On non-multiaffine consistent around the cube lattice equations

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Abstract

We show that integrable involutive maps, due to the fact they admit three integrals in separated form, can give rise to equations which are consistent around the cube and which are not in the multiaffine form assumed in papers [1, 2]. In the examples of maps presented here the equations are related to lattice potential KdV equation by nonlocal transformations (discrete quadratures).

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1 Introduction

For a long time the electromagnetic potentials, Ψ and $\mathbf{A}$ had been considered as convenient purely mathematical concepts without physical significance, so to say phantoms, and only the electromagnetic fields, $\mathbf{E}$ and $\mathbf{B}$, were considered to be physical. It was thought that change of potential that did not affect the fields did not cause measurable effects. The discovery of Aharonov-Bohm effect changed the situation. Investigation of a system of equations together with its potential image is nowadays more than merely searching for convenient way of description.

Here we concentrate on the fact that one of the most important nonlinear difference equations, the lattice potential Korteweg-de Vries (lpKdV) equation (1) (see [3, 4]), arises as a potential version of more fundamental system (26) (the word "potential" in the name of the equation lpKdV is some remnant from classical terminology used in theory of integrable discretizations of integrable systems and should not be confused with the meaning of scalar "potential" of a discrete vector field we use throughout the paper: compare sections 2 and 4). The fact originates from papers where interrelation between difference integrable systems and set theoretical solutions of Yang-Baxter equation (the so called Yang-Baxter maps) were investigated (see section 3 for details and list of references). A surprise and the main result of this paper is that the fundamental system (26) has additional potentials.

To be more specific the n-dimensional system (26) allows us to introduce three potentials, say x, z and f. In the simplest two-dimensional case the potentials obey

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- lattice potential KdV equation denoted in [1] by $H1$

$$(H1): \qquad [x(m+1,n+1)-x(m,n)][x(m+1,n)-x(m,n+1)] = p(m)-q(n), \tag{1}$$

- an equation we denote by $G1$

$$(G1): \qquad z(m+1,n+1) = z(m,n) + [p(m)-q(n)]\frac{z(m,n+1)-z(m+1,n)}{[u(m,n)-v(m,n)]^2}, \tag{2}$$

  where functions u and v are given by

$$[u(m,n)]^2 = z(m+1,n)+z(m,n)+p(m), \quad [v(m,n)]^2 = z(m,n+1)+z(m,n)+q(n)$$

  The equation G1 appeared first in slightly different form in [5].

- and finally an equation we denote by $F1$

$$(F1): \quad f(m+1,n+1) = f(m,n) + [p(m)-q(n)]\left[v(m,n)-u(m,n)+\frac{f(m+1,n)-f(m,n+1)}{[u(m,n)-v(m,n)]^2}+\frac{[p(m)-q(n)]^2}{[u(m,n)-v(m,n)]^3}\right], \tag{3}$$

  where functions u and v are given by

$$\begin{array}{l} [u(m,n)]^3 + a(m)u(m,n) = f(m+1,n)-f(m,n), \\ [v(m,n)]^3 + b(n)v(m,n) = f(m,n+1)-f(m,n) \end{array} \tag{4}$$

  and

$$a(m)-b(n) = 3[q(m)-p(n)].$$

The three cases above can be combined giving rise to family of potentials ψ

$$\begin{array}{l} c_0 + c_1u(m,n) + c_2(-1)^{m+n}\left\{[u(m,n)]^2-p(m)\right\} + c_3\left\{[u(m,n)]^3+p(m)u(m,n)\right\} = \psi(m+1,n)-\psi(m,n), \\ c_0 + c_1v(m,n) + c_2(-1)^{m+n}\left\{[v(m,n)]^2-q(m)\right\} + c_3\left\{[v(m,n)]^3+q(n)v(m,n)\right\} = \psi(m,n+1)-\psi(m,n). \end{array} \tag{5}$$

To emphasise the meaning of the system (26) we suggest to adopt Plato terminology and refer to the system as to Idea system associated with map (19) discussed in section 3. Whereas the systems (30), (33) and (38) (i.e. higher dimensional extensions of equations (1), (2) and (3) respectively) we refer to as idolons of the Idea system (keeping original ancient Greek form of the word idol and having in mind rather idols from Bacon's works than contemporary meaning of the word). Note that in this terminology we do not exclude the case when given idolon participates in several Ideas.

Can we say that equations (2) and (3) are integrable? If yes, in what sense? It turns out that both equations are related to lpKdV equation though dependence is not local. Namely, if x obeys lpKdV then function z given by discrete quadratures

$$\begin{array}{l} z(m+1,n)+z(m,n) = [x(m+1,n)-x(m,n)]^2 - p(m), \\ z(m,n+1)+z(m,n) = [x(m,n+1)-x(m,n)]^2 - q(n) \end{array} \tag{6}$$

satisfies equation (2), whereas function f given by

$$\begin{array}{l} f(m+1,n)-f(m,n) = (x(m+1,n)-x(m,n))^3 + a(m)(x(m+1,n)-x(m,n)), \\ f(m,n+1)-f(m,n) = (x(m,n+1)-x(m,n))^3 + b(n)(x(m,n+1)-x(m,n)) \end{array} \tag{7}$$

is a solution of equation (3). So we understand integrability of (2), (3) as possibility to find their solutions out of known solutions of an integrable system (lpKdV in this case) by solving *linear* equations (6), (7) respectively. We call the equations (6) and (7) non-auto-Bäcklund transformations between (1) and (2), (3) respectively. However, there is a tendency to refer to such transformations as to Miura-type transformations (compare [6, 7]). Note that according to results of paper [6] there exists non-auto-Bäcklund transformations (or Miura transformations) between all equations of the set (H1)–(H3) and (Q1)–(Q3) of paper [1].

Moreover, as we already mentioned, equation (1), and more important equations (2) and (3) which are given on two-dimensional lattice, can be extended to system of compatible equations on n-dimensional lattice (30), (33) and (38). This property (of "compatible extendibility" to multidimension) as it has been pointed out in [8, 9, 10, 11, 1] is a hallmark of integrability. In other words, following the terminology introduced in [1], equations (1), (2) and (3) possess consistency around the cube property.

We start the paper with some basic informations on lpKdV including its consistency around the cube property (section 2). Next we recall some Yang-Baxter maps related to lpKdV (section 3), however we stress, anticipating facts, that Yang-Baxter property is not essential here. The origin of the potentials, namely existence of specific integrals of the maps, is explained in section 3 as well. The main results of the paper are in section 4 and our intention, while writing this letter, was to keep section 4 self-contained.

2 Lattice potential KdV equation

For we are dealing with lattice potential KdV equation, we discuss here some properties of it starting from its continuous counterpart and ending with its consistency around the cube property.

The potential KdV equation is

$$w_t = 6(w_y)^2 - w_{yyy} \tag{8}$$

and it owes its name to the fact the function u given by $u = -w_y$ satisfies Korteweg-de Vries equation [12]. Having a solution w of the equation one can find a solution $w^{(1)}$ of the equation given in quadratures by (see [12])

$$\begin{aligned} w^{(1)}_y &= -w_y - {k_1}^2 + (w^{(1)} - w)^2 \\ w^{(1)}_t &= -w_t + 4[{k_1}^4 + {k_1}^2 w_y + (w_y)^2 + w_{yy}(w^{(1)} - w) + (w_y - {k_1}^2)(w^{(1)} - w)^2] \end{aligned} \tag{9}$$

where k_1 is a constant parameter and (9) is referred to as Bäcklund transformation for potential KdV. Moreover, denoting by $w^{(2)}$ a Bäcklund transform corresponding to parameter k_2 the function $w^{(12)}$ given by

$$w^{(12)} = w + \frac{k_1^2 - k_2^2}{w^{(1)} - w^{(2)}} \tag{10}$$

is a solution of the equation (8) as well [12]. Equation (10) is nothing but nonlinear superposition principle (or Bianchi permutability theorem for Bäcklund transformations) for potential KdV equation.

An ingenious observation was that Bäcklund transformation can be reinterpreted as additional discrete variable [13] and the difference-differential equation obtained this way are integrable, e.g. the first of equations (9) becomes

$$\frac{\partial}{\partial y} w(y, m+1) = -\frac{\partial}{\partial y} w(y, m) - {k_1}^2 + [w(y, m+1) - w(y, m)]^2 \tag{11}$$

and the equation is difference-differential version of potential KdV equation. Then nonlinear superposition principle (10) turns into the fully discrete equation

$$w(m+1,n+1) = w(m,n) + \frac{k_1^2 - k_2^2}{w(m+1,n) - w(m,n+1)} \tag{12}$$

which is referred to as lattice potential KdV [3, 4].

Now we arrive at the point where advantage of difference equations over the differential ones is visible. Namely, a Bäcklund transformation $w \mapsto w^{(3)}$ for equation (12) can be written in the form

$$\begin{aligned} w^{(3)}(m+1,n) &= w(m,n) + \frac{k_1^2 - k_3^2}{w(m+1,n) - w^{(3)}(m,n)} \\ w^{(3)}(m,n+1) &= w(m,n) + \frac{k_2^2 - k_3^2}{w(m,n+1) - w^{(3)}(m,n)} \end{aligned} \tag{13}$$

and when reinterpreted as shift in the third direction is pair of lpKdV equations!

$$\begin{aligned} w(m+1,n,s+1) &= w(m,n,s) + \frac{k_1^2 - k_3^2}{w(m+1,n,s) - w(m,n,s+1)} \\ w(m,n+1,s+1) &= w(m,n,s) + \frac{k_2^2 - k_3^2}{w(m,n+1,s) - w(m,n,s+1)} \end{aligned} \tag{14}$$

It is necessary to mention that both equations (13) are first order fractional linear difference equations on $w^{(3)}$, hence their solving can be reduced to solving linear equations only.

From this place there is one step towards consistency around the cube. If we consider the equations (14) and equation

$$w(m+1,n+1,s) = w(m,n,s) + \frac{k_1^2 - k_2^2}{w(m+1,n,s) - w(m,n+1,s)} \tag{15}$$

as equations on $\mathbb{Z}^3$, there are 3 different ways to obtain $w(m+1,n+1,s+1)$ in terms of $w(m,n,s)$, $w(m+1,n,s)$, $w(m,n+1,s)$, $w(m,n,s+1)$. Due to suitable choice of form of the numerators in formulas (14)-(15), no-matter the path we follow, the resulting value $w(m+1,n+1,s+1)$ is the same (see figure 1). This property is referred as to 3d-consistency or consistency around the cube [1]. We have [12]:

$$w_{123} = \frac{k_1^2 w_1(w_2 - w_3) + k_2^2 w_2(w_3 - w_1) + k_3^2 w_3(w_1 - w_2)}{w_1(w_2 - w_3) + w_2(w_3 - w_1) + w_3(w_1 - w_2)}, \tag{16}$$

where with w_{123} we denoted $w(m+1,n+1,s+1)$, with w_1 we denote $w(m+1,n,s)$, etc.

At this moment the independence w_{123} on $w(m,n,s)$ in formula (16) should be underlined. This property is referred to as tetrahedron property since equation (16) relates four points of the 3D-lattice only. As we shall see equations (2) and (3) does not possess tetrahedron property.

Extensions of the ABS list, interesting structures closely related to consistency around the cube property and solutions of the equations from the list, have been studied since publication of paper [1] (see [14, 15, 11, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 7]).

Recapitulating, for the system (14)-(15) we have

1. Each of the equations include two parameters

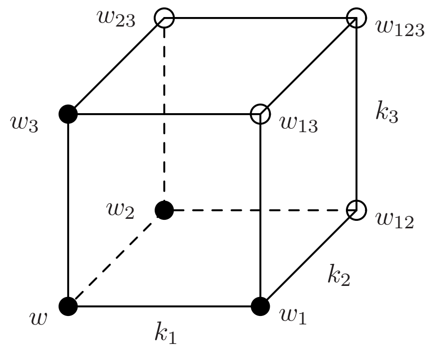

Figure 1: 3-dimensional consistency

2. Due to specific dependence on the parameters the system is compatible (consistent around the cube)

3. On replacing $w(m, n, s+1)$ with $w^{(3)}$ the first two equations turns into fractional linear equation on $w^{(3)}$

4. Formula (16) relates $w(m+1, n+1, s+1)$ with $w(m+1, n, s)$ $w(m, n+1, s)$ and $w(m, n, s+1)$ only, the so called tetrahedron property.

The equations (2), (3) posses properties 1. and 2. only. In other words equations (2), (3) are not in the form assumed in [1, 2], namely are not in the form $Q(x_{12}, x_1, x_2, x) = 0$ where Q is a multiaffine polynomial i.e. polynomial of degree one in each argument and do not possess tetrahedron property.

3 Involutive maps

There are several procedures to obtain integrable mappings from integrable lattice equations. In [31, 32] it was shown that by imposing periodic staircase initial data on an integrable lattice, one can get finite dimensional Liouville integrable maps [33], which is nowadays standard procedure [34, 35]. At the same time, there is another way to get integrable maps from an integrable lattice equation [36, 37, 38]. Using this procedure one can get involutive maps (composition of the map with itself is the identity map) which are set theoretical solutions of the Yang Baxter equation the so called Yang Baxter maps [39, 40, 36, 41]. Let us present this procedure in reverse order, from involutive mappings to lattices.

We confine ourselves to a family of involutive maps $\mathbb{C}^2 \ni (u, v) \mapsto (U, V) \in \mathbb{C}^2$ (see figure 2) which are related to the Yang-Baxter map denoted by F_V on the classification list of quadrirational Yang-Baxter maps given in [36].

$$(F_V): \qquad U = v + \frac{p-q}{u-v}, \qquad V = u + \frac{p-q}{u-v} \tag{17}$$

Apart F_V, we number among the family, the Yang-Baxter map denoted by H_V on the list that appeared in [41] (referred also to as Adler map c.f. [40])

$$(H_V): \qquad U = v + \frac{p-q}{u+v}, \qquad V = u - \frac{p-q}{u+v} \tag{18}$$

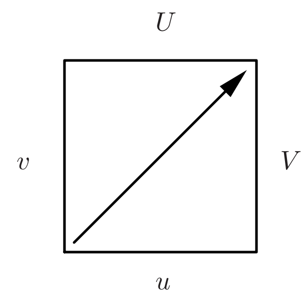

Figure 2: A map on $\mathbb{C}^2$.

and its companion (i.e. the map $(u,v) \to (U,V)$ that arises from mutual replacement of v and V in (18))

$$(cH_V): \qquad U = -v + \frac{p-q}{u-v}, \qquad V = -u + \frac{p-q}{u-v}. \tag{19}$$

Note that (19) is not a Yang-Baxter map [41], and in order to illustrate that Yang-Baxter property is not crucial from the point of view of procedures described in this paper, we deal here with the map (19) mainly.

The standard procedure for reinterpretation of a map as equations on a lattice is based on identification (see fig. 3.)

$$u = u(m,n), \quad v = v(m,n), \quad U = u(m,n+1), \quad V = v(m+1,n) \tag{20}$$

where the function $u(m,n)$ is given on horizontal edges only and $v(m,n)$ is given on vertical ones. For the

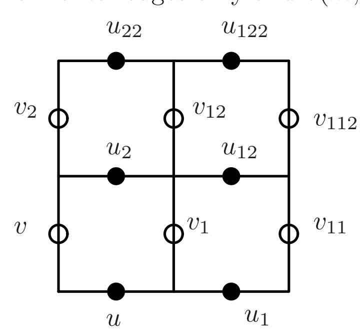

Figure 3: The map as a lattice.

map (19) we have

$$u(m,n+1) = -v(m,n) + \frac{p(m)-q(n)}{u(m,n)-v(m,n)} \quad v(m+1,n) = -u(m,n) + \frac{p(m)-q(n)}{u(m,n)-v(m,n)}. \tag{21}$$

It is essentially important for us to find functions F and G such that

$$F(U) + G(V) = f(u) + g(v) \tag{22}$$

In the case of the map (19) it leads to equation on F and G

$$F''(U)[(U-V)^2 - (p-q)] - 2F'(U)(U-V) - G''(V)[(U-V)^2 + (p-q)] - 2G'(V)(U-V) = 0,$$

which general solution is

$$F(U) + G(V) = c_1(U - V) + c_2(U^2 - V^2) + c_3(U^3 + pU - V^3 - qV) + c_4.$$

As a result we arrive at the following integrals with additively separated variables

$$\mathcal{H}_1(u, v) = u - v, \tag{23}$$

$$\mathcal{H}_3(u, v) = u^3 + pu - v^3 - qv, \tag{24}$$

and a 2-integral with separated variables as well

$$\mathcal{H}_2(u, v) = u^2 - v^2. \tag{25}$$

After the reinterpretation (20) of the map as a system of equations on a lattice, integrals (23), (24) and (25) will give rise to equations that guarantee existence of potentials.

The question arises whether one can extend the system (21) to compatible multidimensional system on $\mathbb{Z}^n$ lattice. The answer is positive and we will take up this issue in the next section.

We end the section with a proposition of equivalence relation in the set of 2D maps. According to the proposition all three maps (17), (18) and (19) are equivalent.

Proposition 3.1 *Two 2D maps are equivalent if their systems arising from identification (20), let say on function $\tilde{u}^i$ and u^i, can be related by an invertible point transformation:*

$$\begin{aligned} &\tilde{u}^i = F^i(u^1, u^2, m, n), \quad i = 1, 2, \\ &\tilde{m} = m, \\ &\tilde{n} = n. \end{aligned}$$

4 Idea I_V and its idolons

We consider $\mathbb{Z}^n = \{(m_1, \dots, m_n) : m_i \in \mathbb{Z}, i = 1, \dots, n\}$ lattice together with its edges i.e. segments connecting two consecutive points. The edges can be viewed as (and described by) pair of vertices. We refer to the elements of the set ordered pairs of points
$\{((m_1, \dots, m_i, \dots, m_n), (m_1, \dots, m_i + 1, \dots, m_n)) : m_i \in \mathbb{Z}, i = 1, \dots, n\}$, as to edges in the i-th direction. Forward shift in the i-th direction T_i we denote by subscript i.e. $T_i f(m_1, \dots, m_i, \dots, m_n) \equiv f_i(m_1, \dots, m_i, \dots, m_n) = f(m_1, \dots, m_i + 1, \dots, m_n)$. We omit independent variables to make the formulas shorter (e.g. $T_i f \equiv f_i$). By Δ_i we denote forward difference operator $\Delta_i = T_i - 1$.

We take into consideration n functions u^i, $i = 1, \dots, n$ (mind we enumerate functions by superscript!). The i-th function u^i is given on edges in the i-th direction only. A function $\mathbf{u} = (u^1, \dots, u^n)$ from set of edges to $\mathbb{R}^n$ can be viewed as vector field on a lattice. We are searching for functions that obey the following system of difference equations

$$(I_V): \qquad u^i_j + u^j = \frac{p^i - p^j}{u^i - u^j} \qquad i, j = 1, \dots, n \qquad i \neq j \tag{26}$$

where p^i are given functions of the i-th argument only ($p^i(m_i)$). We refer to the system (26) as to *Idea system* (or representative of *Idea*, see Proposition 4.1) associated with the map (19) and to underline the connection we denote it by I_V.

The following facts hold

- system (26) is compatible i.e.

$$u^i_{jk} = u^i_{kj} \qquad i,j,k = 1,\ldots,n \qquad i \neq j \neq k \neq i \tag{27}$$

Prescribing the value of each function u^i on i-th initial condition line ($m_j = 0$ for $j \neq i$) and using (26) one can find recursively u^i in the whole domain (singularities can occur because of vanishing of the denominator on right hand side of (26)).

- The following equality holds

$$\Delta_j u^i = \Delta_i u^j \qquad i,j = 1,\ldots,n \qquad i \neq j \tag{28}$$

which implies existence of a potential x (given on vertices of the lattice) such that

$$u^i = \Delta_i x \qquad i = 1,\ldots,n \tag{29}$$

In terms of potential x the system (26) reads

$$(x_{ij} - x)(x_i - x_j) = p^i - p^j \qquad i,j = 1,\ldots,n \qquad i \neq j \tag{30}$$

so we get the system of lpKdV equations. Since the existence of x is guaranteed for any initial data for x (excluding the ones that leads to $u^i - u^j = 0$) on initial lines, system (30) is nD-consistent.

- From the system (26) one infer that

$$(T_j + 1)\left[(u^i)^2 - p^i\right] = (T_i + 1)\left[(u^j)^2 - p^j\right] \qquad i,j = 1,\ldots,n \qquad i \neq j \tag{31}$$

Equations (31) imply existence of a potential z (given on vertices of the lattice) such that

$$(u^i)^2 - p^i = (T_i + 1)z \qquad i = 1,\ldots,n \tag{32}$$

In terms of potential z the system (26) can be rewritten as

$$z_{ij} - z = (p^i - p^j)\frac{z_j - z_i}{(u^i - u^j)^2} \qquad i,j = 1,\ldots,n \qquad i \neq j \tag{33}$$

where u^i should be eliminated by means of (32). Again we get system of equations (33) which is nD-consistent. An explicit manifestation of 3D-consistency is formula

$$z_{ijk} + z = \frac{[p^i u^i (u^j - u^k) + p^j u^j (u^k - u^i) + p^k u^k (u^i - u^j)]^2}{[p^i (u^j - u^k) + p^j (u^k - u^i) + p^k (u^i - u^j)]^2} + \frac{[u^i (p^j - p^k)(p^j + p^k - p^i) + u^j (p^k - p^i)(p^i + p^k - p^j) + u^k (p^i - p^j)(p^i + p^j - p^k)]}{p^i (u^j - u^k) + p^j (u^k - u^i) + p^k (u^i - u^j)}.$$

Finally, we infer that z must satisfy

$$\begin{gathered}(z_{ij} - z)^2 (z_i - z_j + p^i - p^j)^2 + (p^i - p^j)^2 (z_i - z_j)^2 + \\ 2(z_{ij} - z)(p^i - p^j)(z_i - z_j)(z_i + z_j + 2z + p^i + p^j) = 0\end{gathered} \tag{34}$$

compare [5].

- The third identity giving rise to a potential is

$$\Delta_j\left[(u^i)^3 + a^i u^i\right] = \Delta_i\left[(u^j)^3 + a^j u^j\right] \qquad i,j = 1,\dots,n \qquad i \neq j \tag{35}$$

  where a^i are functions of the i-th argument only ($a^i(m_i)$) such that

$$a^i - a^j = 3(p^j - p^i) \qquad i,j = 1,\dots,n \qquad i \neq j \tag{36}$$

  In fact by introduction of a^i functions we tacitly smuggled in the possibility of combination of integrals (23) and (24). Equations (35) imply existence of a potential f (given on vertices of the lattice again) such that

$$(u^i)^3 + a^i u^i = \Delta_i f \qquad i = 1,\dots,n \tag{37}$$

  In terms of potential f the system (26) can be rewritten as

$$f_{ij} - f = (p^i - p^j)\left[\frac{(p^i - p^j)^2}{(u^i - u^j)^3} + \frac{f_i - f_j}{(u^i - u^j)^2} - u^i + u^j\right] \qquad i,j = 1,\dots,n \qquad i \neq j \tag{38}$$

  where u^i, u^j should be replaced by solutions of cubic equations (37). Just like in the previous cases we get system of equations (38) which is nD-consistent and e.g. 3D-consistency manifests explicitly in formula

$$f_{ijk} - f = \frac{(p^i u^i u^{jk} + p^j u^j u^{ki} + p^k u^k u^{ij})^3}{(p^i u^{jk} + p^j u^{ki} + p^k u^{ij})^3} - \frac{3p^{ij}p^{jk}p^{ki}}{p^i u^{jk} + p^j u^{ki} + p^k u^{ij}} + \frac{3\left\{p^i u^i p^{ij} p^{ki}[(u^j)^2 + (u^k)^2] + p^j u^j p^{ij} p^{jk}[(u^k)^2 + (u^i)^2] + p^k u^k p^{jk} p^{ki}[(u^i)^2 + (u^j)^2] + 2u^i u^j u^k[(p^{ij} - p^k)(p^{jk} - p^i)(p^{ki} - p^j) + p^i p^j p^k]\right\}}{[p^i u^{jk} + p^j u^{ki} + p^k u^{ij}]^2}$$

  where for the sake of brevity we introduced notation $u^{ij} := u^i - u^j$ and $p^{ij} := p^i - p^j$.

- One can combine all the cases above by introducing family of potentials ψ

$$c_0 + c_1 u^i + c_2(-1)^{m_1+\dots+m_n}[(u^i)^2 - p^i] + c_3[(u^i)^3 + p^i u^i] = \psi_i - \psi \qquad i = 1,\dots,n \tag{39}$$

  We postpone the investigation of lattice equation on ψ, which includes four parameters c_0, c_1, c_2 and c_3, to a forthcoming paper.

- Eliminating u^i from (32) and from (37) by means of (29) we get

$$\Delta_i z = (\Delta_i x)^2 - p^i \qquad i = 1,\dots,n \tag{40}$$

$$\Delta_i f = (\Delta_i x)^3 + a^i \Delta_i x \qquad i = 1,\dots,n \tag{41}$$

  from which we infer

  – If x obeys $\Delta_i\Delta_j x = 0$ then f given by (41) obeys $\Delta_i\Delta_j f = 0$ as well.
  – If x obeys (30) then f given by (41) obeys (38).
  – If x obeys (30) then z given by (31) obeys (33).

- If we replace (p^i, u^i) with $(-p^i, (-1)^{m_1+\ldots+m_n} u^i)$ we get

$$u^i_j - u^j = \frac{p^i - p^j}{u^i - u^j} \tag{42}$$

In two-dimensional case, after identification (20), one obtains F_V map (17). The whole procedure we have described so far can be repeated using F_V map instead of cH_V map. Since "entities must not be multiplied beyond necessity" we arrive at the proposition.

Proposition 4.1 *Two idea systems, let say on function $\tilde{u}^i$ and u^i, are equivalent iff they are related by invertible point transformation*

$$\begin{array}{l} \tilde{u}^i = F^i(u^1, \ldots, u^n, m_1, \ldots, m_n), \quad i = 1, \ldots, n \\ \tilde{m}_i = m_i, \quad i = 1, \ldots, n \end{array} \tag{43}$$

Equivalence class of idea systems we refer to as Idea.

Idea systems (26) and (42) are representatives of the same Idea, the Idea associated with family of maps discussed in previous section. We denote the Idea by I_V just like its representative (26).

- If we replace u^2 with $-u^2$ we get equations

$$u^i_2 = u^2 + \frac{p^i - p^2}{u^i + u^2}, \qquad u^2_i = u^i - \frac{p^i - p^2}{u^i + u^2} \quad i \neq 2$$

which in two-dimensional case, after identification (20), changes in H_V map (18). However, it is not possible to turn all equations into the form $u^j_i = u^i - \frac{p^i - p^j}{u^i + u^j}$. In this case $u^j_{ik} \neq u^j_{ki}$.

5 Comments

We focused in this paper on two systems (33) and (38) which are multidimensionaly consistent. The systems are related to system of lattice potential KdV equations (30) by non-auto-Bäcklund transformations (40) and (41). Because of lack of the tetrahedron property, for $n > 2$, the systems cannot be related by a point transformation to any system of equations from Adler-Bobenko-Suris list [1]. An open question is whether one can find difference substitution (i.e. generalization of point transformations (43) where functions F^i can depend on functions u^i given in several points) that reduce solving system (33) or system (38) to solving equations from Adler-Bobenko-Suris list or to solving linear equations. We shall pick up this issue in nearby future.

There is no doubt that notion of consistency (compatibility) is important ingredient of theory of integrable systems. However understanding consistency itself as integrability leads to confusion. One of the main feature of integrable systems is that their solvability can be reduced to solving linear equations. Leaving the assumption of multiaffine form of quad-graph equation we face *terra incognita*. One cannot exclude situation when consistent equation cannot be linked to any integrable equation by a linearizible transformation. It can produce equations with essentially nonlinear Bäcklund transformations and that is why conception of consistency around the cube is interesting itself. We take a stand we should not mix it with integrability at this stage state of the art. The class of integrable systems may be identical with the class of equations consistent around the cube or may be not.

For now we are going to confine ourselves to systems that are linked to ABS list by a linearizible Bäcklund or Miura transformation (see e.g. [42, 6]) i.e. by fractional linear difference equation. So we are going to extend the table 3 of paper [42] (see also [6]) with non-multiaffine cases. In the forthcoming paper we deal with the simplest case when the link is given by discrete quadratures (so to say "simplest" fractional linear transformation). It will also cover multiplicative cases f.i. substitution

$$u = x_1 x, \quad v = x_2 x \tag{44}$$

and identification (20) changes lpKdV equation (1) into Yang-Baxter map F_{IV} from the list given in [36]

$$\begin{aligned} V &= u\left(1 + \tfrac{p-q}{u-v}\right), \\ U &= v\left(1 + \tfrac{p-q}{u-v}\right) \end{aligned} \tag{45}$$

The most general function with property (22) for this map is linear combination of functions

$$\begin{aligned} &\log \tfrac{U}{V} = \log \tfrac{v}{u} \\ &U - V = -(u - v + p - q) \\ &U^2 - V^2 + 2pU - 2qV = -(u^2 - v^2 + 2pu - 2qv + p^2 - q^2) \end{aligned} \tag{46}$$

and constant function. It leads to the following potentials x, y and z

$$\begin{aligned} &u = x_1 x, \quad v = x_2 x \\ &2u + p = y_1 + y, \quad 2v + q = y_2 + y \\ &u^2 + 2pu + \tfrac{1}{2}p^2 = z_1 + z, \quad v^2 + 2qv + \tfrac{1}{2}q^2 = z_2 + z \end{aligned} \tag{47}$$

First potential of course satisfies lpKdV equation (1), second potential satisfies equation $H2$ from ABS list [1] and third one gives rise another non-multiaffine equation we denote temptatively denote by $G2$ for equivalence and classification in the class of non-multiaffine equations is yet to be defined.

$$\begin{aligned} (H2):& \quad (y_{12} - y)(y_1 - y_2) - (p - q)(y_{12} + y_1 + y_2 + y) + p^2 - q^2 = 0 \\ (G2):& \quad z_{12} - z = \tfrac{p-q}{u-v}\left[2uv + \tfrac{1}{2}(p+q)(u+v) + uv\tfrac{p-q}{u-v}\right]. \end{aligned} \tag{48}$$

The Idea associated with F_{IV} is (we denote it by I_{IV}).

$$(I_{IV}): \qquad u^i_j = u^j\left(1 + \tfrac{p^i - p^j}{u^i - u^j}\right) \qquad i, j = 1, \ldots, n \qquad i \neq j \tag{49}$$

So we see that lpKdV idolon $H1$ participates both in I_{IV} and I_V Ideas what we illustrate on ending this paper figure 4.

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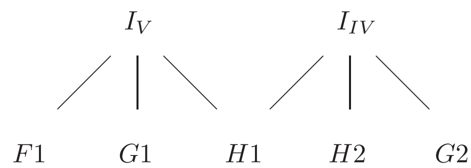

Figure 4: Ideas I_{IV} and I_V and their idolons

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