Learning Item-Attribute Relationship in Q-Matrix Based Diagnostic Classification Models

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June 6, 2011

Abstract

Recent surge of interests in cognitive assessment has led to the developments of novel statistical models for diagnostic classification. Central to many such models is the well-known Q-matrix, which specifies the item-attribute relationship. This paper proposes a principled estimation procedure for the Q-matrix and related model parameters. Desirable theoretic properties are established through large sample analysis. The proposed method also provides a platform under which important statistical issues, such as hypothesis testing and model selection, can be addressed.
Keywords: Cognitive assessment, consistency, DINA model, DINO model, latent traits, model selection, optimization, self-learning, statistical estimation.

1 Introduction

Diagnostic classification models (DCM) are important statistical tools in cognitive diagnosis and have widespread applications in educational measurement, psychiatric evaluation, human resource development, and many other areas in science, medicine, and business. A key component in many such models is the so-called Q-matrix, first introduced by Tatsuoka (1983); see also Tatsuoka (2009) for a detailed coverage. The Q-matrix specifies the item-attribute relationship, so that responses to items can reveal attributes configuration of the respondent. In fact, Tatsuoka (1983, 2009) proposed the rule space method that is simple and easy-to-use.

Flexible and sophisticated statistical models can be built around the Q-matrix. Two such models are the DINA model (Deterministic Input, Noisy Output "AND" gate; see Junker and Sijtsma, 2001) and the DINO model (Deterministic Input, Noisy Output "OR" gate; see Templin, 2006; Templin and Henson, 2006). Other important developments can be found in Tatsuoka (1985); DiBello, Stout, and Roussos (1995); Junker and Sijtsma (2001); Hartz (2002); Tatsuoka (2002); Leighton, Gierl, and Hunka (2004); Templin (2006); Chiu, Douglas, and Li (2009). Rupp, Templin, and Henson (2010) contains a comprehensive summary of many classical and recent developments.

There is a growing literature on the statistical inference of Q-matrix based DCMs that addresses the issues of estimating item parameters when the Q-matrix is prespecified (Rupp, 2002; Henson and Templin, 2005; Roussos, Templin, and Henson, 2007; Stout, 2007). Having a correctly specified Q-matrix is crucial both for parameter estimation (such as the slipping, guessing probability, and the attribute distribution) and for the identification of subjects' underlying attributes. As a result, these approaches are sensitive to the choice of the Q-matrix (Rupp and Templin, 2008; de la Torre, 2008; de la Torre and Douglas, 2004). For instance, a misspecified Q-matrix may lead to substantial lack of fit and, consequently, erroneous attribute identification. Thus, it is desirable to be able to detect misspecification and to obtain a data driven Q-matrix.

In contrast, there has not been much work about estimation of the Q-matrix. To our knowledge, the only rigorous treatment of the subject is given by Liu, Xu, and Ying (2011), which defines an estimator of the Q-matrix under the DINA model assumption and provides regularity conditions under which desirable theoretical properties are established. The work of this paper may be viewed as the continuation of Liu et al. (2011) in the sense that it completes the estimation of the Q-matrix for the DINA model and extends the estimation procedure (as well as the consistency results) to the DINO model. The DINA and the DINO models impose rather different interactions among attributes. However, we show that there exists a duality between the two models. This particular feature is interesting especially for theoretical development, as it allows us to adapt the results and analysis techniques developed for the DINA model to the DINO model without much additional effort. This will be shown in our technical developments.

The main contribution of this paper is two-fold. First, it provides a rigorous analysis of the Q-matrix for the DINA model when both the slipping and guessing parameters are unknown. This is a substantial extension of the results in Liu et al. (2011) which requires a complete knowledge of the guessing parameter. It gives a definitive answer to the estimability of the Q-matrix for the DINA model by presenting a set of sufficient conditions under which a consistent estimator exists. Second, we conduct a parallel analysis (to the analysis for the DINA model) for the DINO model. In particular, a consistent estimator of the Q-matrix for the DINO model and its properties are presented. Thanks to the duality structure, part of the intermediate results developed for the DINA model can be borrowed to the analysis of the DINO model.

One may notice that our estimation procedure is in fact generic in the sense that it is implementable to a large class of DCMs besides the DINA and DINO models. In particular, the procedure is implementable to the NIDA (Noisy Inputs, Deterministic "And" Gate) model and the NIDO (Noisy Inputs, Deterministic "Or" Gate) model among others, though theoretical properties under such model specifications still need to be established. In addition to the estimation of the Q-matrix, we emphasize that the idea behind the derivations forms a principled inference framework. For instance, during the course of the description of the estimation procedure, necessary conditions for a correctly specified Q-matrix are naturally derived. Such conditions can be used to form appropriate statistics for hypothesis testing and model diagnostics. In that connection, additional developments (e.g. the asymptotic distributions of those statistics) are needed, but they are not the focus of the current paper. Therefore, the proposed framework can potentially serve as a principled inference tool for the Q-matrix in diagnostic classification models.

This paper is organized as follows. Section 2 contains the main ingredient: presentation of the estimation procedures for both the DINA and DINO models and the statement of the consistency results. Section 3 includes further discussions of the theorems and various issues. The proofs of the main theorems in Section 2 and several important propositions are given in Section 4. The most technical proofs of two central propositions are given in the Appendix.

2 Main results

2.1 Notation and model specification

The specification of the diagnostic classification models considered in this paper consists of the following concepts.

Attribute: subject's underlying mastery of certain skills or presence of certain mental health conditions. There are k attributes and we use $\mathbf{A} = (A^1, ..., A^k)^\top$ to denote the vector of attributes, where $A^j = 1$ or 0, indicating presence or absence of the j-th attribute, $j = 1, \ldots, k$.

Responses to items: There are m items and we use $\mathbf{R} = (R^1, ..., R^m)$ to denote the vector of responses to them. For simplicity, we assume that $R^j \in \{0, 1\}$ is a binary variable for each $j = 1, ..., m$.

Note that both $\mathbf{A}$ and $\mathbf{R}$ are subject specific. Throughout this paper, we assume that the number of attributes k is known and that the number of items m is always observed.

Q-matrix: the link between the items and the attributes. In particular, $Q = (Q_{ij})_{m\times k}$ is an $m \times k$ matrix with binary entries. For each i and j, $Q_{ij} = 1$ indicates that item i requires attribute j and $Q_{ij} = 0$ otherwise.

We define *capability indicator*, $\xi(\mathbf{A}, Q)$, which indicates if a subject possessing attribute profile $\mathbf{A}$ is capable of providing a positive response to item i if the item-attribute relationship is specified by matrix Q. Different capability indicators give rise to different DCMs. For instance,

$$\xi^i_{DINA}(\mathbf{A}, Q) = \mathbf{1}(A^j \geq Q_{ij} \text{ for all } j = 1, ..., k) \tag{1}$$

is associated with the DINA model, where $\mathbf{1}$ is the usual indicator function. The DINA model assumes conjunctive relationship among attributes, that is, it is necessary to possess all the attributes indicated by the Q-matrix to be capable of providing a positive response to an item. In addition, having additional unnecessary attributes does not compensate for the lack of the necessary attributes. The DINA model is particularly popular in the context of educational testing.

Alternative to the "and" relationship, one may impose an "or" relationship among the attributes, resulting in the DINO model. The corresponding capability indicator takes the following form

$$\xi^i_{DINO}(\mathbf{A}, Q) = \mathbf{1}(\text{there exists a } j \text{ such that } A^j \geq Q_{ij}). \tag{2}$$

That is, one needs to possess at least one of the required attributes to be capable of responding positively to that item.

The last ingredient of the model specification is related to the so-called slipping and guessing parameters. The names "slipping" and "guessing" arise from the educational applications. The slipping parameter is the probability that a subject (with attribute profile $\mathbf{A}$) responds negatively to an item if the capability indicator to that item $\xi_{DINA}(\mathbf{A}, Q) = 1$; similarly, the guessing parameter refers to the probability that a subject's responds positively if his/her capability indicator $\xi_{DINA}(\mathbf{A}, Q) = 0$. We use s to denote the slipping probability and g to denote the guessing probability (with corresponding subscript indicating different items). In the technical development, it is more convenient to work with the complement of the slipping parameter. Therefore, we define $c = 1 - s$ to be the correctly answering probability, with s_i and c_i being the corresponding item-specific notation. Given a specific subject's profile $\mathbf{A}$, the response to item i under the DINA model follows a Bernoulli distribution

$$P(R^i = 1|\mathbf{A}) = c_i^{\xi^i_{DINA}(\mathbf{A},Q)} g_i^{1-\xi^i_{DINA}(\mathbf{A},Q)}. \tag{3}$$

With the same definition of c_i and g_i, the response under the DINO model follows

$$P(R^i = 1|\mathbf{A}) = c_i^{\xi^i_{DINO}(\mathbf{A},Q)} g_i^{1-\xi^i_{DINO}(\mathbf{A},Q)}. \tag{4}$$

In addition, conditional on $\mathbf{A}$, $(R^1, ..., R^m)$ are jointly independent.

Lastly, we use subscripts to indicate different subjects. For instance, $\mathbf{R}_r = (R^1_r, ..., R^m_r)^\top$ is the response vector of subject r. Similarly, $\mathbf{A}_r$ is the attribute vector of subject r. With N subjects, we observe $\mathbf{R}_1, ..., \mathbf{R}_N$ but not $\mathbf{A}_1, ..., \mathbf{A}_N$. Thus, we finished our model specification.

2.2 Estimation of the Q-matrix

In this section, we develop a general approach to the estimation of the Q-matrix and item parameters. We first deal with the DINA model and then, via introducing a duality relation, the DINO model.

2.2.1 DINA model

We need to introduce additional notation and concepts. Throughout the discussion, we use Q to denote the true matrix and Q' to denote a generic $m \times k$ binary matrix.

Attribute distribution. We assume that the subjects are a random sample (of size N) from a designated population so that their attribute profiles, $\mathbf{A}_r$, $r = 1, ..., N$ are i.i.d. random variables, with the following distribution

$$P(\mathbf{A}_r = \mathbf{A}) = p_{\mathbf{A}}, \tag{5}$$

where, for each $\mathbf{A} \in \{0,1\}^k$, $p_{\mathbf{A}} \in [0,1]$ and $\sum_{\mathbf{A}} p_{\mathbf{A}} = 1$. We use $\mathbf{p} = (p_{\mathbf{A}} : \mathbf{A} \in \{0,1\}^k)$ to denote the distribution of the attribute profiles.

The T-matrix. The T-matrix is a non-linear function of the Q-matrix and provides a linear relationship between the attribute distribution and the response distribution. In particular, let $T(Q)$ be a matrix of 2^k columns. Each column of T corresponds to one attribute profile $\mathbf{A} \in \{0,1\}^k$. To facilitate the description, we use binary vectors of length k to label the the columns of $T(Q)$ instead of using ordinal numbers. For instance, the $\mathbf{A}$-th column of $T(Q)$ is the column that corresponds to attribute $\mathbf{A}$.

Let I_i be a generic notation for a positive response to item i. Let "$\wedge$" stand for "and" combination. For instance, $I_{i_1} \wedge I_{i_2}$ denotes positive responses to both item i_1 and i_2. Each row of $T(Q)$ corresponds to one item or one "and" combination of items, for instance, I_{i_1}, $I_{i_1} \wedge I_{i_2}$, or $I_{i_1} \wedge I_{i_2} \wedge I_{i_3}$,... For $T(Q)$ containing all the single items and all "and" combinations, it has $2^m - 1$ rows. We will later say that such a $T(Q)$ is *saturated*.

We now proceed to the description of each row vector of $T(Q)$. We define $B_Q(I_i)$ to be a 2^k dimensional row vector. Using the same labeling system as that of the columns of $T(Q)$, the $\mathbf{A}$-th element of $B_Q(I_i)$ is defined as $\xi^i_{DINA}(\mathbf{A}, Q)$, that is, this element indicates if a subject with attribute $\mathbf{A}$ is capable of responding positively to item i. Thus, $B_Q(I_i)$ is the vector indicating the attribute profiles that is capable of responding positively to item i.

Using a similar notation, we define that

$$B_Q(I_{i_1} \wedge ... \wedge I_{i_l}) = \Upsilon^l_{h=1} B_Q(I_{i_h}), \tag{6}$$

where the operator "$\Upsilon^l_{h=1}$" is element-by-element multiplication from $B_Q(I_{i_1})$ to $B_Q(I_{i_l})$. For instance,

$$W = \Upsilon^l_{h=1} V_h$$

means that $W^j = \prod_{h=1}^l V_h^j$, where $W = (W^1, ..., W^{2^k-1})$ and $V_h = (V_h^1, ..., V_h^{2^k-1})$. Therefore, $B_Q(I_{i_1} \wedge ... \wedge I_{i_l})$ is the vector indicating the attributes that are capable of responding positively to items $i_1, ..., i_l$. The row in $T(Q)$ corresponding to $I_{i_1} \wedge ... \wedge I_{i_l}$ is $B_Q(I_{i_1} \wedge ... \wedge I_{i_l})$.

α-vector. We let α be a column vector whose length is equal to the number of rows in $T(Q)$. Each component in α corresponds to a row vector of $T(Q)$. The element in α corresponding to $I_{i_1} \wedge ... \wedge I_{i_l}$ is $N_{I_{i_1} \wedge ... \wedge I_{i_l}}/N$, where $N_{I_{i_1} \wedge ... \wedge I_{i_l}}$ denotes the number of people with positive responses

to items $i_1, ..., i_l$, that is

$$N_{I_{i_1} \wedge ... \wedge I_{i_l}} = \sum_{r=1}^{N} \prod_{j=1}^{l} R_r^{i_j}.$$

No slipping or guessing. We first consider a simplified situation in which both the slipping and guessing probabilities are zero. Under this special situation, (3) implies that

$$R_r^i = \xi_{DINA}^i(\mathbf{A}_r), \quad i = 1, ..., m; \ r = 1, ..., N.$$

In other words, the probabilistic relationship becomes a certainty relationship. We further let $\hat{\mathbf{p}} = \{\hat{p}_{\mathbf{A}} : \mathbf{A} \in \{0,1\}^k\}$ be the (unobserved) empirical distribution of the attribute profiles, that is,

$$\hat{p}_{\mathbf{A}} = \frac{1}{N} \sum_{r=1}^{N} \mathbf{1}(\mathbf{A}_r = \mathbf{A}).$$

Note that each row vector of $T(Q)$ indicates the attribute profiles that are capable of responding positively to the corresponding item(s). Then, for each set of $i_1, ..., i_l$, we may expect the following identity

$$\frac{N_{i_1 \wedge ... \wedge i_l}}{N} = B_Q(I_{i_1} \wedge .... \wedge I_{i_l})\hat{\mathbf{p}},$$

where B_Q is a row vector and $\hat{\mathbf{p}}$ is a column vector. Therefore, thanks to the construction of $T(Q)$ and vector α, in absence of possibility of slipping and guessing, we may expect the following set of linear equations holds

$$T(Q)\hat{\mathbf{p}} = \alpha.$$

Note that $\hat{\mathbf{p}}$ is not observed. The above display implies that if the Q-matrix is correctly specified and the slipping and guessing probabilities are zero, then the linear equation $T(Q)\mathbf{p} = \alpha$ (with $\mathbf{p}$ being the variable) has at least one solution. For each binary matrix Q', we define that

$$S(Q') = \inf_{\mathbf{p}} |T(Q')\mathbf{p} - \alpha|,$$

where the minimization is subject to the constraints that $p_{\mathbf{A}} \in [0,1]$ and $\sum_{\mathbf{A}} p_{\mathbf{A}} = 1$. Based on the above results, we may expect that $S(Q) = 0$ and therefore Q is one of the minimizers of $S(Q)$. In addition, the empirical distribution $\hat{\mathbf{p}}$ is one of the minimizers of $|T(Q)\mathbf{p} - \alpha|$. Therefore, we just derived a set of necessary conditions for a correctly specified Q-matrix. In our subsequent theoretical developments, we will show that under some circumstances these conditions are also sufficient.

Illustrative example. To aid the understanding of the T-matrix, we provide one simple example. Consider the following 3×2 Q-matrix,

$Q =$ (7)

	addition	multiplication
$2+3$	1	0
5×2	0	1
$(2+3) \times 2$	1	1

and the contingency table of attributes

	multiplication	
addition	$\hat{p}_{00}$	$\hat{p}_{01}$
	$\hat{p}_{10}$	$\hat{p}_{11}$

Note that if the Q-matrix is correctly specified and the slipping and guessing probabilities are all zero we should be able to obtain the following identities

$$N(\hat{p}_{10} + \hat{p}_{11}) = N_{I_1}, \quad N(\hat{p}_{01} + \hat{p}_{11}) = N_{I_2}, \quad N\hat{p}_{11} = N_{I_3}. \tag{8}$$

We then create the corresponding T-matrix and α-vector as follows

$$T(Q) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} N_{I_1}/N \\ N_{I_2}/N \\ N_{I_3}/N \end{pmatrix}. \tag{9}$$

The first column of $T(Q)$ corresponds to the zero attribute profile; the second corresponds to $\mathbf{A} = (1,0)$; the third corresponds to $\mathbf{A} = (0,1)$; and the last corresponds to $\mathbf{A} = (1,1)$. The first row of $T(Q)$ corresponds to item $2+3$, the second to 5×2, the third to $(2+3) \times 2$. In addition, we may further consider combinations such as

$$N\hat{p}_{11} = N_{I_1 \wedge I_2}.$$

The corresponding T-matrix and α-vector should be

$$T(Q) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} N_{I_1}/N \\ N_{I_2}/N \\ N_{I_3}/N \\ N_{I_1 \wedge I_2}/N \end{pmatrix}. \tag{10}$$

Under the DINA model assumption and $g_i = s_i = 1 - c_i = 0$, we obtain that

$$T(Q)\hat{\mathbf{p}} = \alpha.$$

Nonzero slipping and guessing probabilities. We next extend the necessary conditions just derived to nonzero but known slipping and guessing probabilities. To do so, we need to modify the T-matrix. Let $T_{c,g}(Q)$ be a matrix with the same dimension as that of $T(Q)$, with each row vector being defined slightly differently to incorporate the slipping and guessing probability. In particular, let

$$B_{c,g,Q}(I_i) = (c_i - g_i)B_Q(I_i) + g_i\mathbf{E}$$

where $\mathbf{E} = (1, ..., 1)$ is the row vector of ones and c_i is the positive responding probability of item i. In addition, we let

$$B_{c,g,Q}(I_{i_1} \wedge ... \wedge I_{i_l}) = \Upsilon_{h=1}^{l} B_{c,g,Q}(I_{i_h}), \tag{11}$$

Clearly, each element of $B_{c,g,Q}(I_i)$ is the probability of observing a positive response to item i for a certain attribute profile. Likewise, elements of $B_{c,g,Q}(I_{i_1} \wedge ... \wedge I_{i_l})$ indicate the probabilities of positive responses to items i_1,..., i_l. The row in $T_{c,g}(Q)$ corresponding to $I_{i_1} \wedge ... \wedge I_{i_l}$ is $B_{c,g,Q}(I_{i_1} \wedge ... \wedge I_{i_l})$. To facilitate our statement, we define that

$$T_c(Q) = T_{c,\mathbf{0}}(Q), \tag{12}$$

where $\mathbf{0} = (0, ..., 0)^\top$ is the zero vector. That is, $T_c(Q)$ is the matrix $T_{c,g}(Q)$ with guessing probabilities being zero.

Recall that $\mathbf{p}$ is the attribute distribution. Thus,

$$P(R^{i_1} = 1, ..., R^{i_l} = 1) = E(P(R^{i_1} = 1, ..., R^{i_l} = 1|\mathbf{A})) = B_{c,g,Q}(I_{i_1} \wedge ... \wedge I_{i_l})\mathbf{p}.$$

Further, we obtain that

$$E(\alpha) = T_{c,g}(Q)\mathbf{p}.$$

In presence of slipping and guessing, one cannot expect to solve equation $T_{c,g}(Q)\mathbf{p} = \alpha$ exactly the same way as in the case of no guessing and slipping. On the other hand, thanks to the law of large numbers, we obtain that $\alpha \to E(\alpha)$ as $N \to \infty$. Then this equation can be solved asymptotically. Thus, for a generic Q', we defined the loss function

$$S_{c,g}(Q') = \inf_{\mathbf{p}} |T_{c,g}(Q')\mathbf{p} - \alpha|, \tag{13}$$

where the above optimization is subject to the constraint that $p_{\mathbf{A}} \in [0, 1]$ and $\sum_{\mathbf{A}} p_{\mathbf{A}} = 1$ and $|\cdot|$ is the Euclidean normal. In view of the preceding argument, we expect that

$$S_{c,g}(Q) \to 0 \tag{14}$$

almost surely as $N \to \infty$, that is, the true Q-matrix asymptotically minimizes the criterion function $S_{c,g}$. This leads us to propose the following estimator of Q

$$\hat{Q}(c, g) = \arg \inf_{Q'} S_{c,g}(Q'), \tag{15}$$

where (c, g) is included in $\hat{Q}$ to indicate that the resulting estimator requires the knowledge of the correct responding and guessing probabilities.

Situations when c and g are unknown. Suppose that for a given Q', we can construct an estimator $(\hat{c}(Q'), \hat{g}(Q'))$ of (c, g). In addition, suppose that $(\hat{c}(Q), \hat{g}(Q))$ is consistent, that is, $(\hat{c}(Q), \hat{g}(Q)) \to (c, g)$ in probability as $N \to \infty$. Then, we define

$$\hat{Q}_{\hat{c},\hat{g}} = \arg \inf_{Q'} S_{\hat{c}(Q'),\hat{g}(Q')}(Q'), \tag{16}$$

that is, we plug in the estimator of (c, g) into the objective function in (15). We will present one specific choice of $(\hat{c}, \hat{g})$ in Section 2.2.3.

2.2.2 DINO model

We now proceed to the description of the estimation procedure of the DINO model. The DINO can be considered as the dual model of the DINA model. The estimation procedure is similar except that the "AND" relationship needs to be changed to an "OR" relationship. In subsequent technical development, we will provide the precise meaning of the duality. First, we present the construction of the estimator.

The U-matrix. The matrix $U_{c,g}(Q)$ is similar to $T_{c,g}(Q)$ except that it admits an "OR" relationship among items. In particular, first define $F_Q(I_i)$ to be a vector of 2^k dimension and the $\mathbf{A}$-th element is defined as $\xi_{DINO}(\mathbf{A}, Q)$. Therefore, $F_Q(I_i)$ indicates the attribute profiles that are capable of providing positive responses to item i. We use "$\vee$" to denote the "OR" combinations among items and define

$$F_Q(I_{i_1} \vee ... \vee I_{i_l}) = \mathbf{E} - \Upsilon_{j=1}^l (\mathbf{E} - F_Q(I_{i_j})).$$

Thus, $F_Q(I_{i_1} \vee ... \vee I_{i_l})$ is a vector indicating the attribute profiles that are capable of responding positively to at least one of the item(s) i_1,..., i_l. We let the row in $U(Q)$ corresponding to $I_{i_1} \vee ... \vee I_{i_l}$ be $F_Q(I_{i_1} \vee ... \vee I_{i_l})$. In presence of slipping and guessing, we define

$$F_{c,g,Q}(I_i) = (c_i - g_i) F_Q(I_i) + g_i \mathbf{E}$$

and

$$F_{c,g,Q}(I_{i_1} \vee ... \vee I_{i_l}) = \mathbf{E} - \Upsilon_{j=1}^l (\mathbf{E} - F_{c,g,Q}(I_{i_j})).$$

We let the row in $U_{c,g}(Q)$ corresponding to "$I_{i_1} \vee ... \vee I_{i_l}$" be $F_{c,g,Q}(I_{i_1} \vee ... \vee I_{i_l})$.

The β-vector. The vector β plays a similar role as the vector α for the DINA model. Specifically, β is a column vector whose length is equal to the number of rows of $U(Q)$. Each element of β corresponds to one row vector of $U(Q)$. The element of β corresponding to $I_{i_1} \vee ... \vee I_{i_l}$ is defined as

$$N_{I_{i_1} \vee ... \vee I_{i_l}}/N = \frac{1}{N} \sum_{r=1}^N \mathbf{1}(\text{there exists a } j \text{ such that } R_r^{i_j} = 1).$$

With such a construction and a correctly specified Q, one may expect that

$$\beta \to U_{c,g}(Q)\mathbf{p}$$

almost surely as $N \to \infty$. Therefore, we define objective function

$$V_{c,g}(Q) = \inf_{\mathbf{p}} |U_{c,g}(Q)\mathbf{p} - \beta|, \tag{17}$$

where inf subject to $\sum_{\mathbf{A}} p_{\mathbf{A}} = 1$ and $p_{\mathbf{A}} \in [0, 1]$. Furthermore, an estimator of Q can be obtain by

$$\tilde{Q}(c, g) = \arg \inf_{Q'} V_{c,g}(Q'). \tag{18}$$

In cases when parameters c or g are unknown, we may plug in their estimates and define

$$\tilde{Q}_{\hat{c},\hat{g}} \quad = \quad \arg \inf_{Q'} V_{\hat{c}(Q'),\hat{g}(Q')}(Q'). \tag{19}$$

2.2.3 Estimators for the slipping and guessing parameters

To complete our estimation procedure, we provide one generic estimator for (c, g). For the DINA model, we let

$$(\hat{c}(Q), \hat{g}(Q)) = \arg \inf_{c,g \in [0,1]^m} S_{c,g}(Q); \tag{20}$$

and for the DNIO model, we let

$$(\hat{c}(Q), \hat{g}(Q)) = \arg \inf_{c,g \in [0,1]^m} V_{c,g}(Q). \tag{21}$$

We emphasize that $(\hat{c}, \hat{g})$ may not be a consistent estimator of (c, g). To illustrate this, we present one example discussed in Liu et al. (2011). Consider the case of $m = k$ items with k attributes and a complete matrix $Q = \mathcal{I}_k$, the $k \times k$ identity matrix. The degrees of freedom of a k-way binary table is

$2^k - 1$. On the other hand, the dimension of parameters $(\mathbf{p}, c, g)$ is $2^k - 1 + 2k$. Therefore, $\mathbf{p}$, c, and g cannot be consistently identified without additional information. This problem is typically tackled by introducing addition parametric assumptions such as $\mathbf{p}$ satisfying certain functional form or in the Bayesian setting (weakly) informative prior distributions Gelman, Jakulin, Pittau, and Su (2008). Given that the emphasis of this paper is the inference of Q-matrix, we do not further investigate the identifiability of $(\mathbf{p}, c, g)$. Despite the consistency issues, if one adopts the estimators in (20) and (21) for the estimator of Q as in (16) and (19), the consistency results remain even if $(\hat{c}(Q), \hat{g}(Q))$ is inconsistent. We will address this issue in more details in the remarks after the statements of the main theorems.

2.3 Theoretical properties

2.3.1 Notation

To facilitate the statements, we first introduce notation and some necessary conditions that will be referred to in later discussions.

- Linear space spanned by vectors $V_1, ..., V_l$:

$$\mathcal{L}(V_1, ..., V_l) = \left\{ \sum_{j=1}^{l} a_j V_j : a_j \in \mathbb{R} \right\}.$$

- For a matrix M, $M_{1:l}$ denotes the submatrix containing the first l rows and all columns of M.

- Vector e_i denotes a column vector with the i-th element being 1 and the rest being 0. When there is no ambiguity, we omit the length index of e_i.

- Matrix $\mathcal{I}_l$ denotes the $l \times l$ identity matrix.

- For a matrix M, $C(M)$ is the linear space generated by its column vectors. It is usually called the *column space* of M.

- For a matrix M, C_M denotes the set of its column vectors and R_M denotes the set of its row vectors.

- Vector $\mathbf{0}$ denotes the zero vector, $(0, ..., 0)$. When there is no ambiguity, we omit the index of length.

- Define a 2^k dimensional vector

$$\mathbf{p} = \left(p_{\mathbf{A}} : \mathbf{A} \in \{0, 1\}^k \right).$$

- For m dimensional vectors c and g, write $c \succ g$ if $c_i > g_i$ for all $1 \le i \le m$ and $c \ncong g$ if $c_i \neq g_i$ for all $i = 1, ..., m$.

- Matrix Q denotes the true matrix and Q' denotes a generic $m \times k$ binary matrix.

The following definitions will be used in subsequent discussions.

Definition 1 *We say that $T(Q)$ is* saturated *if all combinations of the form $I_{i_1} \wedge ... \wedge I_{i_l}$, for $l = 1, ..., m$, are included in $T(Q)$. Similarly, we say that $U(Q)$ is* saturated *if all combinations of the form $I_{i_1} \vee ... \vee I_{i_l}$, for $l = 1, ..., m$, are included in $U(Q)$.*

Definition 2 *We write $Q \sim Q'$ if and only if Q and Q' have identical column vectors, which could be arranged in different orders; otherwise, we write $Q \nsim Q'$.*

Remark 1 *It is not hard to show that "$\sim$" is an* equivalence relation*. $Q \sim Q'$ if and only if they are identical after an appropriate permutation of the columns. Each column of Q is interpreted as an attribute. Permuting the columns of Q is equivalent to relabeling the attributes. For $Q \sim Q'$, we are not able to distinguish Q from Q' based on data.*

Definition 3 *A Q-matrix is said to be* complete *if $\{e_i : i = 1, ..., k\} \subset R_Q$ (R_Q is the set of row vectors of Q); otherwise, we say that Q is* incomplete*.*

A Q-matrix is complete if and only if for each attribute there exists an item only requiring that attribute. Completeness implies that $m \geq k$. We will show that completeness is among the sufficient conditions to identify Q. In addition, it is pointed out by Chiu et al. (2009) (c.f. the paper for more detailed formulation and discussion) that the completeness of the Q-matrix is a necessary condition for a set of items to consistently identify attributes. Thus, it is always recommended to use a complete Q-matrix unless additional information is available.

Listed below are assumptions which will be used in subsequent development.

C1 Matrix Q is *complete*.

C2 Both $T(Q)$ and $U(Q)$ are *saturated*.

C3 Random vectors $\mathbf{A}_1, ..., \mathbf{A}_N$ are i.i.d. with the following distribution

$$P(\mathbf{A}_r = \mathbf{A}) = p_{\mathbf{A}};$$

We further let $\mathbf{p} = (p_{\mathbf{A}} : \mathbf{A} \in \{0, 1\}^k)$.

C4 The attribute population is *diversified*, that is, $\mathbf{p} \succ \mathbf{0}$.

2.3.2 Consistency results

We first present the consistency results for the DINA model.

Theorem 1 *Under the DINA model, suppose that conditions C1-4 hold, that is, Q is complete, $T(Q)$ is saturated, the attribute the profiles are i.i.d., and $\mathbf{p}$ is diversified. Suppose also that the c and g are known. Let $S_{c,g}(Q')$ be as defined in (13) and*

$$\hat{Q}(c, g) = \arg\inf_{Q'} S_{c,g}(Q').$$

Then,

$$\lim_{N \to \infty} P(\hat{Q}(c, g) \sim Q) = 1.$$

In addition, with an appropriate arrangement of the column order of $\hat{Q}(c, g)$, let

$$\hat{\mathbf{p}} = \arg\inf_{\mathbf{p}'} |T_{c,g}(\hat{Q}(c, g))\mathbf{p}' - \alpha|.$$

Then, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P(|\hat{\mathbf{p}} - \mathbf{p}| > \varepsilon) = 0.$$

Theorem 2 *Under the DINA model, suppose that the conditions in Theorem 1 hold, except that the* c *and* g *are unknown. For any* Q', $\hat{c}(Q')$ *and* $\hat{g}(Q')$ *are estimators for* c *and* g*. When* $Q = Q'$, $(\hat{c}(Q), \hat{g}(Q))$ *is a consistent estimator of* (c, g)*. Let* $\hat{Q}_{\hat{c},\hat{g}}$ *be as defined in* (16)*. Then*

$$\lim_{N\to\infty} P(\hat{Q}_{\hat{c},\hat{g}} \sim Q) = 1.$$

In addition, with an appropriate arrangement of the column order of $\hat{Q}_{\hat{c},\hat{g}}$*, let*

$$\hat{\mathbf{p}} = \arg\inf_{\mathbf{p}'} |T_{\hat{c}(\hat{Q}_{\hat{c},\hat{g}}),\hat{g}(\hat{Q}_{\hat{c},\hat{g}})}(\hat{Q}_{\hat{c},\hat{g}})\mathbf{p}' - \alpha|.$$

Then, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P(|\hat{\mathbf{p}} - \mathbf{p}| > \varepsilon) = 0.$$

In what follows, we present the consistency results for the DINO model.

Theorem 3 *Under the DINO model, suppose that conditions C1-4 hold, that is,* Q *is complete,* $U(Q)$ *is saturated, the attribute profiles are i.i.d., and* $\mathbf{p}$ *is diversifies. Suppose also that the* c *and* g *are known. Let* $V_{c,g}(Q')$ *be defined as in* (17) *and*

$$\tilde{Q}(c, g) = \arg\inf_{Q'} V_{c,g}(Q').$$

Then,

$$\lim_{N\to\infty} P(\tilde{Q}(c, g) \sim Q) = 1.$$

In addition, with an appropriate arrangement of the column order of $\tilde{Q}(c, g)$*, let*

$$\hat{\mathbf{p}} = \arg\inf_{\mathbf{p}'} |U_{c,g}(\tilde{Q}(c, g))\mathbf{p}' - \beta|.$$

Then, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P(|\hat{\mathbf{p}} - \mathbf{p}| > \varepsilon) = 0.$$

Theorem 4 *Under the DINO model, suppose that the conditions in Theorem 3 hold, except that the* c *and* g *are unknown. For any* Q', $\hat{c}(Q')$ *and* $\hat{g}(Q')$ *are estimators for* c *and* g*. When* $Q = Q'$, $\hat{c}(Q)$ *and* $\hat{g}(Q)$ *are consistent estimators of* c *and* g*. Let* $\hat{Q}_{\hat{c},\hat{g}}$ *be defined as in* (19)*. Then*

$$\lim_{N\to\infty} P(\tilde{Q}_{\hat{c},\hat{g}} \sim Q) = 1.$$

In addition, with an appropriate arrangement of the column order of $\tilde{Q}_{\hat{c},\hat{g}}$*, let*

$$\hat{\mathbf{p}} = \arg\inf_{\mathbf{p}'} |U_{\hat{c}(\tilde{Q}_{\hat{c},\hat{g}}),\hat{g}(\tilde{Q}_{\hat{c},\hat{g}})}(\tilde{Q}_{\hat{c},\hat{g}})\mathbf{p}' - \beta|.$$

Then, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P(|\hat{\mathbf{p}} - \mathbf{p}| > \varepsilon) = 0.$$

Remark 2 *It is not hard to verify that "$\sim$" defines a binary equivalence relation on the space of $m \times k$ binary matrices, denoted by $\mathcal{M}_{m,k}$. As previously mentioned, the data do not contain information about the specific meaning of the attributes. Therefore, we do not expect to distinguish Q_1 from Q_2 if $Q_1 \sim Q_2$. Therefore, the identifiability in the theorems is the strongest type that one may expect. The corresponding quotient set is the finest resolution that is possibly identifiable based on the data. Under weaker conditions, such as in absence of completeness of the Q-matrix or the complete diversity of the attribute distribution, the identifiability of the Q-matrix may be weaker, which corresponds to a coarser quotient set.*

Remark 3 *We would like to point out that, when the estimators in* (20) *and* (21) *are chosen, $\hat{Q}_{\hat{c},\hat{g}}$ is always a consistent estimator of Q, even if $(\hat{c},\hat{g})$ is not a consistent estimator for (c,g). This is because the proof of Theorem 2 is based on the fact that $S_{\hat{c}(Q),\hat{g}(Q)}(Q) \to 0$ in probability; when $Q' \nsim Q$, $S_{\hat{c}(Q'),\hat{g}(Q')}(Q')$ is bounded below by some $\delta > 0$. Given that $S_{c,g}(Q) \to 0$ and that $(\hat{c},\hat{g})$ is chosen to minimize the objective function S , $S_{\hat{c}(Q),\hat{g}(Q)}(Q)$ decreases to zero regardless whether or not $(\hat{c},\hat{g})$ is consistent. In addition, the fact that $S_{\hat{c}(Q'),\hat{g}(Q')}(Q')$ is bounded below by some $\delta > 0$ does not require any consistency property of $(\hat{c},\hat{g})$. Therefore, the consistency of $\hat{Q}_{\hat{c},\hat{g}}$ does not rely on the consistency of $(\hat{c},\hat{g})$ if it is of the particular forms as in* (20) *and* (21). *On the other hand, in order to have $\hat{\mathbf{p}}$ being consistent, it is necessary to require the consistency for $(\hat{c},\hat{g})$. Therefore, in the statement of Theorem 2 we require the consistency of $(\hat{c},\hat{g})$, though it is necessary to point out this subtlety. A similar argument applies to Theorem 4 as well.*

3 Discussions and implementation

This paper focuses mostly on the estimation of the Q-matrix. In this section, we discuss several practical issues and a few other usages of the proposed tools.

Computational issues. There are several aspects we would like to address. First, for a given Q, the evaluation of $S_{c,g}(Q)$ only consists of optimization of a quadratic function subject to linear constraint(s). This can be done by quadratic programming type of well established algorithms.

Second, the theories require construction of a saturated T-matrix or U-matrix which is $2^m - 1$ by 2^k. Note that when m is reasonably large, for instance, $m = 20$, a saturated T-matrix has over 1 million rows. One solution is to include part of the combinations and gradually include more combinations if the criterion function admit small values at multiple Q-matrices. Alternatively, we may split the items into multiple groups which we will elaborate in the next paragraph.

The third computational issue is related to minimization of $S_{c,g}(Q)$ with respect to Q. This involves evaluating function S over all the $m \times k$ binary matrices, which has a cardinality of $2^{m\times k}$. Simply searching through such a space is a substantial computation overhead. In practice, one may want to handle such a situation by splitting the Q-matrix in the following manner. Suppose there are m items. We split them into l groups, each of which has m_0 (a computationally manageable number) items. This is equivalent to dividing a large Q-matrix into multiple smaller sub-matrices. When necessary, we may allow different groups to have overlaps of items. Then, we can estimate each sub-matrix separately and merge them into an estimate of the big Q-matrix. Given that the asymptotic results are applicable to each of the sub-matrices, the combined estimate is also consistent. This is similar to the splitting procedure in Chapter 8.6 of Tatsuoka (2009). We emphasize that splitting the parameter space is typically not valid for usual statistical inferences. However, the Q-matrix admits a special structure with which the splitting is feasible and valid. This partially helps to relieve the computation burden related to the proposed procedure. On the other hand, it is always desirable to have a generic efficient algorithm for a general large scale Q-matrix. We leave this as a topic for a future investigation.

Partially specified Q-matrix. It is often reasonable to assume that some entries of the Q-matrix are known. For example, suppose we can separate the attributes into "hard" and "soft" ones. By "hard", we mean those that are concrete and easily recognizable in a given problem and, by "soft", we mean those that are subtle and not obvious. We can then assume that the entry columns which correspond to the "hard" attributes are known. Another instance is that there is a subset of items whose attribute requirements are known and the item-attribute relationships of the other items need to be learnt, such as the scenarios when new items need to be calibrated according to the existing ones. In this sense, even if an estimated Q-matrix may not be sufficient to replace the a priori Q-matrix provided by the "expert" (such as exam makers), it can serve as a validation as well as a source of calibration of the existing knowledge of the Q-matrix.

When such information is available and correct, the computation can be substantially reduced. This is because the optimization, for instance that in (16), can be performed subject to the existing knowledge of the Q-matrix. In particular, once a set of items is known to form a complete Q-matrix, that is, item i is known to only require attribute i for $i = 1, ..., k$, then one can calibrate one item at a time. More specifically, at each time, one can estimate the sub-matrix consisting of items 1 to k as well as one additional item, the computational cost of which is $O(2^k)$. Then the overall computational cost is reduced to $O(m2^k)$, which is typically of a manageable order.

Validation of a Q-matrix. The propose framework is applicable to not only the estimation of the Q-matrix but also validation of an existing Q-matrix. Consider the DINA and DINO models. If the Q-matrix is correctly specified, then one may expect

$$|\alpha - T_{\hat{c},\hat{g}}(Q)\mathbf{p}| \to 0$$

in probability as $N \to \infty$. The above convergence requires no additional conditions (such as completeness or diversified attribute distribution). In fact, it suffices to have that the responses are conditionally independent given the attributes and $(\hat{c}, \hat{g})$ are consistent estimators of (c, g). Then, one may expect that

$$S_{\hat{c},\hat{g}}(Q) \to 0.$$

If the convergence rate of the estimators $(\hat{c}, \hat{g})$ is known, for instance, $(\hat{c} - c, \hat{g} - g) = O_p(n^{-1/2})$, then a necessary condition for a correctly specified Q-matrix is that $S_{\hat{c},\hat{g}}(Q) = O_p(n^{-1/2})$. The asymptotic distribution of S depends on the specific form of $(\hat{c}, \hat{g})$. Consequently, checking the closeness of S to zero forms a procedure for validation of the existing knowledge of the Q-matrix.

4 Proofs of the theorems

4.1 Preliminary results: propositions and lemmas

Proposition 1 *Under the setting of the DINA model, suppose that Q is complete and matrix $T(Q)$ is saturated. Then, we are able to arrange the columns and rows of Q and $T(Q)$ such that $T(Q)_{1:(2^k-1)}$ has rank $2^k - 1$, that is, after removing one zero column this sub-matrix has full column rank.*

Proof of Proposition 1. We let the first column of $T(Q)$ correspond to the zero attribute profile. Then, the first column is a zero vector, which is the column we mean to remove in the statement of the proposition. Provided that Q is complete, without loss of generality we assume that the i-th row vector of Q is $e_i^\top$ for $i = 1, ..., k$, that is, item i only requires attribute i for each $i = 1, ..., k$. The first $2^k - 1$ rows of $T(Q)$ are associated with $\{I_1, ..., I_k\}$. In particular, we let the first k rows correspond to $I_1, ..., I_k$ and the second to the $(k+1)$-th columns of $T(Q)$ correspond to $\mathbf{A}$'s that only have one attribute. We further arrange the next C_2^k rows of $T(Q)$ to correspond to combinations of two items, $I_i \wedge I_j$, $i \neq j$. The next C_2^k columns of $T(Q)$ correspond to $\mathbf{A}$'s that only have two positive attributes. Similarly, we arrange $T(Q)$ for combinations of three, four, and up to k items. Therefore, the first $2^k - 1$ rows of $T(Q)$ admit a block upper triangle form. In addition, we are able to further arrange the columns within each block such that the diagonal matrices are identities, so that $T(Q)$ has form

$$\begin{array}{c} I_1, I_2, ... \\ I_1 \wedge I_2, I_1 \wedge I_3, ... \\ I_1 \wedge I_2 \wedge I_3, ... \\ \vdots \end{array} \left(\begin{array}{cccccc} 0 & \mathcal{I}_k & * & * & * & \cdots \\ 0 & 0 & \mathcal{I}_{C_2^k} & * & * & \\ 0 & 0 & 0 & \mathcal{I}_{C_3^k} & * & \\ \vdots & \vdots & \vdots & \vdots & & \end{array} \right). \tag{22}$$

$T(Q)_{1:(2^k-1)}$ obviously has full rank after removing the zero (first) column. ■

From now on, we assume that $Q_{1:k} = \mathcal{I}_k$ and the first $2^k - 1$ rows of $T(Q)$ are arranged in the order as in (22).

Proposition 2 *Under the DINA model, that is, the ability indicator follows* (1)*, assume that Q is a complete matrix and $T(Q)$ is saturated. Without loss of generality, let $Q_{1:k} = \mathcal{I}_k$. Assume that the first k rows of Q' form a complete matrix. Further, assume that $Q_{1:k} = Q'_{1:k} = \mathcal{I}_k$. If $Q' \neq Q$ and $c \ncong g$, then for all $c' \in \mathbb{R}^m$ there exists at least one column vector of $T_{c,g}(Q)$ not in the column space $C(T_{c'}(Q'))$, where $T_{c'}(Q')$ is as defined in* (12) *being the T-matrix with zero guessing probabilities.*

Proposition 3 *Under the DINA model, that is, the ability indicator follows* (1)*, assume that Q is a complete matrix and $T(Q)$ is saturated. Without loss of generality, let $Q_{1:k} = \mathcal{I}_k$. If $c \ncong g$ and $Q'_{1:k}$ is incomplete, then for all $c' \in \mathbb{R}^m$ there exists at least one nonzero column vector of $T_{c,g}(Q)$ not in the column space $C(T_{c'}(Q'))$.*

In the statement of Propositions 2 and 3, c_i, g_i, and c'_i can be any real numbers and are not restricted to be in $[0, 1]$. Propositions 2 and 3 are the central results of this paper, whose proofs are delayed to the Appendix. To state the next proposition, we define matrix

$$\tilde{T}_{c,g}(Q) = \left(\begin{array}{c} T_{c,g}(Q) \\ \mathbf{E} \end{array} \right), \tag{23}$$

that is, we add one more row of one's to the original T-matrix.

Proposition 4 *Under the DINA model, that is, the ability indicator follows* (1)*, suppose that Q is a complete matrix, $Q' \nsim Q$, T is saturated, and $c \ncong g$. Then, for all $c, g, c', g' \in [0, 1]^m$, there exists one column vector of $\tilde{T}_{c,g}(Q)$ (depending on c, g, c', g') not in $C(\tilde{T}_{c',g'}(Q'))$. In addition, $\tilde{T}_{c,g}(Q)$ is of full column rank.*

Lemma 1 *Consider two matrices T_1 and T_2 of the same dimension. If $C(T_1) \subseteq C(T_2)$, then for any matrix D of appropriate dimension for multiplication, we have*

$$C(DT_1) \subseteq C(DT_2).$$

Conversely, if the l-th column vector of DT_1 does not belong to $C(DT_2)$, then the l-th column vector of T_1 does not belong to $C(T_2)$.

Proof of Lemma 1. Note that DT_i is just a linear row transform of T_i for $i = 1, 2$. The conclusion is immediate by basic linear algebra. ■

Proof of Proposition 4. According to Propositions 2 and 3 and Lemma 1, it is sufficient to show that there exists a matrix D such that

$$D\tilde{T}_{c,g}(Q) = T_{c-g',g-g'}(Q), \quad D\tilde{T}_{c',g'}(Q') = T_{c'-g',\mathbf{0}}(Q') \triangleq T_{c'_{g'}}(Q'),$$

where $c'_{g'} = c' - g'$. Once we obtain such a linear transformation, according to Propositions 2 and 3, there exists a column vector in $T_{c-g',g-g'}(Q)$ that is not in the column space of $T_{c'_{g'}}(Q')$, as long as $Q \nsim Q'$. Then the same column vector in $\tilde{T}_{c,g}(Q)$ is not in the column space of $\tilde{T}_{c',g'}(Q')$. Thereby, we are able to conclude the proof.

In what follows, we construct such a D matrix. Let $g^* = (g_1^*, ..., g_m^*)$. We show that there exists a matrix D_{g^*} only depending on g^* so that $D_{g^*}\tilde{T}_{c,g}(Q) = T_{c-g^*,g-g^*}(Q)$. Note that each row of $D_{g^*}\tilde{T}_{c,g}(Q)$ is just a row linear transform of $\tilde{T}_{c,g}(Q)$. Then, it is sufficient to show that each row vector of $T_{c-g^*,g-g^*}(Q)$ is a linear transform of rows of $\tilde{T}_{c,g}(Q)$ with coefficients only depending on g^*. We prove this by induction.

First, note that

$$B_{c-g^*,g-g^*,Q}(I_i) = B_{c,g,Q}(I_i) - g_i^*\mathbf{E}.$$

Then all row vectors of $T_{c-g^*,g-g^*}(Q)$ of the form $B_{c-g^*,g-g^*,Q}(I_i)$ are inside the row space of $\tilde{T}_{c,g}(Q)$ with coefficients only depending on g^*. Suppose that all the vectors of the form

$$B_{c-g^*,g-g^*,Q}(I_{i_1} \wedge ... \wedge I_{i_l})$$

for all $1 \le l \le j$ can be written linear combinations of the row vectors of $\tilde{T}_{c,g}(Q)$ with coefficients only depending on g^*. Then, we consider

$$B_{c,g,Q}(I_{i_1} \wedge ... \wedge I_{i_{j+1}}) = \Upsilon_{h=1}^{j+1} \left(B_{c-g^*,g-g^*,Q}(I_{i_h}) + g_{i_h}^*\mathbf{E}\right).$$

The left hand side is just a row vector of $\tilde{T}_{c,g}(Q)$. We expand the right hand side of the above display. Note that the last term is precisely

$$B_{c-g^*,g-g^*,Q}(I_{i_1} \wedge ... \wedge I_{i_{j+1}}) = \Upsilon_{h=1}^{j+1} B_{c-g^*,g-g^*,Q}(I_{i_h}).$$

The rest terms are all of the form $B_{c-g^*,g-g^*,Q}(I_{i_1} \wedge ... \wedge I_{i_l})$ for $1 \le l \le j$ multiplied by coefficients only depending on g^*. Therefore, according to the induction assumption, we have that

$$B_{c-g^*,g-g^*,Q}(I_{i_1} \wedge ... \wedge I_{i_{j+1}})$$

can be written as linear combinations of rows of $\tilde{T}_{c,g}(Q)$ with coefficients only depending on g^*. Therefore, we can construct the matrix D_{g^*} accordingly. Lastly, we choose $g^* = g'$ and conclude that

$$D_{g'}\tilde{T}_{c,g}(Q) = T_{c-g',g-g'}(Q), \quad D_{g'}\tilde{T}_{c',g'}(Q') = T_{c'_{g'}}(Q').$$

By Propositions 2 and 3, there exists a column vector of $T_{c-g',g-g'}(Q)$ not in the column space of $T_{c'_{g'}}(Q')$. Furthermore, according to Lemma 1, we conclude the first part of the Proposition.

In addition, consider $D_g\tilde{T}_{c,g}(Q) = T_{c_g}(Q)$ where $c_g = c - g \not\succeq \mathbf{0}$. By construction as in (22), after removing the first zero column, $T_{c_g}(Q)$ is of rank $2^k - 1$. Therefore, the matrix

$$\begin{pmatrix} T_{c_g}(Q) \\ \mathbf{E} \end{pmatrix}$$

is of full rank. Note that each row of the above matrix is a linear transform of $\tilde{T}_{c,g}(Q)$. Thus, $\tilde{T}_{c,g}(Q)$ is a full rank matrix too. Thereby, we conclude the proof of the proposition. ∎

For the DINO model, we define a similar matrix

$$\tilde{U}_{c,g}(Q) = \begin{pmatrix} U_{c,g}(Q) \\ \mathbf{E} \end{pmatrix}, \tag{24}$$

and collect the following proposition.

Proposition 5 *Under the setting of the DINO model, that is, the ability indicator follows* (2)*, suppose that* Q *is a complete matrix,* $Q' \not\sim Q$*,* U *is saturated, and* $c \not\succeq g$*. Then, for all* $c, g, c', g' \in [0,1]^m$*, there exists one column vector of* $\tilde{U}_{c,g}(Q)$ *not in* $C(\tilde{U}_{c',g'}(Q'))$*. In addition,* $\tilde{U}_{c,g}(Q)$ *is of full column rank.*

Lemma 2 *Let* $T(Q)$ *be the* T*-matrix under the DINA model with* $c = 1$ *and* $g = 0$ *and* $U(Q)$ *be the* U*-matrix under DINO model with* $c = 1$ *and* $g = 0$*. We are able to arrange the column order of* $T(Q)$ *and* $U(Q)$ *so that*

$$T(Q) + U(Q) = \mathbb{E},$$

where $\mathbb{E}$ *is a matrix of appropriate dimensions with all entries being one's.*

Proof of Lemma 2. Consider a Q-matrix, an attribute profile $\mathbf{A}$, and an item i. Let $\mathbf{A}^c = \mathbf{E} - \mathbf{A}$ be the complimentary profile. Suppose that $Q_{ij} = 1$ for $1 \le j \le n$ and $Q_{ij} = 0$ for $n < j \le k$. Under the DINO model, $\xi^i_{DINO}(\mathbf{A}, Q) = 1$ if $\mathbf{A}^j = 1$ at least for one $1 \le j \le n$. For the same j, $(\mathbf{A}^c)^j = 0$ and therefore $\xi^i_{DINA}(\mathbf{A}^c, Q) = 0$. That is, $\xi^i_{DINO}(\mathbf{A}, Q) = 1$ implies that $\xi^i_{DINA}(\mathbf{A}^c, Q) = 0$. Similarly we are able to obtain that $\xi^i_{DINO}(\mathbf{A}, Q) = 0$ implies that $\xi^i_{DINA}(\mathbf{A}^c, Q) = 1$. Therefore, if we arrange the columns of $T(Q)$ and $U(Q)$ in such a way that the $\mathbf{A}$-th column of $U(Q)$ and the $\mathbf{A}^c$-th column of $T(Q)$ have the same position, then

$$B_Q(I_i) + F_Q(I_i) = \mathbf{E},$$

for all $1 \le i \le m$. Note that

$$\begin{aligned} B_Q(I_1 \wedge ... \wedge I_l) &= \Upsilon^l_{i=1} B_Q(I_i) \\ &= \Upsilon^l_{i=1}(\mathbf{E} - F_Q(I_i)) \\ &= \mathbf{E} - F_Q(I_1 \vee ... \vee I_l). \end{aligned}$$

Thus, we conclude the proof. ∎

Proof of Proposition 5. Thanks to Propositions 2 and 3 and Lemma 1, it is sufficient to show that with an appropriate order of the columns of $U_{c,g}(Q)$ there exists a matrix $D'_{c'}$ only depending on $c' = (c'_1, ..., c'_m)$ (independent of Q) such that

$$D'_{c'}\tilde{U}_{c,g}(Q) = T_{c'-g,c'-c}(Q)$$

for all $m \times k$ binary matrix Q. To establish that, we only need to show that each row vector of $T_{c'-g,c'-c}(Q)$ can be written as a linear combination of the row vectors of $\tilde{U}_{c,g}(Q)$. In addition, the coefficients only depend on the c' and are independent of c, g, and Q.

We establish this by induction. By construction, we have that for each $i = 1, ..., m$

$$\mathbf{E} - F_{c,g,Q}(I_i) = (1 - c_i)\mathbf{E} + (c_i - g_i)(\mathbf{E} - F_Q(I_i)).$$

Note that each column of U (and T) and each element in $F_Q(I_i)$ (and $B_Q(I_i)$) correspond to one attribute profile $\mathbf{A} \in \{0,1\}^k$. If we arrange the $\mathbf{A}$-th position of $F_Q(I_i)$ and $\mathbf{A}^c$ position of $B_Q(I_i)$ to be the same, then from the proof of Lemma 2 we obtain that $B_Q(I_i) = \mathbf{E} - F_Q(I_i)$. Therefore, $\mathbf{E} - F_{c,g,Q}(I_i) = B_{1-g,1-c,Q}(I_i)$. Similarly, we obtain that

$$\begin{aligned}
\mathbf{E} - F_{c,g,Q}(I_{i_1} \vee ... \vee I_{i_l}) &= \Upsilon_{j=1}^{l}(\mathbf{E} - F_{c,g,Q}(I_{i_j})) \\
&= \Upsilon_{j=1}^{l+1} B_{1-g,1-c,Q}(I_{i_j}) \\
&= B_{1-g,1-c,Q}(I_{i_1} \wedge ... \wedge I_{i_l}),
\end{aligned}$$

where $1 - c = (1 - c_1, ..., 1 - c_m)$. Let $\mathbb{E}$ be the matrix with all entries being one's. We essentially established that

$$\mathbb{E} - U_{c,g}(Q) = T_{1-g,1-c}(Q).$$

We use the matrix D_{g^*} constructed in Proposition 4 and obtain that

$$D_{1-c'} \begin{pmatrix} \mathbb{E} - U_{c,g}(Q) \\ \mathbf{E} \end{pmatrix} = D_{1-c'}\tilde{T}_{1-g,1-c}(Q) = T_{c'-g,c'-c}(Q).$$

Similarly, we have that

$$D_{1-c'} \begin{pmatrix} \mathbb{E} - U_{c',g'}(Q') \\ \mathbf{E} \end{pmatrix} = D_{1-c'}\tilde{T}_{1-g',1-c'}(Q') = T_{c'-g'}(Q').$$

Note that $\mathbf{E}$ is a row vector of both $\tilde{U}_{c,g}(Q)$ and $\tilde{U}_{c',g'}(Q')$. Therefore, one can construct a matrix $D'_{c'}$ so that

$$D'_{c'}\tilde{U}_{c,g}(Q) = T_{c'-g,c'-c}(Q), \quad D'_{c'}\tilde{U}_{c',g'}(Q') = T_{c'-g'}(Q').$$

Thanks to Propositions 2 and 3, there exists a column vector of $T_{c'-g,c'-c}(Q)$ not inside the column space of $T_{c'-g'}(Q')$ whenever $c \ncong g$. Thanks to Lemmas 1 and 2, the corresponding column vector(s) of $\tilde{U}_{c,g}(Q)$ is not inside the column space of $\tilde{U}_{c',g'}(Q')$. In addition, note that

$$\begin{pmatrix} T_{c'-g,c'-c}(Q) \\ \mathbf{E} \end{pmatrix}$$

is of full column rank (Proposition 4) and can be obtained by a row transformation of $\tilde{U}_{c,g}(Q)$. Therefore, $\tilde{U}_{c,g}(Q)$ is also of full column rank. Thereby, we conclude the proof. ∎

4.2 Proof of the theorems

Proof of Theorem 1. Notice that the true parameters c and g form consistent estimators for themselves. Therefore, Theorem 1 is a direct corollary of Theorem 2. ∎

Proof of Theorem 2. By the law of large numbers,

$$|T_{c,g}(Q)\mathbf{p} - \alpha| \to 0$$

almost surely as $N \to \infty$. Therefore,

$$S_{c,g}(Q) \to 0$$

almost surely as $N \to \infty$. Note that $S_{c,g}(Q)$ is a continuous function of (c, g). The consistency of $(\hat{c}(Q), \hat{g}(Q))$ implies that

$$S_{\hat{c}(Q),\hat{g}(Q)}(Q) \to 0,$$

in probability as $N \to \infty$.

For any $Q' \nsim Q$, note that

$$\begin{pmatrix} \alpha \\ 1 \end{pmatrix} \to \tilde{T}_{c,g}(Q)\mathbf{p}$$

According to Proposition 4 and the fact that $\mathbf{p} \succ \mathbf{0}$, there exists $\delta(c', g') > 0$ such that $\delta(c', g')$ is continuous in (c', g') and

$$\inf_{\mathbf{p}'} \left| \tilde{T}_{c',g'}(Q')\mathbf{p}' - \tilde{T}_{c,g}(Q)\mathbf{p} \right| > \delta(c', g').$$

By elementary calculus,

$$\delta \triangleq \inf_{c',g' \in [0,1]^m} \delta(c', g') > 0$$

and

$$\inf_{c',g' \in [0,1]^m, \mathbf{p}'} \left| \tilde{T}_{c',g'}(Q')\mathbf{p}' - \tilde{T}_{c,g}(Q)\mathbf{p} \right| > \delta.$$

Therefore,

$$P\left(\inf_{c',g' \in [0,1]^m, \mathbf{p}'} \left| \tilde{T}_{c',g'}(Q')\mathbf{p}' - \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right| > \delta/2 \right) \to 1,$$

as $N \to \infty$. For the same δ, we have

$$P(S_{\hat{c}(Q'),\hat{g}(Q')}(Q') > \delta/2) \geq P(\inf_{c',g' \in [0,1]^m} S_{c',g'}(Q') > \delta/2) = P\left(\inf_{c',g' \in [0,1]^m, \mathbf{p}'} \left| T_{c',g'}(Q')\mathbf{p}' - \alpha \right| > \delta/2 \right) \to 1.$$

The above minimization in the last probability is subject to the constraint that

$$\sum_{\mathbf{A} \in \{0,1\}^k} p_{\mathbf{A}} = 1.$$

Together with the fact that there are only finitely many $m \times k$ binary matrices, we have

$$P(\hat{Q}_{\hat{c},\hat{g}} \sim Q) = 1.$$

We arrange the columns of $\hat{Q}_{\hat{c},\hat{g}}$ so that $P(\hat{Q}_{\hat{c},\hat{g}} = Q) \to 1$ as $N \to \infty$.

Now we proceed to the proof of consistency for $\hat{\mathbf{p}}$. Note that

$$\left| \tilde{T}_{\hat{c}(\hat{Q}_{\hat{c},\hat{g}}),\hat{g}(\hat{Q}_{\hat{c},\hat{g}})}(\hat{Q}_{\hat{c},\hat{g}})\hat{\mathbf{p}} - \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right| \xrightarrow{p} 0,$$

$$\left| \tilde{T}_{\hat{c}(Q),\hat{g}(Q)}(Q)\mathbf{p} - \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right| \xrightarrow{p} 0.$$

Note that $\tilde{T}_{c,g}(Q)$ is a full column rank matrix, $P(\hat{Q}_{\hat{c},\hat{g}} = Q) \to 1$, $\hat{c}(Q) \to c$, $\hat{g}(Q) \to g$, and $T_{c,g}$ is continuous in (c, g). Then, we obtain that $\hat{\mathbf{p}} \to \mathbf{p}$ in probability. ■

Proof of Theorem 3. Similar to Theorem 1, Theorem 3 is a direct corollary of Theorem 4. ■

Proof of Theorem 4. The proof of Theorem 4 is completely analogous to that of Theorem 2. Therefore, we omit the details. ■

A Technical proofs

Proof of Proposition 2. Note that $Q_{1:k} = Q'_{1:k} = \mathcal{I}_k$. Let $T(\cdot)$ be arranged as in (22). Then, $T(Q)_{1:(2^k-1)} = T(Q')_{1:(2^k-1)}$. Given that $Q \neq Q'$, we have $T(Q) \neq T(Q')$. We assume that $T(Q)_{li} \neq T(Q')_{li}$, where $T(Q)_{li}$ is the entry in the l-th row and i-th column. Since $T(Q)_{1:(2^k-1)} = T(Q')_{1:(2^k-1)}$, it is necessary that $l \geq 2^k$. In addition, we let the l-th row correspond to a single item (not combinations of multiples).

Suppose that the l-th row of the $T(Q')$ corresponds to an item that requires attributes $i_1, ..., i_{l'}$. Then, we consider $1 \leq h \leq 2^k - 1$, such that the h-th row of $T(Q')$ is $B_{Q'}(I_{i_1} \wedge ... \wedge I_{i_{l'}})$. Then, the h-th row vector and the l-th row vector of $T(Q')$ are identical.

Since $T(Q)_{1:(2^k-1)} = T(Q')_{1:(2^k-1)}$, we have $T(Q)_{hj} = T(Q')_{hj} = T(Q')_{lj}$ for $j = 1, ..., 2^k - 1$. If $T(Q)_{li} = 0$ and $T(Q')_{li} = 1$, the matrices $T(Q)$ and $T(Q')$ look like

$$T(Q') = \begin{array}{r} \\ \\ \text{row } h \to \\ \\ \text{row } l \to \\ \\ \end{array} \overset{\substack{\text{column } i \\ \downarrow}}{\begin{pmatrix} 0 & \mathcal{I} & * & \dots & * & \dots \\ \vdots & \vdots & \vdots & & \dots & \dots \\ \vdots & \vdots & \vdots & \mathcal{I} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & * & 1 & * & & \\ 0 & * & * & * & & \end{pmatrix}},$$

and

$$T(Q) = \begin{array}{r} \\ \\ \text{row } h \to \\ \\ \text{row } l \to \\ \\ \end{array} \overset{\substack{\text{column } i \\ \downarrow}}{\begin{pmatrix} 0 & \mathcal{I} & * & \dots & * & \dots \\ \vdots & \vdots & \vdots & & \dots & \dots \\ \vdots & \vdots & \vdots & \mathcal{I} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & * & 0 & * & & \\ 0 & * & * & * & & \end{pmatrix}}.$$

Case 1 The h-th and l-th row vectors of $T_{c'}(Q')$ are nonzero vectors.

Consider the following two submatrices

$$M_1 = \begin{pmatrix} T_{c,g}(Q)_{hi} & T_{c,g}(Q)_{h2^k} \\ T_{c,g}(Q)_{li} & T_{c,g}(Q)_{l2^k} \end{pmatrix}, M_2 = \begin{pmatrix} T_{c'}(Q')_{h1} & ... & T_{c'}(Q')_{h2^k} \\ T_{c'}(Q')_{l1} & ... & T_{c'}(Q')_{l2^k} \end{pmatrix}.$$

By construction that $T(Q')_{hi} = T(Q')_{li}$ for all i, all column vectors of M_2 are proportional to each other. In what follows, we identify one column of $T_{c,g}(Q)$ that is not in the column space of $T_{c'}(Q')$. Also, it is useful to keep in mind that the 2^k-th (last) column of T corresponds to the attribute profile $(1, ..., 1)$.

a1 If $T(Q)_{li} = 0$ and $T(Q)_{hi} = 1$, then $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k}$. Since $c \ncong g$, we obtain that $T_{c,g}(Q)_{li} \neq T_{c,g}(Q)_{l2^k}$. There are two situations:

  b1 $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k} \neq 0$. It is straightforward to see that the column space of M_2 does not contain both column vectors of M_1. This is because $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k} \neq 0$ and $T_{c,g}(Q)_{li} \neq T_{c,g}(Q)_{l2^k}$ imply that the two column vectors of M_1 are not proportional to each other. Then, either the i-th column or the 2^k-th column of $T_{c,g}(Q)$ is not in the column space of $T_{c'}(Q')$.

  b2 $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k} = 0$. $T_{c,g}(Q)_{li} \neq T_{c,g}(Q)_{l2^k}$ implies that at least one of them is nonzero. Suppose that $T_{c,g}(Q)_{li} \neq 0$, then the i-th column of $T_{c,g}(Q)$ is not in the column space of $T_{c'}(Q')$. This is because the h-th row of $T_{c'}(Q')$ is not a zero vector and any vector of the form

$$\begin{pmatrix} 0 \\ nonzero \end{pmatrix} \tag{25}$$

  is not in the column space of the M_2. Similarly, if $T_{c,g}(Q)_{l2^k} \neq 0$, then the 2^k-th column is identified.

a2 If $T(Q)_{li} = 1$ and $T(Q)_{hi} = 0$, then $T_{c,g}(Q)_{li} = T_{c,g}(Q)_{l2^k}$. Note that row h corresponds to a combination of items (or just one item) each of which only requires one attribute. Therefore, we may choose column i such that the corresponding attribute is capable of answering all items in row h except for one. With this construction, if $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k}$, then they must be both zero (most of the time $T_{c,g}(Q)_{hi}$ and $T_{c,g}(Q)_{h2^k}$ are distinct). We consider three situations:

  c1 $T_{c,g}(Q)_{hi} \neq T_{c,g}(Q)_{h2^k}$. Similar to a1, the conclusion is straightforward.

  c2 $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k} = 0$ and $T_{c,g}(Q)_{li} = T_{c,g}(Q)_{l2^k} \neq 0$. Similar to b2, since the h-th row vector of $T_{c'}(Q')$ is nonzero, the statement of the proposition also holds.

  c3 $T_{c,g}(Q)_{hi} = T_{c,g}(Q)_{h2^k} = 0$ and $T_{c,g}(Q)_{li} = T_{c,g}(Q)_{l2^k} = 0$. This situation is slightly complicated, since M_1 is a zero matrix and we have to seek for a different column other than the columns i and 2^k. In what follows, all the item-attribute relationship refers to Q. If the item in the l-th row does not require strictly fewer attributes than the items in row h, then, we are able to find a column as in a1.

  Otherwise, the item in the l-th DOES require strictly fewer attributes than the items in row h. Without loss of generality, assume that the item corresponding to the l-th row requires attribute $1, 2, ..., j$, and the h-th row corresponds to items $1, 2, ..., j, ..., j'$. Suppose that for all $i' = 1, ..., 2^k$ $T_{c,g}(Q)_{li'} = 0$ implies $T_{c,g}(Q)_{hi'} = 0$ (otherwise the i'-th column is not in the column space of $T_{c'}(Q)$, c.f. b2). By slightly abusing notation, we let $c_l = 0$ be the correct answering parameter and $g_l \neq 0$ be the guessing parameter of the item in the l-th row of $T_{c,g}(Q)$. Let $\mathbf{A} = \mathbf{e}_{j'}$. Then, the $\mathbf{A}$-th element of the l-th row is $g_l \neq 0$ (equivalently, $\mathbf{A}$ is NOT able to answer that item).

d1 Suppose that the **A**-th element of the h-th row of $T_{c,g}(Q)$ is non-zero. Let $\mathbf{0} = (0, ..., 0)$ be the (zero) attribute that has precisely one few attribute than **A**. Then, the **0**-th element of the l-th row of $T_{c,g}(Q)$ equals the **A**-th element of that row (being g_l). The **0**-th and the **A**-th elements of the h-th row of $T_{c,g}(Q)$ are different. This is because the **0**-th and the **A**-th elements of the h-th row are

$$\prod_{i'=1}^{j'} g_{i'}, \qquad c_{j'} \prod_{i'=1}^{j'-1} g_{i'}.$$

Thereby, we can identify the vector from either the **A**-th or the **0**-th column vector. (Note that the **0**-th column of $T_{c,g}(Q)$ is the first column, which is not a zero vector.)

d2 Suppose that the **A**-th element of the h-th row of $T_{c,g}(Q)$ is zero. Then, the **A**-th column is not in the column space of $T_{c'}(Q)$, because its l-th element is nonzero and the h-th element is zero (c.f. b2).

Case 2 Either the h-th or l-th row vector of $T_{c'}(Q')$ is a zero vector. Since both the h-th and l-th rows of $T_{c,g}(Q)$ are nonzero vectors, we are always able to identify a column in $T_{c,g}(Q)$ that is not in the column space of $T_{c'}(Q')$.

■

Proof of Proposition 3.

Step 1

We first identify two row vectors such that they are identical in $T(Q')$ but distinct in $T(Q)$. It turns out that we only need to consider the first k items. Consider Q' such that $Q'_{1:k}$ is incomplete. We discuss the following situations.

1. There are two row vectors, say the i-th and j-th row vectors ($1 \leq i, j \leq k$), in $Q'_{1:k}$ that are identical. Equivalently, two items require exactly the same attributes according to Q'. Then, the row vectors in $T(Q')$ corresponding to these two items are identical. All of the first $2^k - 1$ row vectors in $T(Q)$ must be different, because $T(Q)_{1:(2^k-1)}$ has rank $2^k - 1$.

2. No two row vectors in $Q'_{1:k}$ are identical. Then, among the first k rows of Q' there is at least one row vector containing two or more non-zero entries. That is, there exists $1 \leq i \leq k$ such that

$$\sum_{j=1}^{k} Q'_{ij} > 1.$$

This is because if each of the first k items requires only one attribute and $Q'_{1:k}$ is not complete, there are at least two items that require the same attribute. Then, there are two identical row vectors in $Q'_{1:k}$ and it belongs to the first situation. We define

$$a_i = \sum_{j=1}^{k} Q'_{ij},$$

the number of attributes required by item i according to Q'.

Without loss of generality, assume $a_i > 1$ for $i = 1, ..., n$ and $a_i = 1$ for $i = n+1, ..., k$. Equivalently, among the first k items, only the first n items require more than one attribute while the $(n+1)$-th through the k-th items require only one attribute each, all of which are distinct. Without loss of generality, we assume $Q'_{ii} = 1$ for $i = n+1, ..., k$ and $Q_{ij} = 0$ for $i = n+1, ..., k$ and $i \neq j$.

(a) $n = 1$. Since $a_1 > 1$, there exists $i > 1$ such that $Q'_{1i} = 1$. Then, the row vector in $T(Q')$ corresponding to $I_1 \wedge I_i$ (say, the l-th row in $T(Q')$) and the row vector of $T(Q')$ corresponding to I_1 are identical. On the other hand, the first row and the l-th row are different for $T(Q)$ because $T(Q)_{1:(2^k-1)}$ is a full-rank matrix. The above statement can be written as

$$B_{Q'}(I_1 \wedge I_i) = B_{Q'}(I_1), \quad B_Q(I_1 \wedge I_i) \neq B_Q(I_1).$$

(b) $n > 1$ and there exists $j > n$ and $i \leq n$ such that $Q'_{ij} = 1$. Then by the same argument as in (2a), we can find two rows that are identical in $T(Q')$ but different in $T(Q)$. In particular,

$$B_{Q'}(I_j \wedge I_i) = B_{Q'}(I_i), \quad B_Q(I_j \wedge I_i) \neq B_Q(I_i).$$

(c) $n > 1$ and for each $j > n$ and $i \leq n$, $Q'_{ij} = 0$. Let the i^*-th row in $T(Q')$ correspond to $I_1 \wedge ... \wedge I_n$. Let the i_h^*-th row in $T(Q')$ correspond to $I_1 \wedge ... \wedge I_{h-1} \wedge I_{h+1} \wedge ... \wedge I_n$ for $h = 1, ..., n$.

We claim that there exists an h such that the i^*-th row and the i_h^*-th row are identical in $T(Q')$, that is

$$B_{Q'}(I_1 \wedge, ..., \wedge I_{h-1} \wedge I_{h+1} \wedge, ..., \wedge I_n) = B_{Q'}(I_1 \wedge, ..., \wedge I_n). \tag{26}$$

We prove this claim by contradiction. Suppose that there does not exist such an h. This is equivalent to saying that for each $j \leq n$ there exists an α_j such that $Q'_{j\alpha_j} = 1$ and $Q'_{i\alpha_j} = 0$ for all $1 \leq i \leq n$ and $i \neq j$. Equivalently, for each $j \leq n$, item j requires at least one attribute that is not required by other first n items. Consider

$$\mathcal{C}_i = \{j : \text{there exists } i \leq i' \leq n \text{ such that } Q'_{i'j} = 1\}.$$

Let $\#(\cdot)$ denote the cardinality of a set. Since for each $i \leq n$ and $j > n$, $Q'_{ij} = 0$, we have that $\#(\mathcal{C}_1) \leq n$. Note that $Q'_{1\alpha_1} = 1$ and $Q'_{i\alpha_1} = 0$ for all $2 \leq i \leq n$. Therefore, $\alpha_1 \in \mathcal{C}_1$ and $\alpha_1 \notin \mathcal{C}_2$. Therefore, $\#(\mathcal{C}_2) \leq n-1$. By a similar argument and induction, we have that $a_n = \#(\mathcal{C}_n) \leq 1$. This contradicts the fact that $a_n > 1$. Therefore, there exists an h such that (26) is true. As for $T(Q)$, we have that

$$B_Q(I_1 \wedge, ..., \wedge I_{h-1} \wedge I_{h+1} \wedge, ..., \wedge I_n) \neq B_Q(I_1 \wedge, ..., \wedge I_n).$$

Step 2

For the situations 1, 2a, and 2b, the identification of the column vector is completely identical to that of the Proposition 2. For those three situations, we essentially identified one row corresponding to a single item and another row corresponding to a combination of single-attribute items. We need to provide additional proof for situation 2c, that is, the follow-up analysis whence (26) is established. Without loss generality, we assume that

$$B_{Q'}(I_1 \wedge, ..., \wedge I_{n-1}) = B_{Q'}(I_1 \wedge, ..., \wedge I_n), \quad B_Q(I_1 \wedge, ..., \wedge I_{n-1}) \neq B_Q(I_1 \wedge, ..., \wedge I_n). \tag{27}$$

Let h be the row corresponding to $I_1\wedge, ..., \wedge I_{n-1}$ and l be the row to $I_1\wedge, ..., \wedge I_n$.

a Both the h-th and the l-th row of $T_{c'}(Q')$ are nonzero. Among the first 2^n elements of $B_{c,g,Q}(I_1\wedge, ..., \wedge I_{n-1})$ there exists a nonzero element, say corresponding to attribute $\mathbf{A}$. Let $\mathbf{A}'$ be the attribute identical to $\mathbf{A}$ for the first $n-1$ attributes and their n-th elements are different. Then, the $\mathbf{A}$-th and $\mathbf{A}'$-th elements of $B_{c,g,Q}(I_1\wedge, ..., \wedge I_{n-1})$ are identical (and nonzero). The $\mathbf{A}$-th and $\mathbf{A}'$-th elements of $B_{c,g,Q}(I_1\wedge, ..., \wedge I_n)$ must be different. This is because $\mathbf{A}$-th and $\mathbf{A}'$-th elements of $B_{c,g,Q}(I_1\wedge, ..., \wedge I_n)$ are the products of the corresponding elements in $B_{c,g,Q}(I_1\wedge, ..., \wedge I_{n-1})$ with c_l and g_l respectively and $c_l \neq g_l$. Then, either the $\mathbf{A}$-th or the $\mathbf{A}'$-th column of $T_{c,g}(Q)$ is not in the column space of $T_{c'}(Q')$.

b Either the h-th or the l-th row of $T_{c'}(Q')$ is a zero vector. The identification of the column vector is straightforward.

■

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