Miura-type transformation for the Itoh-Narita-Bogoyavlenskii lattice

Andrei K Svinin
Institute for System Dynamics and Control Theory, Siberian Branch of Russian Academy of Sciences, Russia
E-mail: svinin@icc.ru

Abstract. We show that by Miura-type transformation the Itoh-Narita-Bogoyavlenskii lattice, for any $n \geq 1$, is related with some differential-difference (modified) equation. We present corresponding integrable hierarchies in its explicit form. We study elementary Darboux transformation for modified equations.

PACS numbers: 02.30.Ik

Keywords: integrable lattice, Miura-type transformation

1. Introduction

The equation

$$v_i' = -\frac{1}{v_{i-1} - v_{i+1}} \qquad (1)$$

is known to be one of a representative in the class of integrable differential-difference equation of Volterra-type form $v_i' = f(v_{i-1}, v_i, v_{i+1})$ [11]. As is known, the equation (1) is related with the Volterra lattice $a_i' = a_i(a_{i-1} - a_{i+1})$ by Miura-type transformation [5]

$$a_i = \frac{1}{(v_{i-2} - v_i)(v_{i-1} - v_{i+1})}. \qquad (2)$$

The Volterra lattice gives natural integrable discretization of KdV equation. More general class of integrable differential-difference equations sharing this property is given by Itoh-Narita-Bogoyavlenskii (INB) lattice [2, 3, 6]

$$a_i' = a_i \left(\sum_{j=1}^{n} a_{i-j} - \sum_{j=1}^{n} a_{i+j} \right). \qquad (3)$$

The Volterra lattice and its generalization (3) also share the property that it admit integrable hierarchy. This means that the flow defined on suitable phase space by the evolution equation (3) can be included in infinite set of the pair-wise commuting flows. We have shown in [7, 8] that corresponding integrable hierarchies for (3) and for some number of integrable lattices are directly related with KP hierarchy.

The goal of this paper is to present some class of differential-difference equations each of which is related with INB lattice with any $n \geq 1$ by some Miura-type transformation generalizing (2). In the works [9, 10] we showed explicit form for INB lattice hierarchy in terms of some homogeneous polynomials $S_s^l[a]$. Using these results, we give explicit form of modified evolution equations on the field $v = v_i$ governing corresponding integrable hierarchies in terms of some rational functions $\tilde{S}_s^l[v]$. In section 3, we study Darboux transformation for underlying linear equations of modified integrable hierarchies and as a result derive some discrete quadratic equation parameterized by $n \geq 1$ which is shown to relate two solutions $\{v_i\}$ and $\{\bar{v}_i\}$ of modified equation yielding the first flow in corresponding integrable hierarchy. We claim but do not prove in this paper that this algebraic relation is also compatible with higher flows of modified integrable hierarchy. We can interpret quadratic equation $F[v, \bar{v}] = 0$ as a two-dimensional equation on $\{v_{i,j}\}$. We show that this equation can be written in the form of discrete conservation law $I_{i+1,j} = I_{i,j}$. In the case $n = 1$, this two-dimensional equation can be treated as a deautonomization of the discrete KdV equation $I_{i,j} = (v_{i,j} - v_{i+1,j+1})(v_{i+1,j} - v_{i,j+1}) = c$. Similar relation for any $n \geq 2$ gives a generalization of the discrete KdV equation.

2. Miura-type transformation for INB lattice

2.1. The INB lattice and its hierarchy

Let us consider the INB lattice (3) together with auxiliary linear equations on KP wavefunction:

$$z\psi_{i+n} + a_{i+n}\psi_i = z\psi_{i+n+1}, \quad \psi_i' = z\psi_{i+n} - \left(\sum_{j=1}^{n} a_{i+j-1}\right)\psi_i. \tag{4}$$

As was shown in [9], [10], linear equation governing higher flows of the equation (3) can be written as

$$\partial_s\psi_i = z^s\psi_{i+sn} + \sum_{j=1}^{s} z^{s-j}(-1)^j S_{sn-1}^j(i-(j-1)n)\psi_{i+(s-j)n}. \tag{5}$$

Coefficients of this equation are defined by some special polynomials $S_s^l[a]$. More exactly, let

$$S_s^l(y_1, \ldots, y_{s+(l-1)n+1}) = \sum_{1\leq\lambda_l\leq\cdots\leq\lambda_1\leq s+1}\left(\prod_{j=1}^{l} y_{\lambda_j+n(j-1)}\right).$$

Then $S_s^l(i) \equiv S_s^l(a_i, \ldots, a_{i+s+(l-1)n})$. Integrable hierarchy of the flows for INB lattice (3) is defined by the totality of evolution differential-difference equations [9, 10]

$$\partial_s a_i = (-1)^s a_i\left\{S_{sn-1}^s(i-(s-1)n+1) - S_{sn-1}^s(i-sn)\right\} \tag{6}$$

for $s \geq 1$.

2.2. Miura-type transformation

Let us consider following rational noninvertible substitution

$$a_i = \prod_{j=n}^{2n} \frac{1}{v_{i-j} - v_{i-j+n+1}} \tag{7}$$

which is a generalization of (2). By simple but tedious calculations one can check the latter sends solutions of the equation

$$v_i' = -\prod_{j=1}^{n} \frac{1}{v_{i-j} - v_{i-j+n+1}} \tag{8}$$

to solutions of the INB equation (3). Moreover, we have the following.

Proposition 1 *Integrable hierarchy of the lattice (8) is given by evolution differential-difference equations*‡

$$\partial_s v_i = (-1)^s \tilde{S}^{s-1}_{sn-1}(i-(s-2)n) \prod_{j=1}^{n} \frac{1}{v_{i-j} - v_{i-j+n+1}}. \tag{9}$$

Therefore we obtained integrable hierarchy (9) intimately related with INB lattice hierarchy via substitution (7).

2.3. Auxiliary linear equations

Introduce the function γ_i, such that

$$\frac{\gamma_{i+1}}{\gamma_i} = \frac{1}{v_{i-n} - v_{i+1}}. \tag{10}$$

Clearly, in terms of γ_i

$$a_i = \prod_{j=1}^{n+1} \frac{\gamma_{i-n+j}}{\gamma_{i-n+j-1}}.$$

Proposition 2 *In virtue of (9) and (10),*

$$\partial_s \gamma_i = (-1)^s \tilde{S}^{s}_{sn-1}(i-(s-1)n)\gamma_i. \tag{11}$$

Let $\psi_i = \gamma_i \phi_i$. In terms of this new wavefunction, linear equations (4) become

$$z(v_i - v_{i+n+1})\phi_{i+n} + \phi_i = z\phi_{i+n+1}, \tag{12}$$

$$\phi_i' = z\phi_{i+n} \prod_{j=1}^{n} \frac{1}{v_{i-j} - v_{i-j+n+1}}. \tag{13}$$

To derive these equations, we take into account proposition 2, namely, that

$$\gamma_i' = -\tilde{S}^1_{n-1}(i)\gamma_i = -\sum_{j=1}^{n} \left(\prod_{k=n}^{2n} \frac{1}{v_{i-k+j-1} - v_{i-k+j+n}} \right) \gamma_i.$$

‡ It is supposed, in (9), that $\tilde{S}^l_s[v] \equiv S^l_s[a]$, where the field a is expressed via v in virtue of (7).

In a similar way we get the linear equation

$$\partial_s\phi_i = z^s\frac{\gamma_{i+sn}}{\gamma_i}\phi_{i+sn}+\sum_{j=1}^{s-1}(-1)^j z^{s-j}\frac{\gamma_{i+(s-j)n}}{\gamma_i}\tilde{S}^j_{sn-1}(i-(j-1)n)\phi_{i+(s-j)n} \quad (14)$$

governing all the flows (9).

3. Darboux transformation for linear equations (12) and (13)

3.1. Darboux transformation

Let us show that linear equations (12) and (13) admit Darboux transformation

$$\bar{\phi}_i = \phi_{i+1} + (v_{i+1} - \bar{v}_i)\phi_i. \quad (15)$$

It is worth remarking that the latter does not depend on n. It is simple exercise to verify that (15) yields Darboux transformation for linear equation (12) if the relation

$$(v_i - v_{i+n+1})(v_{i+1} - \bar{v}_i) = (v_{i+n+1} - \bar{v}_{i+n})(\bar{v}_i - \bar{v}_{i+n+1}) \quad (16)$$

holds. One can rewrite this equation as

$$(v_i - v_{i+n+1})(v_{i+1} - \bar{v}_{i+n+1}) = (v_i - \bar{v}_{i+n})(\bar{v}_i - \bar{v}_{i+n+1}) \quad (17)$$

and

$$(v_i - \bar{v}_{i+n})(v_{i+1} - \bar{v}_i) = (v_{i+1} - \bar{v}_{i+n+1})(v_{i+n+1} - \bar{v}_{i+n}) \quad (18)$$

and

$$v_iv_{i+1} - v_{i+1}v_{i+n+1} + v_{i+n+1}\bar{v}_{i+n+1} - v_i\bar{v}_i + \bar{v}_i\bar{v}_{i+n} - \bar{v}_{i+n}\bar{v}_{i+n+1} = 0. \quad (19)$$

We have the following.

Proposition 3 *The relation (19) is compatible with the flow given by (8).*

It is natural to expect that the relation (19) in fact is compatible with all the higher flows (9), but this question we leave for subsequent studies. Taking into account proposition (3), one can check that the relation (15) yields elementary Darboux transformation for the linear equation (13).

3.2. Generalized discrete KdV equation

We observe that§

$$I_i = \prod_{k=1}^{n+1}(v_{i+k-1} - \bar{v}_{i+k-2}) \quad (20)$$

is an integral for discrete equation (19). Indeed, making use of (18), one can easily check that $I_{i+1} = I_i$. Identifying $v_i = v_{i,j}$, $\bar{v}_i = v_{i,j+1}$ and $I_i = I_{i,j}$, we can interpret

§ It is supposed in (20) that in all the subscripts of the type $i + r$, the variable r take its values in $\mathbb{Z}/n\mathbb{Z}$.

the relation (19) as two-dimensional discrete equation with an integral (20). In the case $n = 1$ the equation (19) and its integral (20) are specified as

$$v_{i,j}v_{i+1,j} - v_{i+1,j}v_{i+2,j} + v_{i+2,j}v_{i+2,j+1} \\ - v_{i,j}v_{i,j+1} + v_{i,j+1}v_{i+1,j+1} - v_{i+1,j+1}v_{i+2,j+1} = 0 \qquad (21)$$

and

$$I_{i,j} = (v_{i,j} - v_{i+1,j+1})\,(v_{i+1,j} - v_{i,j+1})\,,$$

respectively. Assigning some values: $I_{i,j} = c_j$, we are led to the nonautonomous equation

$$(v_{i,j} - v_{i+1,j+1})\,(v_{i+1,j} - v_{i,j+1}) = c_j$$

which is obviously equivalent to (21). In particular case when $c_j = c$, we come to the well-known discrete KdV equation [4]

$$(v_{i,j} - v_{i+1,j+1})\,(v_{i+1,j} - v_{i,j+1}) = c \qquad (22)$$

also known as the H_1 equation in Adler-Bobenko-Suris classification [1]. Therefore the equation (21) can be treated as the deautonomization of (22). In general case, we have the following equation:

$$\prod_{k=1}^{n+1} (v_{i+k-1,j} - v_{i+k-2,j+1}) = c, \qquad (23)$$

being in a sense a generalization of the discrete KdV equation (22). As can be checked, the deautonomization of this equation given by a change $c \mapsto c_j$ yields the equation (19).

4. Conclusion

We have shown in this paper an infinite number of integrable hierarchies (9) closely related via noninvertible substitution with INB lattice hierarchies given by evolution equations (6). These equations appear as compatibility condition of the linear ones (12) and (14). In section 3, we studied Darboux transformation for linear equations. We have shown that the discrete equation (19) yields elementary Darboux transformation. In the case $n = 1$, it can be obtained by a sort of deautonomization of the discrete KdV equation (22). The same is true in general case. Namely, one can derive the equation (19) as the deautonomization of the generalized discrete KdV equation (23).

Appendix A. Proof of proposition 1

We must to show that in virtue of equation (9) the equation (5) holds. We have

$$\partial_s a_i = \partial_s \left(\prod_{j=n}^{2n} \frac{1}{v_{i-j} - v_{i-j+n+1}} \right) = \prod_{j=n}^{2n} \frac{1}{v_{i-j} - v_{i-j+n+1}} \cdot \sum_{j=n}^{2n} \frac{\partial_s v_{i-j+n+1} - \partial_s v_{i-j}}{v_{i-j} - v_{i-j+n+1}}$$

$$= (-1)^s \prod_{j=n}^{2n} \frac{1}{v_{i-j} - v_{i-j+n+1}} \left\{ \sum_{j=n}^{2n} \tilde{S}^{s-1}_{sn-1}(i-j-(s-3)n+1) \prod_{k=0}^{n} \frac{1}{v_{i-j-k+n} - v_{i-j-k+2n+1}} \right.$$

$$
- \sum_{j=n}^{2n} \tilde{S}^{s-1}_{sn-1}(i-j-(s-2)n) \prod_{k=0}^{n} \frac{1}{v_{i-j-k} - v_{i-j-k+n+1}} \Bigg\}
$$

$$
= (-1)^s a_i \left\{ \sum_{j=n}^{2n} a_{i+2n-j} S^{s-1}_{sn-1}(i-j-(s-3)n+1) \right.
$$

$$
\left. - \sum_{j=n}^{2n} a_{i+n-j} S^{s-1}_{sn-1}(i-j-(s-2)n) \right\}.
$$

To proceed, we need to use the identity [9, 10]

$$
S^l_s(i+1) - S^l_s(i) = a_{i+s+1} S^{l-1}_s(i+n+1) - a_{i+(l-1)n} S^{l-1}_s(i). \tag{A.1}
$$

Taking the latter into account we get

$$
\partial_s a_i = (-1)^s a_i \sum_{j=n}^{2n} \left(S^s_{sn-1}(i-j-(s-2)n+1) - S^s_{sn-1}(i-j-(s-2)n) \right)
$$

$$
= (-1)^s a_i \left\{ S^s_{sn-1}(i-(s-1)n+1) - S^s_{sn-1}(i-sn) \right\}.
$$

Appendix B. Proof of proposition 2

On the one hand

$$
\partial_s \left(\frac{\gamma_{i+1}}{\gamma_i} \right) = \frac{\gamma_{i+1}}{\gamma_i} \left(\frac{\partial_s \gamma_{i+1}}{\gamma_{i+1}} - \frac{\partial_s \gamma_i}{\gamma_i} \right). \tag{B.1}
$$

On the other hand

$$
\partial_s \left(\frac{\gamma_{i+1}}{\gamma_i} \right) = \partial_s \left(\frac{1}{v_{i-n} - v_{i+1}} \right) = \frac{(\partial_s v_{i+1} - \partial_s v_{i-n})}{(v_{i-n} - v_{i+1})^2}
$$

$$
= (-1)^s \frac{1}{v_{i-n} - v_{i+1}} \left\{ \tilde{S}^{s-1}_{sn-1}(i-(s-2)n+1) \prod_{j=0}^{n} \frac{1}{v_{i-j} - v_{i-j+n+1}} \right.
$$

$$
\left. - \tilde{S}^{s-1}_{sn-1}(i-(s-1)n) \prod_{j=0}^{n} \frac{1}{v_{i-j-n} - v_{i-j+1}} \right\}
$$

$$
= (-1)^s \frac{\gamma_{i+1}}{\gamma_i} \left\{ a_{i+n} \tilde{S}^{s-1}_{sn-1}(i-(s-2)n+1) - a_i \tilde{S}^{s-1}_{sn-1}(i-(s-1)n) \right\}
$$

$$
= (-1)^s \frac{\gamma_{i+1}}{\gamma_i} \left\{ \tilde{S}^s_{sn-1}(i-(s-1)n+1) - \tilde{S}^s_{sn-1}(i-(s-1)n) \right\}. \tag{B.2}
$$

Remark that we used here the identity (A.1). Comparing (B.2) with (B.1), we come to (11).

Appendix C. Proof of proposition 3

Differentiating of (19) in virtue of (8) yields

$$
(\bar{v}_i - v_{i+1}) \prod_{j=1}^{n} \frac{1}{v_{i-j} - v_{i-j+n+1}} + (v_{i+n+1} - v_i) \prod_{j=0}^{n-1} \frac{1}{v_{i-j} - v_{i-j+n+1}}
$$

$$+(v_{i+1}-\bar{v}_{i+n+1})\prod_{j=-n}^{-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}+(\bar{v}_{i+n}-v_{i+n+1})\prod_{j=-n}^{-1}\frac{1}{\bar{v}_{i-j}-\bar{v}_{i-j+n+1}}$$

$$+(\bar{v}_{i+n+1}-\bar{v}_i)\prod_{j=-n+1}^{0}\frac{1}{\bar{v}_{i-j}-\bar{v}_{i-j+n+1}}+(v_i-\bar{v}_{i+n})\prod_{j=1}^{n}\frac{1}{\bar{v}_{i-j}-\bar{v}_{i-j+n+1}}=0. \qquad \text{(C.1)}$$

Thus, we must to prove that the relation (C.1) is the identity in virtue of (19). We are in position to prove that in fact two relations

$$(\bar{v}_i-v_{i+1})\prod_{j=1}^{n}\frac{1}{v_{i-j}-v_{i-j+n+1}}+(v_{i+n+1}-v_i)\prod_{j=0}^{n-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}$$

$$+(v_i-\bar{v}_{i+n})\prod_{j=1}^{n}\frac{1}{\bar{v}_{i-j}-\bar{v}_{i-j+n+1}}=0 \qquad \text{(C.2)}$$

and

$$(v_{i+1}-\bar{v}_{i+n+1})\prod_{j=-n}^{-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}+(\bar{v}_{i+n}-v_{i+n+1})\prod_{j=-n}^{-1}\frac{1}{\bar{v}_{i-j}-\bar{v}_{i-j+n+1}}$$

$$+(\bar{v}_{i+n+1}-\bar{v}_i)\prod_{j=-n+1}^{0}\frac{1}{\bar{v}_{i-j}-\bar{v}_{i-j+n+1}}=0 \qquad \text{(C.3)}$$

are both the identities in virtue of (19). Let us prove (C.2). Remark that one can write the second member in (C.2) as

$$G_2=(\bar{v}_{i+1}-\bar{v}_{i-n})\left(\prod_{j=1}^{n-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}\right)\frac{1}{\bar{v}_{i-n}-\bar{v}_{i+1}}.$$

In turn, using step-by-step the relation (17), one can rewrite the third member in (C.2) in the form

$$G_3=(v_{i-n+1}-\bar{v}_{i+1})\left(\prod_{j=1}^{n-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}\right)\frac{1}{\bar{v}_{i-n}-\bar{v}_{i+1}}.$$

Summing these two terms and using (16) we obtain

$$\begin{aligned}G_2+G_3 &= \left(\prod_{j=1}^{n-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}\right)\frac{v_{i-n+1}-\bar{v}_{i-n}}{\bar{v}_{i-n}-\bar{v}_{i+1}}\\ &= \left(\prod_{j=1}^{n-1}\frac{1}{v_{i-j}-v_{i-j+n+1}}\right)\frac{v_{i+1}-\bar{v}_i}{v_{i-n}-v_{i+1}}\\ &= (v_{i+1}-\bar{v}_i)\left(\prod_{j=1}^{n}\frac{1}{v_{i-j}-v_{i-j+n+1}}\right)=-G_1.\end{aligned}$$

Therefore we proved (C.2). The similar reasonings are needed to prove (C.3) and as a result (C.1).

References

[1] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs. The consistency approach *Commun. Math. Phys.* **233** 513-43

[2] Bogoyavlenskii O I 1988 Some constructions of integrable dynamical systems *Math. USSR-Izv.* **31** 47-76

[3] Itoh Y 1975 An H-theorem for a system of competing species *Proc. Japan Acad.* **51** 374-9

[4] Hirota R 1977 Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation *J. Phys. Soc. Japan* **43** 1424-33

[5] Levi D, Petrera M, Scimiterna and Yamilov R I On Miura transformations and Volterra-type equations associated with the Adler-Bobenko-Suris equations *SIGMA* **4** 077 (14 pp)

[6] Narita K 1982 Soliton solutions to extended Volterra equation *J. Phys. Soc. Japan* **51** 1682-5

[7] Svinin A K 2002 Extended discrete KP hierarchy and its reductions from a geometric viewpoint *Lett. Math. Phys.* **61** 231-39

[8] Svinin A K 2004 Invariant submanifolds of the Darboux-Kadomtsev-Petviashvili chain and an extension of the discrete Kadomtsev-Petviashvili hierarchy *Theor. Math.Phys.* **141** 1542-61

[9] Svinin A K 2009 On some class of reductions for the Itoh-Narita-Bogoyavlenskii lattice *J. Phys. A: Math. Theor.* **41** (Art. No. 454021) 15

[10] Svinin A K 2011 On some class of homogeneous polynomials and explicit form of integrable hierarchies of differential-difference equations *J. Phys. A: Math. Theor.* **44** (Art. No. 165206) 16

[11] Yamilov R I 1983 Classification of discrete evolution equations *Uspekhi. Mat. Nauk* **38** 155-6 (in Russian)