MODEL-INDEPENDENT BOUNDS FOR OPTION PRICES: A MASS TRANSPORT APPROACH

MATHIAS BEIGLBÖCK, PIERRE HENRY-LABORDÈRE, AND FRIEDRICH PENKNER

Abstract. In this paper we investigate model-independent bounds for exotic options written on a risky asset using infinite-dimensional linear programming methods.

Using arguments from the theory of Monge-Kantorovich mass-transport we establish a dual version of the problem that has a natural financial interpretation in terms of semi-static hedging.

1. Introduction

Since the introduction of the Black-Scholes paradigm, several alternative models which allow to capture the risk of exotic options have emerged: local volatility models, stochastic volatility models, jump-diffusion models, mixed local stochastic volatility models. These models depend on various parameters which can be calibrated more or less accurately to market prices of liquid options (such as vanilla options). This calibration procedure does not uniquely set the dynamics of forward prices which are only required to be (local) martingales according to the no-arbitrage framework. This could lead to a wide range of prices of a given exotic option when evaluated using different models calibrated to the same market data.

In practice, it would be interesting to know lower and upper bounds for exotic options produced by models calibrated to the same market data, and therefore with similar marginals. If bounds are tight enough, they would be used to detect arbitrage in market prices, provided these bounds have an interpretation as investment strategies. This problem has already been studied in the case of exotic options written on multi-assets $(S_1, \dots, S_T)$ observed at the same time T [BP02, CDDV08, HLW05a, HLW05b, LW05, LW04]. Within the class of models with fixed marginals $\left(\mathrm{Law}(S^1_T), \dots, \mathrm{Law}(S^k_T)\right)$ at T, the search for lower/upper bounds involves infinite-dimensional linear programming issues. Analytical expressions have been obtained in the case of basket options [LW05, LW04]. These correspond to the determination of optimal copulas.

Here we focus on discrete multi-period models. This problem, which has not been extensively considered in the literature as far as we know (a notable exception is [NH00]), is much more involved as we have to impose that the asset price S_t is a discrete time martingale[1]. This additional constraint is more restrictive and therefore allows in principle to obtain tighter bounds.

The problem of determining the interval of consistent prices of a given exotic option can be cast as a (primal) infinite-dimensional linear programming problem. We propose a dual problem that has a practically relevant interpretation in terms of trading strategies and prove that there is no duality gap under rather mild regularity assumptions.

Setting. In the following, we fix an exotic option depending only on the value of a single asset S at discrete times $t_1 < \dots < t_n$ and denote by $\Phi(S_1, \dots, S_n)$ its payoff. In the no-arbitrage framework, the standard approach is to postulate a model, that is, a probability measure $\mathbb{Q}$ on $\mathbb{R}^n$ under which the coordinate process $(S_i)_{i=1}^n$

$$S_i : \mathbb{R}^n \to \mathbb{R},\ S_i(s_1, \dots, s_n) = s_i,\ i = 1, \dots, n,$$

Key words and phrases. Model-independent pricing, Monge-Kantorovich transport problem, option arbitrage.

[1] For the sake of simplicity, we assume zero interest rate and no cash/yield dividends. This assumption can be relaxed by considering the process f_t introduced in [HL09] (see equation 14) which has the property to be a local martingale.

is required to be a (discrete) martingale in its own filtration. The fair value is then given as the expectation of the payoff

$$\mathbb{E}_Q[\Phi] = \int_{\mathbb{R}^n} \Phi(s_1, \dots, s_n)\, dQ(s_1, \dots, s_n).$$

Additionally, we impose that our model is calibrated to a continuum of call options with payoffs $\Phi_{i,K}(S_i) = (S_i - K)^+, K \in \mathbb{R}$ at each date t_i and price

$$C(t_i, K) = \mathbb{E}_Q[\Phi_{i,K}] = \int_{\mathbb{R}^+} (s - K)_+ \, d\mathrm{Law}_{S_i}(s). \tag{1}$$

Plainly (1) is tantamount to prescribing probability measures $\mu_1, \dots, \mu_n$ on the real line[2] so that the *one dimensional marginals* of $\mathbb{Q}$ satisfy

$$\mathbb{Q}^i = \mathrm{Law}_{S_i} = \mu_i \ \text{ for all } i = 1, \dots, n.$$

Primal formulation. For further reference, we denote by $\mathcal{M}(\mu_1, \dots, \mu_n)$ the set of all martingale measures $\mathbb{Q}$ on (the pathspace) $\mathbb{R}^n$ having marginals $\mathbb{Q}^1 = \mu_1, \dots, \mathbb{Q}^n = \mu_n$. Equivalently, we have $\mathbb{Q} \in \mathcal{M}(\mu_1, \dots, \mu_n)$ if and only if $\mathbb{E}_{\mathbb{Q}}[S_i | S_1, \dots, S_{i-1}] = S_{i-1}$ for $i = 2, \dots, n$ and $\mathbb{E}_{\mathbb{Q}}[\Phi_{i,K}] = C(t_i, K)$ for all $K \in \mathbb{R}$ and $i = 1, \dots, n$.

Following the tradition customary in the optimal transport literature we concentrate on the lower [3] bound and consider the *primal problem*

$$P = \inf \{\mathbb{E}_Q[\Phi] : \mathbb{Q} \in \mathcal{M}(\mu_1, \dots, \mu_n)\}. \tag{2}$$

Dual formulation. The dual formulation corresponds to the construction of a *semi-static subhedging strategy* consisting of the sum of a static vanilla portfolio and a delta strategy. More precisely, we are interested in payoffs of the form

$$\Psi_{(u_i),(\Delta_j)}(s_1, \dots, s_n) = \sum_{i=1}^n u_i(s_i) + \sum_{j=1}^{n-1} \Delta_j(s_1, \dots, s_j)(s_{j+1} - s_j), \qquad s_1, \dots, s_n \in \mathbb{R}, \tag{3}$$

where the functions $u_i : \mathbb{R} \to \mathbb{R}$ are μ_i-integrable ($i = 1, \dots, n$) and the functions $\Delta_j : \mathbb{R}^j \to \mathbb{R}$ are assumed to be bounded measurable ($j = 1, \dots, (n-1)$).

If these functions lead to a strategy which is subhedging in the sense

$$\Phi \geq \Psi_{(u_i),(\Delta_j)}$$

we have for every pricing measure $\mathbb{Q} \in \mathcal{M}(\mu_1, \dots, \mu_n)$ the obvious inequality

$$\mathbb{E}_{\mathbb{Q}}[\Phi] \geq \mathbb{E}_{\mathbb{Q}}[\Psi_{(u_i),(\Delta_j)}] = \mathbb{E}_{\mathbb{Q}}\Big[\sum_{i=1}^n u_i(S_i)\Big] = \sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i]. \tag{4}$$

This leads us to consider the *dual problem*

$$D = \sup \Big\{ \sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i] : \exists\, \Delta_1, \dots, \Delta_{n-1} \text{ s.t. } \Psi_{(u_i),(\Delta_j)} \leq \Phi \Big\}; \tag{5}$$

which, by (4), satisfies

$$P \geq D. \tag{6}$$

[2]The cumulative distribution function of μ_i can be read off the call prices through $F_i(K) = \lim_{\varepsilon \downarrow 0} 1/\varepsilon [C(t_i, K - \varepsilon) - C(t_i, K)]$ for $i = 1, \dots, n$.

From a financial perspective it doesn't make much sense to consider marginals which give mass to the negative half-line. However, as this has no effect to our arguments, we prefer not to exclude this case.

[3]Upper bounds can be obtained similarly by replacing Φ with $-\Phi$. However, we point out that the assumptions in our Duality Theorem 1 are sensible to this sign change and seem less satisfying if one is interested in obtaining a tight upper bound.

Semi-static subhedging. The dual formulation corresponds to the construction of a semi-static subhedging portfolio consisting in static vanilla options $u_i(S_i)$ and investments in the risky asset according to the self-financing trading strategy $(\Delta_i(S_1, \dots, S_i))_{i=1}^{n-1}$.

We note the financial interpretation of inequality (6): suppose somebody offers the option Φ at a price $p < D$. Then there exists $(u_i), (\Delta_j)$ with $\Psi_{(u_i),(\Delta_j)} \leq \Phi$ with price $\sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i]$ strictly larger than p. Buying Φ and going short in $\Psi_{(u_i),(\Delta_j)}$, the arbitrage can be locked in.

The crucial question is of course if (6) is sharp, i.e. if every option priced below P allows for an arbitrage by means of semi-static subhedging. We show that this is the case under relatively mild assumptions.

Main result.

Theorem 1. *Assume that $\mu_1, \dots, \mu_n$ are Borel probability measures on $\mathbb{R}$ so that $\mathcal{M}(\mu_1, \dots, \mu_n)$ is non-empty. Let $\Phi : \mathbb{R}^n \to (\infty, \infty]$ be a lower semi-continuous function so that*

$$\Phi(s_1, \dots, s_n) \geq -K \cdot (1 + |s_1| + \dots + |s_n|) \tag{7}$$

on $\mathbb{R}^n$ for some constant K. Then there is no duality gap, i.e. $P = D$. Moreover, the primal value[4] P is attained, i.e. there exists a martingale measure $\mathbb{Q} \in \mathcal{M}(\mu_1, \dots, \mu_n)$ such that $P = \mathbb{E}_Q[\Phi]$.

Our approach to this result is based on the duality theory of optimal transport which is briefly introduced in Section 2; the actual proof will be given in Section 3 with the help of the Min-Max Theorem of decision theory.

We conclude this introductory section by a short discussion of the content of Theorem 1.

The assumption $\mathcal{M}(\mu_1, \dots, \mu_n) \neq \emptyset$ excludes the degenerate case in which no calibrated market model exists. For the existence of a martingale measure having marginals $\mu_1, \dots, \mu_n$ it is necessary and sufficient that these measures possess the same finite first moments and increase in the *convex order*, i.e. $\mathbb{E}_{\mu_1}\phi \leq \dots \leq \mathbb{E}_{\mu_n}\phi$ for each convex function $\phi : \mathbb{R} \to \mathbb{R}$ (cf. [Str65]).[5]

Having financial applications in mind, it is worth noting that (in the setting of Theorem 1) the value D of the dual problem remains unchanged if a smaller set of subhedging strategies $\Psi_{(u_i),(\Delta_j)}$ is used. It is sufficient to consider functions $u_1, \dots, u_n$ which are linear combinations of finitely many call options (plus one position in the bond resp. the stock); at the same time $\Delta_1, \dots, \Delta_{n-1}$ can be taken to be continuous and bounded.

Condition (7) could be somewhat relaxed. For instance it is sufficient to demand that the function Φ is bounded from below by the sum of integrable functions. However, in this case it is necessary to allow for dual strategies that use European options beyond call options and we will not pursue this further.

We conclude this introductory section by noting that an *upper* bound for the price of the option Φ can be given means of *semi-static (super)hedging*. Applying Theorem 1 to the function $-\Phi$ we obtain that this bound is sharp:

Corollary 1.1. *Assume that $\mu_1, \dots, \mu_n$ are Borel probability measures on $\mathbb{R}$ so that $\mathcal{M}(\mu_1, \dots, \mu_n)$ is non-empty. Let $\Phi : \mathbb{R}^n \to [\infty, \infty)$ be an upper semi-continuous function so that*

$$\Phi(s_1, \dots, s_n) \leq K \cdot (1 + |s_1| + \dots + |s_n|) \tag{8}$$

on $\mathbb{R}^n$ for some constant K. Then there is no duality gap

$$\sup\Big\{\mathbb{E}_{\mathbb{Q}}\Phi : \mathbb{Q} \in \mathcal{M}(\mu_1, \dots, \mu_n)\Big\} = \inf\Big\{\sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i] : \exists\, \Delta_1, \dots, \Delta_{n-1} \ s.t.\ \Psi_{(u_i),(\Delta_j)} \geq \Phi\Big\}.$$

The supremum is obtained, i.e. there exists a maximizing martingale measure.

[4]The dual supremum is in general not attained, cf. Example 4.2.

[5]In more financial terms this means that $C(t, K)$ is increasing in t for each fixed $K \in \mathbb{R}$.

2. Optimal Transport

In the usual theory of Monge-Kantorovich optimal transport[6] one considers two probability spaces (X_1, μ_1), (X_2, μ_2) and the problem is to find a "cheap" way of transporting μ_1 to μ_2. Following Kantorovich, a transport plan is formalized as probability measure π on $X_1 \times X_2$ which has X_1-marginal μ_1 and X_2-marginal μ_2.

We will come back to the two dimensional case in Section 4 below; for now we turn to the *multidimensional version* of the transport problem which will be the main tool in our proof of Theorem 1. Subsequently we consider probability measures $\mu_1, \dots, \mu_n$ on the real line[7] which have finite first moments. The set $\Pi(\mu_1, \dots, \mu_n)$ of *transport plans* consists of all Borel probability measures on $\mathbb{R}^n$ with marginals $\mu_1, \dots, \mu_n$. A *cost function* is a measurable function $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ which is bounded from below in the sense that there exist μ_i-integrable functions u_i, $i = 1, \dots, n$ in so that

$$\Phi \geq u_1 \oplus \dots \oplus u_n, \tag{9}$$

where $u_1 \oplus \dots \oplus u_n(x_1, \dots, x_n) := u_1(x_1) + \dots + u_n(x_n)$. Given a cost function Φ and a transport plan π the *cost functional* is defined as

$$I_\pi(\Phi) = \int_{\mathbb{R}^n} \Phi \, d\pi \, . \tag{10}$$

Note that this integral is well defined (assuming possibly the value $+\infty$) by (9). The *primal Monge-Kantorovich problem* is then to minimize $I_\pi(\Phi)$ over the set of all transport plans $\pi \in \Pi(\mu_1, \dots, \mu_n)$.

Given μ_i-integrable functions u_i, $i = 1, \dots, n$, such that

$$\Phi \geq u_1 \oplus \dots \oplus u_n, \tag{11}$$

we have for every transport plan π

$$\int \Phi \, d\pi \geq \int u_1 \oplus \dots \oplus u_n \, d\pi = \int u_1 \, d\mu_1 + \dots + \int u_n \, d\mu_n. \tag{12}$$

The *dual* part of the Monge-Kantorovich problem is to maximize the right side of (12) over a suitable class of functions satisfying (11).

Starting already with Kantorovich, there has been a long line of research on the question in which setting the optimal values of primal and dual problem agree, we refer to the reader to [Vil09, page 88f] for an account of the history of the problem. For our intended application, we need to restrict the dual maximizers to functions in

$$\mathcal{S} = \Big\{ u : \mathbb{R} \to \mathbb{R} : u(x) = a + bx + \sum_{i=1}^m c_i (x - k_i)_+, \ a, b, c_i, k_i \in \mathbb{R} \Big\},$$

i.e., we will employ the following Monge-Kantorovich duality theorem.

Proposition 2.1. *Let $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ be a lower semi-continuous function satisfying*

$$\Phi(s_1, \dots, s_n) \geq -K \cdot (1 + |s_1| + \dots + |s_n|) \tag{13}$$

on $\mathbb{R}^n$ for some constant K and let $\mu_1, \dots, \mu_n$ be probability measures on $\mathbb{R}$ having finite first moments. Then

$$P_{MK}(\Phi) = \inf\{I_\pi(\Phi) : \pi \in \Pi(\mu_1, \dots, \mu_n)\} = \sup\Big\{ \sum_{i=1}^n \int u_i \, d\mu_i : u_1 \oplus \dots \oplus u_n \leq \Phi, \ u_i \in \mathcal{S} \Big\} = D_{MK}(\Phi) \, .$$

We postpone the proof of Proposition 2.1 to the Appendix and continue with our discussion.

The set of transport plans $\Pi(\mu_1, \dots, \mu_n)$ carries a natural topological structure: it is a compact convex subset of the space of finite (signed) Borel measures equipped with the weak topology induced by the bounded continuous functions $C_b(R^n)$. (Compactness of $\Pi(\mu_1, \dots, \mu_n)$ is essentially a consequence of Prohorov's theorem, for a proof we refer the reader to [Vil09, Lemma 4.4].)

[6] See [Vil03, Vil09] for an extensive account on theory of optimal transportation.

[7] Most of the basic results are equally true for polish probability spaces $(X_1, \mu_1), \dots, (X_n, \mu_n)$, but we don't need this generality here.

Subsequently we want to study the set of transport plans which are also martingales. Therefore we will assume from now on that the measures $\mu_1, \ldots, \mu_n$ are in convex order so that $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is a non-empty subset of $\Pi(\mu_1, \ldots, \mu_n)$. It will be crucial for our purposes that also $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is compact in the weak topology. To establish this we need two auxiliary lemmas.

Lemma 2.2. *Let $c : \mathbb{R}^n \to \mathbb{R}$ be continuous and assume that there exists a constant K such that*

$$|c(x_1, \ldots, x_n)| \leq K(1 + |x_1| + \ldots + |x_n|)$$

for all $x_1 \in X_1, \ldots, x_n \in X_n$. Then the mapping

$$\pi \mapsto \int_{\mathbb{R}^n} c \, d\pi$$

is continuous on $\Pi(\mu_1, \ldots, \mu_n)$.

Proof. Since we assume that $\mu_1, \ldots, \mu_n$ have finite first moments, $\int_{\mathbb{R}^n \setminus [-a,a]^n} c \, d\pi$ converges to 0 uniformly in $\pi \in \Pi(\mu_1, \ldots, \mu_n)$ as $a \to \infty$. □

Lemma 2.3. *Let $\pi \in \Pi(\mu_1, \ldots, \mu_n)$. Then the following are equivalent.*

(1) $\mathbb{Q} \in \mathcal{M}(\mu_1, \ldots, \mu_n)$.
(2) *For $2 \leq k \leq n$ and for every continuous bounded function $f : \mathbb{R}^{k-1} \to \mathbb{R}$ we have*

$$\int_{\mathbb{R}^n} f(x_1, \ldots, x_{k-1})(x_k - x_{k-1}) \, d\pi(x_1, \ldots, x_n) = 0.$$

Proof. Plainly, (1) asserts that whenever $A \subseteq R^k$, $k = 1, \ldots, (n-1)$ is Borel measurable, then

$$\int_{\mathbb{R}^n} I_A(x_1, \ldots, x_{k-1})(x_k - x_{k-1}) \, d\pi(x_1, \ldots, x_n) = 0.$$

Using standard approximations techniques one obtains that this is equivalent to (2). □

Proposition 2.4. *The set $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is compact in the weak topology.*

Proof. Since $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is contained in the compact set $\Pi(\mu_1, \ldots, \mu_n)$ it is sufficient to prove that it is closed. By Lemma 2.3, $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is the intersection of the sets

$$\left\{\pi \in \Pi(\mu_1, \ldots, \mu_n) : \int_{\mathbb{R}^n} f(x_1, \ldots, x_k)(x_{k+1} - x_k) \, d\pi(x_1, \ldots, x_n) = 0\right\}, \tag{14}$$

where $k = 1, \ldots, n-1$ and $f : \mathbb{R}^k \to \mathbb{R}$ runs through all continuous bounded support functions. By Lemma 2.2 the sets in (14) are closed. □

3. Proof of Theorem 1

Our argument combines a Monge-Kantorovich duality theorem (in the form of Proposition 2.1) with the following Min-Max theorem of decision theory which we cite here from [Str85, Theorem 45.8].

Theorem 2. *Let K, T be convex subsets of vector spaces V_1 resp. V_2, where V_1 is locally convex and let $f : K \times T \to \mathbb{R}$. If*

(1) *K is compact,*
(2) *$f(., y)$ is continuous and convex on K for every $y \in T$,*
(3) *$f(x, .)$ is concave on T for every $x \in K$*

then

$$\sup_{y \in T} \inf_{x \in K} f(x, y) = \inf_{x \in K} \sup_{y \in T} f(x, y).$$

Proof of Theorem 1. As we want to show that the subhedging portfolios can be formed using just call options, we will restrict ourselves to dual candidates $\Psi_{(u_i),(\Delta_j)}$ satisfying $u_i \in \mathcal{S}, i = 1, \dots, n$ (and $\Delta_j \in C_b(\mathbb{R}^j)$, $j = 1, \dots, n-1$).

If the assertion of Theorem 1 holds true for a function Φ and if $u_1, \dots, u_n \in \mathcal{S}$ then the assertion carries over to $\Phi' = \Phi + u_1 \oplus \dots \oplus u_n$. Therefore we may assume without loss of generality that $\Phi \geq 0$.

Moreover we make the additional assumption that $\Phi \in C_b(\mathbb{R}^n)$.

We will apply Theorem 2 to the compact convex set $K = \Pi(\mu_1, \dots, \mu_n)$, the convex set $E = C_b(\mathbb{R}) \times \dots \times C_b(\mathbb{R}^{n-1})$ of $(n-1)$-tuples of continuous bounded functions on $\mathbb{R}^j$, $j = 1, \dots, (n-1)$ and the function

$$f(\pi, (\Delta_j)) = \int \Phi(x_1, \dots, x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1, \dots, x_j)(x_{j+1} - x_j)\, d\pi(x_1, \dots, x_n). \tag{15}$$

Clearly the assumptions of Theorem 2 are satisfied, the continuity of $f(., (\Delta_j))$ on $\Pi(\mu_1, \dots, \mu_n)$ being a consequence of Lemma 2.2.

We then find

$$D \geq \sup_{\phi_i \in \mathcal{S},\, \Delta_j \in C_b(\mathbb{R}^j),\, \Psi_{(\phi_i),(\Delta_j)} \leq \Phi} \sum_{i=1}^{n} \int \phi_i \, d\mu_i \tag{16}$$

$$= \sup_{\Delta_j \in C_b(\mathbb{R}^j)} \sup_{\phi_i \in \mathcal{S},\, \sum_{i=1}^n \phi_i(x_i) \leq \Phi(x_1,\dots,x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1,\dots,x_j)(x_{j+1}-x_j)} \sum_{i=1}^{n} \int \phi_i \, d\mu_i \tag{17}$$

$$= \sup_{\Delta_j \in C_b(\mathbb{R}^j)} \inf_{\pi \in \Pi(\mu_1,\dots,\mu_n)} \int \Phi(x_1, \dots, x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1, \dots, x_j)(x_{j+1} - x_j)\, d\pi(x_1, \dots, x_n) \tag{18}$$

$$= \inf_{\pi \in \Pi(\mu_1,\dots,\mu_n)} \sup_{\Delta_j \in C_b(\mathbb{R}^j)} \int \Phi(x_1, \dots, x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1, \dots, x_j)(x_{j+1} - x_j)\, d\pi(x_1, \dots, x_n) \tag{19}$$

$$= \inf_{\mathbb{Q} \in \mathcal{M}(\mu_1,\dots,\mu_n)} \int \Phi(x_1, \dots, x_n) = P_M, \tag{20}$$

where Proposition 2.1 (with the cost function $\Phi(x_1, \dots, x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1, \dots, x_i)(x_{j+1} - x_j)$) was used to show the equality between (17) and (18) and the equality of (18) and (19) is guaranteed by Theorem 2.

Next assume that $\Phi : \mathbb{R}^n \to [0, \infty]$ is merely lower semi-continuous and pick a sequence of bounded continuous functions $\Phi_1 \leq \Phi_2 \leq \dots$ such that $\Phi = \sup_{k \geq 0} \Phi_k$. In the following paragraph we will write $P(\Phi), D(\Phi), P(\Phi_k)$, resp. $D(\Phi_k)$ to emphasize the dependence on the cost function. For each k pick $\mathbb{Q}_k \in \Pi(\mu_1, \dots, \mu_n)$ so that

$$P(\Phi_k) \geq \int \Phi \, d\mathbb{Q}_k - 1/k.$$

Passing to a subsequence if necessary, we may assume that $(\mathbb{Q}_k)$ converges weakly to some $\mathbb{Q} \in \Pi(\mu_1, \dots, \mu_n)$. Then

$$\begin{aligned} P(\Phi) \leq \int \Phi \, d\mathbb{Q} &= \lim_{m \to \infty} \int \Phi_m \, d\mathbb{Q} = \lim_{m \to \infty} \left(\lim_{k \to \infty} \int \Phi_m \, d\mathbb{Q}_k \right) \\ &\leq \lim_{m \to \infty} \left(\lim_{k \to \infty} \int \Phi_k \, d\mathbb{Q}_k \right) = \lim_{k \to \infty} P(\Phi_k). \end{aligned} \tag{21}$$

Since $P(\Phi_k) \leq P(\Phi)$ it follows that $D(\Phi) \geq D(\Phi_k) = P(\Phi_k) \uparrow P(\Phi)$.

It remains to prove that the optimal value of the primal problem is attained. To establish this, we use the lower semi-continuity of $\int \Phi \, d\pi$ on $\Pi(\mu_1, \dots, \mu_n)$: if a sequence of measures (π_k) in $\Pi(\mu_1, \dots, \mu_n)$ converges weakly to a measure π, then

$$\liminf_{k\to\infty} \int \Phi \, d\pi_k \geq \int \Phi \, d\pi. \tag{22}$$

We refer the reader to [Vil09, Lemma 4.3] for a proof of this assertion.

If $P = \infty$, the infimum is trivially attained, so assume $P < \infty$ and pick a sequence $(\mathbb{Q}_k)$ in $\mathcal{M}(\mu_1, \ldots, \mu_n)$ such that $P = \lim_k \int \Phi \, d\mathbb{Q}_k$. As $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is compact, $(\mathbb{Q}_k)$ converges to some measure $\mathbb{Q}$ along a subsequence and $\mathbb{Q}$ is a primal minimizer by (22). ◻

4. Further observations in the two dimensional case.

As far as we know, martingale transport plans have not been previously considered in the optimal transport literature. In this section we collect some observations on the primal resp. dual optimization problem which relate to know facts in the classic theory of optimal transport. There the main interest lies in the two dimensional case, hence we focus on the case of just two marginal measures μ_1, μ_2 throughout this section.

For most applications of the theory of optimal transport it is also customary to specify the cost function to be the squared Euclidean distance, i.e. $\Phi(x, y) = (y - x)^2$ in the present setting of probability measures on the real line. We emphasize that this cost function has no significance if one is interested in transport plans that are also martingales: $\int_{\mathbb{R}^2} (y - x)^2 \, d\mathbb{Q}(x, y)$ is constantly equal to $\int_{\mathbb{R}} y^2 \, d\nu(y) - \int_{\mathbb{R}} x^2 \, d\mu(x)$ for *every* martingale measure $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$.

4.1. **A c-convex approach.** In the dual part of usual transport problem it is suffices to maximize over all pairs of functions (u_1, u_2) where u_1 is the conjugate of u_2 with respect to Φ, i.e., satisfies

$$u_1(x) = \inf_y \Phi(x, y) - u_2(y).$$

(We refer the reader to [Vil03, Section 2.4], [Vil09, Chapter 5] for details on this topic.)

To establish an analogue result for the dual problem in our setting we introduce some notation. Given a function $g : \mathbb{R} \to (-\infty, \infty]$, we write g^e for its convex envelope[8]. For $G : \mathbb{R}^2 \to \mathbb{R}$, let $G^e : \mathbb{R}^2 \to \mathbb{R}$ be the function satisfying

$$G^e(x, .) = (G(x, .))^e$$

for every $x \in \mathbb{R}$. (It is straight forward to prove that G^e is Borel measurable resp. lower semi-continuous whenever G is.)

Proposition 4.1. *Let $\Phi : \mathbb{R}^2 \to (-\infty, \infty]$ be a lower semi-continuous function such that $\Phi(x, y) \geq -K(1 + |x| + |y|)$, $x, y \in \mathbb{R}$ and assume that there is some $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ satisfying $\int \Phi \, d\mathbb{Q} < \infty$. Then*

$$P = \sup_{u_2 : \mathbb{R} \to \mathbb{R}, \int |u_2| \, d\mu_2 < \infty} \left\{ \int_{\mathbb{R}} (\Phi - u_2)^e(x, x) \, d\mu_1(x) + \int_{\mathbb{R}} u_2(y) \, d\mu_2(y) \right\}. \tag{23}$$

(In the course of the proof we will see that for every choice of u_2 the first integral in (23) is well defined, assuming possibly the value $-\infty$.)

Proof. We start to show that the primal value P is greater or equal than the right hand side of (23). Let $u_2 : \mathbb{R} \to \mathbb{R}$ be a μ-integrable function. Fix $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ satisfying $\int \Phi \, d\mathbb{Q} < \infty$ and let $(\mathbb{Q}_x)_{x \in \mathbb{R}}$ be a disintegration of $\mathbb{Q}$ with respect to $(\mathbb{R}, \mu)$. Using the abbreviation $(\Phi - u_2)(x, y) := \Phi(x, y) - u_2(y)$, we obtain

$$\int \Phi \, d\mathbb{Q} = \int (\Phi - u_2) \, d\mathbb{Q} + \int u_2 \, d\mu_2 \tag{24}$$

$$\geq \iint (\Phi - u_2)^e(x, y) \, d\mathbb{Q}_x(y) \, d\mu_1(x) + \int u_2 \, d\mu_2 \tag{25}$$

$$\geq \int (\Phi - u_2)^e(x, x) \, d\mu_1(x) + \int u_2 \, d\mu_2, \tag{26}$$

[8] I.e. $g^e : \mathbb{R} \to \mathbb{R}$ is the largest convex function smaller or equal then g.

where $\int(\Phi - u_2)^e(x, y)\, d\mathbb{Q}_x(y) \geq (\Phi - u_2)^e(x, x)$ holds due to Jensen's inequality. This proves the first inequality.

To establish the reverse inequality, we make a simple observation. Let $x \in \mathbb{R}$ and a function $g : \mathbb{R} \to \mathbb{R}$ be fixed. Suppose that for $u_1 \in \mathbb{R}$ there exists $\Delta \in \mathbb{R}$ such that

$$u_1 + \Delta \cdot (y - x) \leq g(y)$$

for all $y \in \mathbb{R}$. Then $u_1 \leq g^e(x)$.

Applying this for $x \in \mathbb{R}$ to the function $y \mapsto g(y) = \Phi(x, y) - u_2(y)$ we obtain

$$\sup_{u_2} \left\{ \int_{\mathbb{R}} (\Phi - u_2)^e(x, x)\, d\mu_1(x) + \int_{\mathbb{R}} u_2(y)\, d\mu_2(y) \right\} \tag{27}$$

$$\geq \sup_{u_2} \left\{ \sup_{u_1 : \exists \Delta, u_1(x) + \Delta(x)(y-x) \leq \Phi(x,y) - u_2(y)} \int u_1(x)\, d\mu_1(x) + \int u_2(y)\, d\mu_2(y) \right\} \tag{28}$$

$$= \sup_{u_1, u_2 : \exists \Delta, \Psi_{u_1, u_2, \Delta} \leq \Phi} \left\{ \int u_1\, d\mu_1 + \int u_2\, d\mu_2 \right\} = D = P\,, \tag{29}$$

where we tacitly assumed that the suprema are taken over μ_i-integrable functions $u_i : \mathbb{R} \to \mathbb{R}, i = 1, 2$ and that $\Delta : \mathbb{R} \to \mathbb{R}$ is bounded measurable. □

A Hamilton-Jacobi-Bellman formulation. We conclude this subsection by *heuristically* rewriting the dual problem in terms of (viscosity) solutions of Hamilton-Jacobi-Bellman equations. A similar Hamilton-Jacobi formulation of the Kantorovich duality can be found in [Vil03, Proposition 5.48]. Let us consider the local martingale

$$dS_t^\sigma = \sigma_t\, dW_t,$$

where W is a scalar Brownian motion defined on a filtered space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\sigma_\cdot \in \mathcal{A}$, the set of $\mathcal{F}_t$-adapted processes valued in $\mathbb{R}$ with finite $L^2(\Omega \times [0, 1))$-norm. Then we introduce the (singular) stochastic control problem defined by

$$u(x, t) \equiv \inf_{\sigma \in \mathcal{A}} \mathbb{E}_x^{\mathbb{P}}[\Phi(x, S_1^\sigma) - u(S_1^\sigma, 1) | \mathcal{F}_t].$$

We have (see [Tou02, Section 2.4.3])

$$u(x, 0) = (\Phi - u(\cdot, 1))^e(x, x).$$

So Equation (23) can be written as

$$P = \sup_u \left\{ \int_{\mathbb{R}} u(x, 0)\, d\mu_1(x) + \int_{\mathbb{R}} u(y, 1)\, d\mu_2(y) \right\}. \tag{30}$$

4.2. **The dual supremum is not necessarily attained.** In the classic optimal transport problem, the optimal value of the dual problem is attained provided that the cost function is bounded ([Kel84, Theorem 2.14]) or satisfies appropriate moment conditions ([AP03, Therorem 2.3]).

This is not the case in our present setting where the subsequent Example 4.2 shows that the dual supremum (5) is not necessarily attained even if Φ is bounded and μ, ν are compactly supported. However, it may be an interesting task for further research to find sufficient conditions which guarantee dual attainment.

Example 4.2. Let $\mu_1 = \mu_2 = \lambda \upharpoonright [0, 1]$ and define $\Phi : \mathbb{R}^2 \to \mathbb{R}$ by $\Phi(x, y) = \max(-|x - y|, -1)$. Then $\mathcal{M}(\mu_1, \mu_2)$ contains a single element $\mathbb{Q}$ which is concentrated on the diagonal of $[0, 1] \times [0, 1]$ and trivially is optimal. Striving for a contradiction, we assume that there exist u_1, u_2 and Δ which form a dual maximizer. It follows that

$$u_1(x) + u_2(y) + \Delta(x)(y - x) \leq -|x - y|$$

for all $x, y \in [0, 1]$ and that equality holds $\mathbb{Q}$-a.s. Thus

$$u_1(x) + u_2(x) = 0$$

for all $x \in I$ where $I \subseteq [0,1]$ is a set of measure 1. Note that

$$J = \{x \in \mathbb{R} : x + n/m \in I \cup [0,1]^c \text{ for all } n \in \mathbb{Z}, m \in \mathbb{N}\}$$

satisfies $\lambda(J) = 1$ and $J + q = J$ for all rational q. For $x \in [0,1]$ and $\delta > 0$ we have

$$u_1(x) + u_2(x - \delta) - \Delta(x)\delta \leq -\delta \tag{31}$$

$$u_1(x) + u_2(x + \delta) + \Delta(x)\delta \leq -\delta. \tag{32}$$

Adding these inequalities, we obtain

$$2u_1(x) + u_2(x - \delta) + u_2(x + \delta) \leq -2\delta. \tag{33}$$

Hence, if $\delta \in \mathbb{Q}^+$ and $x_0, x_0 + \delta, x_0 - \delta \in J \cap [0,1]$ then

$$u_2(x_0 + \delta) \leq 2u_2(x_0) - u_2(x_0 - \delta) - 2\delta.$$

Applying this n times with $\delta = \frac{1}{2m}$, $x = x_0 + \frac{i}{2m}$, $i = 1, \ldots, n$ and adding the resulting inequalities we obtain

$$u_2(x + n\tfrac{1}{2m}) \leq |u_2(x)| + |u_2(x + \tfrac{1}{2m})| - n^2\tfrac{1}{2m}$$

provided that $n\frac{1}{2m} \in [0,1]$. Note also that

$$C(x) := \liminf_{m\to\infty} u_2(x + \tfrac{1}{2m}) < \infty$$

for λ-almost all $x \in \mathbb{R}$. (This holds true for any measurable function.) Consequently, for almost all $x \in (0, 1/2) \cap J$ there are infinitely many $m \in \mathbb{N}$ so that

$$u_2(x + 1/2) = u_2(x + \tfrac{m}{2m}) \leq |u_2(x)| + |u_2(x + \tfrac{1}{2m})| - m^2\tfrac{1}{2m} \leq |u_2(x)| + C(x) + 1 - \tfrac{m}{2}.$$

As the right hand side can be made arbitrarily small, we conclude that $u_2(x) = -\infty$ almost surely on $[1/2, 1]$. This yields the desired contradiction.

Appendix

As a special case of [Kel84, Theorem 2.14] we have the duality equation

$$P_{MK}(\Phi) = \inf\{I_\pi(\Phi) : \pi \in \Pi(\mu_1, \ldots, \mu_n)\} = \sup\Big\{\sum_{i=1}^n \int u_i \, d\mu_i : u_1 \oplus \ldots \oplus u_n \leq \Phi,\ u_i \text{ is } \mu_i\text{-integrable}\Big\}$$

for every lower semi-continuous cost function $\Phi : \mathbb{R}^n \to [0, \infty]$. The main task in the subsequent proof of Proposition 2.1 is to show that the duality equation pertains if one restricts to functions in the class $\mathcal{S}$ in the dual problem.

Proof of Theorem 2.1. As in the proof of Theorem 1, it is sufficient to prove the duality equation in the case $\Phi \geq 0$.

Given a bounded continuous function f and $\varepsilon > 0$, then for every $i = 1, \ldots, n$ there is some $u \in \mathcal{S}$ such that $f \geq u$ and $\int f - u \, d\mu_i < \varepsilon$. Therefore we may change the class of admissible functions from $\mathcal{S}$ to $C_b(\mathbb{R})$, i.e. it suffices to prove

$$P_{MK}(\Phi) = \sup\Big\{\sum_{i=1}^n \int u_i \, d\mu_i : u_1 \oplus \ldots \oplus u_n \leq \Phi,\ u_i \in C_b(\mathbb{R})\Big\}. \tag{34}$$

We will first show this under the additional assumption that $\Phi \in C_c(\mathbb{R}^n)$. By [Kel84, Theorem 2.14] we have that for each $\eta > 0$ there exist μ_i-integrable functions u_i, $i = 1, \ldots, n$ so that

$$P_{MK}(\Phi) - \sum_{i=1}^n \int u_i \, d\mu_i \leq \eta$$

and $u_1 \oplus \ldots \oplus u_n \leq \Phi$. Note that the latter inequality implies that $u_1, \ldots, u_n$ are uniformly bounded since Φ is uniformly bounded from above.

To replace u_1 by a function in C_b we consider $H = \Phi - (u_1 \oplus \ldots \oplus u_n)$ and define

$$\tilde{u}_1(x_1) := \inf_{x_2,\ldots,x_n \in \mathbb{R}} H(x_1, \ldots, x_n) \tag{35}$$

for $x_1 \in \mathbb{R}$. We claim that $\tilde{u}_1$ is (uniformly) continuous. Indeed, as Φ is uniformly continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ so that whenever $x, x' \in \mathbb{R}$, $|x - x'| < \delta$, then

$$|H(x, x_2, \ldots, x_n) - H(x', x_2, \ldots, x_n)| = |\Phi(x, x_2, \ldots, x_n) - \Phi(x', x_2, \ldots, x_n)| < \varepsilon.$$

Thus we obtain

$$|\tilde{u}_1(x) - \tilde{u}_1(x')| = \left| \inf_{x_2,\ldots,x_n \in \mathbb{R}} H(x, x_2, \ldots, x_n) - \inf_{x_2,\ldots,x_n \in \mathbb{R}} H(x', x_2, \ldots, x_n) \right| \leq \varepsilon$$

whenever $|x - x'| < \delta$. By definition $\tilde{u}_1$ is also bounded from below and satisfies $\tilde{u}_1 \geq u_1$ as well as

$$\tilde{u}_1 \oplus u_2 \oplus \ldots \oplus u_n \leq \Phi.$$

Iteratively replacing the functions $u_2, \ldots, u_n$ in the same fashion, we obtain (34) in the case $\Phi \in C_c(\mathbb{R}^n)$.

Using precisely the same argument as in the proof of Theorem 1, we obtain the duality relation in the case of a general, lower semi-continuous function $\Phi : \mathbb{R}^n \to [0, \infty]$.

□

References

[AP03] L. Ambrosio and A. Pratelli. Existence and stability results in the L^1 theory of optimal transportation. In *Optimal transportation and applications (Martina Franca, 2001)*, volume 1813 of *Lecture Notes in Math.*, pages 123–160. Springer, Berlin, 2003.

[BP02] D. Bertsimas and I. Popescu. On the relation between option and stock prices: a convex optimization approach. *Oper. Res.*, 50(2):358–374, 2002.

[CDDV08] X. Chen, G. Deelstra, J. Dhaene, and M. Vanmaele. Static super-replicating strategies for a class of exotic options. *Insurance Math. Econom.*, 42(3):1067–1085, 2008.

[HL09] P. Henry-Labordere. Calibration of local stochastic volatility models to market smiles. *Risk*, September 2009.

[HLW05a] D. Hobson, P. Laurence, and T.H. Wang. Static-arbitrage optimal subreplicating strategies for basket options. *Insurance Math. Econom.*, 37(3):553–572, 2005.

[HLW05b] D. Hobson, P. Laurence, and T.H. Wang. Static-arbitrage upper bounds for the prices of basket options. *Quant. Finance*, 5(4):329–342, 2005.

[Kel84] H.G. Kellerer. Duality theorems for marginal problems. *Z. Wahrsch. Verw. Gebiete*, 67(4):399–432, 1984.

[LW04] P. Laurence and T.H. Wang. What's a basket worth? February 2004.

[LW05] P. Laurence and T.H. Wang. Sharp upper and lower bounds for basket options. *Applied Mathematical Finance*, 12(3):253–282, 2005.

[NH00] A. Neuberger and S.D. Hodges. Rational Bounds on the Prices of Exotic Options. *SSRN eLibrary*, 2000.

[Str65] V. Strassen. The existence of probability measures with given marginals. *Ann. Math. Statist.*, 36:423–439, 1965.

[Str85] H. Strasser. *Mathematical theory of statistics*, volume 7 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. Statistical experiments and asymptotic decision theory.

[Tou02] N. Touzi. Stochastic control and application to finance. 2002.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Vil09] C. Villani. *Optimal Transport. Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.

Mathias Beiglböck
University of Vienna, Department for Mathematics
E-mail address: mathias.beiglboeck@univie.ac.at

Pierre Henry-Labordère
Global Markets Quantitative Research, Société Générale
E-mail address: pierre.henry-labordere@sgcib.com

Friedrich Penkner
University of Vienna, Department for Mathematics
E-mail address: friedrich.penkner@univie.ac.at