Financial Lie groups

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Abstract

In this paper we see the evolution of a capitalized financial event e, with respect to a capitalization factor f, as the exponential map of a suitably defined Lie group $G_{f,e}$, supported by the halfspace of capitalized financial events having the same sign of e. The Lie group $G_{f,e}$ depends on the capitalization factor f and on the event e itself. After the extention of the definition of exponential map of a Lie group, we shall eliminate the dependence on the financial event e, recognizing the precence of essentialy one unique financial Lie semigroup, supported by the entire space of capitalized financial events, determined by the capitalization factor f.

1 Financial preliminaries

Definition (of capitalized financial event). *We call **capitalized financial event** any triple $e = (t, h, c)$ of real numbers. We call **zero financial event** any event of the type $(t, h, 0)$. Moreover we call **credits** (resp. **strict credits**) the financial events with non negative (resp. positive) third component and **debts** (resp. **strict debts**) those with non positive (resp. negative) third component. The event $(t, h, -c)$ is called **the opposite of** $e = (t, h, c)$.*

Financial interpretation. We interpret $e = (t, h, c)$ as a financial object characterized by

1) a *reference time t*;

2) a *capitalization time h*, meaning that the event e exists and is under capitalization (is in the financial market) since the time $t - h$ (we call this time *origin of the capitalized event e*);

3) a *capital c*, considered as the value of the event at the reference time t.

Definition. *If $e = (t, h, c)$ is a capitalized event, we call:*

*1) **reference time of the event** the real number t;*

*2) **capitalization time of** e the real number h;*

*3) **capital (value) of** e (at t) the real c;*

*4) **time origin of** e the time $t - h$;*

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5) **state of the event** e the pair (h, c).

Definition (fibrations of capitalized financial events). *The **state fibration of capitalized events** is the space $\mathbb{R}^3$ endowed with the first canonical projection pr_1. In other terms, we define the state fibration as the trivial fibration $(\mathbb{R}^3, \mathbb{R}, pr_1)$ having as basis the affine space of reference times. This trivial fibration is a vector bundle of type $\mathbb{R}^2$ and, for every time t, the fiber $pr_1^-(t)$ is called **the vector space (fiber) of capitalized financial states**. Moreover we define **capital fibration of capitalized events** the trivial fibration $(\mathbb{R}^3, \mathbb{R}^2, pr_{12})$ having as basis the affine multi-time plane. This trivial fibration is a vector bundle of type $\mathbb{R}$ and, for every pair (t, h), the fiber $pr_{12}^-(t, h)$ is called **the vector space (fiber) of capitals**.*

Definition (of global capitalization factor). *A (global) C^1 **capitalization factor** f is any real function defined over the real line of time displacements enjoing the following properties:*
1) f is positive;
2) f map the zero displacement 0 into 1;
3) $f(-h) = f(h)^{-1}$, for every time displacement h;
4) f is of class C^1.

*The pair $(\mathbb{R}^3, f)$ is called **financial space with capitalization factor f**.*

2 Lie product induced by a capitalization factor

We define the following algebraic product on the space of capitalized financial events.

Definition (Lie product induced by a capitalization factor). *Let f be a capitalization factor over the displacement time line. Let $e = (t, h, c)$ and $e' = (t', h', c')$ be two capitalized financial events, we define their f-**product** ee' to be the capitalized financial event*

$$ee' = [e|e']_f = (t + t', h + h', cf(-h)c'f(-h')f(h + h')).$$

*Analogously, let $s = (h, c)$ and $s' = (h', c')$ be two capitalized financial states, we define their f-**product** ss' to be the capitalized financial state*

$$ss' = [s|s']_f = (h + h', cf(-h)c'f(-h')f(h + h')).$$

*We call the products so defined **Lie products induced by the capitalization factor f**.*

We will use the brief notation ee' (or ss'), instead of the more precise $(ee')_f$ or $[e|e']_f$, when no confusion is possible.

Remark 1. The definition seems to be lacking from the dimensional point of view and for what concerns the addition of two times (it is indeed non proper to add two times) but we must regard the sum $t + t'$ as the sum of the time t with the time duration $h = t' - 0$ (and this is perfectly possible since the time line is a good affine space over the real line of time displacements); moreover we have to consider the product cc' as the product of two capitals divided by 1 monetary unity, in order to obtain a third component of the product with the dimensions of a capital.

3 Lie anti-product induced by a capitalization factor

Definition (Lie anti-product induced by a capitalization factor). *Let f be a capitalization factor over the displacement time line. Let $e = (t, h, c)$ and $e' = (t', h', c')$ be two capitalized financial events, we define their f-**anti-product** $[e|e']_{(f,-)}$ to be the capitalized financial event*

$$[e|e']_{(f,-)} = (t + t', h + h', -cf(-h)c'f(-h')f(h + h')).$$

*Analogously, let $s = (h, c)$ and $s' = (h', c')$ be two capitalized financial states, we define their f-**anti-product** $[s|s']_{(f,-)}$ to be the capitalized financial state*

$$[s|s']_{(f,-)} = (h + h', -cf(-h)c'f(-h')f(h + h')).$$

*We call the products so defined **Lie anti-products induced by the capitalization factor** f.*

Notation. We shall denote the space $\mathbb{R}^3$ endowed by the Lie product induced by the capitalization factor f, that is the structure $(\mathbb{R}^3, [.|.]_f)$, by the symbol $\mathbb{R}_f^3$ and we shall denote the space $\mathbb{R}^3$ endowed by the Lie anti-product induced by the capitalization factor f, that is the structure $(\mathbb{R}^3, [.|.]_{(f,-)})$, by the symbol $\mathbb{R}_{(f,-)}^3$.

4 Financial Lie groups

Theorem 1. *Let f be any C^1 capitalization factor. Then:*

- *the space $\mathbb{R}^3$ of capitalized events is a C^1 Abelian Lie semigroup, with respect to the standard Euclidean differentiable structure on $\mathbb{R}^3$ and to the Lie product $[.|.]_f$ induced by the capitalization factor f;*

- *the orgin of the Lie semigroup $\mathbb{R}_f^3$ is the point $o = (0, 0, 1)$;*

- *the subset of non zero capitalized event is a Lie subgroup of the Lie semi-group $\mathbb{R}^3_f$ and coincides with the subgroup of the invertible elements of the Lie semigroup $\mathbb{R}^3_f$;*

- *denoted the above subgroup, when endowed with the product $[.|.]_f$, by G_f; the Lie group G_f has exactly two connected homeomorphic components, one (the half space of credits C) contains the origin and is a subgroup of G_f and the other one (the half space of debts D) contains the opposite event $(0, 0, -1)$ of the origin of the Lie semigroup $\mathbb{R}^3_f$ and is a group with respect to the anti-product induced by f;*

- *denoted by $G_f^>$ and $G_f^<$ the above groups of credits and debts, respectively (pay attention, only $G_f^>$ is a subgroup of G_f) the mapping sending (t, h, c) into $(t, h, -c)$ is a Lie groups isomorphism of $G_f^>$ onto $G_f^<$.*

Proof. We prove first that the product $[.|.]_f$ is a semigroup product.

1) *Associativity.* Let e, e' and e'' be the events (t, h, c), (t', h', c') and (t'', h'', c''), respectively. Then we obtain (using the brief notation for the Lie product $[.|.]_f$)

$$
\begin{aligned}
(ee')e'' &= (t + t', h + h', cf(-h)c'f(-h')f(h + h'))e'' = \\
&= \left(t + t' + t'', h + h' + h'', \frac{c}{f(h)} \frac{c'}{f(h')} \frac{f(h + h')}{f(h + h')} \frac{c''}{f(h'')} f(h + h' + h'') \right) = \\
&= \left(t + t' + t'', h + h' + h'', \frac{c}{f(h)} \frac{c'}{f(h')} \frac{c''}{f(h'')} \frac{f(h' + h'')}{f(h' + h'')} f(h + h' + h'') \right) = \\
&= e(t' + t'', h' + h'', c'f(-h')c''f(-h'')f(h' + h'')) = \\
&= e(e'e''),
\end{aligned}
$$

as we desired. Note, by the way, that

$$
ee'e'' = \left(t + t' + t'', h + h' + h'', \frac{cc'c''f(h + h' + h'')}{f(h)f(h')f(h'')} \right).
$$

2) *Existence of neutral element.* The neutral element of the magma $\mathbb{R}^3_f$ is (obviously) the capitalized event $(0, 0, 1)$.

3) *Commutativity.* The product $[.|.]_f$ is evidently commutative.

4) *Continuous differentiability of the Lie product.* The Lie product induced by f is C^1 because f is C^1. By the way, note that the Lie product $[.|.]_f$, defined by

$$
[e|e']_f = (t + t', h + h', cf(-h)c'f(-h')f(h + h')),
$$

is differentiable since each of his component is differentiable with respect to each argument; for example, we have

$$
\partial_2([.|.]_f)(e, e') = (0, 1, -cf'(-h)c'f(-h')f'(h + h')),
$$

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and
$$\partial_3([.|.]_f)(e, e') = (0, 0, f(-h)c'f(-h')f(h + h')).$$

5) *Continuous differentiability of the Lie inverse map.* The invertible events are all those events e with non zero capital; indeed, the inverse of a non zero event (t, h, c) is the capitalized event $(-t, -h, 1/c)$. Indeed,

$$(t, h, c)(-t, -h, 1/c) = (0, 0, cf(-h)c^{-1}f(h)f(h - h)) = (0, 0, 1),$$

note that the inverse e_f^{-1} is independent of the capitalization factor f. It is simple to prove that if e is f-invertible then $pr_3(e)$ is different from 0 (otherwise $0 = 1$). The inverse map

$$[.]^{-1} : \mathbb{R}^3 \backslash N \to \mathbb{R}^3 \backslash N : e \mapsto e^{-1},$$

where N is the set of zero events (additive kernel of the third projection), is a group isomorphism (independent of the capitalization factor f) and its derivatives with respect to the arguments are

$$\partial_1([.]^{-1})(e) = (-1, 0, 0), \ \ \partial_2([.]^{-1})(e) = (0, -1, 0), \ \ \partial_3([.]^{-1})(e) = (0, 0, -1/c^2).$$

6) The two connected components of G_f are the sets of strict credits and the set of stricts debts, indeed this two sets are obviously two subgroups of G and they are isomorphic by means of the opposite mapping

$$(t, h, c) \mapsto (t, h, -c).$$

The theorem is proved. ∎

We have the anti-version of the preceding theorem.

Theorem 2. *Let f be any C^1 capitalization factor. Then:*

- *the space $\mathbb{R}^3$ of capitalized events is a C^1 Abelian Lie semigroup, with respect to the standard Euclidean differentiable structure on $\mathbb{R}^3$ and to the Lie anti-product $[.|.]_{(f,-)}$ induced by the capitalization factor f;*

- *the orgin of the Lie semigroup $\mathbb{R}^3_{f,-}$ is the point $-o = (0, 0, -1)$;*

- *the subset of non zero capitalized event is a Lie subgroup of the Lie semigroup $\mathbb{R}^3_{f,-}$ and coincides with the subgroup of the invertible elements of the Lie semigroup $\mathbb{R}^3_{f,-}$;*

- *denoted the above subgroup, when endowed with the product $[.|.]_{f,-}$, by $G_{f,-}$; the Lie group $G_{f,-}$ has exactly two connected homeomorphic components, one (the half space of debts D) contains the origin $-o$ and is a subgroup of $G_{f,-}$ and the other one (the half space of credits C) contains the event $o = (0, 0, 1)$, origin of the Lie semigroup $\mathbb{R}^3_f$, and it is a group with respect to the product induced by f;*

- *denoted by $G_f^>$ and $G_f^\leq$ the above groups of credits and debts, respectively (pay attention, only $G_f^\leq$ is a subgroup of $G_{f,-}$) the mapping sending (t, h, c) into $(t, h, -c)$ is a Lie groups isomorphism of $G_f^\leq$ onto $G_f^>$.*

5   The evolution of the unit event

Consider now the evolution of an event $e_0 = (t_0, h_0, c_0)$ with respect to a capitalization factor f, that is (by definition) the curve

$$\mu_{e_0} : \mathbb{R} \to \mathbb{R}^3 : t \mapsto (t, h + t - t_0, c_0 f(-h_0) f(h_0 + t - t_0)).$$

In particular, if the event e_0 is the unit event $o = (0, 0, 1)$, we have simply

$$\mu_{e_0} : \mathbb{R} \to \mathbb{R}^3 : t \mapsto (t, t, f(t)).$$

Let us see the first resul of the paper on financial dynamical systems.

Theorem 3. *The evolution of the origin o of the semigroup $\mathbb{R}_f^3$ is the exponential map of $\mathbb{R}_f^3$ with respect to the tangent vector $(o, (1, 1, f'(0)))$, a tangent vector to the Lie semigroup $\mathbb{R}_f^3$ at the origin o itself.*

Proof. Note that μ_o is a one parameter group in the Lie group $G_f^>$ of credits (events with positive capital) with respect to the Lie semigroup operation induced by the capitalization factor f. Indeed, we have

$$\mu_o(t + t') = (t + t', t + t', f(t + t')),$$

for every pair (t, t') of times, and

$$
\begin{aligned}
[\mu_o(t)|\mu_o(t')]_f &= (t, t, f(t))(t', t', f(t')) = \\
&= (t + t', t + t', f(t)f(-t)f(t')f(-t')f(t + t')) = \\
&= (t + t', t + t', f(t + t')),
\end{aligned}
$$

again, for every t and t' over the time line. Moreover, the tangent vector at t of the curve μ_o is

$$\mu_o'(t) = (1, 1, f'(t)),$$

so that we have

$$\mu_o'(0) = (1, 1, f'(0)) = (1, 1, \delta_f(0)),$$

where

$$\delta_f : \mathbb{R} \to \mathbb{R} : h \mapsto f'(h)/f(h),$$

is the so called force of interest of the capitalization factor f. Now, as it is well known in Lie Group Theory, there is only one 1-parameter group in a Lie group G having a fixed tangent vector $v \in T_o(G)$ as a tangent vector at 0, and this is the exponential map

$$\exp_v : \mathbb{R} \to G,$$

so we have that

$$\mu_o = \exp_{(1,1,\delta_f(0))},$$

as we claimed. ∎

6 Lie product centered at an event

Now we desire also to see the evolution of any capitalized event as an exponential map.

We define, at this aim, a new product induced by a capitalization f and an event e_0.

Definition (Lie product induced by a capitalization factor and centered at a point). *Let f be a capitalization factor over the real time line and let e_0 be any capitalized event (t_0, h_0, c_0). Let $e = (t, h, c)$ and $e' = (t', h', c')$ two capitalized financial events, we define their (f, e_0)-**Lie product** $[e|e']_{e_0}$ to be the capitalized financial event*

$$[e|e']_{(f,e_0)} = \left(t + (t' - t_0), h + h' - h_0, \frac{c}{f(h)} \frac{c'}{f(h')} \frac{c_0^{-1}}{f(-h_0)} f(h + h' - h_0) \right).$$

We will prove that the above product is indeed a Lie product, that is the following theorem. We could follow the above proof, but we want to follow a totally new and more interesting way, using also the above result.

We need first the concept of translation of a Lie group structure.

7 Translation of a Lie semigroup structure

Theorem 4. *Let G be a commutative Lie semigroup and let e_0 one of its invertible elements, consider the product $[.|.]_{e_0}$ on the supporting set of the semigroup G defined by*

$$[e|e']_{e_0} = ee'e_0^{-1},$$

for every e, e' in G. Then, this new product is a Lie product. Moreover, the neutral element of this new product is the element e_0 and, denoted by G_{e_0} the new Lie semigroup, an element e is invertible in G_{e_0} if and only if it is invertible in the original G and the inverse of an invertible element e in G_{e_0} is the element $e^{-1}e_0^2$, where e^{-1} is the inverse of e in G.

Proof. It is clear that the new operation is also associative and commutative. Indeed, for instance the associativity is given by

$$
\begin{aligned}
[[e|e']_{e_0}|e'']_{e_0} &= [e|e']_{e_0}e''e_0^{-1} = \\
&= (ee'e_0^{-1})e''e_0^{-1} = \\
&= e(e'e_0^{-1}e'')e_0^{-1} = \\
&= [e|[e'|e'']_{e_0}]_{e_0},
\end{aligned}
$$

for every e, e', e'' in G. For what concerns the invertibility, we have

$$
[e|e^{-1}e_0^2]_{e_0} = ee^{-1}e_0^2e_0^{-1} = e_0,
$$

as we claimed. ∎

The new Lie semigroup G_{e_0} is called the translation of G by e_0 and the new product the translation of the product of G by e_0.

8   Financial group translation

Well, we have exactly the following result.

Theorem 5. *The financial product* $[.|.]_{(f,e_o)}$ *is the translation by the event* e_0 *of the financial product* $[.|.]_f$, *that is we have*

$$
[.|.]_{(f,e_o)} = \tau_{e_0}[.|.]_f.
$$

Proof. Indeed, just recalling that

$$
ee'e'' = \left(t + t' + t'', h + h' + h'', \frac{cc'c''f(h+h'+h'')}{f(h)f(h')f(h'')} \right),
$$

we have

$$
\begin{aligned}
\tau_{e_0}[.|.]_f(e, e') &= [e|e']_f e_0^{-1} = \\
&= (ee')(-t_0, -h_0, c_0^{-1}) = \\
&= \left(t + t' - t_0, h + h' - h_0, \frac{cc'c_0^{-1}f(h+h'-h_0)}{f(h)f(h')f(-h_0)} \right) = \\
&= [e|e']_{(f,e_o)},
\end{aligned}
$$

as we claimed. ∎

From which it immediately follows the claimed result about the product $[.|.]_{(f,e_o)}$.

Theorem 6. *The space* $\mathbb{R}^3$ *of capitalized events is a* C^1 *Lie semigroup with respect to the standard Euclidean differentiable structure on* $\mathbb{R}^3$ *and to the centered product* $[.|.]_{(f,e_o)}$, *for every* C^1 *capitalization factor* f *and any event* e_0. *Moreover, the neutral element of this product is the element* e_0.

9 The evolution of an event

Consider again the evolution of an event e_0, that is the curve

$$\mu_{e_0} : \mathbb{R} \to \mathbb{R}^3 : \mu_{e_0}(t) = (t, h_0 + t - t_0, c_0 f(-h_0) f(h_0 + t - t_0)).$$

Theorem 6. *Let* o *be the unit event* $(0,0,1)$ *and let* e_0 *be any other event and let* μ_{e_0} *be its evolution. Then the evolution of* e_0 *is the double translation of the evolution of the origin* o *with respect to the event* e_0 *itself and to the instant of time* t_0, *that is we have*

$$\mu_{e_0} = \tau_{e_0} \circ \tau_{t_0}(\mu_o).$$

Proof. We have

$$
\begin{aligned}
\mu_o(t - t_0)e_0 &= (t - t_0, t - t_0, f(t - t_0))(t_0, h_0, c_0) = \\
&= (t, h_0 + t - t_0, f(t - t_0)f(t_0 - t)c_0 f(-h_0)f(t_0 - t + h_0)) = \\
&= \mu_{e_0}(t),
\end{aligned}
$$

as we claimed. ∎

Now we have the following result.

Theorem 8. *Consider the translation of the usual addition on the real line by a time* t_0 *and denote it by* $+_{t_0}$. *Then, the evolution* μ_{e_0} *of the event* e_0 *is just an homomorphism of the group* $(\mathbb{R}, +_{t_0})$ *into the semigroup* $\mathbb{R}^3_{e_0}$.

First proof. Indeed, we have

$$
\begin{aligned}
[\mu_{e_0}(t)|\mu_{e_0}(t')]_{(f,e_o)} &= \tau_{e_0}[\mu_u(t - t_0)e_0|\mu_u(t' - t_0)e_0]_f = \\
&= [\mu_u(t - t_0)e_0|\mu_u(t' - t_0)e_0]_f e_0^{-1} = \\
&= [\mu_u(t - t_0)|\mu_u(t' - t_0)]_f e_0 = \\
&= \mu_u(t - t_0 + t' - t_0)e_0 = \\
&= \mu_{e_0}(t - t_0 + t'),
\end{aligned}
$$

as we claimed. ■

Second direct proof (more complicated). Indeed, we have

$$\mu_{e_0}(t + t' - t_0) = \left(t + t' - t_0, h_0 + t + t' - 2t_0, c_0 \frac{f(h_0 + t + t' - 2t_0)}{f(h_0)}\right),$$

and consider the capital evolution

$$M : \mathbb{R} \to \mathbb{R} : M(t) = c_0 \frac{f(h_0 + t - t_0)}{f(h_0)},$$

we so have, setting $h := t - t_0$ and $h' := t' - t_0$, that

$$
\begin{aligned}
[\mu_{e_0}(t) | \mu_{e_0}(t')]_{e_0} &= [(t, h_0 + h, M(t)) | (t', h_0 + h', M(t'))]_{(f, e_0)} = \\
&= \left(t + t' - t_0, h_0 + h + h', M(t)M(t') \frac{f(h_0 + h + h')f(h_0)}{c_0 f(h_0 + h)f(h_0 + h')}\right) = \\
&= (t + t' - t_0, h_0 + t + t' - 2t_0, c_0 f(-h_0)f(h_0 + t + t' - 2t_0)),
\end{aligned}
$$

for every t and t' on the time line. ■

Theorem 9. *The evolution of the origin e_0 of the Lie semigroup $\mathbb{R}_{e_0}^3$ is the exponential map of the translated Lie group $\mathbb{R}_{t_0}$ into $\mathbb{R}_{e_0}^3$, with respect to the tangent vector $(e_0, (1, 1, c_0 \delta(h_0)))$, tangent vector to the Lie semigroup $\mathbb{R}_{e_0}^3$ at its own origin e_0.*

Proof. The tangent vector at t of μ_{e_0} is

$$\mu'_{e_0}(t) = (1, 1, c_0 f(-h_0)f'(h_0 + t - t_0)),$$

so that we have

$$\mu'_e(t_0) = (1, 1, c_0 f'(h_0)/f(h_0)) = (1, 1, c_0 \delta_f(h_0)),$$

where

$$\delta_f : \mathbb{R} \to \mathbb{R} : h \mapsto f'(h)/f(h),$$

is the force of interest of the capitalization factor f. Now, there is only one 1-parameter group in $\mathbb{R}_{e_0}^3$ having v as a tangent vector at the origin t_0 of the group $(\mathbb{R}, +_{t_0})$, and it is the exponential map

$$\exp_v : \mathbb{R}_{t_0} \to \mathbb{R}_{e_0}^3,$$

so we have

$$\mu_{e_0} = \exp_{(1, 1, c_0 \delta_f(h_0))},$$

as we claimed. ■

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10 General exponential map on a commutative Lie group

Let G be a Lie group we know that for every tangent vector v at the origin (that is for every element v of its associated Lie algebra) there is a unique homomorphism $\mu : \mathbb{R} \to G$ such that

$$d_0\mu(h) = hv,$$

for every h in $\mathbb{R}$.

Now, for every pair (t_0, e_0) in the product $\mathbb{R} \times G$, consider the double translation

$$\mu_{(t_0,e_0)} : t \mapsto \mu(t - t_0)e_0,$$

it is clear that $\mu_{(t_0,e_0)}$ is a homomorphism from the translated group $\mathbb{R}_{t_0}$ into the translated G_{e_0}. Moreover, we have

$$d_{t_0}\mu_{(t_0,e_0)}(h) = d_0\mu(h)$$

Definition (the general exponential). *We call exponential map of the Lie subgroup G at (t_0, e_0) relative to the tangent vector v in $T_{e_0}(G)$ the unique homomorphism μ from translated group $\mathbb{R}_{t_0}$ into the translated G_{e_0} such that*

$$d_{t_0}\mu(h) = hv,$$

for every h in $\mathbb{R}$ (note that the application $d_{t_0}\mu$ goes from $T_{t_0}(\mathbb{R})$ into $T_{e_0}(G_{e_0})$ since $\mu(t_0)$ is e_0). We denote this μ by

$$\exp_{(t_0,e_0),v}.$$

Remark. Note that the tangent space $T_{e_0}(G_{e_0})$ is the tangent space $T_{e_0}(G)$, indeed a derivation v in $T_{e_0}(G_{e_0})$ is a functional defined on $C^1_{e_0}(G, \mathbb{R})$ and so it is a derivation in $T_{e_0}(G)$.

With the above definition, we can say that

- *the evolution of a capitalized event e_0 is the exponential of the Lie semigroup $\mathbb{R}^3_f$ at the pair (t_0, e_0) with respect to the tangent vector $(e_0, (1, 1, c_0\delta_f(h_0)))$.*

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