The Empirical Edgeworth Expansion for a Studentized Trimmed Mean

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Abstract

We establish the validity of the empirical Edgeworth expansion (EE) for a Studentized trimmed mean, under the sole condition that the underlying distribution function of the observations satisfies a local smoothness condition near the two quantiles where the trimming occurs. A simple explicit formula for the $N^{-1/2}$ term (correcting for skewness and bias; N being the sample size) of the EE will be given. The proof is based on a U-statistic type approximation and also uses a version of Bahadur's [1] representation for sample quantiles.

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1. Introduction

The trimmed mean is a well known estimator of a location parameter. Its asymptotic properties were studied by many authors (see [3], [5], [6], [8], [12], [18], [19], and the references therein). The main reason for applying the trimmed mean is robustness (see [9], [14]). The limit distribution of the trimmed mean for an arbitrary population distribution was found by Stigler [18]. Specifically he has shown that in order for the trimmed mean to be asymptotically normal, it is necessary and sufficient that the sample is trimmed at sample quantiles for which the corresponding population quantiles are uniquely defined [18].

In this paper we study the second-order asymptotic properties of the distribution of the trimmed mean, as well as of the Studentized trimmed mean in view of its practical relevance (construction of confidence intervals, hypothesis tests etc.).

We establish the validity of the empirical Edgeworth expansion (EEE) for a Studentized trimmed mean, under the sole condition that underlying distribution function (df) of the observations satisfies a local smoothness condition near the two quantiles where the trimming occurs. In particular our result supplements previous work by Hall and Padmanabhan [8] and Putter and van Zwet [16]. The existence of an Edgeworth expansion (EE) for a Studentized trimmed mean was also obtained by Hall and Padmanabhan [8], but these authors wrote that the "first term in an Edgeworth expansion is very complex and so it will not be written down explicitly". They suggested to replace analytical difficulties by bootstrap simulation. In contrast, in the present paper we show that our method of proof gives a simple explicit formula for the $N^{-1/2}$- term (correcting for skewness and bias; N being the sample size) of the Edgeworth expansion.

The proof of our result is based on a $U-$statistic type approximation (cf. also Bickel et al [4], Helmers [10]-[11], Putter and van Zwet [16]) and also uses a version of Bahadur's [1] representation for sample quantiles. Our $U-$statistic type approximation is slightly different from the one given by the first two terms of the Hoeffding decomposition and approximates the trimmed mean with a remainder of the classical Bahadur's order $N^{-3/4}\log N^{5/4}$ (cf. (4.4)-(4.5), Sect.4). The first order linear term of our $U-$statistic approximation is a sum of independent identically distributed (i.i.d.) Winsorized random variables. The structure of the quadratic term of the second order is connected with a Bahadur type property of the order statistics close to the sample quantile (cf. lemma 3.2, Sect.3). We will also show (cf. Lemma A.2, Appendix) that our result cannot be obtained as a consequence of a general result on Edgeworth expansions for Studentized symmetric statistics (Theorem 1.2, [16]) of Putter and van Zwet.

In Section 2, we formulate and discuss our main results on EE and EEE. In Section 3, we state and prove Bahadur's type lemmas. Next, in Section 4, we construct $U-$statistic type approximation for the trimmed mean and prove the result on EE for the normalized trimmed mean. In Section 5, the corresponding stochastic approximation for a plug-in estimator, which is used to construct a Studentized trimmed mean is established, and the result on EE for a Studentized trimmed mean is proved. Finally, in Section 6, we prove some lemmas on the consistency of our estimators of the unknown parameters appearing in the formula of one-term EE and establish a rate of convergence. In the Appendix, we establish an asymptotic approximation for the bias of trimmed mean in estimating of the corresponding location parameter, and prove that our results on EE and EEE for a Studentized trimmed mean can not be inferred from results of Putter and van Zwet [16] for Studentized symmetric statistics.

2. The main results

Let $X_1, \dots, X_N$ be i.i.d. real-valued random variables (r.v.) with common *df* F, and let $X_{1:N} \le \dots \le X_{N:N}$ denote the corresponding order statistics. Consider the trimmed mean given by

$$T_N = \frac{1}{([\beta N] - [\alpha N])} \sum_{i=[\alpha N]+1}^{[\beta N]} X_{i:N} \ , \tag{2.1}$$

where $0 < \alpha < \beta < 1$ are any fixed numbers and $[\cdot]$ represents the greatest integer function. Let $F^{-1}(u) = \inf\{x : F(x) \ge u\}$, $0 < u < 1$, denote the left-continuous inverse function of *df* F and put $\frac{d}{du}F^{-1}(u) = 1/f(F^{-1}(u))$ to be its derivative, when the density $f = F'$ exists and $f(F^{-1}(u)) > 0$. Let

$$\xi_\nu = F^{-1}(\nu),$$

$0 < \nu < 1$, be the ν-th quantile of F. Define a function

$$Q(u) = \begin{cases} \xi_\alpha\ , & u \le \alpha, \\ F^{-1}(u)\ , & \alpha < u \le \beta, \\ \xi_\beta\ , & \beta < u. \end{cases}$$

Let W_i, $i = 1, \dots, N$, denote X_i Winsorized outside of $(\xi_\alpha, \xi_\beta]$, that is

$$W_i = \begin{cases} \xi_\alpha\ , & X_i \le \xi_\alpha, \\ X_i\ , & \xi_\alpha < X_i \le \xi_\beta, \\ \xi_\beta\ , & \xi_\beta < X_i. \end{cases} \tag{2.2}$$

Then $W_i \overset{\mathrm{d}}{=} Q(U_i)$, $i = 1, \dots, N$, where U_i are independent r.v.'s with uniform $(0,1)$ distribution. Define

$$\mu_W = \int_0^1 Q(u)\,du, \quad \sigma_W^2 = \int_0^1 (Q(u) - \mu_W)^2\,du, \quad \gamma_{3,W} = \int_0^1 (Q(u) - \mu_W)^3\,du. \tag{2.3}$$

Put

$$\delta_{2,W} = -\alpha^2 \frac{1}{f(\xi_\alpha)}[\mu_W - \xi_\alpha]^2 + (1-\beta)^2 \frac{1}{f(\xi_\beta)}[\mu_W - \xi_\beta]^2. \tag{2.4}$$

Suppose that $\xi_\alpha \ne \xi_\beta$ (that is ξ_α is not an atom with mass at least $(\beta - \alpha)$ for the distribution F), then the W_i's are not degenerate. Define real numbers λ_1 and λ_2 by

$$\lambda_1 = \gamma_{3,W}/\sigma_W^3, \qquad \lambda_2 = \delta_{2,W}/\sigma_W^3. \tag{2.5}$$

We need no moment assumptions about the distribution F and to normalize T_N we use

$$\mu(\alpha, \beta) = \frac{1}{\beta - \alpha} \int_\alpha^\beta F^{-1}(u)\,du \tag{2.6}$$

as a location parameter and $(\beta - \alpha)^{-1}\sigma_W$ (the root of the asymptotic variance, cf.(4.8)) as a scale parameter. Note that T_N often serves as a statistical estimator for the parameter $\mu(\alpha, \beta)$, the population trimmed mean.

Now we show why moments are not needed. Take some fixed $\Delta > 0$ and define auxiliary i.i.d. Winsorized r.v.'s $X_i' = \max(\xi_\alpha - \Delta, \min(X_i, \xi_\beta + \Delta))$. Let $X_{i:N}'$, $i = 1, \dots, N$, denote the corresponding order statistics. Introduce an auxiliary trimmed mean $T_N' = \frac{1}{([\beta N] - [\alpha N])} \sum_{i=[\alpha N]+1}^{[\beta N]} X_{i:N}'$, and note that

$$T_N = T_N' \quad \text{if} \quad \{X_{[\alpha N]+1:N} \ge \xi_\alpha - \Delta\} \cap \{X_{[\beta N]:N} \le \xi_\beta + \Delta\}.$$

If F has a positive and continuous density in neighborhoods of ξ_α and ξ_β, then, by Bernstein's inequality $P(X_{[\alpha N]+1:N} < \xi_\alpha - \Delta) + P(X_{[\beta N]:N} > \xi_\beta + \Delta) = O(exp(-cN))$, as $N \to \infty$, where $c > 0$ is constant independent of N . Therefore

$$\sup_{x \in R} |P(T_N \le x) - P(T_N' \le x)| = O(e^{-cN}) \tag{2.7}$$

and when proving our results we can replace with impunity T_N by T_N', which has finite moments of the arbitrary order.

In absence of any moment assumptions, our formulas for the $N^{-1/2}$ term of the Edgeworth expansions contains a bias term. Define the quantity

$$\beta_N = \frac{1}{N}\Big\{-(\alpha N - [\alpha N])\Big(\mu(\alpha,\beta)-\xi_\alpha\Big) - \frac{1}{2}\alpha(1-\alpha)\frac{1}{f(\xi_\alpha)} + (\beta N - [\beta N])\Big(\mu(\alpha,\beta)-\xi_\beta\Big) + \frac{1}{2}\beta(1-\beta)\frac{1}{f(\xi_\beta)}\Big\}. \tag{2.8}$$

Note that when both αN and βN are integer valued, the bias term has a very simple form: $\beta_N = \frac{1}{2N}\Big\{-\frac{\alpha(1-\alpha)}{f(\xi_\alpha)} + \frac{\beta(1-\beta)}{f(\xi_\beta)}\Big\}$. Moreover, in case $\alpha = 1-\beta$ and $f(\xi_\alpha) = f(\xi_\beta)$ (when the distribution F is symmetric, for example), the bias term vanishes.

We show (cf. Lemma A.1, Appendix) that if the conditions of our Theorem 2.1 are satisfied, then for an arbitrary $\Delta > 0$

$$b_N = (\beta-\alpha)(ET_N' - \mu(\alpha,\beta)) = \beta_N + O(N^{-3/2}) \tag{2.9}$$

as $N \to \infty$.(cf. (2.7)) Note also that the bias term (2.8) does not depend on the auxiliary quantity Δ.

Define

$$F_{T_N}(x) = P\left(\frac{N^{1/2}(T_N - \mu(\alpha,\beta))}{(\beta-\alpha)^{-1}\sigma_W} \le x\right) \tag{2.10}$$

to be the distribution function of the normalized trimmed mean. Using the notation of Putter and van Zwet [16], we shall show that the Edgeworth expansion for $F_{T_N}(x)$ is given by

$$G_N(x) = \Phi(x) - \frac{\phi(x)}{6\sqrt{N}}\left((\lambda_1 + 3\lambda_2)(x^2-1) + 6N\frac{\beta_N}{\sigma_W}\right), \tag{2.11}$$

where Φ is the standard normal distribution function, $\phi = \Phi'$. The quantity $(\lambda_1 + 3\lambda_2)N^{-1/2}$ serves as an approximation to the third cumulant of $\frac{N^{1/2}(T_N' - \mu(\alpha,\beta))}{(\beta-\alpha)^{-1}\sigma_W}$, moreover $\lambda_1 N^{-1/2}$ is the approximation to the third cumulant of the L_2-projection of the normalized trimmed mean, which close to $N^{-1/2}\sigma_W^{-1}\sum_1^N W_i$ - a sum of N i.i.d. Winsorized r.v.'s (cf. Sect.4, below), and $3\lambda_2 N^{-1/2}$ is due to the $U-$statistic type approximation to T_N.

Here is our first result: an Edgeworth expansion for a normalized trimmed mean.

THEOREM 2.1. *Suppose that $f = F'$ exists in neighborhoods of the points ξ_α and ξ_β and satisfies a Lipschitz condition. In addition we assume that $f(\xi_\nu) > 0$, $\nu = \alpha, \beta$. Then*

$$\sup_{x\in R} |F_{T_N}(x) - G_N(x)| = o(N^{-1/2}), \tag{2.12}$$

as $N \to \infty$.

Theorem 2.1 can be viewed as a version of the Edgeworth expansion for the trimmed mean obtained by Bjerve [6] in his unpublished Berkeley Ph.D. thesis (cf. also Helmers

[10]). Our method of proof is completely different from Bjerve's, as he used a conditioning argument to reduce a trimmed mean to a sum of i.i.d. r.v.'s, conditionally given the values of $X_{[\alpha N]+1:N}$ and $X_{[\beta N]:N}$, while in contrast we essentially show that T_N can be approximated by a $U-$statistic U_N; the remainder $T_N - U_N$ can be shown to be of negligible order for our purposes by an application of a version of Bahadur [1] representation for sample quantiles.

Next we state our result on the validity of one-term Edgeworth expansion for the Studentized trimmed mean. Define plug in estimators for μ_W and σ_W^2 by

$$\hat{\mu}_W = \frac{k}{N} X_{k:N} + \frac{1}{N} \sum_{i=k+1}^{m-1} X_{i:N} + \frac{N-m+1}{N} X_{m:N}, \tag{2.13}$$

and

$$S_N^2 = \left(\frac{k}{N} X_{k:N}^2 + \frac{1}{N} \sum_{i=k+1}^{m-1} X_{i:N}^2 + \frac{N-m+1}{N} X_{m:N}^2 \right) - \hat{\mu}_W^2 \tag{2.14}$$

with $k = [\alpha N] + 1$ and $m = [\beta N]$. Let

$$F_{N,S}(x) = P\left(\frac{N^{1/2}(T_N - \mu(\alpha,\beta))}{(\beta-\alpha)^{-1} S_N} \le x \right) \tag{2.15}$$

denote the *df* of a Studentized trimmed mean. Define

$$H_N(x) = \Phi(x) + \frac{\phi(x)}{6\sqrt{N}} \left((2x^2+1)\lambda_1 + 3(x^2+1)\lambda_2 - 6N \frac{\beta_N}{\sigma_W} \right). \tag{2.16}$$

Our main result is:

THEOREM 2.2. *Suppose that the conditions of Theorem 2.1 are satisfied. Then*

$$\sup_{x \in R} |F_{N,S}(x) - H_N(x)| = o(N^{-1/2}), \tag{2.17}$$

as $N \to \infty$.

As already indicated in our introduction the existence of an Edgeworth expansion for $F_{N,S}$ was proved by Hall and Padmanabhan [8]. In (2.16) and (2.17) we give the precise and simple explicit form of the Edgeworth expansion for $F_{N,S}$. In fact formally the form of our H_N (cf.(2.16)) coincides with the one given on p.1545 of Putter and van Zwet [16]. However, our Theorem 2.2 can not be inferred from the result of Putter and van Zwet [16]: the second condition in assumption (1.18) of Putter and van Zwet [16, p.1542], is not satisfied for our T_N, that is, for a Studentized trimmed mean (cf. Lemma A.2, Appendix). Our conjecture is that also the first condition in their assumption (1.18) is not satisfied , but this seems rather difficult to check for a Studentized trimmed mean.

REMARK 2.1. It is clear from the proofs of Theorems 2.1 and 2.2 that the order of the remainder term which we really obtain in relations (2.12) and (2.17) is $O((\log N)^{5/4}/N^{3/4})$, as $N \to \infty$.

To obtain empirical Edgeworth expansions (cf. Helmers [11], Putter and van Zwet [16]) we replace λ_1, λ_2, β_N and σ_W in (2.11) and (2.16) by statistical estimates. The estimation of λ_1 is straightforward. Let us define

$$\hat\lambda_1 = S_N^{-3}\hat\gamma_{3,W}$$

$$= S_N^{-3}\left(\frac{k}{N}(X_{k:N}-\hat\mu_W)^3 + \frac{1}{N}\sum_{i=k+1}^{m-1}(X_{i:N}-\hat\mu_W)^3 + \frac{N-m+1}{N}(X_{m:N}-\hat\mu_W)^3\right)$$

($\hat\mu_W$ and S_N were defined in (2.13) and (2.14)) to be an estimate for λ_1. As to λ_2 and β_N, we first have to estimate the values of density $f(\xi_\alpha)$ and $f(\xi_\beta)$. We shall use kernel estimators with a simple step-like kernel. Put $g(x) = I_{\{|x|\le 1/2\}}$. Take the width of kernel $\delta = N^{-1/4}$ and put $g_\delta(x) = \frac{1}{\delta}g\left(\frac{x}{\delta}\right) = \frac{1}{\delta}I_{\{|x|\le\delta/2\}}$, where $\int_{-\infty}^{\infty} g_\delta(x)\,dx = 1$. Then our estimates for values of density at the quantiles where trimming occurs will be the following:

$$\hat f(\xi_\nu) = \frac{1}{N}\sum_{i=1}^{N} g_\delta(X_i - X_{r:N}) = N^{-3/4}\sum_{i=1}^{N} I_{\{2N^{1/4}|X_i-X_{r:N}|\le 1\}}, \tag{2.18}$$

where $\nu=\alpha$ and $r=k$ or $\nu=\beta$ and $r=m$ respectively. Our estimates of $f(\xi_\alpha)$ and $f(\xi_\beta)$ are rather simple ones and sufficient for our purposes (cf. also Reiss [17, p.262]). One easily obtains the following estimates for λ_2 and β_N:

$$\hat\lambda_2 = S_N^{-3}\left\{-\alpha^2(\hat f(\xi_\alpha))^{-1}[\hat\mu_W - X_{k:N}]^2 + (1-\beta)^2(\hat f(\xi_\beta))^{-1}[\hat\mu_W - X_{m:N}]^2\right\},$$

$$\hat\beta_N = \frac{1}{N}\left\{-(\alpha N - [\alpha N])\Big(T_N - X_{k:N}\Big) - \frac{1}{2}\alpha(1-\alpha)(\hat f(\xi_\alpha))^{-1}\right.$$

$$\left. + (\beta N - [\beta N])\Big(T_N - X_{m:N}\Big) + \frac{1}{2}\beta(1-\beta)(\hat f(\xi_\alpha))^{-1}\right\}.$$

When the conditions of Theorem 2.1 are satisfied, the estimates $\hat\lambda_1$, $\hat\lambda_2$ and $\hat\beta_N$ are consistent estimators of the corresponding quantities λ_1, λ_2 and β_N (cf. Sect.6). Replacing the latter quantities by these estimates in formulas (2.11) and (2.16), we obtain the empirical Edgeworth expansions:

$$\hat G_N(x) = \Phi(x) - \frac{\phi(x)}{6\sqrt{N}}\left((\hat\lambda_1 + 3\hat\lambda_2)(x^2-1) + 6N\frac{\hat\beta_N}{S_N}\right),$$

$$\hat H_N(x) = \Phi(x) + \frac{\phi(x)}{6\sqrt{N}}\left((2x^2+1)\hat\lambda_1 + 3(x^2+1)\hat\lambda_2 - 6N\frac{\hat\beta_N}{S_N}\right).$$

Our result, establishing the validity of the empirical Edgeworth expansions, is given by the following assertion.

THEOREM 2.3. *Suppose that the conditions of Theorem 2.1 hold. Then*

$$\sup_{x\in R}|F_{T_N}(x) - \hat G_N(x)| = o_p\left(\frac{1}{\sqrt{N}}\right), \tag{2.19}$$

$$\sup_{x\in R} |F_{N,S}(x) - \hat{H}_N(x)| = o_p\left(\frac{1}{\sqrt{N}}\right). \tag{2.20}$$

as $N \to \infty$.

Remark 2.2. It is clear from Remark 2.1 and the Lemma's 6.1 and 6.2 that we can strengthen (2.19) and (2.20) to $\sup_{x\in R} |F_{T_N}(x) - \hat{G}_N(x)| = O\left((\log N)^{5/4} N^{-3/4}\right)$ with probability $1 - O\left(N^{-c}\right)$, for every $c > 0$, as $N \to \infty$, and similarly, $\sup_{x\in R} |F_{N,S}(x) - \hat{H}_N(x)| = O\left((\log N)^{5/4} N^{-3/4}\right)$, except on a set with probability $O(N^{-c})$, for every $c > 0$.

To conclude this section we remark that an alternative way of approximating F_{T_N} or $F_{N,S}$ accurately is to use saddlepoint methods. In Helmers et al [12] saddlepoint approximations were established rigorously for the trimmed mean and the Studentized trimmed mean. Compared with the Edgeworth expansions derived in the present paper, the saddlepoint approximations will typically behave better in the far tail of the distribution. An advantage of empirical Edgeworth expansions is that they are much easier to compute.

3. Auxiliary results

Define the binomial r.v. $N_\alpha = \sharp\{i : X_i \le \xi_\alpha\}$, where $0 < \alpha < 1$.

The following lemma is a version of Bahadur's [1] representation (cf. also Theorem 6.3.1, Reiss [17]) for the sample quantile. In this section k denotes an integer satisfying $k = \alpha N + O(1)$, $N \to \infty$.

Lemma 3.1. *Suppose that $f = F'$ exists and is positive and Lipschitz in neighborhood of ξ_α. Let G be a function defined in a neighborhood of ξ_α and $g = G'$ exists and satisfies a Lipschitz condition. Then*

$$G(X_{k:N}) = G(\xi_\alpha) - \frac{N_\alpha - \alpha N}{N} g(\xi_\alpha)/f(\xi_\alpha) + R_N, \tag{3.1}$$

where

$$P(|R_N| > A(\log N/N)^{3/4}) = O(N^{-c}), \tag{3.2}$$

as $N \to \infty$, for every $c > 0$ and some $A > 0$, not depending on N.

We omit the proof because the lemma is essentially known and its proof requires similar arguments,which will also be used in the proof of Lemma 3.2. Our proof of the next lemma will use the following fact: conditional on N_α the order statistics $X_{1:N}, \ldots, X_{N_\alpha:N}$ are distributed as N_α i.i.d. r.v.'s with distribution function $F(x)/\alpha$, $x \le \xi_\alpha$. Though this fact is more or less known, we add a brief explanation of it. Let $U_1, \ldots, U_N$ are independent r.v.'s uniformly distributed on $(0,1)$ and $U_{1,N}, \ldots, U_{N,N}$ denote the corresponding order statistics. Put $N_{\alpha,u} = \sharp\{i : U_i \le \alpha\}$. Since $X_{i:N} \stackrel{\mathrm{d}}{=} F^{-1}(U_{i:N})$ and $N_\alpha \stackrel{\mathrm{d}}{=} N_{\alpha,u}$, it is enough to prove the assertion for the uniform distribution. First consider the case $N_{\alpha,u} = N$. Take arbitrary $0 < u_1 \le \cdots \le u_N < \alpha$ and write

$$\begin{aligned} P(U_{1:N} \le u_1, \ldots, U_{N_{\alpha,u}:N} \le u_N \mid N_{\alpha,u} = N) &= \frac{P(U_{1:N} \le u_1, \ldots, U_{n:N} \le u_N)}{\alpha^N} \\ &= \frac{N!}{\alpha^N} \int_0^{u_1} \int_{u_1}^{u_2} \cdots \int_{u_{N-1}}^{u_N} dx_1 dx_2 \ldots dx_N, \end{aligned}$$

and the latter is $d.f.$ of the order statistics corresponding to the sample of N independent $(0,\alpha)$-uniform distributed r.v.'s. Now let $N_{\alpha,u}=k<N$ and $F_{i,N}(u)=P(U_{i:N}\le u)$ be a df of i-th order statistic, put $P_N(k)=P(N_{\alpha,u}=k)=\binom{N}{k}\alpha^k(1-\alpha)^{N-k}$. Then we can write

$$P(U_{1:N}\le u_1,\dots,U_{N_{\alpha,u}:N}\le u_k \mid N_{\alpha,u}=k)=\frac{P(U_{1:N}\le u_1,\dots,U_{k:N}\le u_k,U_{k+1:N}>\alpha)}{P_N(k)}.$$

The probability in the nominator on the r.h.s. of the latter formula is equal to $\int_\alpha^1 P\big(U_{1:N}\le u_1,\dots,U_{k:N}\le u_k \mid U_{k+1:N}=v\big)\,dF_{k+1,N}(v)$, and by the Markov property of order statistics the latter quantity equals

$$\int_\alpha^1\left(\frac{k!}{v^k}\int_0^{u_1}\int_{u_1}^{u_2}\dots\int_{u_{k-1}}^{u_k}dx_1dx_2\dots dx_k\right)dF_{k+1,N}(v)$$
$$=\frac{k!}{\alpha^k}\left(\int_0^{u_1}\int_{u_1}^{u_2}\dots\int_{u_{k-1}}^{u_k}dx_1dx_2\dots dx_k\right)\times\alpha^k\int_\alpha^1\frac{1}{v^k}\,dF_{k+1,N}(v),$$

and since $\alpha^k\int_\alpha^1\frac{1}{v^k}\,dF_{k+1,N}(v)=\alpha^k\int_\alpha^1\frac{(1-v)^{N-k-1}}{B(k+1,N-k)}\,dv=\binom{N}{k}\alpha^k(1-\alpha)^{N-k}=P_N(k)$, where $B(k+1,N-k)=k!(N-k-1)!/N!$, we obtain that conditional probability we consider is equal to

$$\frac{k!}{\alpha^k}\int_0^{u_1}\int_{u_1}^{u_2}\dots\int_{u_{k-1}}^{u_k}dx_1dx_2\dots dx_k,$$

which corresponds to the $(0,\alpha)$-uniform distribution.

To state next lemma we shall adopt the following notation. Let $\sum_{i=k}^m(.)_i=sign[m-k]\sum_{i=k\wedge m}^{k\vee m}(.)_i$ for all integer k and m.

LEMMA 3.2. *Suppose that the conditions of lemma 3.1 are satisfied. Then*

$$\frac{1}{N}\sum_{i=k}^{N_\alpha}(G(X_{i:N})-G(\xi_\alpha))=-\frac{(N_\alpha-\alpha N)^2}{2N^2}g(\xi_\alpha)/f(\xi_\alpha)+R_N,\tag{3.3}$$

where

$$P(|R_N|>A(\log N/N)^{5/4})=O(N^{-c}),\tag{3.4}$$

as $N\to\infty$ for every $c>0$ with some $A>0$, not depending on N.

This lemma extends and sharpens the relations (3.2) and (3.3) given (for the case $G(x)=x$) in Hall and Padmanabhan [8]. Note also that the factor $(1-\alpha)^{-1}$ in formula (3.2) and $(1-\beta)^{-1}$ in formula (3.3) (see Hall and Padmanabhan [8]) should be omitted. We apply this lemma several times: to approximate the trimmed mean (cf. lemma 4.1), its asymptotic variance (cf. lemma 5.1) and its asymptotic third moment (cf. Thm.2.3 and lemma 6.2).

COROLLARY 3.1. *Suppose that $f=F'$ exists and is positive and Lipschitz in a neighborhood of ξ_α. Then*

$$\frac{1}{N}\sum_{i=k}^{N_\alpha}(X_{i:N}-\xi_\alpha)=-\frac{(N_\alpha-\alpha N)^2}{2N^2}\frac{1}{f(\xi_\alpha)}+R_{N,1},$$

$$\frac{1}{N}\sum_{i=k}^{N_\alpha}\left(X_{i,N}^2-\xi_\alpha^2\right)=-\frac{(N_\alpha-\alpha N)^2}{N^2}\xi_\alpha\frac{1}{f(\xi_\alpha)}+R_{N,2},$$

where $R_{N,i}$, $i=1,2$, *satisfy* (3.4).

PROOF. We begin by writing (cf.(3.3))

$$R_N=\frac{1}{N}\sum_{i=k}^{N_\alpha}(G(X_{i:N})-G(\xi_\alpha))+\frac{(N_\alpha-\alpha N)^2}{2N^2}g(\xi_\alpha)/f(\xi_\alpha). \tag{3.5}$$

Now we will check that R_N satisfies (3.4). Let, as before, $U_1,\dots,U_N$ denote independent r.v.'s uniformly distributed on $(0,1)$ and let $U_{1,N},\dots,U_{N,N}$ denote the corresponding order statistics. Since the joint distribution of $X_{i:N}$, $(i=k,\dots,N_\alpha)$ and N_α coincides with the joint distribution of $F^{-1}(U_{i:N})$ $(i=k,\dots,N_{\alpha,u})$ and $N_{\alpha,u}$, where $N_{\alpha,u}=\sharp\{i: U_i\le\alpha\}$, it of course suffices to verify that

$$\frac{1}{N}\sum_{i=k}^{N_{\alpha,u}}[G(F^{-1}(U_{i:N}))-G(F^{-1}(\alpha))]+\frac{(N_{\alpha,u}-\alpha N)^2}{2N^2}g(\xi_\alpha)/f(\xi_\alpha) \tag{3.6}$$

satisfies (3.4). By our smoothness condition the first term of (3.6) equals

$$\frac{1}{N}g(\xi_\alpha)/f(\xi_\alpha)\sum_{i=k}^{N_{\alpha,u}}(U_{i:N}-\alpha)+R_{N,3}, \tag{3.7}$$

where

$$|R_{N,3}|\le\frac{C}{N}\sum_{i=k\wedge N_{\alpha,u}}^{k\vee N_{\alpha,u}}(U_{i:N}-\alpha)^2\le\frac{C|k-N_{\alpha,u}|}{N}\left[(U_{k:N}-\alpha)^2\vee(U_{N_{\alpha,u},N}-\alpha)^2\right] \tag{3.8}$$

with C is equal to the Lipschitz constant of function $g(F^{-1}(u))/f(F^{-1}(u))$ (we neglect here the event that $U_{k:N}$ does not belong to the neighborhood of α where smoothness conditions hold, as this probability is of the order $O(exp(-cN))$, as $N\to\infty$ for some $c>0$, cf. the introduction). Let us fix an arbitrary $c>0$ and note that

$$P\left((\alpha-U_{N_{\alpha,u},N})^2>A_1\log N/N\right)\le P\left(U_{N_{\alpha,u}+1,N}-U_{N_{\alpha,u},N}>(A_1\log N/N)^{1/2}\right)$$

$$=P\left(U_{1:N}>(A_1\log N/N)^{1/2}\right)=O(N^{-c}). \tag{3.9}$$

Here and elsewhere A_j denote the positive constants which do not depend on N. Besides, by Bernstein's inequality

$$P(|N_{\alpha,u}-k|>(A_2N\log N)^{1/2})=O(N^{-c}), \tag{3.10}$$

with $A_2=2c\alpha(1-\alpha)$, and by lemma 3.1.1, Reiss [17]

$$P((U_{k:N}-\alpha)^2>A_3\log N/N)=O(N^{-c}),$$

as $N\to\infty$. Therefore (3.8) implies that

$$P(|R_{N,3}|>A_4(\log N/N)^{3/2})=O(N^{-c}) \tag{3.11}$$

with $A_4 = CA_2 \max(A_1, A_3)$. Next we consider the dominant term on the r.h.s. of (3.7). By (3.10) we can bound our quantities on the event $E = \{\omega : |N_{\alpha,u} - k| < (A_2 N \log N)^{1/2}\}$. Fix N and $N_{\alpha,u}$ for which the event E holds true. Without loss of generality let $k \le N_{\alpha,u}$. Note that conditional on $N_{\alpha,u}$ the order statistic $U_{i:N}$, $k \le i \le N_{\alpha,u}$, is distributed as i-th order statistic of the sample of size $N_{\alpha,u}$ from the uniform on $(0,\alpha)$ distribution and $E(U_{i:N}|N_{\alpha,u}) = \frac{\alpha i}{N_{\alpha,u}+1}$, for $i = k, \dots, N_{\alpha,u}$. Write

$$\frac{1}{N}\sum_{i=k}^{N_{\alpha,u}}(U_{i:N} - \alpha) = \frac{1}{N}\left[\sum_{i=k}^{N_{\alpha,u}}(U_{i:N} - \frac{\alpha i}{N_{\alpha,u}+1}) + \sum_{i=k}^{N_{\alpha,u}}(\frac{\alpha i}{N_{\alpha,u}+1} - \alpha)\right] \tag{3.12}$$

$$= \frac{1}{N}\sum_{i=k}^{N_{\alpha,u}}(U_{i:N} - \frac{\alpha i}{N_{\alpha,u}+1}) - \frac{\alpha(N_{\alpha,u} - \alpha N)^2}{2NN_{\alpha,u}} + O((\log N)^{1/2}N^{-3/2}).$$

For the second term on the r.h.s. of (3.12) we have

$$-\frac{\alpha(N_{\alpha,u} - \alpha N)^2}{2NN_{\alpha,u}} = -\frac{(N_{\alpha,u} - \alpha N)^2}{2N}\frac{\alpha}{\alpha N + (N_{\alpha,u} - \alpha N)} = -\frac{(N_{\alpha,u} - \alpha N)^2}{2N^2} + R_{N,4}, \tag{3.13}$$

where in view of (3.10)

$$P(|R_{N,4}| > A_5(\log N/N)^{3/2}) = O(N^{-c}) \tag{3.14}$$

as $N \to \infty$ with $A_5 = A_2^3$. For the first term on the r.h.s. of (3.12) we can write

$$\frac{1}{N}\left|\sum_{i=k}^{N_{\alpha,u}}\left(U_{i:N} - \frac{\alpha i}{N_{\alpha,u}+1}\right)\right| \le \frac{N_{\alpha,u} - k + 1}{N}\max_{k \le i \le N_{\alpha,u}}\left|U_{i:N} - \frac{\alpha i}{N_{\alpha,u}+1}\right|. \tag{3.15}$$

Note that we suppose that the event E holds true and (without loss of generality) that $k \le N_{\alpha,u}$ (otherwise a similar argument with respect to $(\alpha, 1)$ instead $(0,\alpha)$) will do). Fix an arbitrary $c_1 > c + 1/2$ and note that conditional on $N_{\alpha,u}$ the variance of the order statistic $U_{i:N}$, $k \le i \le N_{\alpha,u}$, is equal to $\frac{\alpha^2 i(N_{\alpha,u} - i + 1)}{(N_{\alpha,u}+1)^2(N_{\alpha,u}+2)} = O\left((\log N)^{1/2}N^{-3/2}\right)$. By lemma 3.1.1, Reiss [17], we obtain that uniformly for $k \le i \le N_{\alpha,u}$

$$P\left(\left|U_{i:N} - \frac{\alpha i}{N_{\alpha,u}+1}\right| > A_6(\log N/N)^{3/4}|N_{\alpha,u}\right) = O(N^{-c_1}), \tag{3.16}$$

as $N \to \infty$.Relations (3.15) and (3.16) together imply

$$P\left(\frac{1}{N}\left|\sum_{i=k}^{N_{\alpha,u}}(U_{i:N} - \frac{\alpha i}{N_{\alpha,u}+1})\right| > (A_2)^{1/2}A_6\ (\log N/N)^{5/4}\Big|\ N_{\alpha,u}\right) \le \tag{3.17}$$

$$(A_2 N \log N)^{1/2}O(N^{-c_1}) = O(N^{-c}),$$

as $N \to \infty$. Now (3.3) and (3.4) follows from (3.5)–(3.7), (3.11)–(3.14) and (3.17). The lemma is proved. □

4. Proof of Theorem 2.1

To begin with let us note that we can replace T_N (cf. (2.1)) by

$$N^{-1/2}\sum_{i=k}^{m} X_{i:N}, \tag{4.1}$$

where $k = [\alpha N] + 1$, $m = [\beta N]$, $0 < \alpha < \beta < 1$. Note that though T_N in (4.1) is of different order than in (2.1), this will affect only the bias term (see Lemma A.1, Appendix), and we shall take that into account whenever needed. Define $I_\nu(X_i) = I_{\{X_i \le \xi_\nu\}}$, where $\xi_\nu = F^{-1}(\nu)$, $0 < \nu < 1$, and I_A is the indicator of event A. Then for the Winsorized r.v. W_i (cf. (2.2)) we can write

$$W_i = X_i I_\beta(X_i)(1 - I_\alpha(X_i)) + \xi_\alpha I_\alpha(X_i) + \xi_\beta(1 - I_\beta(X_i)). \tag{4.2}$$

Recall that μ_W, σ_W^2, $\gamma_{3,W}$ denote first three cumulants of r.v. W_1 (cf.(2.3)). Define a $U-$statistic of degree 2 by

$$L_N + U_N = \sum_{i=1}^{N} L_{N,i} + \sum_{1\le i}\sum_{<j\le N} U_{N,(i,j)}, \tag{4.3}$$

where

$$L_{N,i} = \frac{1}{\sqrt{N}}(W_i - \mu_W) \tag{4.4}$$

$$= \frac{1}{\sqrt{N}}\Big[\, X_i I_\beta(X_i)(1 - I_\alpha(X_i)) + \xi_\alpha I_\alpha(X_i) + \xi_\beta(1 - I_\beta(X_i)) - \mu_W\Big],$$

$$U_{N,(i,j)} = \frac{1}{N\sqrt{N}}\Big[-\frac{1}{f(\xi_\alpha)}(I_\alpha(X_i) - \alpha)(I_\alpha(X_j) - \alpha) \tag{4.5}$$

$$+ \ \frac{1}{f(\xi_\beta)}(I_\beta(X_i) - \beta)(I_\beta(X_j) - \beta)\Big].$$

Note that

$$EL_{N,i} = 0 \tag{4.6}$$

for all $i = 1, \ldots, N$ and

$$EU_{N,(i,j)} = 0, \qquad E(L_{N,i}U_{N,(i,j)}) = 0 \tag{4.7}$$

for all $i, j = 1, \ldots, N$ $(i \ne j)$. Using (4.4)–(4.7), we easily check that

$$\sigma^2_{L_N+U_N} = E(L_N + U_N)^2 = E(L_N^2) + O(N^{-1}) = \sigma_W^2 + O(N^{-1}), \tag{4.8}$$

and also that

$$E(L_N + U_N)^3 = E(L_N^3) + 3E(L_N^2 U_N) + O(N^{-3/2}) \tag{4.9}$$

$$= \frac{1}{\sqrt{N}}\gamma_{3,W} + 3\frac{1}{\sqrt{N}}\Big\{ -\frac{1}{f(\xi_\alpha)}\Big[\, E((W_1 - \mu_W)(I_\alpha(X_1) - \alpha))\Big]^2$$

$$
+\frac{1}{f(\xi_\beta)}\Big[E((W_1-\mu_W)(I_\beta(X_1)-\beta))\Big]^2\Big\}+O(N^{-3/2})
$$

$$
=\frac{1}{\sqrt{N}}\gamma_{3,W}+3\frac{1}{\sqrt{N}}\left[-\frac{1}{f(\xi_\alpha)}\alpha^2\,[\xi_\alpha-\mu_W]^2+\frac{1}{f(\xi_\beta)}(1-\beta)^2[\xi_\beta-\mu_W]^2\right]
$$

$$
+O(N^{-3/2}).
$$

Relations (4.8) and (4.9) imply that

$$
E\left(\frac{L_N+U_N}{\sigma_{(L_N+U_N)}}\right)^3=\frac{\lambda_1+3\lambda_2}{\sqrt{N}}+O(N^{-3/2}), \tag{4.10}
$$

with λ_1 and λ_2 as in (2.5).

The next lemma ensures that the approximation of T_N by a $U-$statistic of the form (4.3) has a remainder of classical Bahadur's order of magnitude $N^{-3/4}(\log N)^{5/4}$.

Lemma 4.1. *Suppose that the conditions of Theorem 2.1 hold. Then*

$$
P\Big(|T_N-ET'_N-(L_N+U_N)|>A(\log N)^{5/4}N^{-3/4}\Big)=O(N^{-c}) \tag{4.11}
$$

as $N\to\infty$, for every $c>0$ with some $A>0$ independent on N.

Proof of Lemma 4.1. Let $W_{i:N}$, $i=1,\dots,N$, denote the order statistics, corresponding to $W_1,\dots,W_N$. Put $N_\nu=\sharp\{X_i: X_i\le\xi_\nu\},\ 0<\nu<1$. Then

$$
W_{i:N}=\begin{cases}\xi_\alpha\ , & i\le N_\alpha,\\ X_{i:N}\ , & N_\alpha<i\le N_\beta,\\ \xi_\beta\ , & i>N_\beta.\end{cases}
$$

Now note that

$$
T_N-\frac{1}{\sqrt{N}}\sum_{i=1}^N W_i=\frac{1}{\sqrt{N}}\left(\sum_{i=k}^m X_{i:N}-N_\alpha\xi_\alpha-\sum_{i=N_\alpha+1}^{N_\beta}X_{i:N}-(N-N_\beta)\xi_\beta\right)
$$

$$
=\frac{1}{\sqrt{N}}\Bigg\{sign[N_\alpha-(k-1)]\sum_{i=k\wedge(N_\alpha+1)}^{N_\alpha\vee(k-1)}X_{i:N}-sign(N_\beta-m)\sum_{i=(m\wedge N_\beta)+1}^{m\vee N_\beta}X_{i:N}
$$

$$
-\ N_\alpha\xi_\alpha-(N-N_\beta)\xi_\beta\Bigg\}=\frac{1}{\sqrt{N}}\Bigg\{sign[N_\alpha-(k-1)]\sum_{i=k\wedge(N_\alpha+1)}^{N_\alpha\vee(k-1)}(X_{i:N}-\xi_\alpha)
$$

$$
-\,sign(N_\beta-m)\sum_{i=(m\wedge N_\beta)+1}^{m\vee N_\beta}(X_{i:N}-\xi_\beta)-(k-1)\xi_\alpha-(N-m)\xi_\beta\Bigg\}
$$

$$
=-\frac{(N_\alpha-\alpha N)^2}{2N\sqrt{N}}\frac{1}{f(\xi_\alpha)}+\frac{(N_\beta-\beta N)^2}{2N\sqrt{N}}\frac{1}{f(\xi_\beta)}-\frac{k-1}{\sqrt{N}}\xi_\alpha-\frac{N-m}{\sqrt{N}}\xi_\beta+R_N,
$$

where by Lemma 3.2

$$
P\Big(|R_N|>A(\log N)^{5/4}N^{-3/4}\Big)=O(N^{-c}) \tag{4.12}
$$

as $N \to \infty$, for every $c > 0$ with some $A > 0$ independent of N. Define

$$Q_N = -\frac{(N_\alpha - \alpha N)^2}{2N\sqrt{N}} \frac{1}{f(\xi_\alpha)} + \frac{(N_\beta - \beta N)^2}{2N\sqrt{N}} \frac{1}{f(\xi_\beta)}$$
$$= \frac{1}{2N\sqrt{N}} \left\{ -\left[\sum_{i=1}^{N} (I_\alpha(X_i) - \alpha)\right]^2 \frac{1}{f(\xi_\alpha)} + \left[\sum_{i=1}^{N} (I_\beta(X_i) - \beta)\right]^2 \frac{1}{f(\xi_\beta)} \right\}.$$

It is clear that Q_N is a symmetric polynomial of degree two with

$$E(Q_N) = \frac{1}{2\sqrt{N}} \left\{ -\alpha(1-\alpha)\frac{1}{f(\xi_\alpha)} + \beta(1-\beta)\frac{1}{f(\xi_\beta)} \right\}.$$

Note that

$$\text{(4.13)} \qquad E\frac{1}{\sqrt{N}} \sum_{i=1}^{N} W_i = \sqrt{N}\mu_W = \sqrt{N}\Big((\beta - \alpha)\mu(\alpha,\beta) + \alpha\xi_\alpha + (1-\beta)\xi_\beta \Big).$$

Next we can write

$$\text{(4.14)} \qquad T_N = L_N + Q_N - EQ_N + \sqrt{N}\Big((\beta-\alpha)\mu(\alpha,\beta) + \alpha\xi_\alpha + (1-\beta)\xi_\beta \Big)$$
$$-\frac{k-1}{\sqrt{N}}\xi_\alpha - \frac{N-m}{\sqrt{N}}\xi_\beta + \frac{1}{2\sqrt{N}} \left\{ -\alpha(1-\alpha)\frac{1}{f(\xi_\alpha)} + \beta(1-\beta)\frac{1}{f(\xi_\beta)} \right\} + R_N$$
$$= L_N + Q_N - EQ_N + \sqrt{N}(\beta-\alpha)\mu(\alpha,\beta) + \frac{1}{\sqrt{N}} \Big\{ -(k-1-\alpha N)\xi_\alpha$$
$$- \frac{1}{2}\frac{1}{f(\xi_\alpha)}\alpha(1-\alpha) + (m - \beta N)\xi_\beta + \frac{1}{2}\frac{1}{f(\xi_\beta)}\beta(1-\beta) \Big\} + R_N.$$

Let us compare the expression within curly brackets on the r.h.s. of (4.14) with the formula for B_2 (the bias term for $T'_N = N^{-1}\sum_{i=k}^{m} X'_{i:N}$) (cf.(A.3), Appendix). As a result we obtain the following formula:

$$\text{(4.15)} \qquad T_N - \sqrt{N}(\beta-\alpha)\mu(\alpha,\beta) = L_N + Q_N - EQ_N + \sqrt{N}B_2 + R_N,$$

with R_N as in (4.14) plus $O(N^{-1})$ (cf. (A.3), Appendix). Note that R_N satisfies (4.12), and as T'_N is normalized by $N^{-1/2}$ in this lemma, we have $ET'_N - \sqrt{N}(\beta-\alpha)\mu(\alpha,\beta) = \sqrt{N}B_2$ (cf. lemma A.1, Appendix). So, relation (4.15) implies

$$\text{(4.16)} \qquad T_N - ET'_N = L_N + Q_N - EQ_N + R_N.$$

For the quantity $Q_N - EQ_N$ we can write

$$\text{(4.17)} \qquad Q_N - EQ_N = U_N + \frac{\bar{r}_N}{2\sqrt{N}},$$

where U_N as in (4.3) and

$$\bar{r}_N = \frac{1}{N}\sum_{i=1}^{N} \Big\{ -\frac{1}{f(\xi_\alpha)}[(I_\alpha(X_i) - \alpha)^2 - \alpha(1-\alpha)]$$

$$+ \quad \frac{1}{f(\xi_\beta)}[(I_\beta(X_i)-\beta)^2-\beta(1-\beta)]\Big\}.$$

Note that $\bar{r}_N$ is the average of N i.i.d. bounded and centered ($E\bar{r}_N = 0$) r.v.'s, and by Hoeffding's inequality [13] we find that

$$P\left(|\bar{r}_N| > A(\log N/N)^{1/2}\right) = O(N^{-c}) \tag{4.18}$$

for every $c > 0$ and some $A > 0$, not depending on N. Therefore $\frac{1}{2}\bar{r}_N/\sqrt{N}$ on the r.h.s. of (4.17) is negligible for our purposes. Relations (4.12) and (4.16)–(4.18) together imply (4.11). The lemma is proved. □

Remark 4.1. The first linear term of our U-statistic approximation to T_N is a sum of i.i.d. Winsorized r.v.'s W_i. A simple argument involving formula (2.10) for the L_2-projection (i.e. ,the first term of the Hoeffding decomposition) given in [16, p.1548], tells us that our leading term is slightly different from the one given by the Hoeffding decomposition. The same fact holds true for the second quadratic term in our U -statistic approximation to the trimmed mean.

Proof of Theorem 2.1. Using the Lemma 4.1 and Lemma A.1 (cf.Appendix), for the *df* of T_N (cf. (2.1)) defined by (2.10) we can write

$$\begin{aligned} F_{T_N}(x) &= P\left\{\frac{N^{1/2}(\beta-\alpha)(T_N-ET_N')}{\sigma_W} \le x - \frac{N^{1/2}\beta_N + O(N^{-1})}{\sigma_W}\right\} \\ &= P\left\{\frac{L_N+U_N}{\sigma_W} \le \frac{[\beta N]-[\alpha N]}{(\beta-\alpha)N}\left(x-\frac{N^{1/2}\beta_N}{\sigma_W}+O(N^{-1})\right)-\frac{R_N}{\sigma_W}\right\} \\ &= P\left\{\frac{L_N+U_N}{\sigma_W} \le x(1+O(N^{-1})) - \frac{N^{1/2}\beta_N}{\sigma_W} - \frac{R_N}{\sigma_W} + O(N^{-1})\right\}, \end{aligned} \tag{4.19}$$

where $L_N + U_N$ (cf.(4.3)) is U-statistic of degree two with the canonical functions

$$\begin{aligned} g_N(x) &= E(L_N+U_N|X_1=x) \\ &= \frac{1}{\sqrt{N}}[xI_\beta(x)(1-I_\alpha(x)) + \xi_\alpha I_\alpha(x) + \xi_\beta(1-I_\beta(x)) - \mu_W], \\ \psi_N(x,y) &= E(L_N+U_N|X_1=x, X_2=y) - g_N(x) - g_N(y) \\ &= \frac{1}{N\sqrt{N}}\Big[-(I_\alpha(x)-\alpha)(I_\alpha(y)-\alpha)\frac{1}{f(\xi_\alpha)} + (I_\beta(x)-\beta)(I_\beta(y)-\beta)\frac{1}{f(\xi_\beta)}\Big], \end{aligned}$$

where

$$E(g_N(X_1)) = 0, \qquad E(\psi_N(X_1,X_2)) = 0,$$
$$E(\psi_N(X_1,X_2)|X_2) = 0 \quad a.s.$$

The local smoothness assumption of our theorem directly yields that the distribution of r.v. $g_N(X_1) = \frac{1}{\sqrt{N}}(W_1 - \mu_W)$ has a nontrivial absolutely continuous component and Cramér's condition

$$(C) \qquad \limsup_{|t|\to\infty} |E\exp\{it\sqrt{N}g_N(X_1)\}| < 1$$

is satisfied. Since the functions $\sqrt{N}g_N(x)$ and $N^{3/2}\psi_N(x,y)$ are both bounded, we trivially have that

$$\beta_4 = E\left(\sqrt{N}g_N(X_1)\right)^4 < \infty,$$

$$\gamma_3 = E\left|N^{3/2}\psi_N(X_1,X_2)\right|^3 < \infty.$$

Therefore, we can apply Thm.1.2 of Bentkus, Götze and van Zwet [2] (note that the quantity Δ_3^2 appearing in Thm.1.2 of Bentkus et. all [2] is zero in our case). Define $F_N(x) = \Phi(x) - \phi(x)\frac{\lambda_1+3\lambda_2}{6\sqrt{N}}(x^2-1)$, where λ_1 and λ_2 as in (2.5) (cf.also (4.10)). Then by Thm.1.2 (Bentkus et.all [2])

$$\sup_{x\in R}\left|P\left\{\frac{L_N+U_N}{\sigma_W} \le x\right\} - F_N(x)\right| = O(N^{-1}).$$

For R_N we have the bound (4.12), that is $|R_N| = O((\log N)^{5/4}N^{-3/4})$ with probability $1-o(N^{-c})$ for every $c>0$. Therefore, as $F_N'(x)$ and $xF_N'(x)$ are bounded functions, we obtain on the r.h.s. of (4.19)

$$F_N(x) - \frac{\sqrt{N}\beta_N}{\sigma_W}\phi(x) + O((\log N)^{5/4}N^{-3/4})$$

$$= G_N(x) + O((\log N)^{5/4}N^{-3/4}).$$

This proves (2.12) and Theorem 2.1. □

5. Proof of Theorem 2.2

Let S_N^2 be (cf.(2.14)) the plug in estimator for σ_W^2 (cf.(2.3)). The following lemma is a modification of Lemma 4.3 of Putter and van Zwet [16], appropriate for our purposes.

LEMMA 5.1. *Suppose that the assumptions of Theorem 2.1 are satisfied. Then*

(5.1) $$P\left(|S_N^2 - \sigma_W^2 - V_N| > A(\log N/N)^{3/4}\right) = O(N^{-c})$$

as $N\to\infty$ *for every* $c>0$ *and some* $A>0$*, not depending on* N*, where*

(5.2) $$V_N = V_{N,1} + V_{N,2}\ ,$$

$$V_{N,1} = 2\alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha-\alpha N}{N}[\mu_W-\xi_\alpha] + 2(1-\beta)\frac{1}{f(\xi_\beta)}\frac{N_\beta-\beta N}{N}[\mu_W-\xi_\beta],$$

$$V_{N,2} = \frac{1}{N}\sum_{i=1}^{N}\left[(W_i-\mu_W)^2 - \sigma_W^2\right]\ .$$

Moreover,

(5.3) $$E(V_N) = 0\ ;\quad E(V_N^2) = O(N^{-1})$$

as $N\to\infty$*.*

This lemma essentially asserts that the difference between σ_W^2 and its estimator S_N^2 can be expressed as a sum of i.i.d. r.v.'s plus a remainder term which is of negligible order for our purposes.

PROOF. Define the auxiliary quantity

$$S_W^2 = \frac{1}{N}\sum_{i=1}^{N} W_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N} W_i\right)^2$$

$$= \frac{N_\alpha}{N}\xi_\alpha^2 + \frac{1}{N}\sum_{i=N_\alpha+1}^{N_\beta} X_{i:N}^2 + \frac{N-N_\beta}{N}\xi_\beta^2 - \left(\frac{N_\alpha}{N}\xi_\alpha + \frac{1}{N}\sum_{i=N_\alpha+1}^{N_\beta} X_{i:N} + \frac{N-N_\beta}{N}\xi_\beta\right)^2.$$

First we prove that

$$S_N^2 = S_W^2 + V_{N,1} + R_{N,1}, \tag{5.4}$$

Here and elsewhere $R_{N,1}$, $R_{N,1}^{(r)}$, $r = 1, 2, \dots$ denote the remainder terms of Bahadur's order, satisfying (3.2). We have

$$\begin{aligned} S_N^2 - S_W^2 = \Bigg[& \frac{k}{N}X_{k:N}^2 + \frac{1}{N}\sum_{i=k+1}^{m-1} X_{i:N}^2 + \frac{N-m+1}{N}X_{m:N}^2 \\ & - \frac{N_\alpha}{N}\xi_\alpha^2 - \frac{1}{N}\sum_{i=N_\alpha+1}^{N_\beta} X_{i:N}^2 - \frac{N-N_\beta}{N}\xi_\beta^2\Bigg] + \left[\left(\frac{1}{N}\sum_{i=1}^{N} W_i\right)^2 - \hat\mu_W^2\right]. \end{aligned} \tag{5.5}$$

Rewrite the term within the first square brackets on the r.h.s. of (5.5) as

$$\frac{k}{N}(X_{k:N}^2 - \xi_\alpha^2) + sign(N_\alpha - k)\frac{1}{N}\sum_{i=(k\wedge N_\alpha)+1}^{k\vee N_\alpha} (X_{i:N}^2 - \xi_\alpha^2)$$

$$+\frac{N-m+1}{N}(X_{m:N}^2 - \xi_\beta^2) - sign(N_\beta - m + 1)\frac{1}{N}\sum_{i=m\wedge(N_\beta+1)}^{(m-1)\vee N_\beta} (X_{i:N}^2 - \xi_\beta^2)$$

with $sign(0) = 0$ (cf. the proof of Lemma 4.1). By Lemmas 3.1 and 3.2 this expression is equal to

$$\begin{aligned} & -2\alpha\xi_\alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha - \alpha N}{N} - \frac{(N_\alpha - \alpha N)^2}{N^2}\xi_\alpha\frac{1}{f(\xi_\alpha)} + R_{N,1}^{(1)} - \\ & -2(1-\beta)\xi_\beta\frac{1}{f(\xi_\beta)}\frac{N_\beta - \beta N}{N} + \frac{(N_\beta - \beta N)^2}{N^2}\xi_\beta\frac{1}{f(\xi_\beta)} + R_{N,1}^{(2)}, \end{aligned} \tag{5.6}$$

and by Bernstein's inequality for the binomial r.v.'s N_α and N_β the latter formula reduces to

$$-2\alpha\xi_\alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha - \alpha N}{N} - 2(1-\beta)\xi_\beta\frac{1}{f(\xi_\beta)}\frac{N_\beta - \beta N}{N} + R_{N,1}^{(3)}. \tag{5.7}$$

Now we consider the term within the second square brackets on the r.h.s. of (5.5). Arguing as before, we can rewrite this expression as

$$\left(\frac{2}{N}\sum_{i=1}^{N} W_i - \alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha - \alpha N}{N} - (1-\beta)\frac{1}{f(\xi_\beta)}\frac{N_\beta - \beta N}{N} + R_{N,1}^{(4)}\right) \tag{5.8}$$

$$\cdot\left(\alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha-\alpha N}{N}+(1-\beta)\frac{1}{f(\xi_\beta)}\frac{N_\beta-\beta N}{N}+R_{N,1}^{(5)}\right)$$

$$=\frac{2}{N}\left(\sum_{i=1}^N W_i\right)\left(\alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha-\alpha N}{N}+(1-\beta)\frac{1}{f(\xi_\beta)}\frac{N_\beta-\beta N}{N}\right)+R_{N,1}^{(6)}\ .$$

The relations (5.6)–(5.8) together imply that

(5.9) $$S_N^2-S_W^2=V_{N,1}+R_N+R_{N,1}^{(7)},$$

where

$$R_N=2\left[\alpha\frac{1}{f(\xi_\alpha)}\frac{N_\alpha-\alpha N}{N}+(1-\beta)\frac{1}{f(\xi_\beta)}\frac{N_\beta-\beta N}{N}\right]\frac{1}{N}\sum_{i=1}^N(W_i-\mu_W)\ .$$

Note that the W_i, $i=1,\ldots,N$, are bounded i.i.d. *r.v.*'s. Therefore by Hoeffding's inequality $\frac{1}{N}\left|\sum_{i=1}^N(W_i-\mu_W)\right|=O\left((\log N/N)^{1/2}\right)$ as $N\to\infty$ with probability $1-o(N^{-c})$ for every $c>0$. Combining the latter bound with Bernstein's inequality for the binomial r.v.'s N_α and N_β, we obtain that $|R_N|=O(\log N/N)$ with probability $1-o(N^{-c})$ for every $c>0$. Therefore (5.9) implies (5.4).

Next we prove that

(5.10) $$S_W^2=\sigma_W^2+V_{N,2}+R_{N,2},$$

where $|R_{N,2}|=O(\log N/N)$ with probability $1-o(N^{-c})$ for every $c>0$. We have

$$S_W^2-\sigma_W^2-V_{N,2}=S_W^2-\frac{1}{N}\sum_{i=1}^N(W_i-\mu_W)^2=-(\overline{W}-\mu_W)^2=R_{N,2}.$$

An application of Hoeffding's inequality to the bounded i.i.d. *r.v.*'s W_i (cf. [13]) proves (5.10). Relations (5.4) and (5.10) together imply (5.1). The lemma is proved. □

Now we turn to the proof of our result concerning the Studentized version of trimmed mean.

Proof of Theorem 2.2. Our proof of this theorem closely resembles the proof of Theorem 1.2 of Putter and van Zwet [16]. For the *df* $F_{N,S}(x)$ (cf.(2.15)) of a Studentized trimmed mean we have

(5.11) $$F_{N,S}(x)=P\left\{\frac{L_N+U_N}{S_N}\le\left(1+O(N^{-1})\right)\left[x-\frac{N^{1/2}\beta_N+O(N^{-1})}{S_N}\right]+\frac{R_{N,1}}{S_N}\right\}$$

(cf.(4.19)). Here and elsewhere $R_{N,1}$ denotes a remainder, which satisfies (4.12) and which can be different from line to line. Lemma 5.1 and Hoeffding's inequality for r.v. V_N together imply that $\left|\frac{1}{S_N}-\frac{1}{\sigma_W}\right|=O((\log N/N)^{1/2})$ with probability $1-O(N^{-c})$ as $N\to\infty$ for every $c>0$ (cf.also Lemma 6.2, below). Therefore, the r.h.s. of (5.11) equals to

(5.12) $$P\left\{\frac{L_N+U_N}{S_N}\le\left(1+O(N^{-1})\right)\left[x-\frac{N^{1/2}\beta_N}{\sigma_W}\right]+R_{N,1}\right\}.$$

Our aim now is to prove that

$$\sup_{x\in R}|F_{N,S}(x)-H_N(x)|=O\left((\log N)^{5/4}/N^{3/4}\right) \tag{5.13}$$

as $N\to\infty$ (this implies (2.17)). Define $\tilde{H}_N(x)=H_N(x)+\sigma_W^{-1}\sqrt{N}\beta_N\phi(x)$ (i.e. $\tilde{H}_N(x)$ is $H_N(x)$ without bias term). Since $H_N'(x)$ and $xH_N'(x)$ are bounded, relations (5.11) and (5.12) imply that it is sufficient to show that

$$\sup_{x\in R}|F_{(L_N+U_N)/S_N}(x)-\tilde{H}_N(x)|=O\left((\log N)^{5/4}/N^{3/4}\right), \tag{5.14}$$

where $F_{(L_N+U_N)/S_N}(x)=P\left((L_N+U_N)/S_N\le x\right)$. An application of the Lemma 5.1 yields that

$$F_{(L_N+U_N)/S_N}(x)=P\left(\frac{L_N+U_N}{\sigma_W}\le x\frac{(\sigma_W^2+V_N+R_N)^{1/2}}{\sigma_W}\right),$$

where R_N is a remainder of Bahadur's order (i.e. satisfying (3.2)). Since $x\tilde{H}_N'(x)$ is bounded, it is sufficient to prove (5.14) with $F_{(L_N+U_N)/S_N}(x)$ replaced by

$$P\left(\frac{L_N+U_N}{\sigma_W}\le x\frac{(\sigma_W^2+V_N)^{1/2}}{\sigma_W}\right)=P\left(\frac{L_N+U_N}{\sigma_W}-x\left\{\left(1+\frac{V_N}{\sigma_W^2}\right)^{1/2}-1\right\}\le x\right).$$

Following Putter and van Zwet [16], we also use the inequality $1+\frac{z}{2}-\frac{z^2}{4}\le(1+z)^{1/2}\le 1+\frac{z}{2}$ ($|z|\le\frac{4}{5}$) to find that $\frac{V_N}{2\sigma_W^2}-\frac{V_N^2}{4\sigma_W^4}\le\left(1+\frac{V_N}{\sigma_W^2}\right)^{1/2}-1\le\frac{V_N}{2\sigma_W^2}$ (with probability $1-O(N^{-c})$, $c>0$). Since by Hoeffding's inequality $V_N^2=O(\log N/N)$ with probability $1-O(N^{-c})$ for every $c>0$, we can replace $F_{(L_N+U_N)/S_N}(x)$ in (5.14) by $P\left(\frac{L_N+U_N}{\sigma_W}-x\frac{V_N}{2\sigma_W^2}\le x\right)$. Now it remains to show that

$$\sup_{x\in R}\left|P\left(\frac{L_N+U_N}{\sigma_W}-x\frac{V_N}{2\sigma_W^2}\le x\right)-\tilde{H}_N(x)\right|=O\left((\log N)^{5/4}/N^{3/4}\right), \tag{5.15}$$

as $N\to\infty$. First we prove (5.15), taking supremum for $x:|x|<\log N$ (cf. [16]). Note that $U_{Nx}=\frac{L_N+U_N}{\sigma_W}-x\frac{V_N}{2\sigma_W^2}$ is a centered U-statistic of degree two with bounded (uniformly for all x: $|x|<\log N$) kernel. Moreover, U_{Nx} has a nontrivial absolutely continuous component and Cramér's condition is satisfied. Theorem 1.1 of Bentkus, Götze and van Zwet [2] now yields that

$$\sup_{|x|<\log N}\left|P\left(\frac{L_N+U_N}{\sigma_W}-\frac{xV_N}{2\sigma_W^2}\le x\right)-\tilde{G}_N(x)\right|=O\left(N^{-1}\right), \tag{5.16}$$

where $\tilde{G}_N(x)=\Phi\left(\frac{x}{\sigma_x}\right)-\frac{k_{3x}}{6\sigma_x^3}\left[\left(\frac{x}{\sigma_x}\right)^2-1\right]\phi\left(\frac{x}{\sigma_x}\right)$ with $\sigma_x^2=Var(U_{Nx})=E\left(\frac{L_N+U_N}{\sigma_W}-\frac{xV_N}{2\sigma_W^2}\right)^2$ and $k_{3x}=E\left(\frac{L_N+U_N}{\sigma_W}-\frac{xV_N}{2\sigma_W^2}\right)^3$. Using the formulas (4.3)–(4.5) and the relations (5.2)–(5.3), we find that $\sigma_x^2=1+O\left(\frac{\log N}{\sqrt{N}}\right)$ and $k_{3x}=\frac{\lambda_1+3\lambda_2}{\sqrt{N}}+O\left(\frac{\log N}{N}\right)$. Therefore

$$\tilde{G}_N(x)=\Phi\left(\frac{x}{\sigma_x}\right)-\frac{\lambda_1+3\lambda_2}{6\sqrt{N}}(x^2-1)\phi(x)+O\left(\frac{\log N}{N}\right) \tag{5.17}$$

(for $|x| < \log N$), that is σ_x influences the form of EE only through the term $\Phi\left(\frac{x}{\sigma_x}\right)$ (cf. [16]). For σ_x^2 we can write $\sigma_x^2 = E\left(\frac{L_N+U_N}{\sigma_W} - \frac{xV_N}{2\sigma_W^2}\right)^2 = 1 - x\sigma_W^{-3}E[(L_N+U_N)V_N] + O\left(\frac{\log^2 N}{N}\right)$. As U_N and V_N are uncorrelated, using formulas (4.3)–(4.4) and (5.2), we can write $E[(L_N+U_N)V_N] = E(L_N V_N) = \frac{1}{\sqrt{N}}(\gamma_{3,W} + 2\delta_{2,W})$. Thus, we obtain that $\sigma_x^2 = 1 - \frac{x(\lambda_1+2\lambda_2)}{\sqrt{N}} + O\left(\frac{\log^2 N}{N}\right)$ (cf. notations (2.3)–(2.5)). This implies that

$$\Phi\left(\frac{x}{\sigma_x}\right) = \Phi(x) + \phi(x)\frac{1}{2}\frac{x^2(\lambda_1+2\lambda_2)}{\sqrt{N}} + O\left(\frac{\log^2 N}{N}\right). \tag{5.18}$$

Relations (5.17) and (5.18) together yield that $\tilde{G}_N(x) = \tilde{H}_N(x) + O\left(\frac{\log^2 N}{N}\right)$ for $|x| < \log N$. To treat the case $|x| \geq \log N$, we use the same arguments as in [16, p. 1561] to find that $\sup_{x\in R}\left|P\left(\frac{L_N+U_N}{\sigma_W} - \frac{xV_N}{2\sigma_W^2} \leq x\right) - \tilde{H}_N(x)\right| = O\left(\frac{\log^2 N}{N}\right)$. This proves (5.15) and the theorem. □

6. Proof of Theorem 2.3

In this section we state and prove two lemmas on the consistency of the estimators for λ_1, λ_2 and β_N. The validity of Theorem 2.3 follows directly from Theorems 2.1, 2.2 and these lemmas. In the first lemma we obtain the rate of convergence for our kernel estimates of the density evaluated at given quantiles, defined by (2.18).

Lemma 6.1. *Suppose that $f = F'$ exists in a neighborhood of ξ_α and satisfies a Lipschitz condition. In addition we assume that $f(\xi_\alpha) > 0$. Then*

$$P\left(|\hat{f}(\xi_\alpha) - f(\xi_\alpha)| > A(\log N)^{1/2}/N^{1/4}\right) = O(N^{-c}) \tag{6.1}$$

as $N \to \infty$, for every $c > 0$ and some $A > 0$, not depending on N.

Proof. Define random quantities

$$\nu_{k,N} = \sharp\left\{X_i : |X_i - X_{k:N}| \leq N^{-1/4}/2\right\}, \quad \nu_{\alpha,N} = \sharp\left\{X_i : |X_i - \xi_\alpha| \leq N^{-1/4}/2\right\}. \tag{6.2}$$

Note that $E\nu_{\alpha,N} = N\int_{\xi_\alpha - N^{-1/4}/2}^{\xi_\alpha+N^{-1/4}/2} f(x)\,dx$, and one can write

$$\begin{aligned} \hat{f}(\xi_\alpha) - f(\xi_\alpha) &= N^{-3/4}\nu_{k,N} - f(\xi_\alpha) \\ &= N^{-3/4}\nu_{\alpha,N} + N^{-3/4}(\nu_{k,N} - \nu_{\alpha,N}) - f(\xi_\alpha) = Q_{1,N} + Q_{2,N} + Q_{3,N}, \end{aligned} \tag{6.3}$$

where

$$Q_{1,N} = N^{-3/4}(\nu_{\alpha,N} - E\nu_{\alpha,N}), \qquad Q_{2,N} = N^{-3/4}(\nu_{k,N} - \nu_{\alpha,N}),$$

$$Q_{3,N} = N^{1/4}\int_{\xi_\alpha - N^{-1/4}/2}^{\xi_\alpha+N^{-1/4}/2}(f(x) - f(\xi_\alpha)\,dx.$$

For $Q_{1,N}$ we can write $Q_{1,N} = N^{1/4}(\bar{\nu}_{\alpha,N} - E\bar{\nu}_{\alpha,N})$, where $\bar{\nu}_{\alpha,N} = \frac{1}{N}\sum_{i=1}^N I_{\{2N^{1/4}|X_i-\xi_\alpha|\leq 1\}}$ is a mean of i.i.d. bounded r.v.'s. Therefore, by Hoeffding's inequality

$$P\left(|Q_{1,N}| > A_1(\log N)^{1/2}/N^{1/4}\right) = O(N^{-c}) \tag{6.4}$$

for every $c > 0$, as $N \to \infty$. Here and elsewhere A_i, $i = 1, 2, \dots$ denote positive constants, not depending on N. Since $P(|X_{k:N} - \xi_\alpha| > A_2(\log N/N)^{1/2}) = O(N^{-c})$, for $Q_{2,N}$ we have with probability $1 - O(N^{-c})$

$$|Q_{2,N}| \le N^{-3/4}(\nu_{l,N} + \nu_{r,N}), \tag{6.5}$$

where $\nu_{l,N} = \sharp\left\{X_i : |X_i - \xi_\alpha + N^{-1/4}/2| \le A_2(\log N/N)^{1/2}\right\}$, $\nu_{r,N} = \sharp\left\{X_i : |X_i - \xi_\alpha - N^{-1/4}/2| \le A_2(\log N/N)^{1/2}\right\}$. Since $(\nu_{l,N} + \nu_{r,N})$ is a Binomial r.v. with parameter $p_N = O\left((\log N/N)^{1/2}\right)$ and $E(\nu_{l,N} + \nu_{r,N}) = O\left(N^{1/2}(\log N)^{1/2}\right)$, $\sigma_{\nu_{l,N}+\nu_{r,N}} = O\left(N^{1/4}(\log N)^{1/4}\right)$, by Bernstein inequality, with probability $1 - O(N^{-c})$, we have the following bound

$$|Q_{2,N}| \le A_3 N^{-1/4}(\log N)^{1/2}. \tag{6.6}$$

Finally for $Q_{3,N}$ the Lipschitz condition directly yields that

$$|Q_{3,N}| \le C N^{1/4} \int_{\xi_\alpha - N^{-1/4}/2}^{\xi_\alpha + N^{-1/4}/2} |x - \xi_\alpha|\, dx = \frac{1}{4} C N^{-1/4}, \tag{6.7}$$

where C is the Lipschitz constant. Relations (6.3)–(6.7) imply (6.1). The lemma is proved. □

Let $\mu_{r,W} = EW_i^r = \int_0^1 Q^r(u)\, du$ denotes the r-th moment of W_i for any positive integer r and let $\hat{\mu}_{r,W} = \frac{k}{N} X_{k:N}^r + \frac{1}{N}\sum_{i=k+1}^{m-1} X_{i:N}^r + \frac{N-m+1}{N} X_{m:N}^r$ be the plug in estimator for $\mu_{r,W}$.

Lemma 6.2. *Suppose that $f = F'$ exists in neighborhoods of ξ_α and ξ_β and satisfies a Lipschitz condition. In addition we assume that $f(\xi_\nu) > 0$, $\nu = \alpha, \beta$. Then*

$$P\left(|\hat{\mu}_{r,W} - \mu_{r,W}| > A(\log N/N)^{1/2}\right) = O(N^{-c}) \tag{6.8}$$

as $N \to \infty$ for every $c > 0$ with some $A > 0$, not depending on N.

Proof. Put $\bar{W}_r = \frac{1}{N}\sum_{i=1}^N W_i^r$, where W_i is defined by (2.2), and note that similarly when proving of lemma 5.1 we can write

$$\bar{W}_r = \frac{N_\alpha}{N}\xi_\alpha^r + \frac{1}{N}\sum_{i=N_\alpha+1}^{N_\beta} X_{i:N}^r + \frac{N - N_\beta}{N}\xi_\beta^r.$$

We have

$$\hat{\mu}_{r,W} - \mu_{r,W} = (\hat{\mu}_{r,W} - \bar{W}_r) + (\bar{W}_r - \mu_{r,W}). \tag{6.9}$$

Note that $E\bar{W}_r = \mu_{r,W}$, therefore by Hoeffding inequality for the average of i.i.d. bounded r.v.'s we have $|\bar{W}_r - \mu_{r,W}| = O\left((\log N/N)^{1/2}\right)$ with probability $1 - O(N^{-c})$ for every $c > 0$. For $(\hat{\mu}_{r,W} - \bar{W}_r)$ on the r.h.s. of (6.9) we have

$$\hat{\mu}_{r,W} - \bar{W}_r = \frac{k}{N}(X_{k:N}^r - \xi_\alpha^r) + sign(N_\alpha - k)\frac{1}{N}\sum_{i=(k\wedge N_\alpha)+1}^{k\vee N_\alpha}(X_{i:N}^r - \xi_\alpha^r)$$

$$+\frac{N-m+1}{N}(X^r_{m:N}-\xi^r_\beta)-sign(N_\beta-m+1)\frac{1}{N}\sum_{i=m\wedge(N_\beta+1)}^{(m-1)\vee N_\beta}(X^r_{i:N}-\xi^r_\beta).$$

(cf.(5.6)). By Lemmas 3.1 and 3.2 the last expression equals to

$$-\alpha r\xi_\alpha^{r-1}\frac{N_\alpha-\alpha N}{N}\frac{1}{f(\xi_\alpha)}-\frac{(N_\alpha-\alpha N)^2}{2N^2}r\xi_\alpha^{r-1}\frac{1}{f(\xi_\alpha)}$$
$$-(1-\beta)r\xi_\beta^{r-1}\frac{N_\beta-\beta N}{N}\frac{1}{f(\xi_\beta)}+\frac{(N_\beta-\beta N)^2}{2N^2}r\xi_\beta^{r-1}\frac{1}{f(\xi_\beta)}+R_N, \tag{6.10}$$

where R_N is a remainder term of the Bahadur's order (cf. (3.2)). Thus, by Bernstein inequality we find that

$$|\hat{\mu}_{r,W}-\bar{W}_r|=O\left((\log N/N)^{1/2}\right) \tag{6.11}$$

with probability $1-O(N^{-c})$ for every $c>0$. Relations (6.9)–(6.11) together imply (6.8). The lemma is proved. □

Appendix

In this appendix we first establish an asymptotic approximation for the bias of T'_N (cf. (2.9)) in estimating of $\mu(\alpha,\beta)$. Secondly we prove that our Theorem 2.2 can not be inferred from Theorem 1.2 of Putter and van Zwet [16] for Studentized symmetric statistics..

Lemma A.1. *Suppose the conditions of Theorem 2.1 are satisfied. Then*

$$b_N=\beta_N+O(N^{-3/2}), \tag{A.1}$$

with b_N and β_N as in (2.8) and (2.9).

Proof. To begin with we note that b_N (cf. (2.9) can be written as B_1+B_2 where $B_1=(\beta-\alpha)ET'_N-E\left(\frac{1}{N}\sum_{i=[\alpha N]+1}^{[\beta N]}X'_{i:N}\right)$ and $B_2=E\left(\frac{1}{N}\sum_{i=[\alpha N]+1}^{[\beta N]}X'_{i:N}\right)-(\beta-\alpha)\mu(\alpha,\beta)$. First we consider B_2. By a simple conditioning argument we have that B_2 equals (with $k=[\alpha N]+1$, $m=[\beta N]$)

$$\frac{1}{N}E\left(F^{-1}(U_{k:N})+F^{-1}(U_{m:N})+(m-k-1)\frac{\int_{U_{k:N}}^{U_{m:N}}F^{-1}(u)\,du}{U_{m:N}-U_{k:N}}\right)-(\beta-\alpha)\mu(\alpha,\beta). \tag{A.2}$$

Define

$$I(v_1,v_2)=\frac{\int_{v_1}^{v_2}F^{-1}(u)\,du}{v_1-v_2},\qquad I(\alpha,\beta)=\mu(\alpha,\beta).$$

The first and second partial derivatives are given by

$$\left.\frac{\partial I}{\partial v_1}\right|_{(\alpha,\beta)}=\frac{-\xi_\alpha+\mu(\alpha,\beta)}{\beta-\alpha},\qquad \left.\frac{\partial I}{\partial v_2}\right|_{(\alpha,\beta)}=\frac{\xi_\beta-\mu(\alpha,\beta)}{\beta-\alpha}\ ,$$

$$\left.\frac{\partial^2 I}{\partial v_1^2}\right|_{(\alpha,\beta)}=-\frac{2}{\beta-\alpha}\left[\frac{1}{2f(\xi_\alpha)}-\frac{\mu(\alpha,\beta)-\xi_\alpha}{\beta-\alpha}\right],$$

$$\left.\frac{\partial^2 I}{\partial v_2^2}\right|_{(\alpha,\beta)} = \frac{2}{\beta-\alpha}\left[\frac{1}{2f(\xi_\beta)} + \frac{\mu(\alpha,\beta)-\xi_\beta}{\beta-\alpha}\right],$$

$$\left.\frac{\partial^2 I}{\partial v_1 \partial v_2}\right|_{(\alpha,\beta)} = \frac{\xi_\alpha+\xi_\beta-2\mu(\alpha,\beta)}{(\beta-\alpha)^2}.$$

A Taylor expansion argument now yields that (A.2) reduces to

$$\frac{1}{N}E\left(F^{-1}(U_{k:N}) + F^{-1}(U_{m:N})\right) + \frac{m-k-1}{N}\Bigg\{\mu(\alpha,\beta)$$
$$+\frac{-\xi_\alpha+\mu(\alpha,\beta)}{\beta-\alpha}\left(\frac{k}{N+1}-\alpha\right) + \frac{\xi_\beta-\mu(\alpha,\beta)}{\beta-\alpha}\left(\frac{m}{N+1}-\beta\right)$$
$$-\frac{1}{\beta-\alpha}\left[\frac{1}{2f(\xi_\alpha)} - \frac{\mu(\alpha,\beta)-\xi_\alpha}{\beta-\alpha}\right]\frac{\frac{k}{N+1}(1-\frac{k}{N+1})}{N+2}$$
$$+\frac{1}{\beta-\alpha}\left[\frac{1}{2f(\xi_\beta)} + \frac{\mu(\alpha,\beta)-\xi_\beta}{\beta-\alpha}\right]\frac{\frac{m}{N+1}(1-\frac{m}{N+1})}{N+2}$$
$$+\left[\frac{\xi_\alpha+\xi_\beta-2\mu(\alpha,\beta)}{(\beta-\alpha)^2}\right]\frac{\frac{k}{N+1}(1-\frac{m}{N+1})}{N+2} + O(N^{-3/2})\Bigg\} - (\beta-\alpha)\mu(\alpha,\beta),$$

which easily leads to

$$\frac{1}{N}\Bigg\{\xi_\alpha(\alpha N-[\alpha N]) - \xi_\beta(\beta N-[\beta N])$$
$$-\frac{1}{2f(\xi_\alpha)}\alpha(1-\alpha) + \frac{1}{2f(\xi_\beta)}\beta(1-\beta)\Bigg\} + O(N^{-3/2}). \tag{A.3}$$

For B_1 we have

$$B_1 = \frac{(\beta N-[\beta N])-(\alpha N-[\alpha N])}{[\beta N]-[\alpha N]}E\left(\frac{1}{N}\sum_{i=k}^{m}X'_{i:N}\right)$$
$$= \frac{(\beta N-[\beta N])-(\alpha N-[\alpha N])}{[\beta N]-[\alpha N]}\left((\beta-\alpha)\mu(\alpha,\beta)+\beta_N+O(N^{-3/2})\right)$$
$$= \frac{1}{N}\left((\beta N-[\beta N])-(\alpha N-[\alpha N])\right)\mu(\alpha,\beta)+O(N^{-2}).$$

This together with (A.2)–(A.3) implies (A.1). The lemma is proved. □

Consider a trimmed mean T_N as in (4.1). Let $T_{N\Omega_k}$ is defined as in (1.8) of Putter and van Zwet [16]. We prove the following assertion.

LEMMA A.2. *Suppose that the conditions of Theorem 2.1 hold. Then*

$$\sum_{k=3}^{N}\binom{N-2}{k-2}ET^2_{N\Omega_k} = N^{-3}\left(\frac{\alpha^2(1-\alpha)^2}{f^2(\xi_\alpha)} + \frac{\beta^2(1-\beta)^2}{f^2(\xi_\beta)}\right) + o(N^{-3}) \tag{A.4}$$

as $N\to\infty$.

Relation (A.4) directly yields that in the second condition of (1.18) in Theorem 1.2 of Putter and van Zwet [16] is not satisfied for a Studentized trimmed mean, as Putter and van Zwet [16] require that the l.h.s. of (A.4) is of order $N^{-7/2}$, instead of N^{-3} as in our relation (A.4).

PROOF. In Putter's Ph.D thesis [15] it was proved that if T_N is a linear combination of order statistics, then

$$(A.5)\qquad \sum_{k=3}^{N}\binom{N-2}{k-2}ET^2_{N\Omega_k} = E(Z_N - E(Z_N|U_{N-1},U_N))^2$$

$$= EZ_N^2 - E(T_{N,(1,2)})^2,$$

(cf. (3.5.17), Putter [15]), where $T_{N\Omega_k}$, $T_{N,(1,2)}$ are defined as in (1.8) of Putter and van Zwet [16], Z_N is a r.v. defined as in (4.21) of van Zwet [20], $U_1,\dots,U_N$ are uniformly on (0,1) distributed r.v.'s. Let R_j denotes the rank of U_j among $U_1,\dots,U_N$, $K_1 = R_{N-1}\wedge R_N$, $K_2 = R_{N-1}\vee R_N$. Take $X_{0:N} = -\infty$, $X_{N+1:N} = +\infty$ (cf. van Zwet [20]). Let the functions G, H, M are defined as in (4.17) of van Zwet [20], and define in addition the functions G_1 and H_1 by $G_1(x) = \int_{-\infty}^{x} F^2(y)\,dy$, $H_1(x) = \int_x^{\infty}(1-F(y))^2\,dy$. Then formula (4.21) of van Zwet [20] reduces to

$$N^{1/2}Z_N = -\sum_{j=1}^{K_1}(c_{j+1}-c_j)(G_1(X_{j:N}) - G_1(X_{j-1:N}))$$

$$+\sum_{j=K_1}^{K_2-1}(c_{j+1}-c_j)(M(X_{j+1:N}) - M(X_{j:N})) - \sum_{j=K_2}^{N}(c_j - c_{j-1})(H_1(X_{j:N}) - H_1(X_{j+1:N}))$$

(cf. Gribkova [7]), where in the trimmed mean case ($c_j = 1$ for $k\le j\le m$ and $c_j = 0$ for $j<k$, $j>m$) there are only two nonzero summands, which depend on K_1 and K_2 (the event $\{K_1 = k-1 \text{ or } K_2 = m+1\}$ is negligible for our aims because its probability is $O(N^{-1})$, cf. below). For instance, when $K_2 < k$ (which happens with probability $P(K_2<k) = \alpha^2 + O(N^{-1})$), the value of $N^{1/2}Z_N$ equals

$$-[H_1(X_{k:N}) - H_1(X_{k+1:N})] + [H_1(X_{m+1:N}) - H_1(X_{m+2:N})] \stackrel{d}{=}$$

$$-[H_1\circ F^{-1}(U_{k:N}) - H_1\circ F^{-1}(U_{k+1:N})] + [H_1\circ F^{-1}(U_{m+1:N}) - H_1\circ F^{-1}(U_{m+2:N})],$$

where $U_{i:N}$ are order statistics of r.v.'s U_i, $i = 1,\dots,N$. Application of a two term Taylor expansion of the function $H_1\circ F^{-1}$ in neighborhoods of α and β respectively, together with the well-known facts that $E(s_i^2) = \frac{2}{(N+2)(N+1)}$, $E(s_is_j) = \frac{1}{(N+2)(N+1)}(i\neq j)$, where $s_i = U_{i:N} - U_{i-1:N}$, $i = 1,\dots,N+1$, $U_{0:N} = 0$, $U_{N+1:N} = 1$, yields that $E(Z_N^2|K_2<k) = \frac{2}{N^3}\left(\frac{(1-\alpha)^4}{f^2(\alpha)} + \frac{(1-\beta)^4}{f^2(\beta)} - \frac{(1-\alpha)^2(1-\beta)^2}{f(\alpha)f(\beta)}\right) + o(N^{-3})$, where $P(K_2<k) = \alpha^2 + O(1/N)$. Analyzing in similar fashion the other possibilities for K_1 and K_2, we find that

$$(A.6)\qquad EZ_N^2 = \frac{2}{N^3}\left(\frac{\alpha^2(1-\alpha)^2}{f^2(\xi_\alpha)} - \frac{\alpha^2(1-\beta)^2}{f(\xi_\alpha)f(\xi_\beta)} + \frac{\beta^2(1-\beta)^2}{f^2(\xi_\beta)}\right) + o(N^{-3}),$$

as $N \to \infty$. Next we consider $T_{N,(1,2)}$. By formula (2.11) of Putter and van Zwet [16] we have

$$N^{1/2} T_{N,(1,2)} = -\int_0^1 (I_{[U_1,1)}(t) - t)(I_{[U_2,1)}(t) - t) \binom{N-2}{k-2} t^{k-2}(1-t)^{N-k}\, dF^{-1}(t)$$
$$+ \int_0^1 (I_{[U_1,1)}(t) - t)(I_{[U_2,1)}(t) - t) \binom{N-2}{m-1} t^{m-1}(1-t)^{N-m-1}\, dF^{-1}(t).$$

Define $\Delta F_{i,N}(x) = F_{i-1,N}(x) - F_{i,N}(x)$, where $F_{i,N}(x) = P(X_{i:N} \le x)$. The latter relation implies that $E(T_{N,(1,2)})^2$ equals to

$$(A.7) \quad \frac{2}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{z} \Big[-\int_{-\infty}^{\infty} (I_{[y,+\infty)}(x) - F(x))(I_{[z,+\infty)}(x) - F(x)) \Delta F_{k-1,N-2}(x)\, dx$$
$$+ \int_{-\infty}^{\infty} (I_{[y,+\infty)}(x) - F(x))(I_{[z,+\infty)}(x) - F(x)) \Delta F_{m,N-2}(x)\, dx \Big]^2 \, dF(y)\, dF(z)$$
$$= \frac{2}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{z} [I_{k-1}(y,z)]^2 \, dF(y)\, dF(z) + \frac{2}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{z} [I_m(y,z)]^2 \, dF(y)\, dF(z)$$
$$- \frac{4}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{z} [I_{k-1}(y,z) I_m(y,z)] \, dF(y)\, dF(z),$$

where $I_r(y,z)$, $r = k-1, m$, is defined by

$$\int_{-\infty}^{y} \Delta F_{r,N-2}(x)\, dG_1(x) - \int_{y}^{z} \Delta F_{r,N-2}(x)\, dM(x) - \int_{z}^{\infty} \Delta F_{r,N-2}(x)\, dH_1(x).$$

Consider the first term at the r.h.s. of (A.7) (the treatment of the second and third term is similar). Integrating by parts, we reduce it to

$$\frac{2}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{z} \Big[(G_1(y) + M(y)) \Delta F_{k-1,N-2}(y) + (H_1(z) - M(z)) \Delta F_{k-1,N-2}(z)$$
$$- \int_{-\infty}^{y} G_1(x)\, d(\Delta F_{k-1,N-2}(x)) + \int_{y}^{z} M(x) d(\Delta F_{k-1,N-2}(x))$$
$$+ \int_{z}^{\infty} H_1(x) d(\Delta F_{k-1,N-2}(x)) \Big]^2 \, dF(y)\, dF(z).$$

Note that the 'basic' support of the function $\Delta F_{k-1,N-2}(x) = F_{k-2,N-2}(x) - F_{k-1,N-2}(x)$ is some interval $I_\alpha(A) = [\xi_\alpha - A(\log N/N)^{1/2}, \xi_\alpha + A(\log N/N)^{1/2}]$ in the sense that for every $c > 2$ we have the following bound: $\sup_{y \in R \setminus I_\alpha(A)} \Delta F_{k-1,N-2}(y) = O\left(P\left(|U_{k:N} - \alpha| > (\log N/N)^{1/2}\right)\right) = O(N^{-c})$, where $A > 0$ is some constant, depending only on c, α and $f(\xi_\alpha)$. Moreover, smoothness conditions imply that $\sup_{y \in I_\alpha(A)} \Delta F_{k-1,N-2}(y) = O(N^{-1})$ as $N \to \infty$. Thus, the last expression reduces to

$$(A.8) \qquad \frac{2}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{z} \Big[-\int_{-\infty}^{y} G_1(x)\, d(\Delta F_{k-1,N-2}(x)) + \int_{y}^{z} M(x) d(\Delta F_{k-1,N-2}(x))$$
$$+ \int_{z}^{\infty} H_1(x) d(\Delta F_{k-1,N-2}(x)) \Big]^2 \, dF(y)\, dF(z) + o(N^{-3}),$$

as $N \to \infty$. Consider the integrand in (A.8) and note that if $I_\alpha(A) \subset (-\infty, y)$, then the integrand equals to $[E(G_1(X_{k-2:N-2}) - G_1(X_{k-1:N-2}))]^2 + o(N^{-2}) = \frac{\alpha^4}{N^2}\frac{1}{f^2(\xi_\alpha)} + o(N^{-2})$, and the corresponding part of the integral in (A.8) (in the domain where $Y = \min(X_1, X_2) \geq \xi_\alpha$) equals to $\frac{(1-\alpha)^2\alpha^4}{f^2(\xi_\alpha)}N^{-3} + o(N^{-3})$. Arguing similarly for the cases $I_\alpha(A) \subset (y, z)$ and $I_\alpha(A) \subset (z, +\infty)$ (the cases $y \in I_\alpha(A)$ or $z \in I_\alpha(A)$ are negligible) we obtain that the quantity (A.8), and hence the first term at the r.h.s. in (A.7), equals to $\left(\frac{\alpha^4(1-\alpha)^2}{f^2(\xi_\alpha)} + 2\frac{\alpha^2(1-\alpha)^2}{f^2(\xi_\alpha)} + \frac{(1-\alpha)^4\alpha^2}{f^2(\xi_\alpha)}\right)N^{-3} + o(N^{-3}) = \frac{\alpha^2(1-\alpha)^2}{f^2(\xi_\alpha)}N^{-3} + o(N^{-3})$. Similarly for the second term at the r.h.s. of (A.7) we get $\frac{\beta^2(1-\beta)^2}{f^2(\xi_\beta)}N^{-3} + o(N^{-3})$, and for the third one we obtain $-2\frac{\alpha^2(1-\beta)^2}{f(\xi_\alpha)f(\xi_\beta)}N^{-3} + o(N^{-3})$. Together these results give us

$$(A.9) \qquad E(T_{N,(1,2)})^2 = N^{-3}\left(\frac{\alpha^2(1-\alpha)^2}{f^2(\xi_\alpha)} - 2\frac{\alpha^2(1-\beta)^2}{f(\xi_\alpha)f(\xi_\beta)} + \frac{\beta^2(1-\beta)^2}{f^2(\xi_\beta)}\right) + o(N^{-3})$$

as $N \to \infty$. The relations (A.5), (A.6) and (A.9) together imply (A.4) and the lemma is proved. □

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