq \mathcal{F}_T = \mathcal{F}$, where each sigma-algebra $\mathcal{F}_k$ corresponds to the information revealed at each period $k \in \{0, ..., T\}$. In the present paper, adaptedness and predictability is always meant with respect to the filtration $(\mathcal{F}_k)_{k=0,\ldots,T}$. We will use the following notations $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$. The economy in our model is inhabited by N (types of) agents labelled by $i = 1, ..., N$. The preferences of each agent i are characterized by an exponential utility function $u_i(x) = -e^{-\gamma_i x}$, defined on $\mathbb{R}_+$, for a given risk aversion coefficient $\gamma_i > 0$. Each agent i receives a random income $(\epsilon^i_k)_{k=0,\ldots,T}$. This process, which will be referred to as the agent's endowments stream, is assumed to be non-negative and adapted. We assume that prices of payoffs are determined according to a certain pricing functional (or, state price density) represented by a strictly positive adapted process $(\xi_k)_{k=0,\ldots,T}$. More precisely, given a non-negative random variable X that represents a certain payoff at the maturity date T, the price at time j assigned to X is given by

$$E\left[\frac{\xi_T}{\xi_j} X \middle| \mathcal{F}_j\right],$$

for all $0 \leq j \leq T$. We will not distinguish between non-normalized pricing kernels $(\xi_k)_{k=0,\ldots,T}$ and the corresponding normalized ones $(\xi_k/\xi_0)_{k=0,\ldots,T}$. Each agent i solves the following utility maximization problem from consumption

$$\sup_{(c^i_0,\ldots,c^i_T)} \sum_{k=0}^{T} e^{-\rho_i k} E\left[u_i(c^i_k)\right], \tag{2.1}$$

under the constraints: $c^i_k \in L^+_0(\mathcal{F}_k)$ for all $k = 0, ..., T$, and

$$\sum_{k=0}^{T} E\left[\xi_k c^i_k\right] = \sum_{k=0}^{T} E\left[\xi_k \epsilon^i_k\right]. \tag{2.2}$$

Here, $\rho_i \geq 0$ stands for the degree of impatience of agent i, and $L^+_0(\mathcal{F}_k)$ denotes the space of all non-negative $\mathcal{F}_k$-measurable random variables. Below, we introduce the notion of Arrow-Debreu equilibrium[1].

[1] Under certain regularity conditions, the whole model can be implemented by a complete security market with a unique state price density process $(\xi_k)_{k=0,\ldots,T}$. Since time is discrete and the probability space is not assumed to be finite, there might be infinitely many securities that complete the market. The Arrow-Debreu equilibrium then becomes an equilibrium of Radner type; see Duffie and Huang (1986) for a detailed treatment of these issues.

Definition 2.1. *An Arrow-Debreu equilibrium (or, equilibrium for short) is a pair of processes* $(c_k^i)_{k=0,...,T;i=1,...,N}$*, and* $(\xi_k)_{k=1,...,T}$ *such that:*

(a) The process $(\xi_k)_{k=1,...,T}$ *is a state price density, and* $(c_k^i)_{k=0,...,T}$ *is the optimal consumption stream of each agent i, i.e., solves (2.1) under the constraints* $c_k^i \in L_0^+(\mathcal{F}_k)$ *for all* $k = 0, ..., T$*, and (2.2).*

(b) The market clearing condition holds

$$\sum_{i=1}^{N} c_k^i = \epsilon_k := \sum_{i=1}^{N} \epsilon_k^i, \tag{2.3}$$

for all $k = 0, ..., T$.

3. Optimal Consumption and Existence of an Equilibrium

3.1. **Optimal Consumption Streams.** We study an individual's exponential utility maximization problem with the constraint of non-negative consumption policies. This is a typical situation where the use of the convex conjugate and some related notions is efficient in characterizing the corresponding controls; see Rockafellar (1970)). Based on those standard ideas from convex analysis, we derive a formula for the optimal consumption stream.

Theorem 3.1. *Consider i-th agent's utility maximization problem (2.1) under the constraints:* $c_k^i \in L_0^+(\mathcal{F}_k)$*, for all* $k = 0, ..., T$*, and (2.2). Assume further that* $\sum_{k=0}^{T} E\left[\xi_k \epsilon_k^i\right] > 0$*, for all* $i = 1, ..., N$*. Set*

$$I_i(y) = \frac{1}{\gamma_i} \log\left(\frac{\gamma_i}{y}\right) \vee 0, \tag{3.1}$$

for all $y > 0$*. Then, there exists a unique optimal consumption stream given by*

$$c_k^i = I_i(\lambda^* e^{\rho_i k} \xi_k) = \frac{1}{\gamma_i}\left(\log\left(\frac{\gamma_i}{\lambda^*}\right) - \rho_i k - \log(\xi_k)\right)^+, \tag{3.2}$$

where λ^* *is determined as the unique positive solution of the equation*

$$\sum_{k=0}^{T} E\left[\xi_k I_i(\lambda e^{\rho_i k} \xi_k)\right] = \sum_{k=0}^{T} E\left[\xi_k \epsilon_k^i\right]. \tag{3.3}$$

We first proof the following auxiliary lemma:

Lemma 3.2. *Consider the function* $\psi^i : \mathbb{R}_{++} \to \mathbb{R}_+$ *defined by*

$$\psi^i(\lambda) = \sum_{k=0}^{T} E\left[\xi_k I_i(\lambda e^{\rho_i k} \xi_k)\right].$$

Then, $\psi^i(\lambda)$ *is a decreasing continuous function of the following form: If* $\psi^i(b) > 0$ *for some* $b \in \mathbb{R}_{++}$*, then* $\psi^i(a) > \psi^i(b)$ *for all* $0 < a < b$*. Furthermore,* $\lim_{\lambda \to 0} \psi^i(\lambda) = \infty$ *and* $\lim_{\lambda \to +\infty} \psi^i(\lambda) = 0$.

Proof of Lemma 3.2. First observe that $E\left[\xi_k I_i(c\xi_k)\right] < +\infty$ for all $c > 0$ since

$$E\left[\xi_k I_i(c\xi_k)\right] = \frac{1}{\gamma_i} E\left[\xi_k \left(\log\left(\frac{\gamma_i}{c}\right) - \log(\xi_k)\right) \mathbf{1}_{\{\xi_k \le \frac{\gamma_i}{c}\}}\right] \le \frac{1}{c} + \frac{1}{\gamma_i} E\left[\xi_k\right] < \infty, \tag{3.4}$$

where the first inequality follows from the fact that $1 + \log t \le t$, for all $t \ge 1$. This shows that $\psi^i(\lambda)$ is well defined for all $\lambda \in \mathbb{R}_{++}$. Next, we prove that ψ^i is continuous. Let λ_n be a sequence such that $\lambda_n \uparrow \lambda$, hence, $\lim_{n\to\infty} \xi_k I_i(\lambda_n e^{\rho_i k}\xi_k) = \xi_k I_i(\lambda e^{\rho_i k}\xi_k)$, $P-$a.s, and thus the dominated convergence theorem implies that $\lim_{n\to\infty} \psi^i(\lambda_n) = \psi^i(\lambda)$. The same argument holds for a sequence $\lambda_n \downarrow \lambda$. Now, we treat the limits. Consider an arbitrary increasing sequence $\lambda_n \to \infty$, then, $\xi_k I_i(\lambda_n e^{\rho_i k}\xi_k) \to 0$ and $\xi_k I_i(\lambda_1 e^{\rho_i k}\xi_k) \ge \xi_k I_i(\lambda_n e^{\rho_i k}\xi_k)$, for all n. Therefore, dominated convergence implies that $\lim_{n\to\infty} \psi^i(\lambda_n) = 0$. Next, pick an arbitrary sequence $\lambda_n \downarrow 0$ and note that $\lim_{n\to\infty} \xi_k I_i(\lambda_n e^{\rho_i k}\xi_k) = +\infty$, $P-$a.s. Therefore, Fatou's lemma implies that

$$\liminf_{n\to\infty} E\left[\xi_k I(\lambda_n e^{\rho k}\xi_k)\right] \ge E\left[\liminf_{n\to\infty} \xi_k I(\lambda_n e^{\rho k}\xi_k)\right] = +\infty.$$

This shows that $\lim_{\lambda\to 0} \psi^i(\lambda) = \infty$. Next, note that $\lambda \mapsto E\left[\xi_k I_i\left(\lambda e^{\rho_i k}\xi_k\right)\right]$ is a decreasing function for each $k = 1, ..., T$ since I_i is decreasing. Therefore, $\psi^i(\lambda)$ is decreasing. Finally, assume in a contrary that $\psi^i(b) > 0$ and $\psi^i(a) = \psi^i(b)$, for some $a < b$. By the previous observation, it follows that $E\xi_k\left[I_i\left(ae^{\rho_i k}\xi_k\right)\right] = E\left[\xi_k I_i\left(be^{\rho_i k}\xi_k\right)\right]$, for each $k = 0, ..., T$. By definition, $I_i(y)$ is a strictly decreasing function for $0 < y \le \gamma_i$, thus $I_i\left(ye^{\rho_i k}\xi_k\right)\mathbf{1}_{\{ye^{\rho_i k}\xi_k \le \gamma_i\}} < I_i\left(xe^{\rho_i k}\xi_k\right)\mathbf{1}_{\{xe^{\rho_i k}\xi_k \le \gamma_i\}}$. This is a contradiction, finishing the proof. □

Proof of Theorem 3.1. First, observe that Lemma 3.2 yields the existence of a unique solution to equation (3.3) denoted by $\lambda^* > 0$. Next, consider the Legendre transform of $u_i(x) = -e^{-\gamma_i x}$, defined for $y > 0$ by

$$v_i(y) = \sup_{x\in\mathbb{R}_+} \left(u_i(x) - xy\right).$$

Observe that $\frac{\partial}{\partial x}(u_i(x) - xy) = \gamma_i e^{-\gamma_i x} - y$. Therefore, if $y < \gamma_i$, then $v_i(y) = u_i\left(\frac{1}{\gamma_i}\log\left(\frac{\gamma_i}{y}\right)\right) - \frac{y}{\gamma_i}\log\left(\frac{\gamma_i}{y}\right)$. If $y \ge \gamma_i$, then $v_i(y) = u_i(0)$. Thus, we can rewrite $v_i(y) = u_i(I_i(y)) - I_i(y)y$, for all $y > 0$. Now, it follows that

$$\begin{aligned} v_i(\lambda^* e^{\rho_i k}\xi_k) &= u_i(I_i(\lambda^* e^{\rho_i k}\xi_k)) - \lambda^* e^{\rho_i k}\xi_k I_i(\lambda^* e^{\rho_i k}\xi_k) \\ &\ge u(X_k) - \lambda^* e^{\rho k}\xi_k X_k, \end{aligned}$$

for all $X_k \in L_0^+(\mathcal{F}_k)$. Hence,

$$e^{-\rho_i k} u_i(I_i(\lambda^* e^{\rho_i k}\xi_k)) \ge e^{-\rho_i k} u_i(X_k) + \lambda^* \xi_k (I_i(\lambda^* e^{\rho_i k}\xi_k) - X_k).$$

In particular, we have that

$$\sum_{k=0}^{T} e^{-\rho_i k} E\left[u_i(I_i(\lambda^* e^{\rho_i k}\xi_k))\right]$$
$$\geq \sum_{k=0}^{T} e^{-\rho_i k} E\left[u_i(X_k)\right] + \sum_{k=0}^{T} \lambda^* E\left[\xi_k(I_i(\lambda^* e^{\rho k}\xi_k) - X_k)\right],$$

for all $X_k \in L_0^+(\mathcal{F}_k)$ such that $E[\xi_k X_k] < \infty$. These considerations combined with the fact that $\sum_{k=0}^{T} E\left[\xi_k(I_i(\lambda^* e^{\rho_i k}\xi_k)\right] = \sum_{k=0}^{T} E\left[\xi_k \epsilon_k^i\right]$ imply that

$$\sum_{k=0}^{T} e^{-\rho_i k} E\left[u_i(I_i(\lambda^* e^{\rho_i k}\xi_k))\right] \geq \sum_{k=0}^{T} e^{-\rho_i k} E\left[u_i(X_k)\right],$$

for all $X_0, ..., X_T$ such that $X_k \in L_0^+(\mathcal{F}_k)$ and $\sum_{k=0}^{T} E\left[\xi_k X_k\right] = \sum_{k=0}^{T} E\left[\xi_k \epsilon_k^i\right]$. We conclude the proof by remarking that uniqueness follows from the strict concavity of the utility function. $\square$

For a sufficiently large level of endowments, and under some regularity assumptions on the state price density, the optimal consumption stream is strictly positive. In particular, it coincides with the optimal consumptions corresponding to the setting of unconstrained exponential utility.

Corollary 3.3. *Assume that* $\xi_k < C$ *and* $\epsilon_k^i > \frac{1}{\gamma_i}\left(\log\left(\frac{C}{\xi_k}\right) + \rho_i(T-k)\right)$, $P-a.s.$, *for all* $k = 0, ..., T$, *where* $C > 0$ *is some positive constant. Then, the optimal consumption stream of the* $i-th$ *agent is strictly positive and is given by*

$$c_k^i = \frac{1}{\gamma_i}\left(\frac{\sum_{l=0}^{T} E\left[\xi_l\left(\gamma_i \epsilon_l^i + \log \xi_l + \rho_i l\right)\right]}{\sum_{l=0}^{T} E\left[\xi_l\right]} - \rho_i k - \log \xi_k\right) > 0, \tag{3.5}$$

for all $k = 0, ..., T$.

Proof of Corollary 3.3. Fix $z = \frac{\gamma_i}{C e^{\rho_i T}}$ and observe that $\frac{\gamma_i}{z e^{\rho_i k}\xi_k} = \frac{C}{\xi_k} e^{\rho_i (T-k)} > 1$, for all $k = 0, ..., T$. Therefore, by applying the same argument as in the proof of Lemma 3.2, one concludes that $\psi^i(\lambda)$ is a strictly decreasing function on the interval $(0, z)$. Moreover, note that

$$\psi^i(z) = \frac{1}{\gamma_i}\sum_{k=0}^{T} E\left[\xi_k\left(\log\left(\frac{C}{\xi_k}\right) + \rho_i(T-k)\right)\right] < \sum_{k=0}^{T} E\left[\epsilon_k^i \xi_k\right],$$

by assumption. Therefore, a solution λ^* to equation (3.3) is attained for some $\lambda^* \in (0, z)$, and thus $\frac{\gamma_i}{\lambda^* e^{\rho_i k}\xi_k} > \frac{\gamma_i}{z e^{\rho_i k}\xi_k} > 1$. In view of (3.2), we obtain that the optimal consumption stream admits the form

$$c_k^i = \frac{1}{\gamma_i}\log\left(\frac{\gamma_i}{\lambda^* e^{\rho_i k}\xi_k}\right) > 0.$$

By plugging this back into the budget constraints equation (3.3), one can solve this equation explicitly and verify the validity of (3.5). $\square$

3.2. **Equilibrium.** In the current subsection we show that there exists an equilibrium under the assumption that all endowments are positive. Moreover, we describe the set of all feasible equilibrium state price densities and the associated optimal consumption policies. For this purpose we introduce the following quantities. For each vector $(\lambda_1, ..., \lambda_N) \in \mathbb{R}^N_{++}$, we define

$$\beta_i(k) = \beta_i^{(\lambda_i)}(k) = \frac{\gamma_i}{\lambda_i e^{\rho_i k}}, \tag{3.6}$$

for all $i = 1, ..., N$ and all $k = 0, ..., T$. For a fixed $k = 0, ..., T$, let $i_1(k), ..., i_N(k)$ denote the order statistics of $\beta_1(k), ..., \beta_N(k)$, that is, $\{i_1(k), ..., i_N(k)\} = \{1, ..., N\}$ and $\beta_{i_1(k)}(k) \leq ... \leq \beta_{i_N(k)}(k)$. We set $\beta_{i_0(k)}(k) = 0$, for all $k = 0, ..., T$. With the preceding notations, we denote

$$\eta_j(k) = \eta_j^{(\lambda_1,...,\lambda_N)}(k) \; = \sum_{l=j+1}^{N} \frac{\log\left(\beta_{i_l(k)}(k)\right) - \log\left(\beta_{i_j(k)}(k)\right)}{\gamma_{i_l(k)}} \geq 0. \tag{3.7}$$

Note that $\eta_0(k) = +\infty$ and $\eta_N(k) = 0$, for all $k = 0, ..., T$. At last, we introduce a candidate for the equilibrium state price density

$$\xi_k\left(\lambda_1, ..., \lambda_N\right) = \tag{3.8}$$

$$\sum_{j=1}^{N} \left(\prod_{l=j}^{N} \beta_{i_l(k)}^{\left(\sum_{m=j}^{N} (\gamma_{i_l(k)}/\gamma_{i_m(k)})\right)^{-1}} \exp\left(-\frac{\epsilon_k}{\sum_{l=j}^{N} 1/\gamma_{i_l(k)}} \right) \right) \mathbf{1}_{\{\eta_j(k) \leq \epsilon_k < \eta_{j-1}(k)\}},$$

for all $k = 1, ..., T$.

Theorem 3.4. *Assume that $\epsilon_k^i > 0$ for each period k and each agent i, $P-$a.s. Then, every equilibrium state price density is given by $\left(\xi_k\left(\lambda_1^*, ..., \lambda_N^*\right)\right)_{k=0,...,T}$, where $\lambda_1^*, ..., \lambda_N^* \in \mathbb{R}_{++}$ are constants that solve the following system of equations*

$$\sum_{k=0}^{T} E\left[\xi_k(\lambda_1, ..., \lambda_N) I_i(\lambda_i e^{\rho_i k} \xi_k(\lambda_1, ..., \lambda_N))\right] = \sum_{k=0}^{T} E\left[\xi_k\left(\lambda_1, \lambda_2, ..., \lambda_N\right) \epsilon_k^i\right], \tag{3.9}$$

for $i = 1, ..., N$. The optimal consumption stream of agent i is given by

$$c_k^i = I_i(\lambda_i^* e^{\rho_i k} \xi_k(\lambda_1^*, ..., \lambda_N^*)), \tag{3.10}$$

for all $k = 0, ..., T$.

Remark. Theorem 3.4 constitutes a preliminary tool for proving the existence of an equilibrium (see Theorem 3.6) and yields a characterization of all feasible equilibrium pricing kernels.

Proof of Theorem 3.4. Let $(c_k^i)_{k=0,...,T}$ denote the optimal consumption stream of agent i. Recall that, by (3.2), we have that $c_k^i = I_i\left(\lambda_i^* e^{\rho_i k} \xi_k\right)$ for some $\lambda_i^* > 0$.

Plugging this into the market clearing condition (2.3), we obtain that the following holds in equilibrium:

$$\sum_{i=1}^{N} I_i\left(\lambda_i^* e^{\rho_i k}\xi_k\right) = \epsilon_k, \tag{3.11}$$

for all $k = 0, ..., T$. Here, $\lambda_1^*, ..., \lambda_N^*$ are constants that will be derived from the budget constraints in the sequel. Using the explicit form of I_i, this is equivalent to

$$\sum_{i=1}^{N} \log\left[\left(\frac{\gamma_i}{\gamma_i \mathbf{1}_{\left\{\lambda_i^* e^{\rho_i k}\xi_k > \gamma_i\right\}} + \lambda_i^* e^{\rho_i k}\xi_k \mathbf{1}_{\left\{\lambda_i^* e^{\rho_i k}\xi_k \le \gamma_i\right\}}}\right)^{1/\gamma_i}\right] = \epsilon_k,$$

a further transformation yields,

$$\prod_{i=1}^{N}\left(\gamma_i \mathbf{1}_{\{\xi_k > \beta_i(k)\}} + \lambda_i^* e^{\rho_i k}\xi_k \mathbf{1}_{\{\xi_k \le \beta_i(k)\}}\right)^{1/\gamma_i} = \prod_{i=1}^{N}\gamma_i^{1/\gamma_i}\exp(-\epsilon_k),$$

where $\beta_i(k) = \beta_i^{(\lambda_i^*)}(k)$ was defined in (3.6). This is equivalent to

$$\sum_{j=1}^{N} Y_j^{(k)} \mathbf{1}_{\left\{\beta_{i_{j-1}(k)}(k) < \xi_k \le \beta_{i_j(k)}(k)\right\}} + \prod_{i=1}^{N}\gamma_i^{1/\gamma_i}\mathbf{1}_{\left\{\xi_k > \beta_{i_N(k)}(k)\right\}} = \prod_{i=1}^{N}\gamma_i^{1/\gamma_i}\exp\left(-\epsilon_k\right), \tag{3.12}$$

where

$$Y_j^{(k)} = \prod_{l=1}^{j-1}\gamma_{i_l(k)}^{1/\gamma_{i_l(k)}}\prod_{l=j}^{N}(\lambda_{i_l(k)}^*)^{1/\gamma_{i_l(k)}}\exp\left(k\sum_{l=j}^{N}\frac{\rho_{i_l(k)}}{\gamma_{i_l(k)}}\right)\xi_k^{\sum_{l=j}^{N}\frac{1}{\gamma_{i_l(k)}}}.$$

The strict-positivity assumption on the endowments implies that $\epsilon_k > 0$, P-a.s, and thus $\xi_k \le \beta_{i_N}(k)$ holds $P-$a.s. Next, for each $k = 0, ..., T$, we have that

$$Y_i^{(k)}\mathbf{1}_{\left\{\beta_{i_{j-1}(k)}(k) < \xi_k \le \beta_{i_j(k)}(k)\right\}} = \prod_{i=1}^{N}\gamma_i^{1/\gamma_i}\exp\left(-\epsilon_k\right)\mathbf{1}_{\left\{\beta_{i_{j-1}(k)}(k) < \xi_k \le \beta_{i_j(k)}(k)\right\}},$$

which implies that the following holds on each set $\left\{\beta_{i_{j-1}(k)}(k) < \xi_k \le \beta_{i_j(k)}(k)\right\}$:

$$\xi_k = \prod_{l=j}^{N}(\beta_{i_l(k)}(k))^{\left(\sum_{m=j}^{N}\gamma_{i_l(k)}/\gamma_{i_m(k)}\right)^{-1}}\exp\left(-\frac{\epsilon_k}{\sum_{l=j}^{N}1/\gamma_{i_l(k)}}\right).$$

In particular, one checks that $\xi_k \le \beta_{i_j(k)}(k)$ is equivalent to $\epsilon_k \ge \eta_j(k)$ and $\beta_{i_{j-1}(k)}(k) < \xi_k$ is equivalent to $\epsilon_k < \eta_{j-1}(k)$, where, $\eta_j(k) = \eta_j^{\lambda_1^*,...,\lambda_N^*}(k)$ is given in (3.7). Now, one can revise the above identity in terms of $\lambda_1^*, ..., \lambda_N^*$ and conclude that every equilibrium state price density is of the form (3.8) for some $\lambda_1^*, ..., \lambda_N^*$. At last, observe that due to the budget constraints (2.2), in equilibrium, the constants $\lambda_1^*, ..., \lambda_N^*$ solve the system of equations (3.9). This concludes the proof. □

The following result is essential for proving an existence of equilibrium.

Lemma 3.5. *For each $i = 1, ..., N$, consider the excess demand function $g^i : \mathbb{R}^N_{++} \to \mathbb{R}$ defined by*

$$g^i(\lambda_1, ..., \lambda_N) = \tag{3.13}$$

$$\lambda_i^{-1} \sum_{k=0}^{T} \left(E\left[\xi_k(\lambda_1^{-1}, ..., \lambda_N^{-1}) I_i\left(\lambda_i^{-1} e^{\rho_i k} \xi_k(\lambda_1^{-1}, ..., \lambda_N^{-1})\right)\right] - E\left[\xi_k(\lambda_1^{-1}, ..., \lambda_N^{-1}) \epsilon^i_k\right] \right),$$

where $\xi_k(\lambda_1, ..., \lambda_N)$ is defined in (3.8). Then, the following properties are satisfied:

(1) Each function $g^i(\lambda_1, ..., \lambda_N)$ is a homogeneous function of degree 0.
(2) We have

$$\sum_{i=1}^{N} \lambda_i g^i(\lambda_1, ..., \lambda_N) = 0,$$

for all $(\lambda_1, ..., \lambda_N) \in \mathbb{R}^N_{++}$.
(3) Each function $g^i(\lambda_1, ..., \lambda_N)$ is continuous.
(4) (i) The following limit holds

$$\lim_{\lambda_i \to 0} g^i(\lambda_1, ..., \lambda_N) = -\infty.$$

(ii) Each function $g^i(\lambda_1, ..., \lambda_N)$ is bounded from above.

Proof of Lemma 3.5. (1) This follows from the identity $\xi_k((c\lambda_1)^{-1}, ..., (c\lambda_N)^{-1}) = c\xi_k(\lambda_1^{-1}, ..., \lambda_N^{-1})$, for all $c > 0$, which follows from the fact that $\eta_j^{(\lambda_1, ..., \lambda_N)} = \eta_j^{(c\lambda_1, ..., c\lambda_N)}$ and (3.8).
(2) By the construction of the pricing kernels in the proof of Theorem 3.4 (see (3.11)), it follows that $c^i_k = I_i\left(\lambda_i^{-1} e^{\rho_i k} \xi_k\left(\lambda_1^{-1}, ..., \lambda_N^{-1}\right)\right)$ satisfies the market clearing condition (2.3), for all $(\lambda_1^{-1}, ..., \lambda_N^{-1}) \in \mathbb{R}^N_{++}$. By changing the order of summation, the claim becomes equivalent to

$$\sum_{k=0}^{T} E\left[\xi_k(\lambda_1^{-1}, ..., \lambda_N^{-1}) \left(\sum_{i=1}^{N} I_i\left(\lambda_i^{-1} e^{\rho_i k} \xi_k\left(\lambda_1^{-1}, ..., \lambda_N^{-1}\right)\right) - \sum_{i=1}^{N} \epsilon^i_k\right)\right] = 0,$$

which follows from the latter observation concerning the market clearing condition.
(3) Consider some $x = (x_1, ..., x_N) \in \mathbb{R}^N_{++}$ and a sequence $x_m = (x^m_1, ..., x^m_N) \in \mathbb{R}^N_{++}$ such that $x_m \to x$. One checks that the random function $\xi_k(\lambda_1, ..., \lambda_N) : \mathbb{R}^N_{++} \to \mathbb{R}_{++}$ is $P-$a.s continuous, and hence $\lim_{m\to\infty} \xi_k(x_m) = \xi_k(x)$, and consequently $\lim_{m\to\infty} \xi_k(x_m)\, I_i(x^m_i e^{\rho_i k} \xi_k(x_m)) = \xi_k(x) I_i(x_i e^{\rho_i k} \xi_k(x))$. Therefore, it suffices to show uniform integrability in order to obtain L^1-convergence. Thus, it is enough to check that

$$\sup_{m\geq 1} E\left[\left(\xi_k(x_m) I_i(x^m_i e^{\rho_i k} \xi_k(x_m))\right)^p\right] < +\infty \quad , \quad \sup_{m\geq 1} E\left[(\xi_k(x_m)\epsilon_k)^p\right] < +\infty,$$

for some $p > 1$. Since $x_i > 0$, for all $i = 1, ..., N$, we can assume that there exists $\varepsilon > 0$ such that $x^m_i > \varepsilon$, for all $i = 1, ..., N$ and all m. Therefore, we can estimate $\xi_k(x_m) \le Ke^{-M\epsilon_k}$, for all m and some constants $K, M > 0$. Hence, $\sup_{m\ge1} E\left[(\xi_k(x_m)\epsilon_k)^p\right] \le K^p E\left[e^{-Mp\epsilon_k}\epsilon_k^p\right] < \left(\frac{K}{Me}\right)^p$. Next, one checks that

$$\begin{aligned}
&\sup_{m\ge1} E\left[\left(\xi_k(x_m) I_i(x_i^m e^{\rho_i k}\xi_k(x_m))\right)^p\right] \\
&\le \frac{1}{\gamma_i^p}\sup_{m\ge1} E\left[\left(\frac{\gamma_i}{x_i^m e^{\rho_i k}} - \xi_k(x_m)\right)^p \mathbf{1}_{\left\{\frac{\gamma_i}{x_i^m e^{\rho_i k}} - \xi_k(x_m)\ge0\right\}}\right] \\
&\le \frac{1}{\gamma_i^p}\sup_{m\ge1} E\left[\left(\frac{\gamma_i}{x_i^m e^{\rho_i k}} + Ke^{-M\epsilon_k}\right)^p\right] < \frac{1}{\gamma_i^p}\left(\frac{\gamma_i}{\varepsilon e^{\rho_i k}} + K\right)^p,
\end{aligned}$$

where the first inequality follows from the fact that $1+\log t \le t$, for all $t \ge 1$. This shows that $g(\lambda_1, ..., \lambda_N)$ is continuous.

(4) (i) For each $i = 1, ..., N$ and $k = 0, ..., T$, consider the random function (depending on $\omega \in \Omega$) $f^i(k, \lambda_1, ..., \lambda_N) : \mathbb{R}^N_{++} \to \mathbb{R}$ given by

$$f^i(k, \lambda_1, ..., \lambda_N) = \lambda_i \xi_k(\lambda_1, ..., \lambda_N)\left(I_i\left(\lambda_i e^{\rho_i k}\xi_k(\lambda_1, ..., \lambda_N)\right) - \epsilon_k^i\right).$$

The claim is equivalent to showing that $\lim_{\lambda_i\to\infty} E\left[f^i(k, \lambda_1, ..., \lambda_N)\right] = -\infty$. First, we show that $\lim_{\lambda_i\to\infty} \xi_k(\lambda_1, ..., \lambda_N)\left(I_i\left(\lambda_i e^{\rho_i k}\xi_k(\lambda_1, ..., \lambda_N)\right) - \epsilon_k^i\right) < 0$, $P-$a.s. Assume that λ_i is sufficiently large so that $i = i_1(k)$ and $\lambda_i = \lambda_{i_1(k)} > \max\{\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_N\}$. One checks that $\lim_{\lambda_i\to\infty} \eta_1(k) = \infty$. This implies that

$$\lim_{\lambda_i\to\infty} \xi_k\left(\lambda_1, ..., \lambda_N\right) \tag{3.14}$$

$$= \prod_{l=j}^{N} \beta_{i_l(k)}^{\left(\sum_{m=j}^N \gamma_{i_l(k)}/\gamma_{i_m(k)}\right)^{-1}} \exp\left(-\frac{\epsilon_k}{\sum_{l=j}^N 1/\gamma_{i_l(k)}}\right)\mathbf{1}_{\{\eta_2(k)\le\epsilon_k<+\infty\}} +$$

$$\sum_{j=3}^{N}\left(\prod_{l=j}^{N} \beta_{i_l(k)}^{\left(\sum_{m=j}^N \gamma_{i_l(k)}/\gamma_{i_m(k)}\right)^{-1}} \exp\left(-\frac{\epsilon_k}{\sum_{l=j}^N 1/\gamma_{i_l(k)}}\right)\right)\mathbf{1}_{\{\eta_j(k)\le\epsilon_k<\eta_{j-1}(k)\}} > 0,$$

$P-$a.s. Next, we claim that $\lim_{\lambda_i\to\infty} I_i\left(\lambda_i e^{\rho_i k}\xi_k(\lambda_1, ..., \lambda_N)\right) = 0$. Indeed, recall that

$$I_i\left(\lambda_i e^{\rho_i k}\xi_k(\lambda_1, ..., \lambda_N)\right) = \frac{1}{\gamma_i}\log\left(\frac{\gamma_i}{\lambda_i e^{\rho_i k}\xi_k(\lambda_1, ..., \lambda_N)}\right)\mathbf{1}_{\left\{\frac{\gamma_i}{e^{\rho_i k}\xi_k(\lambda_1,...,\lambda_N)}\ge\lambda_i\right\}},$$

and note that, by (3.14),

$$\mathbf{1}_{\left\{\frac{\gamma_i}{e^{\rho_i k}\xi_k(\lambda_1,...,\lambda_N)}\ge\lambda_i\right\}} = 0$$

is satisfied identically P-a.s, for sufficiently large λ_i. This yields that

$$\lim_{\lambda_i\to\infty} \xi_k(\lambda_1, ..., \lambda_N)\left(I_i\left(\lambda_i e^{\rho_i k}\xi_k(\lambda_1, ..., \lambda_N)\right) - \epsilon_k^i\right) < 0.$$

Now, consider an arbitrary sequence $(a_n)_{n=1}^\infty$ such that $a_n \to +\infty$, and denote $b_n = (\lambda_1, ..., \lambda_{i-1}, a_n, \lambda_{i+1}, ..., \lambda_N) \in \mathbb{R}^N_{++}$. The same arguments as in (3) can be applied to show that the sequence $\left\{\xi_k(b_n)\left(I_i\left(a_n e^{\rho_i k}\xi_k(b_n)\right)-\epsilon_k^i\right)\right\}_{n=1}^{\infty}$ is uniformly integrable. Therefore,

$$\lim_{\lambda_i\to\infty} E\left[f^i(k,\lambda_1,...,\lambda_N)\right]$$
$$=\lim_{\lambda_i\to\infty}\lambda_i E\left[\lim_{\lambda_i\to\infty}\xi_k(\lambda_1,...,\lambda_N)\left(I_i\left(\lambda_i e^{\rho_i k}\xi_k(\lambda_1,...,\lambda_N)\right)-\epsilon_k^i\right)\right]=-\infty.$$

This accomplishes the proof of part (i).
(ii) This Follows by the fact that the function $h:\mathbb{R}_{++}\to\mathbb{R}$ given by $h(x)=x\big((\log 1/x)^+-a\big)$ is bounded from above, for any fixed $a>0$. □

The next statement establishes the existence of an equilibrium.

Theorem 3.6. *Assume that $\epsilon_k^i>0$ for each period k and each agent i, $P-$a.s. Then, there exists an equilibrium. Furthermore, a process $(\xi_k')_{k=0,...,T}$ is an equilibrium state price density if and only if $\xi_k'=\xi_k(\lambda_1^*,...,\lambda_N^*)$ for some $(\lambda_1^*,...,\lambda_N^*)\in\mathbb{R}_{++}^N$ that solves the system of equations (3.9). The optimal consumption stream is given by (3.10).*

Proof of Theorem 3.6. By Theorem (3.4) it is sufficient to prove that $\xi_k(\lambda_1^*,...,\lambda_N^*)$ and $I_i(\lambda_i^* e^{\rho_i k}\xi_k(\lambda_1^*,...,\lambda_N^*))$ is an equilibrium. Therefore, it is left to check that the system of equations (3.9) has a solution, or equivalently, the system of equations $g^i(\lambda_1,...,\lambda_N)=0$, for $i=1,...,N$ (see (3.13)) has a solution. The existence of a solution follows by properties (1)-(4) in Lemma 3.5 and fixed point arguments similar to those appearing in Theorem 17.C.1 in Mas-Colell et al. (1995). □

As will be shown in the next section, the equilibrium state price density is, in general, not unique. Nevertheless, when the economy is homogeneous, the utility maximization problem is similar to the one in the non-constrained setting for consumption, and in particular assures uniqueness.

Example: Homogeneous Economy. In an economy populated only by an agent i that holds a strictly positive endowment stream $(\epsilon_k^i)_{k=0,...,T}$, there exists a unique equilibrium and the corresponding homogeneous state price density process $\{\xi_k^i\}_{k=0,...,T}$ is given by

$$\xi_k^i=e^{-\rho_i\left(\epsilon_k^i-\epsilon_0^i\right)},$$

for all $k=1,...,T$. The optimal consumptions obviously coincide with the endowments: $c_k^i=\epsilon_k^i$, for all $k=0,...,T$.

4. Non-Uniqueness of The Equilibrium

4.1. **Non-Uniqueness with Positive Endowments.** It is evident that the system of equations (3.9) is related to the uniqueness of the equilibrium. The system of equations (3.9) is somewhat cumbersome in certain aspects: the functions

$\xi_k(\lambda_1, ..., \lambda_N)$ are not differentiable with respect to each variable λ_i; it might happen that (3.9) has infinitely many different solutions but the equilibrium is still unique; the property of "gross substitution" (see Definition 3.1 in Dana (1993b)), which would be a sufficient condition for the uniqueness of the equilibrium, is not necessarily satisfied. The next example demonstrates the existence of multiple equilibria.

Example: Non-Uniqueness of Equilibrium. We assume a one period market with $\mathcal{F}_0 = \mathcal{F}_1 = \{\Omega, \emptyset\}$. Consider two agents $i = 1, 2$ represented by $u_1(x) = u_2(x) = -e^{-x}$ and $\rho_1 = \rho_2 = 0$. The agents hold different endowments $\epsilon_0^1, \epsilon_1^1$ and $\epsilon_0^2, \epsilon_1^2$ respectively. Let ϵ_0 and ϵ_1 denote the aggregate endowments. By Theorem 3.4, every equilibrium state price density is of the form

$$\xi_1(x, y) = \frac{1}{\min\{x, y\}} \mathbf{1}_{\{\epsilon_1 < \log \frac{\max\{x,y\}}{\min\{x,y\}}\}} e^{-\epsilon_1} + \frac{1}{\sqrt{xy}} \mathbf{1}_{\{\log \frac{\max\{x,y\}}{\min\{x,y\}} \le \epsilon_1\}} e^{-\epsilon_1/2},$$

where x and y are to be determined by the budget constraints. One can check that it is possible to rewrite it as

$$\xi_1(x, y) = \begin{cases} \frac{1}{e^{\epsilon_1}} \frac{1}{x} & \text{if } 0 < x < \frac{y}{e^{\epsilon_1}}, \\ \frac{1}{\sqrt{y} e^{\epsilon_1/2}} \frac{1}{\sqrt{x}} & \text{if } y e^{-\epsilon_1} \le x \le y e^{\epsilon_1}, \\ \frac{1}{y e^{\epsilon_1}} & \text{if } x > y e^{\epsilon_1}. \end{cases}$$

The positive arguments x, y solve equations (3.9) which take the form:

$$(1)\ \log(1/y) \mathbf{1}_{\{y \le 1\}} + \xi_1(x, y) \log\left(\frac{1}{y \xi_1(x, y)}\right) \mathbf{1}_{\{x \xi_1(x,y) \le 1\}} = \epsilon_0^1 + \epsilon_1^1 \xi_1(x, y),$$

$$(2)\ \log(1/x) \mathbf{1}_{\{x \le 1\}} + \xi_1(x, y) \log\left(\frac{1}{x \xi_1(x, y)}\right) \mathbf{1}_{\{x \xi_1(x,y) \le 1\}} = \epsilon_0^2 + \epsilon_1^2 \xi_1(x, y).$$

Let us note that we work with a normalized state price density, i.e., $\xi_0 = 1$. We denote

$$h(x, y) = \xi_1(x, y) \left(\log(\left(\frac{1}{y \xi_1(x, y)}\right) \mathbf{1}_{\{x \xi_1(x,y) \le 1\}} - \epsilon_1^1 \right),$$

and

$$g(x, y) = \xi_1(x, y) \left(\log\left(\frac{1}{x \xi_1(x, y)}\right) \mathbf{1}_{\{x \xi_1(x,y) \le 1\}} - \epsilon_1^2 \right).$$

Observe that

$$h(x, y) = \begin{cases} -\epsilon_1^1 \frac{1}{e^{\epsilon_1}} \frac{1}{x} & \text{if } 0 < x < \frac{y}{e^{\epsilon_1}}, \\ \frac{1}{\sqrt{y} e^{\epsilon_1/2}} \frac{1}{\sqrt{x}} \left(\log\left(\frac{\sqrt{x} e^{\epsilon_1/2}}{\sqrt{y}}\right) - \epsilon_1^1 \right) & \text{if } y e^{-\epsilon_1} \le x \le y e^{\epsilon_1}, \\ \frac{1}{y e^{\epsilon_1}} \left(\epsilon_1 - \epsilon_1^1\right) & \text{if } x > y e^{\epsilon_1}. \end{cases}$$

and that

$$g(x, y) = \begin{cases} -\epsilon_1^2 \frac{1}{e^{\epsilon_1}} \frac{1}{x} & \text{if } 0 < x < \frac{y}{e^{\epsilon_1}}, \\ \frac{1}{\sqrt{y} e^{-1/2\epsilon_1}} \frac{1}{\sqrt{x}} \left(\log\left(\frac{\sqrt{y} e^{1/2\epsilon_1}}{\sqrt{x}}\right) - \epsilon_1^2 \right) & \text{if } y e^{-\epsilon_1} \le x \le y e^{\epsilon_1}, \\ \frac{1}{y e^{\epsilon_1}} \left(\epsilon_1 - \epsilon_1^2\right) & \text{if } x > y e^{\epsilon_1}. \end{cases}$$

Hence, equations (1) and (2) can be rewritten as

$$(1')\ \log(1/y)\mathbf{1}_{\{y\leq 1\}} + h(x,y) = \epsilon_0^1,$$

$$(2')\ \log(1/x)\mathbf{1}_{\{x\leq 1\}} + g(x,y) = \epsilon_0^2.$$

We are going to define the endowments in a particular way that will yield two distinct solutions to (1′) and (2′). More precisely, we are going to construct two solutions (x_1, y_1) and (x_2, y_2) such that $y_l < 1$ and $0 < x_l < y_l e^{-\epsilon_1}$, for $l = 1, 2$. Set $\epsilon_1^1 = \epsilon_1^2$, $e^{2\epsilon_1^1 - 1} > \epsilon_1^1$ and $\epsilon_1^1 < e^{-1}$. We start by treating equation (2′). Consider the function $\phi(x) = \log(1/x) - \frac{\epsilon_1^1}{e^{\epsilon_1}}\frac{1}{x}$ on the interval $[0, ye^{-\epsilon_1}]$, for arbitrary $y > 0$. Note that $\phi'(x) = \frac{1}{x^2}\frac{\epsilon_1^1}{e^{\epsilon_1}} - \frac{1}{x}$, and hence $\phi'(x) > 0$ if $x < \frac{\epsilon_1^1}{e^{\epsilon_1}}$, and $\phi'(x) < 0$ if $x > \frac{\epsilon_1^1}{e^{\epsilon_1}}$, which implies that $\frac{\epsilon_1^1}{e^{\epsilon_1}}$ is a maximum of ϕ. Furthermore, y_l (to be determined explicitly in the sequel) will satisfy $\epsilon_1^1 < y_l$, and this will guarantee that the maximum is indeed in the domain of definition of ϕ, i.e., $\frac{\epsilon_1^1}{e^{\epsilon_1}} \in [0, y_l e^{-\epsilon_1}]$. Next, note that $\phi(\frac{\epsilon_1^1}{e^{\epsilon_1}}) = \log\left(\frac{e^{\epsilon_1}}{\epsilon_1^1}\right) - 1 > 0$ due to the assumption $e^{2\epsilon_1^1 - 1} > \epsilon_1^1$. Now let $\delta > 0$ be some small quantity to be determined below. One can pick ϵ_0^2 such that the equation $\phi(x) = \epsilon_0^2$ has exactly two solutions x_1 and x_2 in the interval $[\frac{\epsilon_1^1}{e^{\epsilon_1}} - \delta, \frac{\epsilon_1^1}{e^{\epsilon_1}} + \delta]$. Now, equation (1′) has two solutions denoted by y_1 and y_2 (depending on ϵ_0^1) corresponding to x_1 and x_2 that are given by

$$y_l = \exp\left(-\epsilon_0^1 - \frac{\epsilon_1^1}{e^{\epsilon_1}}\frac{1}{x_l}\right),$$

for $l = 1, 2$. Obviously, $y_1, y_2 < 1$. It is left to check that $\max\{x_l, \frac{\epsilon_1^2}{e^{\epsilon_1}}\} < y_l e^{-\epsilon_1}$, for $l = 1, 2$, that is,

$$\max\{x_l, \frac{\epsilon_1^1}{e^{\epsilon_1}}\} < \exp\left(-\epsilon_1 - \epsilon_0^1 - \frac{\epsilon_1^1}{e^{\epsilon_1}}\frac{1}{x_l}\right).$$

Since $x_l \in [\frac{\epsilon_1^2}{e^{\epsilon_1}} - \delta, \frac{\epsilon_1^2}{e^{\epsilon_1}} + \delta]$, it suffices to verify that

$$e^{\epsilon_1}\left(\frac{\epsilon_1^1}{e^{\epsilon_1}} + \delta\right) < \exp\left(-\epsilon_0^1 - \frac{\epsilon_1^1}{e^{\epsilon_1}}\frac{1}{\frac{\epsilon_1^1}{e^{\epsilon_1}} - \delta}\right),$$

for an appropriate choice of $\delta > 0$ and ϵ_0^1. This inequality is equivalent to

$$\epsilon_1^1 + \delta e^{\epsilon_1} < \exp\left(-\epsilon_0^1 - \frac{\epsilon_1^1}{\epsilon_1^1 - \delta e^{\epsilon_1}}\right).$$

By continuity, it suffices to prove this inequality for $\delta = 0$ and $\epsilon_0^1 = 0$, which becomes $\epsilon_1^1 e < 1$, and follows from the assumptions imposed on ϵ_1^1. □

4.2. **Non-Uniqueness with Vanishing Endowments.** In Theorems 3.4 and 3.6 we have assumed that $P(\epsilon_k^i > 0) = 1$, for all $k = 1, ..., T$ and all $i = 1, ..., N$. This assumption was crucial for proving that every equilibrium state price density is of the form (3.8). It turns out that once this assumption is relaxed, there exist necessarily infinitely many equilibria all of the same canonical form.

Theorem 4.1. *Assume that all the assumptions of Theorem 3.4 hold, except that $P(\epsilon_k = 0) > 0$ and $P(\cup_{k=0}^{T}\{\epsilon_k^i > 0\}) > 0$, for all $i = 1, ..., N$. Then, there exist infinitely many equilibria. Every equilibrium state price density $(\widetilde{\xi}_k)_{k=0,...,T}$ is of the form*

$$\widetilde{\xi}_k(\widetilde{\lambda}_1, ..., \widetilde{\lambda}_N) = \xi_k(\widetilde{\lambda}_1, ..., \widetilde{\lambda}_N)\mathbf{1}_{\{\epsilon_k \neq 0\}} + X_k \mathbf{1}_{\{\epsilon_k = 0\}}, \tag{4.1}$$

*for all $k = 1, ..., T$, where $\xi_k(\widetilde{\lambda}_1, ..., \widetilde{\lambda}_N)$ is given by (3.8) and X_k is an **arbitrary** $\mathcal{F}_k$-measurable random variable that satisfies $E[X_k] < \infty$ and $X_k > \beta_{i_N(k)}^{(\widetilde{\lambda}_{i_N(k)})}(k)$, P-a.s, where $\beta_{i_N(k)}^{(\widetilde{\lambda}_{i_N(k)})}(k)$ is given in (3.6). The constants $\widetilde{\lambda}_1, ..., \widetilde{\lambda}_N$ are determined by the budget constraints*

$$\sum_{k=0}^{T} \left(E\left[\widetilde{\xi}_k(\lambda_1, ..., \lambda_N)) I_i\left(\lambda_i e^{\rho_i k} \widetilde{\xi}_k(\lambda_1, ..., \lambda_N)\right)\right] - E\left[\widetilde{\xi}_k(\lambda_1, ..., \lambda_N)\epsilon_k^i\right]\right) = 0$$

for $i = 1, ..., N$.

Proof of Theorem 4.1. The proof is identical to the proofs of Theorem 3.4 and Theorem 3.6 apart from a slight modification as follows. Consider equation (3.12) and note that in the current context this equation admits the form $\mathbf{1}_{\{\widetilde{\xi}_k > \beta_{i_N(k)}(k)\}} = 1$ on the set $\{\epsilon_k = 0\}$, which implies that $\widetilde{\xi}_k$ is of the form (4.1). The rest follows by similar arguments to those found in Section 3. □

We illustrate the above phenomenon in the following elementary example.

Example: Infinitely Many Equilibria. Let $(\Omega, \mathcal{F}_1, P)$ be a probability space where $\Omega = \{\omega_1, \omega_2\}$, $P(\{\omega_1\}), P(\{\omega_2\}) > 0$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_1 = 2^{\Omega}$. Consider a one period homogeneous economy with an individual represented by the utility function $u(x) = -e^{-x}$ and $\rho = 0$. The endowments of the agent are denoted by ϵ_0 and ϵ_1. For the sake of transparency, we analyze the following two simple cases directly by using the definition of equilibrium rather than by using Theorem 4.1.
(i) Let $\epsilon_0 = 0$ and ϵ_1 be an arbitrary $\mathcal{F}_1-$measurable positive random variable. Theorem 3.1 implies that the optimal consumption policies c_0 and c_1 are given by $c_0 = -\log\left(\mathbf{1}_{\{\lambda>1\}} + \lambda\mathbf{1}_{\{\lambda\leq1\}}\right)$ and $c_1 = -\log\left(\mathbf{1}_{\{\lambda\xi_1>1\}} + \lambda\xi_1\mathbf{1}_{\{\lambda\xi_1\leq1\}}\right)$. The market clearing condition $c_0 = 0$ and $c_1 = \epsilon_1$ implies that $\xi_1 = \frac{e^{-\epsilon_1}}{\lambda}$ is an equilibrium state price density, for all $\lambda \geq 1$. Note that the budget constraints of the type (2.2) are automatically satisfied due to the fact that the market clears and due to the homogeneity of the economy. We stress out that for the corresponding unconstrained problem

$$\sup_{c_0\in\mathbb{R}, c_1\in L^0(\mathcal{F}_1)} -e^{-c_0} - E[e^{-c_1}],$$

under the budget constraint

$$c_0 + E[\xi_1 c_1] = \epsilon_0 + E[\xi_1 \epsilon_1],$$

there exists a unique equilibrium corresponding to $\lambda = 1$, that is, $\xi_1 = e^{-\epsilon_1}$.
(ii) Let $\epsilon_0 > 0$ be arbitrary, $\epsilon_1(\omega_1) > 0$ and $\epsilon_1(\omega_2) = 0$. Then, by Theorem 3.1 we obtain that $c_0 = \log(1/\lambda) = \epsilon_0$ and $c_1 = \max\{\log(\frac{1}{\lambda\xi_1}), 0\} = \epsilon_1$. It follows that $\lambda = e^{-\epsilon_0}$, and that there are infinitely many equilibrium state price densities of the form $\xi_1(y) = e^{\epsilon_0 - \epsilon_1}\mathbf{1}_{\{\epsilon_1 \neq 0\}} + y\mathbf{1}_{\{\epsilon_1 = 0\}}$, for every $y > e^{\epsilon_0}$. As in (i), the market clearing condition is redundant due to the homogeneity of the economy. □

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